Gerd Baumann

Mathematics for Engineers II

Calculus and Linear Algebra

With CD-ROM

Oldenbourg

Mathematics for Engineers II

Calculus and Linear Algebra

by
Gerd Baumann

Oldenbourg Verlag München

Prof. Dr. Gerd Baumann is head of Mathematics Department at German University Cairo (GUC). Before he was Professor at the Department of Mathematical Physics at the University of Ulm.

Rosenheimer Straße 145, D-81671 München
Telefon: (089) 45051-0
oldenbourg.de

Editor: Kathrin Mönch
Producer: Anna Grosser
Cover design: Kochan & Partner, München
Printed on acid-free and chlorine-free paper
Printing: Druckhaus „Thomas Müntzer“ GmbH, Bad Langensalza

ISBN 978-3-486-59040-1

Preface

Theory without Practice is empty,
Practice without Theory is blind.

The current text *Mathematics for Engineers* is a collection of four volumes covering the first three up to the fifth terms in undergraduate education. The text is mainly written for engineers but might be useful for students of applied mathematics and mathematical physics, too.

Students and lecturers will find more material in the volumes than a traditional lecture will be able to cover. The organization of each of the volumes is done in a systematic way so that students will find an approach to mathematics. Lecturers will select their own material for their needs and purposes to conduct their lecture to students.

For students the volumes are helpful for their studies at home and for their preparation for exams. In addition the books may be also useful for private study and continuing education in mathematics. The large number of examples, applications, and comments should help the students to strengthen their knowledge.

The volumes are organized as follows: Volume I treats basic calculus with differential and integral calculus of single valued functions. We use a systematic approach following a bottom-up strategy to introduce the different terms needed. Volume II covers series and sequences and first order differential equations as a calculus part. The second part of the volume is related to linear algebra. Volume III treats vector calculus and differential equations of higher order. In Volume IV we use the material of the previous volumes in numerical applications; it is related to numerical methods and practical calculations. Each of the volumes is accompanied by a CD containing the *Mathematica* notebooks of the book.

As prerequisites we assume that students had the basic high school education in algebra and geometry. However, the presentation of the material starts with the very elementary subjects like numbers and introduces in a systematic way step by step the concepts for functions. This allows us to repeat most of the material known from high school in a systematic way, and in a broader frame. This way the reader will be able to use and categorize his knowledge and extend his old frame work to a new one.

The numerous examples from engineering and science stress on the applications in engineering. The idea behind the text concept is summarized in a three step process:

Theory → Examples → Applications

When examples are discussed in connection with the theory then it turns out that the theory is not only valid for this specific example but useful for a broader application. In fact, usually theorems or a collection of theorems can even handle whole classes of problems. These classes are sometimes completely separated from this introductory example; e.g. the calculation of areas to motivate integration or the calculation of the power of an engine, the maximal height of a satellite in space, the moment of inertia of a wheel, or the probability of failure of an electronic component. All these problems are solvable by one and the same method, integration.

However, the three-step process is not a feature which is always used. Some times we have to introduce mathematical terms which are used later on to extend our mathematical frame. This means that the text is not organized in a historic sequence of facts as traditional mathematics texts. We introduce definitions, theorems, and corollaries in a way which is useful to create progress in the understanding of relations. This way of organizing the material allows us to use the complete set of volumes as a reference book for further studies.

The present text uses *Mathematica* as a tool to discuss and to solve examples from mathematics. The intention of this book is to demonstrate the usefulness of *Mathematica* in everyday applications and calculations. We will not give a complete description of its syntax but demonstrate by examples the use of its language. In particular, we show how this modern tool is used to solve classical problems and to represent mathematical terms.

We hope that we have created a coherent way of a first approach to mathematics for engineers.

Acknowledgments Since the first version of this text, many students made valuable suggestions. Because the number of responses are numerous, I give my thanks to all who contributed by remarks and enhancements to the text. Concerning the historical pictures used in the text, I acknowledge the support of the http://www-gapdcs.st-and.ac.uk/~history/ webserver of the University of St Andrews, Scotland. The author deeply appreciates the understanding and support of his wife, Carin, and daughter, Andrea, during the preparation of the books.

Cairo

Gerd Baumann

To Carin and Andrea

Contents

1 Outline

1.1. Introduction

During the years in circles of engineering students the opinion grew that calculus and higher mathematics is a simple collection of recipes to solve standard problems in engineering. Also students believed that a lecturer is responsible to convey these recipes to them in a nice and smooth way so that they can use it as it is done in cooking books. This approach of thinking has the common short coming that with such kind of approach only standard problems are solvable of the same type which do not occur in real applications.

We believe that calculus for engineers offers a great wealth of concepts and methods which are useful in modelling engineering problems. The reader should be aware that this collection of definitions, theorems, and corollaries is not the final answer of mathematics but a first approach to organize knowledge in a systematic way. The idea is to organize methods and knowledge in a systematic way. This text was compiled with the emphasis on understanding concepts. We think that nearly everybody agrees that this should be the primary goal of calculus instruction.

This first course of Engineering Mathematics will start with the basic foundation of mathematics. The basis are numbers, relations, functions, and properties of functions. This first chapter will give you tools to attack simple engineering problems in different fields. As an engineer you first have to understand the problem you are going to tackle and after that you will apply mathematical tools to solve the problem. These two steps are common to any kind of problem solving in engineering as well as in science. To understand a problem in engineering needs to be able to use and apply engineering knowledge and engineering procedures. To solve the related mathematical problem needs the knowledge of the basic steps how mathematics is working. Mathematics gives you the frame to handle a problem in a systematic way and use the procedure and knowledge mathematical methods to derive a solution. Since mathematics sets up the frame for the solution of a problem you should be able to use it efficiently. It is not to apply recipes to solve a problem but to use the appropriate concepts to solve it.

Mathematics by itself is for engineers a tool. As for all other engineering applications working with tools you must know how they act and react in applications. The same is true for mathematics. If you know how a mathematical procedure (tool) works and how the components of this tool are connected by each other you will understand its application. Mathematical tools consist as engineering tools of components. Each component is usually divisible into other components until the basic component (elements) are found. The same idea is used in mathematics. There are basic elements from mathematics you should know as an engineer. Combining these basic elements we are able to set up a mathematical frame which incorporates all those elements which are needed to solve a problem. In other words, we use always basic ideas to derive advanced structures. All mathematical thinking follows a simple track which tries to apply fundamental ideas used to handle more complicated situations. If you remember this simple concept you will be able to understand advanced concepts in mathematics as well as in engineering.

1.2. Concept of the Text

Concepts and conclusions are collected in definitions and theorems. The theorems are applied in examples to demonstrate their meaning. Every concept in the text is illustrated by examples, and we included more than 1,000 tested exercises for assignments, class work and home work ranging from elementary applications of methods and algorithms to generalizations and extensions of the theory. In addition, we included many applied problems from diverse areas of engineering. The applications chosen demonstrate concisely how basic calculus mathematics can be, and often must be, applied in real life situations.

During the last 25 years a number of symbolic software packages have been developed to provide symbolic mathematical computations on a computer. The standard packages widely used in academic applications are *Mathematica*®, Maple® and Derive®. The last one is a package which is used for basic calculus while the two other programs are able to handle high sophisticated calculations. Both *Mathematica* and Maple have almost the same mathematical functionality and are very useful in symbolic calculations. However the author's preference is *Mathematica* because the experience over the last 25 years showed that *Mathematica*'s concepts are more stable than Maple's one. The author used both of the programs and it turned out during the years that programs written in *Mathematica* some years ago still work with the latest version of *Mathematica* but not with Maple. Therefore the book and its calculations are based on a package which is sustainable for the future.

Having a symbolic computer algebra program available can be very useful in the study of techniques used in calculus. The results in most of our examples and exercises have been generated using problems for which exact values can be determined, since this permits the performance of the calculus method to be monitored. Exact solutions can often be obtained quite easily using symbolic computation. In addition, for many techniques the analysis of a problem requires a high amount of laborious steps, which can be a tedious task and one that is not particularly instructive once the techniques of calculus have been mastered. Derivatives and integrals can be quickly obtained symbolically with computer algebra systems, and a little insight often permits a symbolic computation to aid in understanding the process as well.

We have chosen *Mathematica* as our standard package because of its wide distribution and reliability. Examples and exercises have been added whenever we felt that a computer algebra system would be of significant benefit.

1.3. Organization of the Text

The book is organized in chapters which continues to cover the basics of calculus. We examine first series and their basic properties. We use series as a basis for discussing infinite series and the related convergence tests. Sequences are introduced and the relation and conditions for their convergence is examined. This first part of the book completes the calculus elements of the first volume. The next chapter discusses applications of calculus in the field of differential equations. The simplest kind of differential equations are examined and tools for their solution are introduced. Different symbolic solution methods for first order differential equations are discussed and applied. The second part of this volume deals with linear algebra. In this part we discus the basic elements of linear algebra such as vectors and matrices. Operations on these elements are introduced and the properties of a vector space are examined. The main subject of linear algebra is to deal with solutions of linear systems of equations. Strategies for solving linear systems of equations are discussed and applied to engineering applications. After each section there will be a test and exercise subsection divided into two parts. The first part consists of a few test questions which examines the main topics of the previous section. The second part contains exercises related to applications and advanced problems of the material discussed in the previous section. The level of the exercises ranges from simple to advanced.

The whole material is organized in four chapters where the first of this chapter is the current introduction. In Chapter 2 we deal with series and sequences and their convergence. The series and sequences are discussed for the finite and infinite case. In Chapter 3 we examine first order ordinary differential equations. Different solution approaches and classifications of differential equations are discussed. In Chapter 4 we switch to linear algebra. The subsections of this chapter cover vectors, matrices, vector spaces, linear systems of equations, and linear transformations.

1.4. Presentation of the Material

Throughout the book we will use the traditional presentation of mathematical terms using symbols, formulas, definitions, theorems, etc. to set up the working frame. This representation is the classical mathematical part. In addition to these traditional presentation tools we will use *Mathematica* as a symbolic, numeric, and graphic tool. *Mathematica* is a computer algebra system (CAS) allowing hybrid calculations. This means calculations on a computer are either symbolic or/and numeric. *Mathematica* is a tool allowing us in addition to write programs and do automatic calculations. Before you use such kind of tool it is important to understand the mathematical concepts which are used by *Mathematica* to derive symbolic or numeric results. The use of *Mathematica* allows you to minimize the calculations but you should be aware that you will only understand the concept if you do your own calculations by pencil and paper. Once you have understood the way how to avoid errors in calculations and concepts you are ready to use the symbolic calculations offered by *Mathematica*. It is important for your understanding that you make errors and derive an improved understanding from these errors. You will never reach a higher level of understanding if you apply the functionality of *Mathematica* as a black box solver of your problems. Therefore I recommend to you first try to understand by using pencil and paper calculations and then switch to the computer algebra system if you have understood the concepts.

You can get a test version of *Mathematica* directly from Wolfram Research by requesting a download address from where you can download the trial version of *Mathematica*. The corresponding web address to get *Mathematica* for free is:

http://www.wolfram.com/products/mathematica/experience/request.cgi

2

Power Series

2.1 Introduction

In this chapter we will be concerned with infinite series, which are sums that involve infinitely many terms. Infinite series play a fundamental role in both mathematics and engineering — they are used, for example, to approximate trigonometric functions and logarithms, to solve differential equations, to evaluate difficult integrals, to create new functions, and to construct mathematical models of physical laws. Since it is important to add up infinitely many numbers directly, one goal will be to define exactly what we mean by the sum of infinite series. However, unlike finite sums, it turns out that not all infinite series actually have a finite value or in short a sum, so we will need to develop tools for determining which infinite series have sums and which do not. Once the basic ideas have been developed we will begin to apply our work.

2.2 Approximations

In Vol. I Section 3.5 we used a tangent line to the graph of a function to obtain a linear approximation to the function near the point of tangency. In the current section we will see how to improve such local approximations by using polynomials. We conclude the section by obtaining a bound on the error in these approximations.

2.2.1 Local Quadratic Approximation

Recall that the local linear approximation of a function f at x_0 is

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0). \tag{2.1}$$

In this formula, the approximating function

$$p(x) = f(x_0) + f'(x_0)(x - x_0) \tag{2.2}$$

is a first-degree polynomial satisfying $p(x_0) = f(x_0)$ and $p'(x_0) = f'(x_0)$. Thus the local linear approximation of f at x_0 has the property that its value and the values of its first derivatives match those of f at x_0. This kind of approach will lead us later on to interpolation of data points (see Vol. IV)

If the graph of a function f has a pronounced bend at x_0, then we can expect that the accuracy of the local linear approximation of f at x_0 will decrease rapidly as we progress away from x_0 (Figure 2.1)

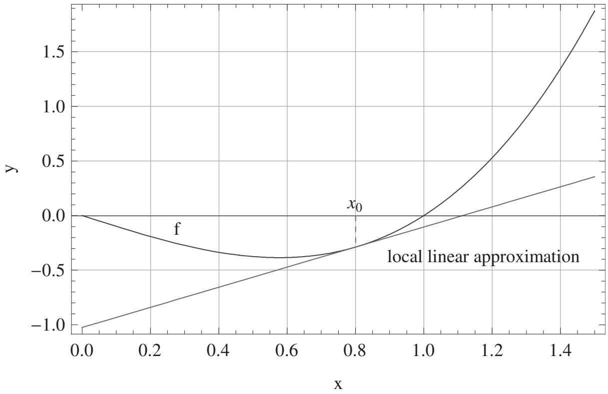


Figure 2.1. Graph of the function $f(x) = x^3 - x$ and its linear approximation at x_0.▲

One way to deal with this problem is to approximate the function f at x_0 by a polynomial of degree 2 with the property that the value of p and the values of its first two derivatives match those of f at x_0. This ensures that the graphs of f and p not only have the same tangent line at x_0, but they also bend in the same direction at x_0. As a result, we can expect that the graph of p will remain close to the graph of f over a larger interval around x_0 than the graph of the local linear approximation. Such a polynomial p is called the local quadratic approximation of f at $x = x_0$.

To illustrate this idea, let us try to find a formula for the local quadratic approximation of a function f at $x = 0$. This approximation has the form

$$f(x) \approx c_0 + c_1 x + c_2 x^2 \tag{2.3}$$

which reads in *Mathematica* as

$$\mathbf{eq11 = f(x) == c2\, x^2 + c1\, x + c0}$$

$$f(x) == \mathrm{c2}\, x^2 + \mathrm{c1}\, x + \mathrm{c0}$$

where c_0, c_1, and c_2 must be chosen such that the values of

$$p(x) = c_0 + c_1\,x + c_2\,x^2 \tag{2.4}$$

and its first two derivatives match those of f at 0. Thus, we want

$$p(0) = f(0),\;\; p'(0) = f'(0),\;\; p''(0) = f''(0). \tag{2.5}$$

In *Mathematica* notation this reads

eq1 = {p0 == f(0), pp0 == f'(0), ppp0 == f''(0)}; TableForm[eq1]

p0 == 0
pp0 == −1
ppp0 == 0

where p0, pp0, and ppp0 is used to represent the polynomial, its first and second order derivative at $x = 0$. But the values of $p(0)$, $p'(0)$, and $p''(0)$ are as follows:

p(x_) := c0 + c1 x + c2 x^2

$p(x)$

c0 + c1 x + c2 x^2

The determining equation for the term p0 is

eqh1 = p(0) == p0

c0 == p0

The first order derivative allows us to find a relation for the second coefficient

pd = $\dfrac{\partial\, p(x)}{\partial x}$

c1 + 2 c2 x

This relation is valid for $x_0 = 0$ so we replace x by 0 in *Mathematica* notation this is /. x −> 0

eqh2 = (pd /. $x \to 0$) == pp0

c1 == pp0

The second order derivative in addition determines the higher order coefficient

pdd = $\dfrac{\partial^2 p(x)}{\partial x\, \partial x}$

2 c2

eqh3 = (pdd /. $x \to 0$) == ppp0

2 c2 == ppp0

Knowing the relations among the coefficients allows us to eliminate the initial conditions for the p coefficients which results to

sol = Flatten[Solve[Eliminate[Flatten[{eq1, eqh1, eqh2, eqh3}], {p0, pp0, ppp0}], {c0, c1, c2}]]

$$\left\{\text{c0} \to f(0), \text{c1} \to f'(0), \text{c2} \to \frac{f''(0)}{2}\right\}$$

and substituting these in the representation of the approximation of the function yields the following formula for the local quadratic approximation of f at $x = 0$.

eq11 /. sol

$$f(x) = \frac{1}{2} x^2 f''(0) + x f'(0) + f(0)$$

Remark 2.1. Observe that with $x_0 = 0$. Formula (2.1) becomes $f(x) \approx f(0) + f'(0)\, x$ and hence the linear part of the local quadratic approximation if f at 0 is the local linear approximation of f at 0.

Example 2.1. Approximation

Find the local linear and quadratic approximation of e^x at $x = 0$, and graph e^x and the two approximations together.

Solution 2.1. If we let $f(x) = e^x$, then $f'(x) = f''(x) = e^x$; and hence $f(0) = f'(0) = f''(0) = e^0 = 1$

Thus, the local quadratic approximation of e^x at $x = 0$ is

$$e^x \approx 1 + x + \frac{1}{2} x^2 \tag{2.6}$$

and the actual linear approximation (which is the linear part of the quadratic approximation) is

$$e^x \approx 1 + x. \tag{2.7}$$

The graph of e^x and the two approximations are shown in the following Figure 2.2. As expected, the local quadratic approximation is more accurate than the local linear approximation near $x = 0$.

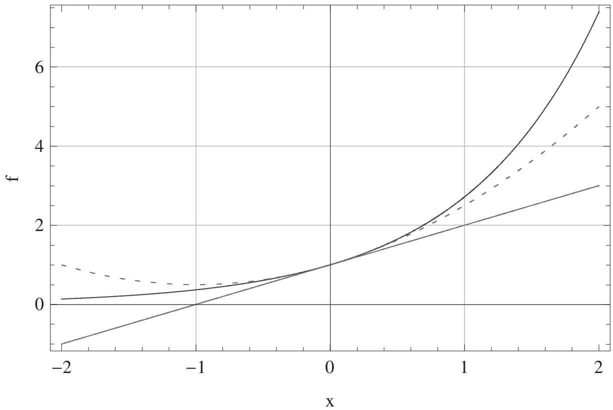


Figure 2.2. Linear and quadratic approximation of the function $f(x) = e^x$.The quadratic approximation is plotted as a dashed line.▲

2.2.2 Maclaurin Polynomial

It is natural to ask whether one can improve on the accuracy of a local quadratic approximation by using a polynomial of order 3. Specifically, one might look for a polynomial of degree 3 with the property that its value and values of its first three derivatives match those of f at a point; and if this provides an improvement in accuracy, why not go on polynomials of even higher degree? Thus, we are led to consider the following general problem.

Given a function f that can be differentiated n times at $x = x_0$, find a polynomial p of degree n with the property that the value of p and the values of its first n derivatives match those of f at x_0.

We will begin by solving this problem in the case where $x_0 = 0$. Thus, we want a polynomial

$$p(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \ldots + c_n x^n \tag{2.8}$$

such that

$$f(0) = p(0), \;\; f'(0) = p'(0), \;\; \ldots, \;\; f^{(n)}(0) = p^{(n)}(0). \tag{2.9}$$

But

$$p(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \ldots + c_n x^n$$

$$p'(x) = c_1 + 2 c_2 x + 3 c_3 x^2 + \ldots + n c_n x^{n-1}$$

$$p''(x) = 2 c_2 + 3 \times 2 c_3 x + \ldots + n (n-1) c_n x^{n-2}$$

$$\vdots$$

$$p^{(n)}(x) = n\,(n-1)\,(n-2)\,\ldots\,(1)\,c_n$$

Thus, to satisfy (2.9), we must have

$$f(0) = p(0) = c_0$$

$$f'(0) = p'(0) = c_1$$

$$f''(0) = p''(0) = 2\,c_2 = 2!\,c_2$$

$$f'''(0) = p'''(0) = 2 \times 3\,c_3 = 3!\,c_3$$

$$\vdots$$

$$f^{(n)}(0) = p^{(n)}(0) = n\,(n-1)\,(n-2)\ldots(1)\,c_n = n!\,c_n$$

which yields the following values for the coefficients of $p(x)$:

$$c_0 = f(0),\ \ c_1 = f'(0),\ \ c_2 = \frac{f''(0)}{2!},\ \ldots,\ \ c_n = \frac{f^{(n)}(0)}{n!}.$$

The polynomial that results by using these coefficients in (2.8) is called the nth Maclaurin polynomial for f.

Definition 2.1. *Maclaurin Polynomial*

If f can be differentiated n times at $x = 0$, then we define the nth Maclaurin polynomial for f to be

$$p_n(x) = f(0) + f'(0)\,x + \frac{f''(0)}{2!}\,x^2 + \ldots + \frac{f^{(n)}(0)}{n!}\,x^n.$$

The polynomial has the property that its value and the values of its first n derivatives match the values of f and its first n derivatives at $x = 0$.∎

Remark 2.2. Observe that $p_1(x)$ is the local linear approximation of f at 0 and $p_2(x)$ is the local quadratic approximation of f at $x_0 = 0$.

Example 2.2. Maclaurin Polynomial

Find the Maclaurin polynomials p_0, p_1, p_2, p_3, and p_n for e^x.

Solution 2.2. For the exponential function, we know that the higher order derivatives are equal to the exponential function. We generate this by the following table

$$\mathbf{Table}\left[\frac{\partial^n e^x}{\partial x^n}, \{n, 1, 8\}\right]$$

$$\{e^x, e^x, e^x, e^x, e^x, e^x, e^x, e^x\}$$

and thus the expansion coefficients of the polynomial defined as $f^{(n)}(0)$ follow by replacing x with 0 (/. $x \to 0$)

$$\text{Table}\left[\frac{\partial^n e^x}{\partial x^n}, \{n, 1, 8\}\right] /. x \to 0$$

{1, 1, 1, 1, 1, 1, 1, 1}

Therefore, the polynomials of order 0 up to 5 are generated by summation which is defined in the following line

$$p(\text{n\_}, \text{x\_}, \text{f\_}) := \text{Fold}\left[\text{Plus}, 0, \text{Table}\left[\frac{x^m \left(\frac{\partial^m f}{\partial x^m} /. x \to 0\right)}{m!}, \{m, 0, n\}\right]\right]$$

The different polynomial approximations are generated in the next line by using this function.

TableForm[Table[$p(n, x, e^x)$, {n, 0, 5}]]

$\{1\}$

$\{x+1\}$

$\left\{\frac{x^2}{2}+x+1\right\}$

$\left\{\frac{x^3}{6}+\frac{x^2}{2}+x+1\right\}$

$\left\{\frac{x^4}{24}+\frac{x^3}{6}+\frac{x^2}{2}+x+1\right\}$

$\left\{\frac{x^5}{120}+\frac{x^4}{24}+\frac{x^3}{6}+\frac{x^2}{2}+x+1\right\}$

Figure 2.3 shows the graph of e^x and the graphs of the first four Maclaurin polynomials. Note that the graphs of $p_1(x)$, $p_2(x)$, and $p_3(x)$ are virtually indistinguishable from the graph of e^x near $x = 0$, so that these polynomials are good approximations of e^x for x near 0.

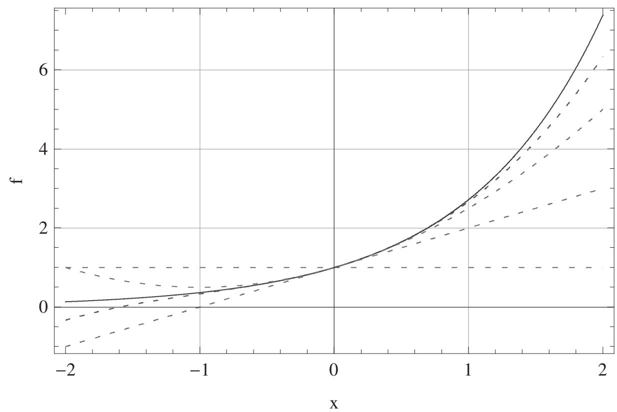


Figure 2.3. Maclaurin approximations of the function $f(x) = e^x$.The approximations are shown by dashed curves.▲

However, the farther x is from 0, the poorer these approximations become. This is typical of the Maclaurin polynomials for a function $f(x)$; they provide good approximations of $f(x)$ near 0, but the accuracy diminishes as x progresses away from 0. However, it is usually the case that the higher the degree of the polynomial, the larger the interval on which it provides a specified accuracy.

2.2.3 Taylor Polynomial

Up to now we have focused on approximating a function f in the vicinity of $x = 0$. Now we will consider the more general case of approximating f in the vicinity of an arbitrary domain value x_0. The basic idea is the same as before; we want to find an nth-degree polynomial p with the property that its first n derivatives match those of f at x_0. However, rather than expressing $p(x)$ in powers of x, it will simplify the computations if we express it in powers of $x - x_0$; that is,

$$p(x) = c_0 + c_1(x - x_0) + c_2(x - x_0)^2 + \ldots + c_n(x - x_0)^n. \tag{2.10}$$

We will leave it as an exercise for you to imitate the computations used in the case where $x_0 = 0$ to show that

$$c_0 = f(x_0),\ \ c_1 = f'(x_0),\ \ c_2 = \frac{f''(x_0)}{2!},\ \ldots,\ c_n = \frac{f^{(n)}(x_0)}{n!}. \tag{2.11}$$

Substituting these values in (2.10), we obtain a polynomial called the nth Taylor polynomial about $x = x_0$ for f.

Definition 2.2. *Taylor Polynomial*

If f can be differentiated n times at x_0, then we define the nth Taylor polynomial for f about $x = x_0$ to be

$$p_n(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \ldots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n. \blacksquare$$

Remark 2.3. Observe that the Maclaurin polynomials are special cases of the Taylor polynomials; that is, the nth-order Maclaurin polynomial is the nth-order Taylor polynomial about $x_0 = 0$. Observe also that $p_1(x)$ is the local linear approximation of f at $x = x_0$ and $p_2(x)$ is the local quadratic approximation of f at $x = x_0$.

Example 2.3. Taylor Polynomial

Find the five Taylor polynomials for $\ln(x)$ about $x = 2$.

Solution 2.3. In *Mathematica* Taylor series are generated with the function Series[]. Series[] uses the formula given in the definition to derive the approximation. The following line generates a table for the first five Taylor polynomials at $x = 2$

TableForm[Table[Normal[Series[ln(*x*), {*x*, 2, *m*}]], {*m*, 0, 4}]]

$\ln(2)$

$\frac{x-2}{2} + \ln(2)$

$-\frac{1}{8}(x-2)^2 + \frac{x-2}{2} + \ln(2)$

$\frac{1}{24}(x-2)^3 - \frac{1}{8}(x-2)^2 + \frac{x-2}{2} + \ln(2)$

$-\frac{1}{64}(x-2)^4 + \frac{1}{24}(x-2)^3 - \frac{1}{8}(x-2)^2 + \frac{x-2}{2} + \ln(2)$

The graph of $\ln(x)$ and its first four Taylor polynomials about $x = 2$ are shown in Figure 2.4. As expected these polynomials produce the best approximations to $\ln(x)$ near 2.

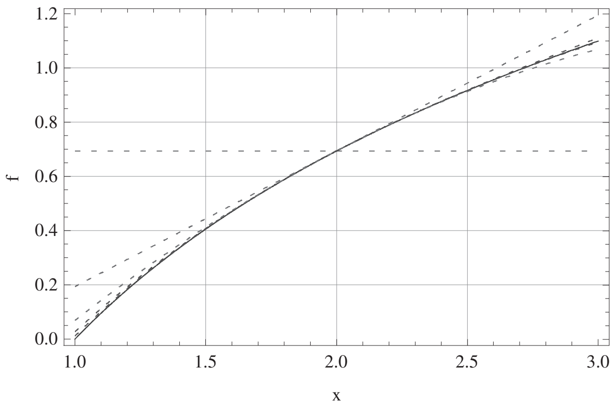


Figure 2.4. Taylor approximations of the function $f(x) = \ln(x)$.The approximations are shown by dashed curves.▲

2.2.4 Σ-Notation

Frequently, we will want to express the sums in the definitions given in sigma notation. To do this, we use the notation $f^{(k)}(x_0)$ to denote the kth derivative of f at $x = x_0$, and we make the convention that $f^{(0)}(x_0)$ denotes $f(x_0)$. This enables us to write

$$\sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \ldots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n. \quad (2.12)$$

In particular, we can write the nth-order Maclaurin polynomial for $f(x)$ as

$$\sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k = f(0) + f'(0)\,x + \frac{f''(0)}{2!}x^2 + \ldots + \frac{f^{(n)}(0)}{n!}x^n. \quad (2.13)$$

Example 2.4. Approximation with Polynomials

Find the nth Maclaurin polynomials for $\sin(x)$, $\cos(x)$ and $1/(1-x)$.

Solution 2.4. We know that Maclaurin polynomials are generated by the function

$$\textbf{MaclaurinPolynomial(n\_, f\_, x\_) := Fold}\left[\textbf{Plus, 0, Table}\left[\frac{x^m \left(\frac{\partial^m f}{\partial x^m} \ /.\ x \to 0\right)}{m\,!}, \{m, 0, n\}\right]\right]$$

A table of Maclaurin polynomials can thus be generated by

TableForm[Table[MaclaurinPolynomial(m, sin(x), x), {m, 0, 5}]]

0

x

x

$x - \frac{x^3}{6}$

$x - \frac{x^3}{6}$

$\frac{x^5}{120} - \frac{x^3}{6} + x$

In the Maclaurin polynomials for $\sin(x)$, only the odd powers of x appear explicitly. Due to the zero terms, each even-order Maclaurin polynomial is the same as the preceding odd-order Maclaurin polynomial. That is

$$p_{2k+1}(x) = p_{2k+2}(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots + (-1)^k \frac{x^{2k+1}}{(2k+1)!} \qquad k = 0, 1, 2, 3, \ldots. \tag{2.14}$$

The graph and the related Maclaurin approximations are shown in Figure 2.5

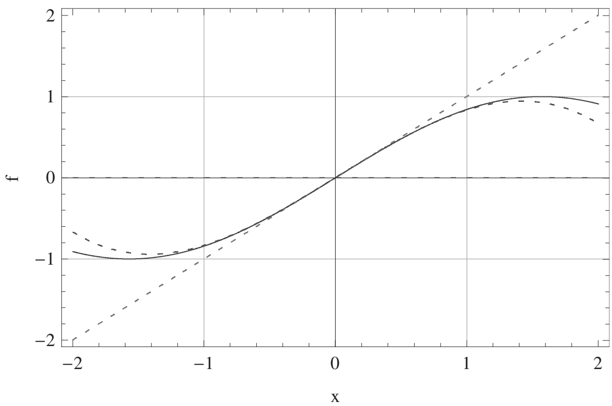


Figure 2.5. Maclaurin approximations of the function $f(x) = \sin(x)$.The approximations are shown by dashed curves.

In the Maclaurin polynomial for $\cos(x)$, only the even powers of x appear explicitly; the computation is similar to those for the $\sin(x)$. The reader should be able to show the following:

TableForm[Table[MaclaurinPolynomial(m, cos(x), x), {m, 0, 5}]]

1
1
$1-\frac{x^2}{2}$
$1-\frac{x^2}{2}$
$\frac{x^4}{24}-\frac{x^2}{2}+1$
$\frac{x^4}{24}-\frac{x^2}{2}+1$

The graph of cos(x) and the approximations are shown in Figure 2.6.

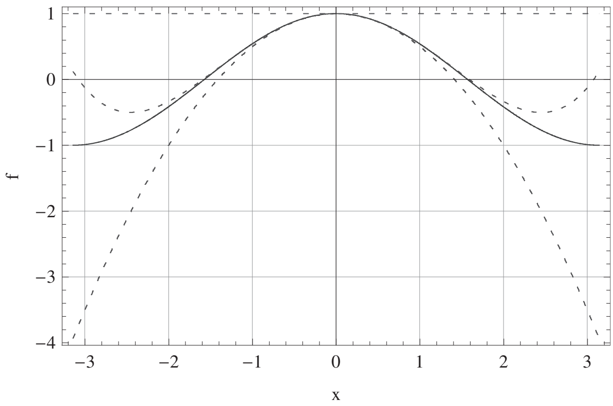


Figure 2.6. Maclaurin approximations of the function $f(x) = \cos(x)$.The approximations are shown by dashed curves.

Let $f(x) = 1/(1-x)$. The values of f and its derivatives at $x = 0$ are collected in the results

TableForm$\left[\right.$Table$\left[\right.$MaclaurinPolynomial$\left(m, \frac{1}{1-x}, x\right)$, {$m$, 0, 5}$\left.\right]\left.\right]$

1
$x+1$
x^2+x+1
x^3+x^2+x+1
$x^4+x^3+x^2+x+1$
$x^5+x^4+x^3+x^2+x+1$

From this sequence the Maclaurin polynomial for $1/(1-x)$ is

$$p_n(x) = \sum_{k=0}^{n} x^k = 1 + x + x^2 + \ldots + x^n, \qquad n = 0, 1, 2, \ldots . \blacktriangle$$

Example 2.5. Approximation with Taylor Polynomials

Find the nth Taylor polynomial for $1/x$ about $x = 1$.

Solution 2.5. Let $f(x) = 1/x$. The computations are similar to those done in the last example. The results are

$$\textbf{Series}\left[\frac{\mathbf{1}}{\boldsymbol{x}}, \{\boldsymbol{x}, \mathbf{1}, \mathbf{5}\}\right]$$

$$1 - (x-1) + (x-1)^2 - (x-1)^3 + (x-1)^4 - (x-1)^5 + O\left((x-1)^6\right)$$

This relation suggest the general formula

$$\sum_{k=0}^{n} (-1)^k (x-1)^k = 1 - (x-1) + (x-1)^2 + \ldots + (-1)^n (x-1)^n . \blacktriangle$$

2.2.5 n^{th}-Remainder

The nth Taylor polynomial p_n for a function f about $x = x_0$ has been introduced as a tool to obtain good approximations to values of $f(x)$ for x near x_0. We now develop a method to forecast how good these approximations will be.

It is convenient to develop a notation for the error in using $p_n(x)$ to approximate $f(x)$, so we define $R_n(x)$ to be the difference between $f(x)$ and its nth Taylor polynomial. That is

$$R_n(x) = f(x) - p_n(x) = f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k. \tag{2.15}$$

This can also be written as

$$f(x) = p_n(x) + R_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k + R_n(x) \tag{2.16}$$

which is called Taylor's formula with remainder.

Finding a bound for $R_n(x)$ gives an indication of the accuracy of the approximation $p_n(x) \approx f(x)$. The following Theorem 2.1 given without proof states

Theorem 2.1. *Remainder Estimation*

If the function f can be differentiated $n+1$ times on an interval I containing the number x_0, and if M is an upper bound for $\left|f^{(n+1)}(x)\right|$ on I, that is $\left|f^{(n+1)}(x)\right| \leq M$ for all x in I, then

$$\left|R_n(x)\right| \leq \frac{M}{(n+1)!}\left|x - x_0\right|^{n+1}$$

for all x in I. ■

Example 2.6. Remainder of an Approximation

Use the nth Maclaurin polynomial for e^x to approximate e to five decimal-place accuracy.

Solution 2.6. We note first that the exponential function e^x has derivatives of all orders for every real number x. The Maclaurin polynomial is

$$\sum_{k=0}^{n} \frac{1}{k!} x^k = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!}$$

from which we have

$$e = e^1 \approx \sum_{k=0}^{n} \frac{1^k}{k!} = 1 + 1 + \frac{1}{2!} + \ldots + \frac{1}{n!}.$$

Thus, our problem is to determine how many terms to include in a Maclaurin polynomial for e^x to achieve five decimal-place accuracy; that is, we want to choose n so that the absolute value of nth remainder at $x = 1$ satisfies

$$|R_n(1)| \leq 0.000005.$$

To determine n we use the Remainder Estimation Theorem 2.1 with $f(x) = e^x$, $x = 1$, $x_0 = 0$, and I being the interval [0,1]. In this case it follows from the Theorem that

$$\left|R_n(1)\right| \leq \frac{M}{(n+1)!} \tag{2.17}$$

where M is an upper bound on the value of $f^{(n+1)}(x) = e^x$ for x in the interval [0, 1]. However, e^x is an increasing function, so its maximum value on the interval [0, 1] occurs at $x = 1$; that is, $e^x \leq e$ on this interval. Thus, we can take $M = e$ in (2.17) to obtain

$$\left|R_n(1)\right| \leq \frac{e}{(n+1)!}. \tag{2.18}$$

Unfortunately, this inequality is not very useful because it involves e which is the very quantity we are trying to approximate. However, if we accept that $e < 3$, then we can replace (2.18) with the following less precise, but more easily applied, inequality:

$$\left|R_n(1)\right| \leq \frac{3}{(n+1)!}. \tag{2.19}$$

Thus, we can achieve five decimal-place accuracy by choosing n such that

$$\frac{3}{(n+1)!} \leq 0.000005 \qquad \text{or} \qquad (n+1)! \geq 600\,000.$$

Since 9! = 362880 and 10!=3628800, the smallest value of n that meets this criterion is n = 9. Thus, by five decimal-place accuracy

$$e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} + \frac{1}{7!} + \frac{1}{8!} + \frac{1}{9!} \approx 2.71828$$

As a check, a calculator's twelve-digits representation of e is $e \approx 2.71828182846$, which agrees with the preceding approximation when rounded to five decimal places.▲

2.2.6 Tests and Exercises

The following two subsections serve to test your understanding of the last section. Work first on the test examples and then try to solve the exercises.

2.2.6.1 Test Problems

T1. Describe the term approximation in mathematical terms.

T2. What kind of approximation polynomials do you know?

T3. How are Maclaurin polynomials defined?

T4. Describe the difference between Maclaurin and Taylor polynomials.

T5. How can we estimate the error in an approximation?

T6. What do we mean by truncation error?

2.2.6.2 Exercises

E1. Find the Maclaurin polynomials up to degree 6 for $f(x) = \cos(x)$. Graph f and these polynomials on a common screen. Evaluate f and these polynomials at $x = \pi/4$, $\pi/2$, and π. Comment on how the Maclaurin polynomials converge to $f(x)$.

E2. Find the Taylor polynomials up to degree 3 for $f(x) = 1/x$. Graph f and these polynomials on a common screen. Evaluate f and these polynomials at $x = 0.8$ and $x = 1.4$. Comment on how the Taylor polynomials converge to $f(x)$.

E3. Find the Taylor polynomial $T_n(x)$ for the function f at the number x_0. Graph f and $T_3(x)$ on the same screen for the following functions:

- **a.** $f(x) = x + e^{-x}$, at $x_0 = 0$,
- **b.** $f(x) = 1/x$, at $x_0 = 2$,
- **c.** $f(x) = e^{-x}\cos(x)$, at $x_0 = 0$,
- **d.** $f(x) = \arcsin(x)$, at $x_0 = 0$,
- **e.** $f(x) = \ln(x)/x$, at $x_0 = 1$,
- **f.** $f(x) = x\, e^{-x^2}$, at $x_0 = 0$,
- **g.** $f(x) = \sin(x)$, at $x_0 = \pi/2$.

E4. Use a computer algebra system to find the Taylor polynomials $T_n(x)$ centered at x_0 for $n = 2, 3, 5, 6$. Then graph these polynomials and on the same screen the following functions:

- **a.** $f(x) = \cot(x)$, at $x_0 = \pi/4$,
- **b.** $f(x) = \sqrt[3]{1 + x^2}$, at $x_0 = 0$,
- **c.** $f(x) = 1/\cosh(x)^2$, at $x_0 = 0$.

Approximate f by a Taylor polynomial with degree n at the number x_0. Use Taylor's Remainder formula to estimate the accuracy of the approximation $f(x) \approx T_n(x)$ when x lies in the given interval. Check your result by graphing $|R_n(x)|$.

a. $f(x) = x^{-2}$, $x_0 = 1$, $n = 2$, $0.9 \leq x \leq 1.1$,

b. $f(x) = \sqrt{x}$, $x_0 = 4$, $n = 2$, $4.1 \leq x \leq 4.3$,

c. $f(x) = \sin(x)$, $x_0 = \pi/3$, $n = 3$, $0 \leq x \leq \pi$,

d. $f(x) = e^{-x^2}$, $x_0 = 0$, $n = 3$, $-1 \leq x \leq 1$,

e. $f(x) = \ln(1 + x^2)$, $x_0 = 1$, $n = 3$, $0.5 \leq x \leq 1.5$,

f. $f(x) = x \sinh(4x)$, $x_0 = 2$, $n = 4$, $1 \leq x \leq 2.5$.

E6. Use Taylor's Remainder formula to determine the number of terms of the Maclaurin series for e^x that should be used to estimate $e^{0.1}$ to within 0.000001.

E7. Let f (x) have derivatives through order n at $x = x_0$. Show that theTaylor polynomial of order n and its first n derivatives have the same values that f and its first n derivatives have at $x = x_0$.

E8. For approximately what values of x can you replace $\sin(x)$ by $x - (x^3/6)$ with an error of magnitude no greater than 5×10^{-4}? Give reasons for your answer.

E9. Show that if the graph of a twice-differentiable function $f(x)$ has an inflection point at $x = x_0$ then the linearization of f at $x = x_0$ is also the quadratic approximation of f at $x = x_0$. This explains why tangent lines fit so well at inflection points.

E10 Graph a curve $y = 1/3 - x^2/5$ and $y = (x - \arctan(x))/x^3$ together with the line $y = 1/3$. Use a Taylor series to explain what you see. What is

$$\lim_{x\to 0} \frac{x - \arctan(x)}{x^3} \tag{1}$$

2.3 Sequences

In everyday language, the term sequence means a succession of things in a definite order — chronological order, size order, or logical order, for example. In mathematics, the term sequence is commonly used to denote a succession of numbers whose order is determined by a rule or a function. In this section, we will develop some of the basic ideas concerning sequences of numbers.

2.3.1 Definition of a Sequence

An infinite sequence, or more simply a sequence, is an unending succession of numbers, called terms. It is understood that the terms have a definite order; that is, there is a first term a_1, a second term a_2, a third term a_3, a fourth term a_4, and so forth. Such a sequence would typically be written as

$$a_1,\ a_2,\ a_3,\ a_4,\ \ldots$$

where the dots are used to indicate that the sequence continues indefinitely. Some specific examples are

$$1,\ 2,\ 3,\ 4,\ 5,\ 6,\ \ldots$$

$$1,\ \frac{1}{2},\ \frac{1}{3},\ \frac{1}{4},\ \frac{1}{5},\ \ldots$$

$$2,\ 4,\ 6,\ 8,\ \ldots$$

$$1,\ -1,\ 1,\ -1,\ 1,\ -1,\ \ldots.$$

Each of these sequences has a definite pattern that makes it easy to generate additional terms if we assume that those terms follow the same pattern as the displayed terms. However, such patterns can be deceiving, so it is better to have a rule of formula for generating the terms. One way of doing this is to look for a function that relates each term in the sequence to its term number. For example, in the sequence

$$2,\ 4,\ 6,\ 8,\ \ldots$$

each term is twice the term number; that is, the nth term in the sequence is given by the formula $2\,n$. We denote this by writing the sequence as

$$2,\ 4,\ 6,\ 8,\ \ldots,\ 2\,n,\ \ldots.$$

We call the function $f(n) = 2\,n$ the general term of this sequence. Now, if we want to know a specific term in the sequence, we just need to substitute its term number into the formula for the general term. For example, the 38th term in the sequence is $2 \cdot 38 = 76$.

Example 2.7. Sequences

In each part find the general term of the sequence.

$$a)\ \frac{1}{2},\ \frac{2}{3},\ \frac{3}{4},\ \frac{4}{5},\ \ldots$$

$$b)\ \frac{1}{2},\ \frac{1}{4},\ \frac{1}{8},\ \frac{1}{16},\ \ldots$$

$$c)\ \frac{1}{2},\ -\frac{2}{3},\ \frac{3}{4},\ -\frac{4}{5},\ \ldots$$

$$d)\ 1,\ 3,\ 5,\ 7,\ \ldots$$

Solution 2.7. a) To find the general formula of the sequence, we create a table containing the term numbers and the sequence terms themselves.

n	f(n)
1	$\frac{1}{2}$
2	$\frac{2}{3}$
3	$\frac{3}{4}$
4	$\frac{4}{5}$

On the left we start counting the term number and on the right there are the results of the counting. We see that the numerator is the same as the term number and denominator is one greater than the term number. This suggests that the nth term has numerator n and denominator $n+1$. Thus the sequence can be generated by the following general expression

$$f(\text{n\_}) := \frac{n}{n+1}$$

We introduce this *Mathematica* expression for further use. The application of this function to a sequence of numbers shows the equivalence of the sequences

Table[$f(n)$, {n, 1, 6}]

$$\left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}\right\}$$

b) The same procedure is used to find the general term expression for the sequence

n	f(n)
1	$\frac{1}{2}$
2	$\frac{1}{4}$
3	$\frac{1}{8}$
4	$\frac{1}{16}$

Here the numerator is always the same and equals to 1. The denominator for the four known terms can be expressed as powers of 2. From the table we see that the exponent in the denominator is the same as the term number. This suggests that the denominator of the nth term is 2^n. Thus the sequence can be expressed by

$$f(\text{n\_}) := \frac{1}{2^n}$$

The generation of a table shows the agreement of the first four terms with the given numbers

Table[$f(n)$, {n, 1, 6}]

$$\left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \frac{1}{64}\right\}$$

c) This sequence is identical to that in part a), except for the alternating signs.

n	f(n)
1	$\frac{1}{2}$
2	$-\frac{2}{3}$
3	$\frac{3}{4}$
4	$-\frac{4}{5}$

Thus the nth term in the sequence can be obtained by multiplying the nth term in part a) by $(-1)^{n+1}$. This factor produces the correct alternating signs, since its successive values, starting with $n = 1$ are $1, -1, 1, -1, \ldots$. Thus, the sequence can be written as

$$\boldsymbol{f(\mathrm{n\_}) := \frac{(-1)^{n+1}\, n}{n+1}}$$

and the verification shows agreement within the given numbers

Table[*f*(*n*), {*n*, 1, 6}]

$$\left\{\frac{1}{2}, -\frac{2}{3}, \frac{3}{4}, -\frac{4}{5}, \frac{5}{6}, -\frac{6}{7}\right\}$$

d) For this sequence, we have the table

n	f(n)
1	1
2	3
3	5
4	7

from which we see that each term is one less than twice its term number. This suggests that the nth term in the sequence is $2\,n - 1$. Thus we can generate the sequence by

$$\boldsymbol{f(\mathrm{n\_}) := 2\, n - 1}$$

and show the congruence by

Table[*f*(*n*), {*n*, 1, 6}]

{1, 3, 5, 7, 9, 11}

▲

When the general term of a sequence

$$a_1,\, a_2,\, a_3,\, \ldots,\, a_n,\, \ldots \tag{2.20}$$

is known, there is no need to write out the initial terms, and it is common to write only the general term enclosed in braces. Thus (2.20) might be written as

$$\{a_n\}_{n=1}^{+\infty}. \tag{2.21}$$

For example here are the four sequences from Example 2.7 expressed in brace notation and the corresponding results. Fo the first sequence we write

$$\left\{\frac{n}{n+1}\right\}_{n=1}^{6}$$

$$\left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}\right\}$$

Here we introduced a definition in *Mathematica* (s. Appendix) which allows us to use the same notation as introduced in (2.21). For the second sequence we write

$$\left\{\frac{1}{2^n}\right\}_{n=1}^{6}$$

$$\left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \frac{1}{64}\right\}$$

The third sequence is generated by

$$\left\{\frac{(-1)^{n+1}\, n}{n+1}\right\}_{n=1}^{6}$$

$$\left\{\frac{1}{2}, -\frac{2}{3}, \frac{3}{4}, -\frac{4}{5}, \frac{5}{6}, -\frac{6}{7}\right\}$$

and finally the last sequence of Example 2.7 is derived by

$$\{2\, n - 1\}_{n=1}^{6}$$

$$\{1, 3, 5, 7, 9, 11\}$$

The letter n in (2.21) is called the index of the sequence. It is not essential to use n for the index; any letter not reserved for another purpose can be used. For example, we might view the general term of the sequence a_1, a_2, a_3 … to be the kth term, in which case we would denote this sequence as $\{a_k\}_{k=1}^{+\infty}$. Moreover, it is not essential to start the index at 1; sometimes it is more convenient to start at 0. For example, consider the sequence

$$1,\ \frac{1}{2},\ \frac{1}{2^2},\ \frac{1}{2^3},\ \ldots$$

One way to write this sequence is

$$\left\{\frac{1}{2^{n-1}}\right\}_{n=1}^{5}$$

$$\left\{1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}\right\}$$

However, the general term will be simpler if we think of the initial term in the sequence as the zeroth term, in which case we write the sequence as

$$\left\{\frac{1}{2^{n}}\right\}_{n=0}^{6}$$

$$\left\{1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \frac{1}{64}\right\}$$

Remark 2.4. In general discussions that involve sequences in which the specific terms and the starting point for the index are not important, it is common to write $\{a_n\}$ rather than $\{a_n\}_{n=1}^{\infty}$. Moreover, we can distinguish between different sequences by using different letters for their general terms; thus $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ denote three different sequences.

We began this section by describing a sequence as an unending succession of numbers. Although this conveys the general idea, it is not a satisfactory mathematical definition because it relies on the term *succession*, which is itself an undefined term. To motivate a precise definition, consider the sequence

$$2,\ 4,\ 6,\ 8,\ \ldots,\ 2\,n,\ \ldots$$

If we denote the general term by $f(n) = 2\,n$, then we can write this sequence as

$$f(1),\ f(2),\ f(3),\ \ldots,\ f(n),\ \ldots.$$

which is a list of values of the function

$$f(n) = 2\,n \qquad n = 1,\ 2,\ 3,\ \ldots.$$

whose domain is the set of positive integer. This suggests the following Definition 2.3.

Definition 2.3. *Sequence*

A sequence is a function whose domain is a set of integers. Specifically, we will regard the expression $\{a_n\}_{n=1}^{\infty}$ to be an alternative notation for the function $f(n) = a_n$, $n = 1, 2, 3, \ldots$. ■

2.3.2 Graphs of a Sequence

Since sequences are functions, it makes sense to talk about graphs of a sequence. For example, the graph of a sequence $\{1/n\}_{n=1}^{\infty}$ is the graph of the equation

$$y = \frac{1}{n} \qquad \text{for } n = 1, 2, 3, 4, \ldots.$$

Because the right side of this equation is defined only for positive integer values of n, the graph consists of a succession of isolated points (Figure 2.7)

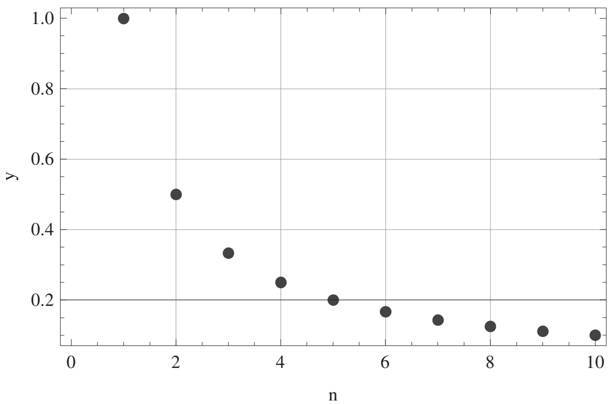


Figure 2.7. Graph of the sequence $\left\{\frac{1}{n}\right\}_{n=1}^{10}$.

This graph is resembling us to the graph of

$$y = \frac{1}{x} \qquad \text{for } x > 1$$

which is a continuous curve in the (x, y)-plane.

2.3.3 Limit of a Sequence

Since Sequences are functions, we can inquire about their limits. However, because a sequence $\{a_n\}$ is only defined for integer values of n, the only limit that makes sense is the limit of a_n as $n \to \infty$. In Figure 2.8 we have shown the graph of four sequences, each of which behave differently as $n \to \infty$

- The terms in the sequence $\{n + 1\}$ increases without bound.
- The terms in the sequence $\left\{(-1)^{n+1}\right\}$ oscillate between -1 and 1.
- The terms in the sequence $\left\{\frac{n}{n+1}\right\}$ increases toward a limiting value of 1.
- The terms in the sequence $\left\{1 + \left(-\frac{1}{2}\right)^n\right\}$ also tend toward a limiting value of 1, but do so in an oscillatory fashion.

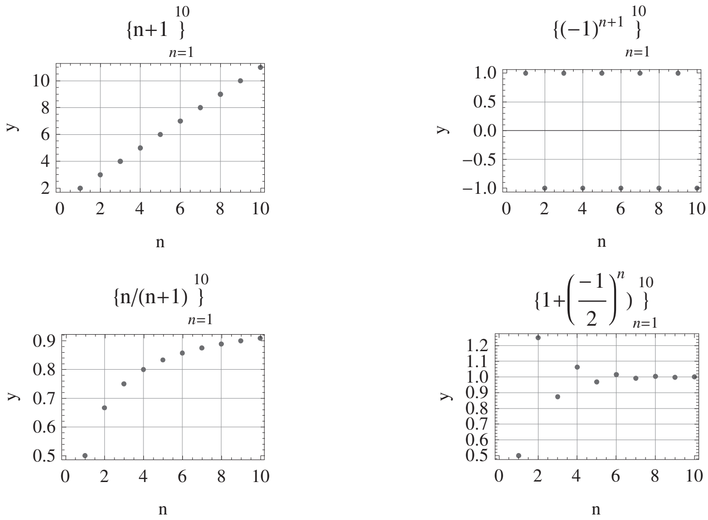


Figure 2.8. Graph of four different sequences.

Informally speaking, the limit of a sequence $\{a_n\}$ is intended to describe how a_n behaves as $n \to \infty$. To be more specific, we will say that a sequence $\{a_n\}$ approaches a limit L if the terms in the sequence eventually become arbitrarily close to L. Geometrically, this means that for any positive number ϵ there is a point in the sequence after which all terms lie between the lines $y = L - \epsilon$ and $y = L + \epsilon$

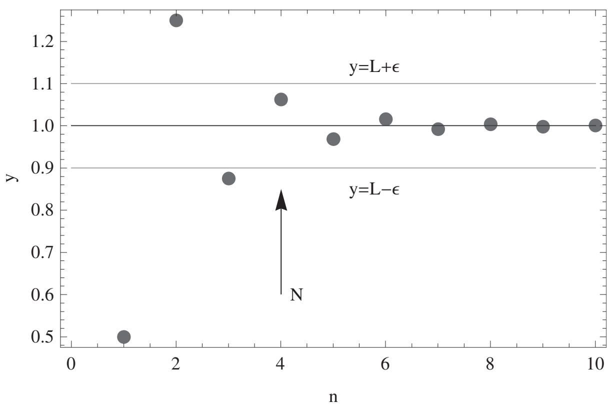


Figure 2.9. Limit process of a sequence.

The following Definition 2.4 makes these ideas precise.

Definition 2.4. *Limit of a Sequence*

A sequence $\{a_n\}$ is said to converge to the limit L if given any $\epsilon > 0$, there is a positive integer N such that $|a_n - L| < \epsilon$ for $n \geq N$. In this case we write

$\lim_{n\to\infty} a_n = L.$

A sequence that does not converge to some finite limit is said to diverge. ■

Example 2.8. Limit of a Sequence

The first two sequences in Figure 2.8 diverge, and the second two converge to 1; that is

$$\lim_{n\to\infty} \frac{n}{n+1}$$

```
1
```

and

$$\lim_{n\to\infty} \left(\left(-\frac{1}{2}\right)^n + 1\right)$$

```
1
```

The following Theorem 2.2, which we state without proof, shows that the familiar properties of limits apply to sequences. This theorem ensures that the algebraic techniques used to find limits of the form $\lim_{x\to\infty}$ can be used for limits of the form $\lim_{n\to\infty}$.

Theorem 2.2. *Rules for Limits of Sequences*

Suppose that the sequences $\{a_n\}$ and $\{b_n\}$ converge to the limits L_1 and L_2, respectively, and c is a constant. Then

a) $\lim_{n\to\infty} c = c$

b) $\lim_{n\to\infty} c\, a_n = c \lim_{n\to\infty} a_n = c\, L_1$

c) $\lim_{n\to\infty} (a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n = L_1 + L_2$

d) $\lim_{n\to\infty} (a_n - b_n) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n = L_1 - L_2$

e) $\lim_{n\to\infty} (a_n\, b_n) = \lim_{n\to\infty} a_n \lim_{n\to\infty} b_n = L_1\, L_2$

f) $\lim_{n\to\infty} \left(\frac{a_n}{b_n}\right) = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} = \frac{L_1}{L_2}$ if $L_2 \neq 0$. ■

Example 2.9. Calculating Limits of Sequences

In each part, determine whether the sequence converges or diverges. If it converges, find the limit

$a)\left\{\dfrac{n}{2\,n+1}\right\}_{n=1}^{\infty},$

$b)\left\{(-1)^{n}\,\dfrac{1}{n}\right\}_{n=1}^{\infty},$

$c)\,\{8-2\,n\}_{n=1}^{\infty}.$

Solution 2.9. a) Dividing numerator and denominator by n yields

$$\lim_{n\to\infty}\frac{n}{2\,n+1}=\lim_{n\to\infty}\frac{1}{2+\frac{1}{n}}=\frac{\lim_{n\to\infty}1}{\lim_{n\to\infty}2+\lim_{n\to\infty}\frac{1}{n}}=\frac{1}{2+0}=\frac{1}{2}$$

which agrees with the calculation done by *Mathematica*

$$\lim_{n\to\infty}\frac{n}{2\,n+1}$$

$$\frac{1}{2}$$

b) Since the $\lim_{n\to\infty}\frac{1}{n}=0$, the product $(-1)^{n+1}\,(1/n)$ oscillates between positive and negative values, with the odd-numbered terms approaching 0 through positive values and the even-numbered terms approaching 0 through negative values. Thus,

$$\lim_{n\to\infty}(-1)^{n+1}\,\frac{1}{n}=0$$

which is also derived via

$$\lim_{n\to\infty}\frac{(-1)^{n+1}}{n}$$

$$0$$

so the sequence converges to 0.

c) $\lim_{n\to\infty}(8-2\,n)=-\infty$, so the sequence $\{8-2\,n\}_{n=1}^{\infty}$ diverges.▲

If the general term of a sequence is $f(n)$, and if we replace n by x, where x can vary over the entire interval $[1,\,\infty)$, then the values of $f(n)$ can be viewed as sample values of $f(x)$ taken at the positive integers. Thus, if $f(x)\to L$ as $x\to\infty$, then it must also be true that $f(n)\to L$ as $n\to\infty$ (see Figure 2.10). However, the converse is not true; that is, one cannot infer that $f(x)\to L$ as $x\to\infty$ from the fact that $f(n)\to L$ as $n\to\infty$ (see Figure 2.11).

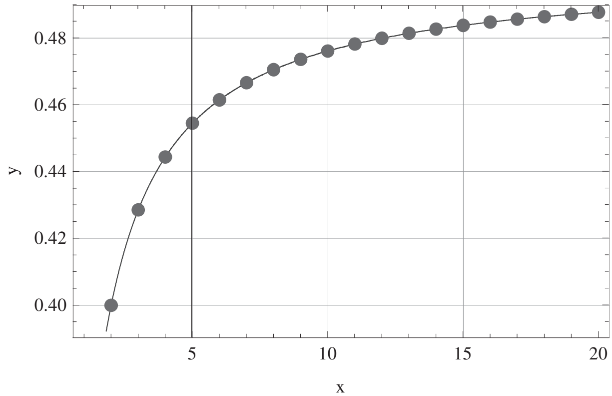


Figure 2.10. Replacement of a sequence $\left\{\frac{n}{2n+1}\right\}_{n=1}^{20}$ by a function $f(x) = \frac{x}{2x+1}$.

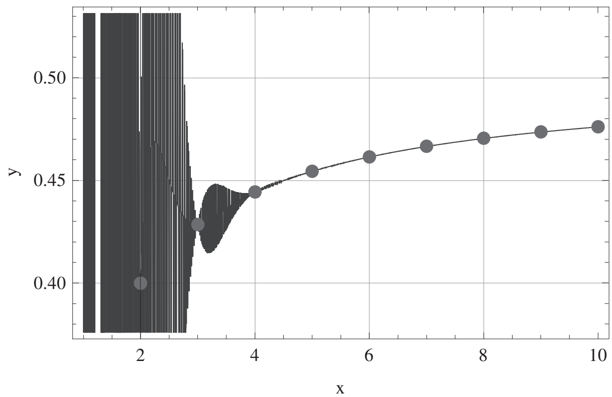


Figure 2.11. Replacement of a function $f(x) = \frac{x}{2x+1}$ by a sequence $\left\{\frac{n}{2n+1}\right\}_{n=1}^{10}$.

Example 2.10. L'Hopital's Rule

Find the limit of the sequence $\{n / e^n\}_{n=1}^{\infty}$.

Solution 2.10. The expression n / e^n is an indeterminate form of type ∞ / ∞ as $n \to \infty$, so L'Hopital's rule is indicated. However, we cannot apply this rule directly to n / e^n because the functions n and e^n have been defined here only at the positive integers, and hence are not differentiable functions. To circumvent this problem we extend the domains of these functions to all real numbers, here implied by replacing n by x, and apply L'Hopital's rule to the limit of the quotient x / e^x. This yields

$$\lim_{x\to\infty} \frac{x}{e^x} = \lim_{x\to\infty} \frac{1}{e^x} = 0$$

from which we can conclude that

$$\lim_{n\to\infty} \frac{n}{e^n} = 0$$

which can be verified by

$$\lim_{n\to\infty} \frac{n}{e^n}$$

0

▲

Sometimes the even-numbered and odd-numbered terms of a sequence behave sufficiently differently; so it is desirable to investigate their convergence separately. The following Theorem 2.3, whose proof is omitted, is helpful for that purpose.

Theorem 2.3. *Even and Odd Sequences*

A sequence converges to a limit L if and only if the sequences of even-numbered terms and odd-numbered terms both converge to L.■

Example 2.11. Even and Odd Sequences

The sequence

$$\frac{1}{2}, \frac{1}{3}, \frac{1}{2^2}, \frac{1}{3^2}, \frac{1}{2^3}, \ldots$$

converges to 0, since the odd-numbered terms and the even-numbered terms both converge to 0, and the sequence

$$1, \frac{1}{2}, 1, \frac{1}{3}, 1, \frac{1}{4}, 1, \ldots$$

diverges, since the odd-numbered terms converge to 1 and the even-numbered terms converge to 0.▲

2.3.4 Squeezing of a Sequence

The following Theorem 2.4, which we state without proof, is an adoption of the Squeezing Theorem to sequences. This theorem will be useful for finding limits of sequences that cannot be obtained directly.

Theorem 2.4. *The Squeezing Theorem for Sequences*

Let $\{a_n\}$, $\{b_n\}$, *and* $\{c_n\}$ *be sequences such that*

$$a_n \leq b_n \leq c_n \qquad \text{for all values of } n \text{ beyond some index } N.$$

If the sequence $\{a_n\}$ *and* $\{c_n\}$ *have a common limit* L *as* $n \to \infty$, *then* $\{b_n\}$ *also has the limit* L *as* $n \to \infty$.∎

Example 2.12. Squeezing

Use numerical evidence to make a conjecture about the limit of the sequence

$$\left\{\frac{n!}{n^n}\right\}_{n=1}^{\infty}$$

and then confirm that your conjecture is correct.

Solution 2.12. The following table shows the sequence for n from 1 to 12.

TableForm$\left[N\left[\textbf{Table}\left[\frac{n!}{n^n}, \{n, 1, 12\}\right]\right]\right]$

1.
0.5
0.222222
0.09375
0.0384
0.0154321
0.0061199
0.00240326
0.000936657
0.00036288
0.000139906
0.0000537232

The values suggest that the limit of the sequence may be 0. To confirm this we have to examine the limit of

$$a_n = \frac{n!}{n^n} \qquad \text{as } n \to \infty.$$

Although this is an indeterminate form of type ∞/∞, L'Hopital's rule is not helpful because we have no definition of $x!$ for values of x that are not integers. However, let us write out some of the initial terms and the general term in the sequence

$$a_1 = 1, \qquad a_2 = \frac{1\times 2}{2\times 2}, \qquad a_3 = \frac{1\times 2\times 3}{3\times 3\times 3},$$

$$\ldots, \qquad a_n = \frac{1\times 2\times 3\times \ldots \times n}{n\times n\times n\times \ldots \times n}, \; \ldots.$$

We can rewrite the general term as

$$a_n = \frac{1}{n}\,\frac{(2\times 3\times \ldots \mathrm{n})}{n\times n\times \ldots \times n}$$

from which it is evident that

$$0 \le a_n \le \frac{1}{n}.$$

However, the two outside expressions have a limit of 0 as $n \to \infty$; thus, the Squeezing Theorem 2.4 for sequences implies that $a_n \to 0$ as $n \to \infty$, which confirms our conjecture.▲

The following Theorem 2.5 is often useful for finding the limit of a sequence with both positive and negative terms—it states that the sequence $\{|a_n|\}$ that is obtained by taking the absolute value of each term in the sequence $\{a_n\}$ converges to 0, then $\{a_n\}$ also converges to 0.

Theorem 2.5. *Magnitude Sequences*

If $\lim_{n\to\infty} |a_n| = 0$*, then* $\lim_{n\to\infty} a_n = 0$.■

Example 2.13. Magnitude Sequence

Consider the sequence

$$1,\ -\frac{1}{2},\ \frac{1}{2^2},\ -\frac{1}{2^3},\ \ldots,\ (-1)^n\,\frac{1}{2^n},\ \ldots.$$

If we take the absolute value of each term, we obtain the sequence

$$1,\ \frac{1}{2},\ \frac{1}{2^2},\ \frac{1}{2^3},\ \ldots,\ \frac{1}{2^n},\ \ldots.$$

which converges to 0. Thus from Theorem 2.5 we have

$$\lim_{n\to\infty}\left((-1)^n\,\frac{1}{2^n}\right) = 0.\ ▲$$

2.3.5 Recursion of a Sequence

Some sequences did not arise from a formula for the general term, but rather form a formula or set of formulas that specify how to generate each term in the sequence from terms that precede it; such sequences are said to be defined recursively, and the defining formulas are called recursion formulas. A good example is the mechanic's rule of approximating square roots. This can be done by

$$x_1 = 1, \qquad x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right) \tag{2.22}$$

describing the sequence produced by Newton's Method to approximate $\sqrt{a}$ as a root of the function $f(x) = x^2 - a$. The following table shows the first few terms of the approximation of $\sqrt{2}$

```
tab1 = N[Transpose[{{i}_{i=1}^9, NestList[1/2 (#1 + 2/#1) &, 1, 8]}]];
(TableForm[#1, TableHeadings → {{}, {"n", "x_n"}}] &)[N[tab1]]
```

n	x_n
1.	1.
2.	1.5
3.	1.41667
4.	1.41422
5.	1.41421
6.	1.41421
7.	1.41421
8.	1.41421
9.	1.41421

It would take us to far afield to investigate the convergence of sequences defined recursively. Thus we let this subject as a challenge for the reader. However, we note that sequences are important in numerical procedures like Newton's root finding method (see Vol. IV).

2.3.6 Tests and Exercises

The following two subsections serve to test your understanding of the last section. Work first on the test examples and then try to solve the exercises.

2.3.6.1 Test Problems

T1. What is an infinite sequence? What does it mean for such a sequence to converge? To diverge? Give examples.

T2. What theorems are available for calculating limits of sequences? Give examples.

T3. What is a non decreasing sequence? Under what circumstances does such a sequence have a limit? Give examples.

T4. What is the limit of a sequence? Give examples.

T5. How does squeezing work with sequences? Give examples.

2.3.6.2 Exercises

E1. Each of the following examples gives a formula for the nth term a_n of a sequence $\{a_n\}$. Find the values of a_1, a_3, a_6.

a. $a_n = \frac{1}{n!}$,

b. $a_n = \frac{2^n - 1}{2^n}$,

c. $a_n = \frac{2^n}{2^{n+1}}$,

d. $a_n = \frac{(-1)^{2n+1}}{3n-1}$.

E2. List the first five terms of the sequence.

a. $a_n = 1 - \left(\frac{2}{10}\right)^n$,

b. $a_n = 5\,\frac{(-1)^n}{n!}$,

$a_n = \frac{n+1}{5\,n-2}$,

d. $\{2, 4, 6, \ldots, (2\,n)\}$,

e. $a_1 = -1$, $a_{n+1} = a_n (2 - 2^n)$.

E3. List the first six terms of the sequence defined by

$$a_n = \frac{n}{3\,n+1}. \tag{1}$$

Does the sequence appear to have a limit? If so, find it.

E4. Each of the following examples gives the first term or two of a sequence along with a recursion formula for the remaining terms. Write out the first ten terms of the sequence.

a. $a_1 = 1$, $a_{n+1} = a_n + 2^{-n}$,

b. $a_1 = 2$, $a_{n+1} = a_n / (2\,n+1)$,

c. $a_1 = -2$, $a_{n+1} = (-1)^{n+2}\, a_n$,

d. $a_1 = 1$, $a_{n+1} = n\, a_n - n^2\, a_n$,

e. $a_1 = a_2 = 1$, $a_{n+2} = a_{n+1} - a_n$,

f. $a_1 = 1$, $a_2 = -1$, $a_{n+2} = a_n + 2^{-n}\, a_{n+1}$.

E5. Find a formula for the nth term of the sequences:

a. $\{1, -1, 1, -1, 1, -1, \ldots\}$,

b. $\{-1, 1, -1, 1, -1, 1, \ldots\}$,

c. $\{1, -4, 9, -16, 25, \ldots\}$,

d. $\left\{1, \frac{-1}{4}, \frac{1}{9}, \frac{-1}{16}, \frac{1}{25}, \ldots\right\}$,

e. $\{1, 5, 9, 13, 17, \ldots\}$,

f. $\{1, 0, 1, 0, 1, 0, \ldots\}$,

g. $\{2, 5, 11, 17, 23, \ldots\}$.

Do the sequences converge? If so, to what value? In each case, begin by identifying the function f that generates the sequence.

E6. The following sequences come from the recursion formula for Newton's method,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \tag{2}$$

Do the sequences converge? If so, to what value? In each case, begin by identifying the function f that generates the sequence.

a. $x_0 = 1$, $x_{n+1} = x_n - \frac{x_n^2-2}{2\,x_n} = \frac{x_n}{2} + \frac{1}{x_n}$,

b. $x_0 = 1$, $x_{n+1} = x_n - 1$,

c. $x_0 = 1$, $x_{n+1} = x_n - \frac{\tan(x_n)-1}{\sec(x_n)^2}$.

E7. Is it true that a sequence $\{a_n\}$ of positive numbers must converge if it is bounded from above? Give reasons for your answer.

E8. Which of the following sequences converge, and which diverge? Give reasons for your answers.

a. $a_n = 1 - \frac{1}{n}$,

b. $a_n = n - \frac{1}{n}$,

c. $a_n = \frac{2^n-1}{5^n}$,

d. $a_n = \frac{2^n-3}{2^n}$,

$a_n = ((-1)^n + 1)\left(\frac{n+1}{n^2}\right)$.

E9. Determine if the sequence is non decreasing and if it is bounded from above.

a. $a_n = 2 - \frac{2}{n} - 2^{-n}$,

b. $a_n = \frac{2^n\, 3^n}{n!}$,

c. $a_n = \frac{(2\,n+3)!}{(n+1)!}$,

d. $a_n = \frac{3\,n+1}{n+1}$.

E10 Find the limit of the sequence

$$\left\{\sqrt{5}\,,\ \sqrt{5\sqrt{5}}\,,\ \sqrt{5\sqrt{5\sqrt{5}}}\ ,\ \ldots\right\} \tag{3}$$

2.4 Infinite Series

The purpose of this section is to discuss sums that contain infinitely many terms. The most familiar example of such sums occur in the decimal representation of real numbers. For example, when we write $\frac{1}{3}$ in the decimal form $\frac{1}{3} = 0.3333333\ldots$, we mean

$$\frac{1}{3} = 0.3 + 0.03 + 0.003 + 0.0003 + \ldots$$

which suggests that the decimal representation of $\frac{1}{3}$ can be viewed as a sum of infinitely many real numbers.

2.4.1 Sums of Infinite Series

Our first objective is to define what is meant by the sum of infinite many real numbers. We begin with some terminology.

Definition 2.5. *Infinite Sum*

An infinite series is an expression that can be written in the form

$$\sum_{k=1}^{\infty} u_k = u_1 + u_2 + u_3 + \ldots + u_k + \ldots. \tag{2.23}$$

The numbers u_1, u_2, u_3,… are called the terms of the series. ■

Since it is impossible to add infinitely many numbers together directly, sums of infinite series are defined and computed by an indirect limiting process. To motivate the basic idea, consider the decimal

$$0.33333\ldots. \tag{2.24}$$

This can be viewed as the infinite series

$$0.3 + 0.03 + 0.003 + 0.0003 + \ldots \tag{2.25}$$

or equivalently

$$\frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \frac{3}{10\,000} + \ldots. \tag{2.26}$$

Since (2.25) is the decimal expansion of $\frac{1}{3}$, any reasonable definition for the sum of an infinite series should yield $\frac{1}{3}$ for the sum of (2.26). To obtain such a definition, consider the following sequence of finite sums:

$$s_1 = \frac{3}{10} = 0.3$$

$$s_2 = \frac{3}{10} + \frac{3}{100} = 0.33$$

$$s_3 = \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} = 0.333$$

$$\vdots$$

The sequence of numbers $s_1, s_2, s_3, \ldots$ can be viewed as a succession of approximations of the sum of the infinite series, which we want to be $\frac{1}{3}$. As we progress through the sequence, more and more terms of the infinite series are used, and the approximations get better and better, suggesting that the desired sum of $\frac{1}{3}$ might be the limit of this sequence of approximations. To see that this is so, we must calculate the limit of the general term in the sequence of approximations, namely

$$s_n = \frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} + \frac{3}{10^4} + \ldots + \frac{3}{10^n} \tag{2.27}$$

The problem of calculating

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty}\left(\frac{3}{10} + \frac{3}{10^2} + \ldots + \frac{3}{10^n}\right) \tag{2.28}$$

is complicated by the fact that both the last term and the number of terms in the sum change with n. It is best to rewrite such limits in a closed form in which the number of terms does not vary, if possible. To do this, we multiply both sides of (2.27) by $\frac{1}{10}$ to obtain

$$\frac{1}{10} s_n = \frac{3}{10^2} + \frac{3}{10^3} + \frac{3}{10^4} + \frac{3}{10^5} + \ldots + \frac{3}{10^{n+1}} \tag{2.29}$$

and then subtract (2.27) from (2.29) to obtain

$$s_n - \frac{1}{10} s_n = \frac{3}{10} - \frac{3}{10^{n+1}}$$

which simplifies to

$$\frac{9}{10} s_n = \frac{3}{10} \left(1 - \frac{1}{10^n}\right)$$

$$s_n = \frac{1}{3} \left(1 - \frac{1}{10^n}\right)$$

Since $1 / 10^n \to 0$ as $n \to \infty$, it follows that

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{1}{3} \left(1 - \frac{1}{10^n}\right) = \frac{1}{3}$$

which we denote by writing

$$\frac{1}{3} = \frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} + \frac{3}{10^4} + \ldots + \frac{3}{10^n} + \ldots.$$

Motivated by the preceding example, we are now ready to define the general concept of the sum of an infinite series

$$u_1 + u_2 + u_3 + \ldots + u_k + \ldots.$$

We begin with some terminology: Let s_n denote the sum of the initial terms of the series, up to and including the term with index n. Thus,

$$s_1 = u_1$$

$$s_2 = u_1 + u_2$$

$$s_3 = u_1 + u_2 + u_3$$

$$\vdots$$

$$s_n = u_1 + u_2 + u_3 + \ldots + u_n = \sum_{k=1}^{n} u_k.$$

The number s_n is called the partial sum of the series and the sequence $\{s_n\}_{n=1}^{\infty}$ is called the sequence of partial sums.

As n increases, the partial sum $s_n = u_1 + u_2 + u_3 + \ldots + u_n$ includes more and more terms of the series.

Thus if s_n tends toward a limit as $n \to \infty$, it is reasonable to view this limit as the sum of all the terms in the series. This suggests the following Definition 2.6.

Definition 2.6. *Convergent Infinite Sum*

Let $\{s_n\}$ be the sequence of partial sums of the series

$$u_1 + u_2 + u_3 + \ldots + u_k + \ldots$$

If the sequence $\{s_n\}$ converges to a limit S, then the series is said to converge to S, and S is called the sum of the series. We denote this by writing

$$S = \sum_{k=1}^{\infty} u_k.$$

If the sequence of partial sums diverges, then the series is said to diverge. A divergent series has no sum. ■

Remark 2.5. Sometimes it will be desirable to start the summation index in an infinite series at $k = 0$ rather than $k = 1$, in which case we will view u_0 as the zeroth term and $s_0 = u_0$ as the zeroth partial sum. It can be proved that changing the starting value for the index has no effect on the convergence or divergence of an infinite series.

Example 2.14. Divergent Series

Determine whether the series

$$1 - 1 + 1 - 1 + 1 - 1 + 1 - \ldots$$

converges or diverges. If it converges, find the sum.

Solution 2.13. It is tempting to conclude that the sum of series is zero by arguing that the positive and negative terms cancel one another. However, this is not correct; the problem is that algebraic operations that hold for finite sums do not apply over to infinite series in all cases. Later we will discuss conditions under which familiar algebraic operations can be applied to infinite series, but for the example we directly turn to Definition 2.6. The partial sums are

$$s_1 = 1$$

$$s_2 = 1 - 1 = 0$$

$$s_3 = 1 - 1 + 1 = 1$$

$$s_4 = 1 - 1 + 1 - 1 = 0$$

and so forth. Thus, the sequence of partial sums is

$$1, 0, 1, 0, 1, 0, 1, \ldots.$$

Since this is a divergent sequence, the given series diverges and consequently has no sum.▲

2.4.2 Geometric Series

In many important series, each term is obtained by multiplying the preceding term by some fixed constant. Thus, if the initial term of the series is a and each term is obtained by multiplying the preceding term by r, then the series has the form

$$\sum_{k=0}^{\infty} a\, r^k = a + a\, r + a\, r^2 + a\, r^3 + \ldots + a\, r^k + \ldots \qquad \text{with } a \neq 0.$$

Such series are called geometric series, and the number r is called the ratio for the series. Geometric series are very important because a lot of applications can be cast into this type of series. Here are some examples:

$$1 + 2 + 4 + 8 + \ldots + 2^k + \ldots \qquad \text{with } a = 1 \text{ and } r = 2$$

$$\frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} + \frac{3}{10^4} + \ldots + \frac{3}{10^k} + \ldots \qquad \text{with } a = 3 \text{ and } r = \frac{1}{10}$$

$$\frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \ldots + (-1)^k \frac{1}{2^k} + \ldots \qquad \text{with } a = \frac{1}{2} \text{ and } r = -\frac{1}{2}$$

$$1 + 1 + 1 + 1 + 1 + 1 + \ldots \qquad \text{with } a = 1 \text{ and } r = 1$$

$$1 - 1 + 1 - 1 + 1 - 1 + \ldots \qquad \text{with } a = 1 \text{ and } r = -1$$

$$1 + x + x^2 + x^3 + \ldots + x^k + \ldots \qquad \text{with } a = 1 \text{ and } r = x.$$

The following theorem is the fundamental result on convergence of geometric series.

Theorem 2.6. *Geometric Series*

A geometric series

$$\sum_{k=0}^{\infty} a\, r^k = a + a\, r + a\, r^2 + a\, r^3 + \ldots + a\, r^k + \ldots \qquad \text{with } a \neq 0$$

converges if $|r| < 1$ *and diverges if* $|r| \geq 1$*. If the series converges, then the sum is*

$$\sum_{k=0}^{\infty} a\, r^k = \frac{a}{1-r}.\blacksquare$$

The proof of this theorem is left as an exercise.

Example 2.15. Geometric Series

The series

$$\sum_{k=0}^{\infty} \frac{5}{4^k}$$

is a geometric series with $a = 5$ and $r = 1/4$. Since $|r| = 1/4 < 1$, the series converges and the sum is

$$\frac{a}{1-r} = \frac{5}{1-\frac{1}{4}} = \frac{5}{\frac{3}{4}} = 5 \times \frac{4}{3} = \frac{20}{3}. \blacktriangle$$

Example 2.16. Geometric Series

In each part determine whether the series converges, and if so find its sum.
a) $\sum_{k=1}^{\infty} 3^{2k}\, 5^{1-k}$ and b) $\sum_{k=0}^{\infty} x^k$

Solution 2.16.

a) This is a geometric series in a concealed form since we can rewrite it as

$$\sum_{k=1}^{\infty} 3^{2k}\, 5^{1-k} = \sum_{k=1}^{\infty} 9^k\, 5^{1-k} = \sum_{k=1}^{\infty} 9^k \frac{1}{5^{k-1}} = \sum_{k=1}^{\infty} 9 \left(\frac{9}{5}\right)^{k-1}$$

Since $a = 9$ and $r = 9/5 > 1$ the series diverges.

b) The series of case b) is a geometric series with $a = 1$ and $r = x$, so it converges if $|x| < 1$ and diverges otherwise. When the series converges its sum is

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}. \blacktriangle$$

In addition to geometric series there are a few series which are important in applications. The next subsections will discuss these series and give some useful properties. The series discussed are telescoping sums, harmonic series, and hyper harmonic series.

2.4.3 Telescoping Sums

Telescoping sums are sums which collapse, like an old-fashioned collapsing telescope, into just two terms. How this works let us examine in the next example.

Example 2.17. Telescoping Sum

Determine whether the series

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)} = \frac{1}{1\times 2} + \frac{1}{2\times 3} + \ldots$$

converges or diverges. If it converges, find the sum.

Solution 2.17. The nth partial sum of the series is

$$s_n = \sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{1}{1\times 2} + \frac{1}{2\times 3} + \frac{1}{3\times 4} + \ldots + \frac{1}{n(n+1)}.$$

To calculate the limit $\lim_{n\to\infty} s_n$ we will rewrite s_n in closed form. This can be accomplished by using the method of partial fractions to obtain

$$\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$$

from which we obtain the telescoping sum

$$\begin{aligned} s_n &= \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1}\right) = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \ldots + \left(\frac{1}{n} - \frac{1}{n+1}\right) \\ &= 1 + \left(-\frac{1}{2} + \frac{1}{2}\right) + \left(-\frac{1}{3} + \frac{1}{3}\right) + \ldots + \left(-\frac{1}{n} + \frac{1}{n}\right) - \frac{1}{n+1} \\ &= 1 - \frac{1}{n+1} \end{aligned}$$

so

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)} = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \left(1 - \frac{1}{n+1}\right) = 1. \blacktriangle$$

2.4.4 Harmonic Series

One of the most important of all divergent series is the harmonic series.

$$\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \ldots$$

which arises in connection with the overtones produced by a vibrating musical string. It is not immediately evident that this series diverges. However, the divergence will become apparent when we examine the partial sum in detail. Because the terms in the series are all positive, the partial sums are

$$\begin{aligned} s_1 &= 1 \\ s_2 &= 1 + \frac{1}{2} = \frac{3}{2} \\ s_3 &= 1 + \frac{1}{2} + \frac{1}{3} = \frac{11}{6} \\ &\vdots \end{aligned}$$

from a strictly increasing sequence

$$s_1 < s_2 < s_3 < \ldots < s_n < \ldots.$$

Thus it is obvious that the partial sums grow to an unlimited number. In conclusion harmonic series do not converge.

2.4.5 Hyperharmonic Series

Hyperharmonic series are a special class of series which are represented as

$$\sum_{k=1}^{\infty} \frac{1}{k^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \ldots + \frac{1}{k^p} + \ldots \tag{2.30}$$

The following theorem tells when a hyperharmonic series converges.

Theorem 2.7. *Convergence of Hyperharmonic Series*

The series

$$\sum_{k=1}^{\infty} \frac{1}{k^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \ldots + \frac{1}{k^p} + \ldots$$

converges if $p > 1$ and diverges if $0 < p \leq 1$.■

2.4.6 Tests and Exercises

The following two subsections serve to test your understanding of the last section. Work first on the test examples and then try to solve the exercises.

2.4.6.1 Test Problems

T1. What is the difference between a sequence and a series?

T2. What is a convergent series? What is a divergent series?

T3. Explain what it means to say that $\sum_{i=1}^{\infty} a_n = 2$.

T4. What is a geometric series? When does such a series converge? Diverge? When it does converge, what is its sum? Give examples.

T5. Besides geometric series, what other convergent and divergent series do you know?

T6. How do you reindex a series? Why might you want to do this?

2.4.6.2 Exercises

E1. Find a formula for the nth partial sum of each series and use it to find the series' sum if the series converges.

a. $2 + \frac{2}{3} + \frac{2}{9} + \frac{2}{27} + \ldots + \frac{2}{3^{n-1}} + \ldots,$

b. $\frac{7}{100} + \frac{7}{100^2} + \frac{7}{100^3} + \ldots + \frac{7}{100^n} + \ldots$

c. $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \ldots + (-1)^{n-1} \frac{1}{2^{n-1}} + \ldots$

d. $\frac{1}{2\times 3} + \frac{1}{3\times 4} + \frac{1}{4\times 5} + \ldots + \frac{1}{(n+1)(n+2)} + \ldots$

E2. Find at least 10 partial sums of the series. Graph both the sequence of terms and the sequence of partial sums on the same screen. Does it appear that the series is convergent or divergent? If it is convergent, find the sum. If it is divergent, explain why.

$\sum_{n=1}^{\infty} \frac{2n^2-1}{n^2+1}$,

b. $\sum_{n=2}^{\infty} \frac{1}{n(n+2)}$,

c. $\sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}} \right)$,

d. $\sum_{n=1}^{\infty} \tan(n)$,

e. $\sum_{n=1}^{\infty} \frac{16}{(-4)^n}$.

E3. Write out the first few terms of each series to show how the series starts. Then find the sum of the series.

a. $\sum_{n=1}^{\infty} \frac{5}{4^n}$,

b. $\sum_{n=0}^{\infty} \left(\frac{5}{2^n} - \frac{1}{3^n} \right)$,

c. $\sum_{n=0}^{\infty} \left(\frac{1}{2^n} + \frac{(-1)^{2n}}{5^n} \right)$,

d. $\sum_{n=0}^{\infty} \left(\frac{5}{2^n} + \frac{1}{3^n} \right)$,

e. $\sum_{n=0}^{\infty} (-1)^n \frac{6}{5^n}$.

E4. Find the values of x for which the given geometric series converges. Also, find the sum of the series (as a function of x) for those values of x.

a. $\sum_{n=0}^{\infty} 2^n x^n$,

b. $\sum_{n=0}^{\infty} (-1)^n x^{-2n}$,

c. $\sum_{n=0}^{\infty} (-1)^n (x+1)^n$,

d. $\sum_{n=0}^{\infty} \sin(x)^n$,

e. $\sum_{n=0}^{\infty} \ln(x)^n$.

E5. Let $a_n = 2n/(3n+1)$.

a. Determine whether $\{a_n\}_{n=1}^{\infty}$ is convergent.

b. Determine whether $\sum_{n=1}^{\infty} a_n$ is convergent.

E6. Determine whether the series is convergent or divergent by expressing as a telescoping sum. If it is convergent, find its sum.

a. $\sum_{n=2}^{\infty} \frac{2}{n^2-1}$,

b. $\sum_{n=1}^{\infty} \frac{2}{n^2+4n+3}$,

c. $\sum_{n=1}^{\infty} \frac{3}{n(n+3)}$,

d. $\sum_{n=1}^{\infty} \ln\left(\frac{n}{n+1}\right)$.

E7. Express the number as a ratio of integers.

a. $0.\overline{2} = 0.22222\ldots$,

b. $0.\overline{73} = 0.73737373\ldots$,

c. $3.\overline{417} = 3.417417417\ldots$,

d. $7.\overline{12345}$.

E8. Make up an infinite series of nonzero terms whose sum is

a. 1,

b. −3,

c. 0.

E9. Find the value of b for which

$$1 + e^b + e^{2b} + e^{3b} + \ldots = 9. \tag{1}$$

E10 Use the partial fraction command on your CAS to find a convenient expression for the partial sum, and then use this expression to find the sum of the series. Check your answer by using the CAS to sum the series directly.

a. $\sum_{n=1}^{\infty} \frac{3\,n^2+3\,n+1}{\left(n^2+n\right)^3}$,

b. $\sum_{n=2}^{\infty} \frac{1}{n^3-n}$,

c. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n!\,n^2\,e^{-n}}$

E11 The Cantor set, named after the German mathematician Georg Cantor (1845–1918), is constructed as follows. We start with the closed interval [0, 1] and remove the open interval $\left(\frac{1}{3}, \frac{2}{3}\right)$. That leaves the two intervals $\left[0, \frac{1}{3}\right]$ and $\left[\frac{2}{3}, 1\right]$ and we remove the open middle third of each. Four intervals remain and again we remove the open middle third of each of them. We continue this procedure indefinitely, at each step removing the open middle third of every interval that remains from the preceding step. The Cantor set consists of the numbers that remain in [0, 1] after all those intervals have been removed. Show that the total length of all the intervals that are removed is 1. Despite that, the Cantor set contains infinitely many numbers. Give examples of some numbers in the Cantor set.

2.5 Convergence Tests

In Section 2.4 we showed how to find the sum of a series by finding a closed form for the nth partial sum and taking its limit. However, it is relatively rare that one can find a closed form for the nth partial sum of a series, so alternative methods are needed for finding the sum of a series. One possibility is to prove that the series converges, and then to approximate the sum by a partial sum with sufficiently many terms to achieve the desired degree of accuracy. In this section we will develop various tests that can be used to determine whether a given series converges or diverges.

2.5.1 The Divergent Test

In stating general results about convergence or divergence of series, it is convenient to use the notation Σu_k as a generic template for a series, thus avoiding the issue of whether the sum begins with $k = 0$ or $k = 1$ or some other value. Indeed, we will see shortly that the starting index value is irrelevant to the issue of convergence. The kth term in an infinite series Σu_k is called the general term of the series. The following Theorem 2.8 establishes a relationship between the limit of the general term and the convergence properties of a series.

Theorem 2.8. *Divergence Test*

a) If $\lim_{k\to+\infty} u_k \neq 0$, then the series Σu_k diverges.

b) If $\lim_{k\to+\infty} u_k = 0$, then the series Σu_k may either converge or diverge. ■

The alternative form of part a) given in the preceding theorem is sufficiently important that we state it separately for future reference.

Theorem 2.9. *Convergence*

If the series $\Sigma\, u_k$ converges, then $\lim_{k\to+\infty} u_k = 0$. ■

Example 2.18. Divergent Series

The series

$$\sum_{k=1}^{\infty} \frac{k}{k+1} = \frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \ldots + \frac{k}{k+1} + \ldots$$

diverges since

$$\lim_{k\to+\infty} \frac{k}{k+1} = \lim_{k\to+\infty} \frac{1}{1+\frac{1}{k}} = 1 \neq 0. \blacktriangle$$

Warning: The converse of Theorem 2.9 is false. To prove that a series converges it does not suffice to show that $\lim_{k\to\infty} u_k = 0$, since this property may hold for divergent as well as convergent series.

Remark 2.6. Convergent series behave algebraically as any other object. Sums and differences are calculated as usual.

2.5.2 The Integral Test

The expressions

$$\sum_{k=1}^{\infty} \frac{1}{k^2} \quad \text{and} \quad \int_1^{\infty} \frac{1}{x^2}\, dx$$

are related in that the integrand in the improper integral results when the index k in the general term of the series is replaced by x and the limits of summation in the series are replaced by the corresponding limits of integration. The following Theorem 2.10 shows that there is a relationship between the convergence of the series and the integral.

Theorem 2.10. *Integral Test*

Let Σu_k be a series with positive terms, and let $f(x)$ be the function that results where k is replaced by x in the general term of the series. If f is decreasing and continuous on the interval $[a, +\infty)$, then

$$\sum_{k=1}^{\infty} u_k \quad \text{and} \quad \int_a^{\infty} f(x)\, dx$$

both converge or both diverge. ■

Example 2.19. Integral Test

Use the integral test to determine whether the following series converge or diverge.

$$a)\ \sum_{k=1}^{\infty} \frac{1}{k}$$

$$b)\ \sum_{k=1}^{\infty} \frac{1}{k^2}$$

Solution 2.19. a) We know that this series is a divergent harmonic series, so the integral test will simply provide another way of establishing the divergence. If we replace k by x in the general term $1/k$, we obtain the function $f(x) = 1/x$, which is decreasing and continuous for $x \geq 1$ (as required to apply the integral test with $a = 1$). Since

$$\int_1^{\infty} \frac{1}{x}\, dx$$

```
Integrate::idiv : Integral of 1/x does not converge on {1, ∞}. More…
```

$$\int_1^{\infty} \frac{1}{x}\, dx$$

we replace this by

$$\mathbf{int} = \int_1^{\beta} \frac{1}{x}\, dx$$

$$(\beta - 1)\,\mathrm{If}\left[\mathrm{Re}(\beta) \geq 0 \bigvee \mathrm{Im}(\beta) \neq 0,\ \frac{\ln(\beta)}{\beta - 1},\right.$$

$$\left.\mathrm{Integrate}\left[\frac{1}{x\,(\beta - 1) + 1},\ \{x, 0, 1\},\ \mathrm{Assumptions} \to \neg\,(\mathrm{Re}(\beta) \geq 0 \bigvee \mathrm{Im}(\beta) \neq 0)\right]\right]$$

which represents the general solution under the condition that the real part of β is greater than zero or the imaginary part is not equal to zero. In this case an anti derivative exists which is given by $\ln(\beta)$. The limit for $\beta \to \infty$ shows

Limit[int, {$\beta \to \infty$}]

$\{\infty\}$

This means that the integral diverges and consequently so does the series.

b) If we replace k by x in the general term $1/k^2$, we obtain the function $f(x) = 1/x^2$, which is decreasing and continuous for $x \geq 1$. Since

$$\int_1^{\infty} \frac{1}{x^2}\, dx$$

1

the integral converges and consequently the series converges by the integral test with $a = 1$.▲

Remark 2.7. In part b) of the last example, do not erroneously conclude that the sum of the series is 1 because the value of the corresponding integral is 1. It can be proved that the sum of the series is actually $\pi^2 / 6$ and, indeed, the first two terms alone exceeds 1.

$$\sum_{k=1}^{\infty} \frac{1}{k^2}$$

$$\frac{\pi^2}{6}$$

2.5.3 The Comparison Test

We will discuss in this section a test that is useful in its own right and is also the building block for other important convergence tests. The underlying idea of this test is to use the known convergence or divergence of a series to deduce the convergence or divergence of another series.

Theorem 2.11. *Comparison Test*

Let $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ be series with non negative terms and suppose that

$$a_1 \leq b_1,\ a_2 \leq b_2,\ a_3 \leq b_3,\ \ldots,\ a_k \leq b_k,\ \ldots$$

a) If the bigger series $\sum b_k$ converges, then the smaller series $\sum a_k$ also converge.

b) If the smaller series $\sum a_k$ diverges, then the bigger series $\sum b_k$ also diverges.■

The proof of the theorem is given visually in Figure 2.12. The proof is based on the interpretation of the terms in the series as areas of rectangles. The comparison test states that if the total area $\sum b_k$ is finite, then the total area $\sum a_k$ must be also be finite; and if the total area $\sum a_k$ is infinite, then the total area $\sum b_k$ must be also be infinite.

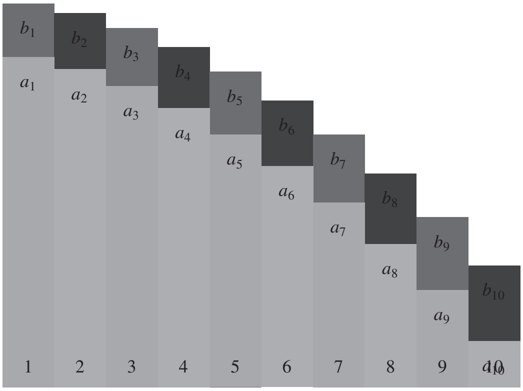


Figure 2.12. Comparison test.

Remark 2.8. As one would expect, it is not essential in Theorem 2.11 that the condition $a_k \leq b_k$ hold for all k, as stated; the conclusions of the theorem remain true if this condition is eventually true.

There are two steps required for using the comparison test to determine whether a series $\sum u_k$ with positive terms converges:

- Guess at whether the series $\sum u_k$ converges or diverges.
- Find a series that proves the guess to be correct. That is, if the guess is divergent, we must find a divergent series whose terms are smaller than the corresponding therms of $\sum u_k$, and if the guess is convergent, we must find a convergent series whose terms are bigger than the corresponding terms of $\sum u_k$.

In most cases, the series $\sum u_k$ being considered will have its general term u_k expressed as a fraction. To help with the guessing process in the first step, we have formulated two principles that are based on the form of the denominator for u_k. These principles sometimes suggest whether a series is likely to converge or diverge. We have called these informal principles or corollaries because they are not intended as formal theorems. In fact, we will not guarantee that they always work. However, they work often enough to be useful.

Proposition 2.1. *Constants*

Constant summands in the denominator of u_k can usually be deleted without affecting the convergence or divergence of a series. ■

Proposition 2.2. *Polynomial*

If a polynomial in k appears as a factor in the numerator or denominator of u_k, all but the leading term in the polynomial can usually be discarded without affecting the convergence or divergence of the series. ■

Example 2.20. Comparison Test

Use the comparison test to determine whether the following series converge or diverge

$$a) \sum_{k=1}^{\infty} \frac{1}{\sqrt{k} - \frac{1}{2}}$$

$$b) \sum_{k=1}^{\infty} \frac{1}{2\,k^2 + k}$$

Solution 2.20. a) According to Proposition 2.1, we should be able to drop the constant in the denominator without affecting the convergence or divergence. Thus, the series is likely to behave like

$$\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$$

which is a divergent hyperharmonic series because $p = 1/2$.

Remark 2.9. Power series $\sum_{k=1}^{\infty} \frac{1}{k^p}$ converge for $p > 1$ and diverge for $0 < p \leq 1$.

Thus, we will guess that the given series diverges and try to prove this by finding a divergent series that is smaller than the given series. However, the simplified series already does the trick since

$$\frac{1}{\sqrt{k} - \frac{1}{2}} > \frac{1}{\sqrt{k}} \qquad \text{for } k = 1,\ 2,\ 3,\ \ldots.$$

Thus, we have proved that the given series diverges.

b) According to Proposition 2.2, we should be able to discard all but the leading term in the polynomial without affecting the convergence or divergence. Thus, the given series is likely to behave like

$$\sum_{k=1}^{\infty} \frac{1}{2\,k^2} = \frac{1}{2} \sum_{k=1}^{\infty} \frac{1}{k^2}$$

which converges since it is a convergent hyperharmonic series with $p = 2$. Thus, we will guess that the given series converges and try to prove this by finding a convergent series that is bigger than the given series. However, the derived simplified series does the trick since

$$\frac{1}{2\,k^2 + k} < \frac{1}{2\,k^2} \qquad \text{for } k = 1,\ 2,\ 3,\ \ldots.$$

Thus, we have proved that the given series converges.▲

2.5.4 The Limit Comparison Test

In the last example the Proposition 2.1 and 2.2 provided the guess about convergence or divergence as well as the series needed to apply the comparison test. Unfortunately, it is not always so straightforward to find the series required for comparison, so we will now consider an alternative to the comparison test that is usually easier to apply.

Theorem 2.12. *Limit Comparison Test*

Let $\sum a_k$ and $\sum b_k$ be series with positive terms and suppose that

$$\rho = \lim_{k\to\infty} \frac{a_k}{b_k}.$$

If ρ is finite and $\rho > 0$, then the series both converge or both diverge. ■

Example 2.21. Limit Comparison Test

Use the limit comparison test to determine whether the following series converge or diverge

$$a) \sum_{k=1}^{\infty} \frac{1}{\sqrt{k} - 1}$$

$$b) \sum_{k=1}^{\infty} \frac{1}{2k^2 + k}$$

$$c) \sum_{k=1}^{\infty} \frac{3k^3 - 2k^2 + 4}{k^7 - k^3 + 2}$$

Solution 2.21. a) As in Example 2.20, Proposition 2.1 suggests that the series is likely to behave like the divergent hypergeometric series. To prove that the given series diverges, we will apply the limit comparison test with

$$a_k = \frac{1}{\sqrt{k} - 1} \text{ and } b_k = \frac{1}{\sqrt{k}}$$

We obtain

$$\rho = \lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{\sqrt{k}}{\sqrt{k} - 1} = \lim_{k\to\infty} \frac{1}{1 - \frac{1}{\sqrt{k}}} = 1.$$

Since ρ is finite and positive it follows from Theorem 2.12 that the given series diverges.

b) As in Example 2.20, Proposition 2.2 suggests that the series is likely to behave like the convergent hypergeometric series. To prove that the given series converges, we will apply the limit comparison test with

$$a_k = \frac{1}{2\,k^2 + k} \qquad \text{and} \qquad b_k = \frac{1}{2\,k^2}$$

We obtain

$$\rho = \lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{2\,k^2}{2\,k^2 + k} = \lim_{k\to\infty} \frac{2}{2 + \frac{1}{k}} = 1$$

Since ρ is finite and positive, it follows from Theorem 2.12 that the given series converges, which agree with the conclusion reached in Example 2.20 using the comparison test.

c) From Proposition 2.1, the series is likely to behave like

$$\sum_{k=1}^{\infty} \frac{3\,k^3}{k^7} = \sum_{k=1}^{\infty} \frac{3}{k^4} \tag{2.31}$$

which converges since it is a constant times a convergent hypergeometric series. Thus the given series is likely to converge. To prove this, we will apply the limit comparison test to series (2.31) and the given series in c) from above. We obtain

$$\rho = \lim_{k\to\infty} \frac{\frac{3\,k^3 - 2\,k^2 + 4}{k^7 - k^3 + 2}}{\frac{3}{k^4}} = \lim_{k\to\infty} \frac{3\,k^7 - 2\,k^6 + 4\,k^4}{3\,k^7 - 3\,k^3 + 6} = 1.$$

Since ρ is finite and nonzero, it follows from Theorem 2.12 that the given series converges, since (2.31) converges.▲

2.5.5 The Ratio Test

The comparison test and the limit comparison test hinge on first making a guess about convergence and then finding an appropriate series for comparison, both of which can be difficult tasks in cases where the Proposition 2.1 and 2.2 cannot be applied. In such cases the next test can often be used, since it works exclusively with the terms of the given series—it requires neither an initial guess about convergence nor the discovery of a series for comparison.

Theorem 2.13. *Ratio Test*

Let $\sum u_k$ be a series with positive terms and suppose that

$$\rho = \lim_{k\to\infty} \frac{u_{k+1}}{u_k}$$

a) If $\rho < 1$, the series converges.

b) If $\rho > 1$ or $\rho = +\infty$, the series diverges.

c) If $\rho = 1$, the series may converge or diverge, so that another test must be tried.■

Example 2.22. Ratio Test

Use the ratio test to determine whether of the following series converge or diverge.

a) $\sum_{k=1}^{\infty} \frac{1}{k!}$

b) $\sum_{k=1}^{\infty} \frac{k}{2^k}$

c) $\sum_{k=1}^{\infty} \frac{k^k}{k!}$

d) $\sum_{k=1}^{\infty} \frac{(2k)!}{4^k}$

e) $\sum_{k=1}^{\infty} \frac{1}{2k-1}$

Solution 2.22. a) The series converges, since

$$\rho = \lim_{k\to\infty} \frac{u_{k+1}}{u_k} = \lim_{k\to\infty} \frac{1/(k+1)!}{1/k!} = \lim_{k\to\infty} \frac{k!}{(k+1)!} = \lim_{k\to\infty} \frac{1}{(k+1)} = 0 < 1.$$

b) The series converges, since

$$\rho = \lim_{k\to\infty} \frac{u_{k+1}}{u_k} = \lim_{k\to\infty} \frac{k+1}{2^{k+1}} \frac{2^k}{k} = \frac{1}{2} \lim_{k\to\infty} \frac{k+1}{k} = \frac{1}{2} < 1.$$

c) The series diverges, since

$$\rho = \lim_{k\to\infty} \frac{u_{k+1}}{u_k} = \lim_{k\to\infty} \frac{(k+1)^{k+1}}{(k+1)!} \frac{k!}{k^k} = \lim_{k\to\infty} \frac{(k+1)^k}{k^k} = \lim_{k\to\infty} \left(1 + \frac{1}{k}\right)^k = e > 1.$$

d) The series diverges, since

$$\rho = \lim_{k\to\infty} \frac{u_{k+1}}{u_k} =$$

$$\lim_{k\to\infty} \frac{(2(k+1))!}{4^{k+1}} \frac{4^k}{(2k)!} = \lim_{k\to\infty} \left(\frac{(2k+2)!}{(2k)!} \frac{1}{4}\right) = \frac{1}{4} \lim_{k\to\infty} (2k+2)(2k+1) = \infty.$$

e) The ratio test is of no help since

$$\rho = \lim_{k\to\infty} \frac{u_{k+1}}{u_k} = \lim_{k\to\infty} \frac{1}{2(k+1)-1} \frac{2k-1}{1} = \lim_{k\to\infty} \frac{2k-1}{2k+1} = 1.$$

However, the integral test proves that the series diverges since

$$\int_1^\infty \frac{1}{2x-1} \, dx$$

```
Integrate::idiv :
 Integral of    1     does not converge on {1, ∞}. More…
             -1 + 2 x
```

$$\int_1^\infty \frac{1}{2x-1} \, dx$$

Both of the comparison test and the limit comparison test would also have worked here.▲

2.5.6 The Root Test

In cases where it is difficult or inconvenient to find the limit required for the ratio test, the next test is sometimes useful.

Theorem 2.14. *Root Test*

Let $\sum u_k$ be a series with positive terms and suppose that

$$\rho = \lim_{k\to\infty} \sqrt[k]{u_k} = \lim_{k\to\infty} (u_k)^{1/k}$$

a) If $\rho < 1$, the series converges.

b) If $\rho > 1$ or $\rho = +\infty$, the series diverges.

c) If $\rho = 1$, the series may converge or diverge, so that another test must be tried.■

Example 2.23. Root Test

Us the root test to determine whether the following series converge or diverge.

$$a) \sum_{k=1}^{\infty} \left(\frac{4k-5}{2k+1}\right)^k$$

$$b) \sum_{k=1}^{\infty} \frac{1}{(\ln(k+1))^k}$$

Solution 2.23. a) The series diverges, since

$$\rho = \lim_{k\to\infty} (u_k)^{1/k} = \lim_{k\to\infty} \frac{4\,k-5}{2\,k+1} = 2 > 1.$$

b) The series converges, since

$$\rho = \lim_{k\to\infty} (u_k)^{1/k} = \lim_{k\to\infty} \frac{1}{\ln(k+1)} = 0 < 1. \blacktriangle$$

2.5.7 Tests and Exercises

The following two subsections serve to test your understanding of the last sections. Work first on the test examples and then try to solve the exercises.

2.5.7.1 Test Problems

T1. What is the nth-Term Test for Divergence? What is the idea behind the test?

T2. What can be said about term-by-term sums and differences of convergent series? About constant multiples of convergent and divergent series?

T3. What happens if you add a finite number of terms to a convergent series? A divergent series? What happens if you delete a finite number of terms from a convergent series? A divergent series?

T4. Under what circumstances will an infinite series of non negative terms converge? Diverge? Why study series of non negative terms?

T5. What is the Integral Test? What is the reasoning behind it? Give an example of its use.

2.5.7.2 Exercises

E1. Which of the series converge, and which diverge? Give reasons for your answers. (When you check an answer, remember that there may be more than one way to determine the series' convergence or divergence.)

a. $\sum_{n=2}^{\infty} \frac{\ln(n)}{n}$,

b. $\sum_{n=1}^{\infty} \frac{-8}{n}$,

c. $\sum_{n=1}^{\infty} \frac{3}{\sqrt{n}}$,

d. $\sum_{n=1}^{\infty} e^{-n}$,

e. $\sum_{n=1}^{\infty} \frac{1}{10^n}$,

f. $\sum_{n=1}^{\infty} \frac{6}{n+1}$,

g. $\sum_{n=1}^{\infty} -\frac{1}{8^n}$,

h. $\sum_{n=1}^{\infty} -\frac{8}{n}$.

E2. Draw a picture to show that

$$\sum_{n=2}^{\infty} \frac{1}{n^{1.3}} < \int_1^{\infty} \frac{1}{x^{1.3}}\,dx \tag{1}$$

What can you conclude about the series?

E3. Determine whether the series is convergent or divergent.

a. $1 + \frac{1}{8} + \frac{1}{27} + \frac{1}{64} + \frac{1}{125} + \ldots$,

b. $1 + \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} + \frac{1}{4\sqrt{4}} + \ldots$,

c. $\frac{1}{5} + \frac{1}{8} + \frac{1}{11} + \frac{1}{14} + \frac{1}{17} + \ldots$,

$\sum_{n=1}^{\infty} \frac{n^2}{n^3+1}$,

e. $\sum_{n=1}^{\infty} \frac{5-2\sqrt{n}}{n^3}$,

f. $\sum_{n=1}^{\infty} \frac{\ln(n)}{n^3}$,

g. $\sum_{n=3}^{\infty} \frac{n^2}{e^n}$.

E4. Find all values of α for which the following series converges.

$$\sum_{n=1}^{\infty} \left(\frac{\alpha}{n} - \frac{1}{n+1}\right). \tag{2}$$

E5. Determine whether the series converges or diverges.

a. $\sum_{n=1}^{\infty} \frac{\cos(n)^2}{n^2+1}$,

b. $\sum_{n=0}^{\infty} \frac{1+\sin(x)}{10^n}$,

c. $\sum_{n=1}^{\infty} \frac{2+(-1)^n}{n\sqrt{n}}$,

d. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3+1}}$,

e. $\sum_{n=1}^{\infty} \frac{n+4^n}{n+6^n}$,

f. $\sum_{n=1}^{\infty} \frac{\arctan(n)}{n^{1.5}}$.

E6. Calculate the first 10 partial sums of the series and graph both the sequence of terms and the sequence of partial sums on the same screen. Estimate the error in using the 10th partial sum to approximate the total sum.

a. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^{3/2}}$,

b. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3}$.

E7. Are there any values of x for which $\sum_{n=1}^{\infty} 1/(n\,x)$ converges? Give reasons for your answer.

E8. Neither the Ratio nor the Root Test helps with p-series. Try them on

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \tag{3}$$

and show that both tests fail to provide information about convergence.

E9. Let

$$a_n = \begin{cases} n/2^n & \text{if } n \text{ is } a \text{ prime number} \\ 1/2^n & \text{otherwise} \end{cases} \tag{4}$$

Does Σa_n converge? Give reasons for your answer.

E10 Approximate the sum of the series correct to four decimal places.

a. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^6}$,

b. $\sum_{n=1}^{\infty} \frac{(-1)^n}{3^n\,(n-1)!}$,

c. $\sum_{n=1}^{\infty} \frac{(-1)^n\, n^2}{n!\, e^n}$.

2.6 Power Series and Taylor Series

2.6.1 Definition and Properties of Series

In Section 2.2.2 we defined the n^{th} Maclaurin polynomial for a function f as

$$\sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k = f(0) + f'(0)\, x + \frac{f''(0)}{2!} x^2 + \ldots + \frac{f^{(n)}(0)}{n!} x^n \tag{2.32}$$

and the n^{th} Taylor polynomial for f about $x = x_0$ as

$$\sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!} (x - x_0)^2 + \ldots + \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n. \tag{2.33}$$

Since then we have gone on to consider sums with an infinite number of terms, so it is not a big step to extend the notion of Maclaurin and Taylor polynomials to series by not stopping the summation index at n. Thus we have the following Definition 2.7.

Definition 2.7. *Taylor and Maclaurin Series*

If f has derivatives of all order at x_0, then we call the series

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!} (x - x_0)^2 + \ldots + \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k + \ldots.$$

the Taylor series for f about $x = x_0$. In the special case where $x_0 = 0$, this series becomes

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = f(0) + f'(0)\, x + \frac{f''(0)}{2!} x^2 + \ldots + \frac{f^{(k)}(0)}{k!} x^k + \ldots$$

in which case we call it the Maclaurin series of f.■

Note that the n^{th} Maclaurin and Taylor polynomials are the n^{th} partial sums for the corresponding Maclaurin and Taylor series.

A generalization of Maclaurin and Taylor series is achieved by replacing the expansion coefficients by arbitrary constants $c_0, c_1, c_2, c_3, \ldots$. The related series are so-called power series about $x_0 = 0$ or about a certain finite value x_0 which have representations as

Definition 2.8. *Power Series*

A series with arbitrary expansion coefficients c_0, c_1, c_2, … is called a power series about $x_0 = 0$ or about an arbitrary value of x_0 if the series is given as

$$\sum_{k=0}^{\infty} c_k\,(x-x_0)^k = c_0 + c_1(x-x_0) + c_2(x-x_0)^2 + \ldots + c_k(x-x_0)^k + \ldots.$$

or

$$\sum_{k=0}^{\infty} c_k\,x^k = c_0 + c_1\,x + c_2\,x^2 + \ldots + c_k\,x^k + \ldots.$$

Power series reduce to Taylor or Maclaurin series if the expansion coefficients are $c_k = \frac{f^{(k)}(x_0)}{k!}$ for Taylor or $c_k = \frac{f^{(k)}(0)}{k!}$ for Maclaurin series, respectively. ■

If a numerical value is substituted for x in a power series $\sum_{k=0}^{\infty} c_k\,x^k$, then the resulting series of numbers may either converge or diverge. This leads to the problem of determining the set of x-values for which a given power series converges; this is called its **convergence set**.

The main result on convergence of a power series in $x - x_0$ can be formulated as follows.

Theorem 2.15. *Radius of Convergence*

For a power series $\sum_{k=0}^{\infty} c_k(x-x_0)^k$, exactly one of the following statements is true:

a) The series converges only for $x = x_0$.

b) The series converges absolutely (and hence converges) for all real values of x.

c) The series converges absolutely (and hence converges) for all x in some finite open interval $(x_0 - R,\, x_0 + R)$ and diverges if $x < x_0 - R$ or $x > x_0 + R$. At either of the values $x = x_0 - R$ or $x = x_0 + R$, the series may converge absolutely, converge conditionally, or diverge, depending on the particular series. ■

Remark 2.10. The same theorem is valid for $x_0 = 0$ which is related to power series of the Maclau rin type.

It follows from Theorem 2.15 that the set of values for which a power series in $x - x_0$ converges is always an interval centered at $x = x_0$; we call this the **interval of convergence**. In part a) of Theorem 2.15 the interval of convergence reduces to the single value $x = x_0$, in which case we say that the series has radius of convergence $R = 0$; in part b) the interval of convergence is infinite (the entire real line), in which case we say that the series has **radius of convergence $R = \infty$**; and in part c) the interval extends between $x_0 - R$ and $x_0 + R$, in which case we say that the series has **radius of convergence R**.

Example 2.24. Convergence of a Power Series

Find the interval of convergence and radius of convergence of the series

$$\sum_{k=1}^{\infty} \frac{(x-5)^k}{k^2}.$$

Solution 2.24. We apply the ratio test for absolute convergence.

$$\rho = \lim_{n\to\infty} \left| \frac{u_{k+1}}{u_k} \right| = \lim_{k\to\infty} \left| \frac{(x-5)^{k+1}}{(k+1)^2} \frac{k^2}{(x-5)^k} \right| = \lim_{k\to\infty} \left(\left| x-5 \right| \left(\frac{k}{k+1} \right)^2 \right) =$$

$$\left| x-5 \right| \lim_{k\to\infty} \left(\frac{1}{1+\frac{1}{k}} \right)^2.$$

Thus, the series converges absolutely if $|x-5| < 1$, $\Leftrightarrow -1 < x-5 < 1$, $\Leftrightarrow 4 < x < 6$. The series diverges if $x < 4$ or $x > 6$.

To determine the convergence behavior at the endpoints $x = 4$ and $x = 6$, we substitute these values in the given series. If $x = 6$, the series becomes

$$\sum_{k=1}^{\infty} \frac{1^k}{k^2} = \sum_{k=1}^{\infty} \frac{1}{k^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots$$

which is a convergent hyperharmonic series with $p = 2$. If $x = 4$, the series becomes

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2} = -1 + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} \ldots$$

Since this series converges absolutely, the interval of convergence for the given series is [4, 6]. The radius of convergence is $R = 1$.▲

If a function f is expressed as a power series on some interval, then we say that f is represented by the power series on that interval. Sometimes new functions actually originate as power series, and the properties of the function are developed by working with their power series representations. For example, the functions

$$J_0(x) = \sum_{k=1}^{\infty} \frac{(-1)^k}{2^{2k}\,(k!)^2} x^{2k} = 1 - \frac{x^2}{2^2\,(1!)^2} + \frac{x^4}{2^4\,(2!)^2} - \frac{x^6}{2^6\,(3!)^2} + \ldots$$

and

$$J_1(x) = \sum_{k=1}^{\infty} \frac{(-1)^k}{2^{2(k+1)}(k+1)!\; k!} x^{2k+1} = \frac{x}{2} - \frac{x^3}{2^3\,(1!)\,(2!)} + \frac{x^5}{2^5\,(2!)\,(3!)} - \ldots$$

which is called the Bessel function in honor of the German mathematician and astronomer Friedrich Wilhelm Bessel (1784-1846), arise naturally in the study of planetary motion and in various problems that involve heat flow.

To find the domain of these functions, we must determine where their defining power series converge. For example, in the case $J_0(x)$ we have

$$\rho = \lim_{k\to\infty} \left| \frac{u_{k+1}}{u_k} \right| = \lim_{k\to\infty} \left| \frac{x^{2(k+1)}}{2^{2(k+1)}((k+1)!)^2} \frac{2^{2k}((k)!)^2}{x^{2k}} \right| = \lim_{k\to\infty} \left| \frac{x^2}{4(k+1)^2} \right| = 0 < 1$$

so that the series converge for all x; that is, the domain of $J_0(x)$ is $(-\infty, +\infty)$. We leave it as an exercise to show that the power series for $J_1(x)$ also converges for all x.

2.6.2 Differentiation and Integration of Power Series

We begin considering the following problem: Suppose that a function f is represented by a power series on an open interval. How can we use the power series to find the derivative of f on that interval? The other question we deal with is how can f be integrated?

The solution to the first problem can be motivated by considering the Maclaurin series for $\sin(x)$:

$$\sin(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{(2k+1)} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots \quad -\infty < x < +\infty. \tag{2.34}$$

Of course, we already know that the derivative of $\sin(x)$ is $\cos(x)$; however, we are concerned here with using the Maclaurin series to deduce this. The solution is easy—all we need to do is differentiating the Maclaurin series term by term and observe that the resulting series is a Maclaurin series for $\cos(x)$:

$$\begin{gathered}\frac{d}{dx}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots\right) = \\ 1 - \frac{3x^2}{3!} + \frac{5x^4}{5!} - \frac{7x^6}{7!} + \ldots = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots = \cos(x)\end{gathered} \tag{2.35}$$

Here is another example of a function, the exponential e^x. The series expansion up to order 5 gives us

exponential = Series[e^x, {x, 0, 5}]

$$1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + O(x^6)$$

The differentiation of this series yields

$$\frac{\partial \textbf{exponential}}{\partial x}$$

$$1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + O\left(x^5\right)$$

which is again the series of the exponential function.

The preceding computations suggest that if a function f is represented by a power series on an open interval, then a power series representation of f' on that interval can be obtained by differentiating the power series for f term by term. This is stated more precisely in the following theorem, which we give without proof.

Theorem 2.16. *Differentiation of Power Series*

Suppose that a function f is represented by a power series in $x - x_0$ that has a nonzero radius of convergence R; that is

$$f(x) = \sum_{k=0}^{\infty} c_k (x - x_0)^k \qquad (x_0 - R < x < x_0 + R).$$

Then:

a) The function f is differentiable on the interval $(x_0 - R,\, x_0 + R)$.

b) If the power series representation for f is differentiated term by term, then the resulting series has radius of convergence R and converges to f' on the interval $(x_0 - R,\, x_0 + R)$; that is

$$f'(x) = \sum_{k=0}^{\infty} \frac{d}{\mathrm{dx}} \left(c_k (x - x_0)^k\right) \qquad x_0 - R < x < x_0 + R. \blacksquare$$

This theorem has an important implication about the differentiability of functions that are represented by power series. According to Theorem 2.16, the power series for f' has the same radius of convergence as the power series for f, and this means that the theorem can be applied to f' as well as f. However, if we do this, then we conclude that f' is differentiable on the interval $(x_0 - R,\, x_0 + R)$, and the power series for f'' has the same radius of convergence as the power series for f and f'. We can now repeat this process ad infinity, applying the theorem successively to f'', f''',…, $f^{(n)}$,… to conclude that f has derivatives of all orders in the interval $(x_0 - R,\, x_0 + R)$. Thus, we have established the following result.

Theorem 2.17. *Order of Derivatives for Power Series*

If a function f can be represented by a power series in $x - x_0$ with a nonzero radius of convergence R, then f has derivatives of all orders on the interval $(x_0 - R,\, x_0 + R)$. $\blacksquare$

In short, it is only the most well-behaved functions that can be represented by power series; that is, if a function f does not possess derivatives of all orders on an interval $(x_0 - R, x_0 + R)$, then it cannot be represented by a power series in $x - x_0$ on that interval.

Example 2.25. Derivatives of Power Series

We defined the Bessel function $J_0(x)$ as

$$J_0(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{2k}(k!)^2} x^{2k} \tag{2.36}$$

with an infinite radius of convergence. Thus $J_0(x)$ has derivatives of all orders on the interval $(-\infty, +\infty)$, and these can be obtained by differentiating the series term by term. For example, if we write (2.36) as

$$J_0(x) = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{2^{2k}(k!)^2} x^{2k}$$

and differentiate term by term, we obtain

$$J_0'(x) = \sum_{k=0}^{\infty} \frac{(-1)^k (2k)}{2^{2k}(k!)^2} x^{2k-1} = \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{2k-1}\, k!\,(k-1)!} x^{(2k-1)}.\blacktriangle$$

Remark 2.11. The computations in this example use some techniques that are worth noting. Firs when a power series is expressed in sigma notation, the formula for the general term of the series will often not be of a form that can be used for differentiating the constant term. Thus, if the series has a nonzero constant term, as here, it is usually a good idea to split it off from the summation before differentiating. Second, observe how we simplified the final formula by canceling the factor k from the factorial in the denominator. This is a standard simplification technique.

Since the derivative of a function that is represented by a power series can be obtained by differentiating the series term by term, it should not be surprising that an anti derivative of a function represented by a power series can be obtained by integrating the series term by term. For example, we know that $\sin(x)$ is an anti derivative of $\cos(x)$. Here is how this result can be obtained by integrating the Maclaurin series for $\cos(x)$ term by term:

$$\int \cos(x)\, dx = \int \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots\right) dx =$$

$$\left(x - \frac{x^3}{(3)\,2!} + \frac{x^5}{(5)\,4!} - \frac{x^7}{(7)\,6!} + \ldots\right) + C == \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots\right) + C = \sin(x) + C.$$

The same idea applies to definite integrals. For example, by direct integration we have

$$\int_0^1 \frac{1}{x^2+1}\,dx = \int_0^1 \text{Series}\left[\frac{1}{x^2+1}, \{x, 0, 6\}\right] dx$$

$$\frac{\pi}{4} = \int_0^1 \left(1 - x^2 + x^4 - x^6 + O\left(x^7\right)\right) dx$$

The preceding computations are justified by the following Theorem 2.18, which we give without proof.

Theorem 2.18. *Integration of Power Series*

Suppose that a function f is represented by a power series in $x - x_0$ that has a nonzero radius of convergence R; that is

$$f(x) = \sum_{k=0}^{\infty} c_k (x - x_0)^k \qquad x_0 - R < x < x_0 + R.$$

a) If the power series representation of f is integrated term by term, then the resulting series has radius of convergence R and converges to an anti derivative for $f(x)$ on the interval $(x_0 - R, x_0 + R)$; that is

$$\int f(x)\,dx = \sum_{k=0}^{\infty} \left(\frac{c_k}{k+1} (x - x_0)^{k+1}\right) + C \qquad x_0 - R < x < x_0 + R$$

b) If α and β are points in the interval $(x_0 - R, x_0 + R)$, and if the power series representation of f is integrated term by term from α to β, then the resulting series converges absolutely on the interval $(x_0 - R, x_0 + R)$ and

$$\int_\alpha^\beta f(x)\,dx = \sum_{k=0}^{\infty} \left(\int_\alpha^\beta c_k (x - x_0)^k\,dx\right). \blacksquare$$

Based on Theorems 2.18 and 2.17 there are different practical ways to generate power series for functions.

2.6.3 Practical Ways to Find Power Series

In this section we discuss different examples demonstrating how power series can be calculated in practice.

Example 2.26. Power Series for ArcTan

Find the Maclaurin series for $\arctan(x)$.

Solution 2.26. It would be tedious to find the Maclaurin series directly. A better approach is to start with the formula

$$\textbf{arcTan} = \int \frac{1}{x^2+1}\,dx$$

$\tan^{-1}(x)$

and integrate the Maclaurin series

$$\textbf{serexp} = \textbf{Series}\left[\frac{1}{x^2+1}, \{x, 0, 7\}\right]$$

$1 - x^2 + x^4 - x^6 + O(x^8)$

term by term. This yields

$$\textbf{intArcTan} = \int \textbf{Normal[serexp]}\,dx$$

$$-\frac{x^7}{7} + \frac{x^5}{5} - \frac{x^3}{3} + x$$

or

arcTan == intArcTan

$$\tan^{-1}(x) = -\frac{x^7}{7} + \frac{x^5}{5} - \frac{x^3}{3} + x$$

which is defined for $-1 < x < 1$.▲

Remark 2.12. Observe that neither Theorem 2.18 nor Theorem 2.17 addresses what happens at the endpoints of the interval of convergence. However, it can be proven that if the Taylor series for f about $x = x_0$ converge to $f(x)$ for all x in the interval $(x_0 - R, x_0 + R)$, and if the Taylor series converges at the right endpoint $x_0 + R$, then the value that it converges to at that point is the limit of $f(x)$ as $x \to x_0 + R$ from the left; and if the Taylor series converges at the left endpoint $x_0 - R$, then the value that it converges to at that point is the limit of $f(x)$ as $x \to x_0 - R$.

Taylor series provide an alternative to Simpson's rule and other numerical methods for approximating definite integrals.

Example 2.27. Approximation of Integrals

Approximate the integral

$$\int_0^1 e^{-x^2}\,dx$$

to three decimal place accuracy by expanding the integrand in a Maclaurin series and integrating term by term.

Solution 2.27. The simples way to obtain the Maclaurin series foe e^{-x^2} is to replace x by $-x^2$ in the Maclaurin series of the exponential function e^x.

exp = Normal[Series[e^x, {x, 0, 5}]]

$$\frac{x^5}{120}+\frac{x^4}{24}+\frac{x^3}{6}+\frac{x^2}{2}+x+1$$

The replacement is done by substituting $-x^2$ for x in the next line

reExp = exp /. $x \to -x^2$

$$-\frac{x^{10}}{120}+\frac{x^8}{24}-\frac{x^6}{6}+\frac{x^4}{2}-x^2+1$$

Therefore in the integral we find

intReRxp = $\int$reExp dx

$$-\frac{x^{11}}{1320}+\frac{x^9}{216}-\frac{x^7}{42}+\frac{x^5}{10}-\frac{x^3}{3}+x$$

This can be generally be written as

$$\int_0^1 e^{-x^2}\,dx=\sum_{k=0}^{\infty}\frac{(-1)^k}{(2\,k+1)\,k\,!}$$

which represents a convergent series. The requirement that the accuracy should be three decimal places requires that the remainder is smaller than 0.0005. To find a formula for such an estimation we represent the largest part of the remainder as the difference between the integral and the n^{th} partial sum as

$$\int_0^1 e^{-x^2}\,dx-s_n<\frac{1}{(2\,(n+1)+1)\,(n+1)\,!}=0.0005$$

To determine the n value for which this condition is satisfied, we solve the following relation

FindRoot$\left[(2\,(n+1)+1)\,(n+1)\,!=\frac{1}{0.0005},\ \{n,\,5\}\right]$

{$n \to 4.21824$}

which delivers the value 4.21. Since n should be an integer we select $n = 5$ as a sufficient value thus the approximation of the integral delivers

***N*[intReRxp /. $x \to 1$]**

0.746729

▲

The following examples illustrate some algebraic techniques that are sometimes useful for finding Taylor series.

Example 2.28. Multiplication of Series

Find the first three nonzero terms in the Maclaurin series for the function $f(x) = e^{-x^2} \arctan(x)$.

Solution 2.28. Using the series for e^{-x^2} and $\arctan(x)$ gives

exp = Normal$\left[\textbf{Series}\left[e^{-x^2}, \{x, 0, 7\}\right]\right]$

$$-\frac{x^6}{6} + \frac{x^4}{2} - x^2 + 1$$

arcTan = Normal$\left[\textbf{Series}\left[\tan^{-1}(x), \{x, 0, 7\}\right]\right]$

$$-\frac{x^7}{7} + \frac{x^5}{5} - \frac{x^3}{3} + x$$

The product is

Expand[arcTan exp]

$$\frac{x^{13}}{42} - \frac{11\,x^{11}}{105} + \frac{94\,x^9}{315} - \frac{71\,x^7}{105} + \frac{31\,x^5}{30} - \frac{4\,x^3}{3} + x$$

More terms in the series can be obtained by including more terms in the factors. Moreover, one can prove that a series obtained by this method converges at each point in the intersection of the intervals of convergence of the factors. Thus we can be certain that the series we have obtained converges for all x in the interval $-1 \leq x \leq 1$.▲

2.6.4 Tests and Exercises

The following two subsections serve to test your understanding of the last section. Work first on the test examples and then try to solve the exercises.

2.6.4.1 Test Problems

T1. What is a power series?

T2. What is the radius of convergence of a power series? How do you find it?

T3. What is the interval of convergence of a power series? How do you find it?

T4. How do you test a power series for convergence? What are the possible outcomes?

2.6.4.2 Exercises

E1. Find the first four terms of the binomial series for the functions:

a. $f(x) = \sqrt{1+x}$,

b. $f(x) = \left(1 + \frac{x}{2}\right)^{-2}$,

c. $f(x) = \left(1 - \frac{x}{2}\right)^{-2}$,

$f(x) = 1 \Big/ \sqrt{1 + x^2}$.

E2. Find the radius of convergence and interval of convergence of the series.

a. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n!} x^n$,

b. $\sum_{n=1}^{\infty} n^n x^n$,

c. $\sum_{n=1}^{\infty} \frac{10^n x^n}{n^3}$,

d. $\sum_{n=1}^{\infty} \frac{(x-3)^n}{n^2+1}$,

e. $\sum_{n=1}^{\infty} \frac{x^n}{5^n n^5}$,

f. $\sum_{n=1}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$.

E3. If k is a positive integer, find the radius of convergence of the series

$$\sum_{n=0}^{\infty} \frac{(n!)^k}{(k\,n)!} x^n. \tag{1}$$

E4. Graph the first several partial sums $s_n(x)$ of the series $\sum_{n=0}^{\infty} x^n$, together with the sum function $f(x) = 1/(1-x)$, on a common screen. On what interval do these partial sums appear to be converging to $f(x)$?

E5. Use series to evaluate the limits.

a. $\lim_{x \to 0} \frac{e^x - e^{-x}}{x}$,

b. $\lim_{x \to 0} \frac{e^x - (1+x)}{x^2}$,

c. $\lim_{x \to 0} \frac{x^2 - 4}{\ln(x-1)}$,

d. $\lim_{x \to 0} \frac{\ln(1+x^2)}{1-\cos(x)}$.

E6. Find the binomial series for the functions

a. $(1+x)^4$,

b. $\left(1+x^2\right)^3$,

c. $(1-2\,x)^3$,

d. $\left(1-\frac{x}{2}\right)^4$.

E7. Use series to estimate the integrals' values with an error of magnitude less than 10^{-3}. Use yous CAS to verify your results.

a. $\int_0^{0.4} \sin(x)^2 \, dx$,

b. $\int_0^{1/4} \sqrt{1+x^2} \, dx$,

c. $\int_0^{3/10} \frac{e^{-x}-1}{x} \, dx$,

d. $\int_0^{1/20} \frac{1}{\sqrt{1+x^4}} \, dx$.

E8. Find the first six nonzero terms of the Taylor series for

$$\operatorname{arcsinh}(x) = \int_0^x \frac{1}{\sqrt{1+t^2}} \, dt. \tag{2}$$

E9. The English mathematician Wallis discovered the formula

$$\frac{\pi}{4} = \frac{2 \times 4 \times 4 \times 6 \times 6 \times 8 \ldots}{3 \times 3 \times 5 \times 5 \times 7 \times 7 \ldots} \tag{3}$$

Find π to two decimal places with this formula.

E10 Show that the function

$$f(x) = \sum_{k=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \tag{4}$$

is a solution of the differential equation

$$f''(x) + f(x) = 0. \tag{5}$$

3 Differential Equations and Applications

The theory of differential equations is the most important discipline in the whole modern mathematics.

S. Lie, 1895

3.1 Introduction

Many of the principles in science and engineering concern relationships between changing quantities. Since rate of change are represented mathematically by derivatives, it should not be surprising that such principles are often expressed in terms of differential equations. We will discuss some important mathematical models that involve differential equations, and we will discuss some methods for solving and approximating solutions of some of the basic types of differential equations. However, we will only be able to touch the surface of this topic, leaving many important topics in differential equations to courses that are devoted completely to the subject.

Central to the subject of differential equations are not only the application of derivatives but also anti derivatives. Derivatives and anti derivatives are one of the main achievements mankind ever invented. Based on these two methods of calculus we are able to model nature and formulate laws which govern engineering applications. The connection between derivatives and anti derivatives is created by the most important theorem of calculus the Fundamental Theorem of Calculus, (see Chapter 5.2 Vol. I). This theorem stated in an earlier chapter links the two concepts.

Due to the importance of this theorem let us shortly recall the main ideas. Let us assume that f is a continuous function on the interval $I = [a, b]$ and A is the area under the graph of f along the interval $[a, b]$, then A is given as

$$A = \int_a^b f(x)\, dx. \tag{3.1}$$

This relation is shown in Figure 3.1

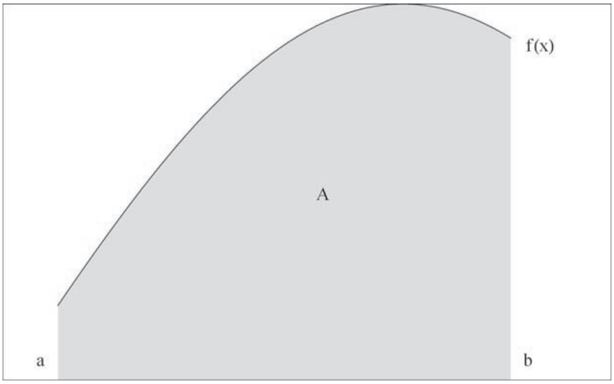


Figure 3.1. Representation of the integral as area.

Recall that the discussion of the anti derivative method suggested that if $A(x)$ is the area under the graph f from a to x (see Figure 3.2) then $A'(x) = f(x)$ and $A(a) = 0$ the area under the curve from a to a is the area above the single point a, and hence is zero. $A(b) = A$ represents the total area. The meaning of the variation of x in $A(x)$ is visualized in the following animation (animations are only effective in notebooks).

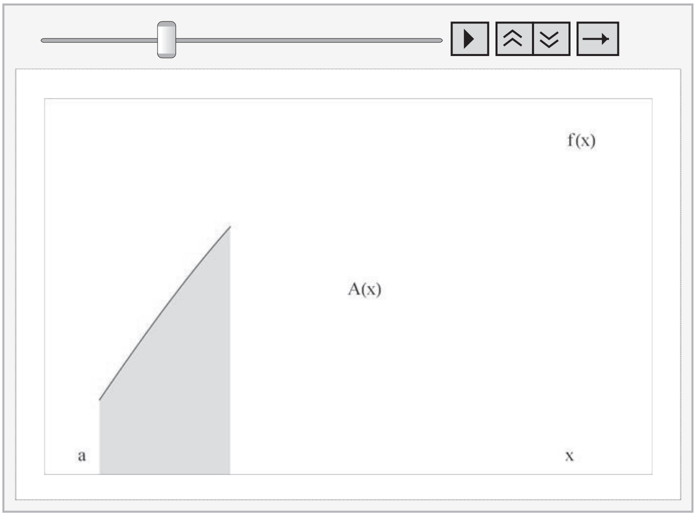


Figure 3.2. Representation of the integral as area where the upper boundary is not fixed.

The formula $A'(x) = f(x)$ states that $A(x)$ is an anti derivative of $f(x)$, which implies that every other anti derivative of $f(x)$ on $[a, b]$ can be obtained by adding a constant to $A(x)$. Accordingly, let

$$F(x) = A(x) + C \tag{3.2}$$

be any anti derivative of $f(x)$, and consider what happens when we subtract $F(a)$ from $F(b)$

$$F(b) - F(a) = A(b) + C - A(a) - C = A(b) - A(a) = A - 0 = A \tag{3.3}$$

Hence (3.1) can be expressed as

$$\int_a^b f(x)\, dx = F(b) - F(a). \tag{3.4}$$

In words this equation states:

Remark 3.1. The definite integral can be evaluated by finding any anti derivative of the integrand and then subtracting the value of this anti derivative at the lower limit of integration from its value at the upper limit of integration.

If on the other hand $A(x)$ is the area under the graph $y = f(x)$ over the interval $[a, b]$, then $A'(x) = f(x)$. But $A(x)$ can be expressed as the definite integral

$$A(x) = \int_a^x f(t)\, dt. \tag{3.5}$$

Thus the relationship $A'(x) = f(x)$ can be expressed as

$$\frac{d}{\mathrm{dx}}\left(\int_a^x f(t)\,dt\right) = f(x). \tag{3.6}$$

Which allows us to formulate the Fundamental Theorem of Calculus as:

Theorem 3.1. *Fundamental Theorem of Calculus*

If f is continuous on an interval $I = [a, b]$ and F is any anti derivative of f on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a). \tag{3.7}$$

In particular, if a is any number in $I = [a, b]$, then the function F defined by

$$F(x) = \int_a^x f(t)\,dt \tag{3.8}$$

is an anti derivative of f on I; that is $F'(x) = f(x)$ for each x in I, or in an alternative notation

$$\frac{d}{\mathrm{dx}}\left(\int_a^x f(t)\,dt\right) = f(x).\blacksquare \tag{3.9}$$

This theorem is the basis of introducing differential equations.

3.1.1 Tests and Exercises

The following two subsections serve to test your understanding of the last section. Work first on the test examples and then try to solve the exercises.

3.1.1.1 Test Problems

T1. Formulate the Fundamental Theorem of Calculus in your own words.

T2. Tell about applications of the Fundamental Theorem of Calculus.

3.1.1.2 Exercises

E1. Use the Fundamental Theorem of Calculus to find the derivative of the function.

a. $y = \int_0^{\tan(x)} \sqrt{t + \sqrt{t}}\;dt$,

b. $y = \int_{e^x}^{\infty} \sin(t)^5\,dt$,

c. $y = \int_{1-3x}^{1} \frac{v^3}{1+v^2}\,dv$,

d. $y = \int_1^{\cos(x)} \left(1 + u^4\right)^{12}\,du$,

E2. Evaluate the integral

a. $\int_0^{10} \frac{x-1}{\sqrt{x}}\,dx$,

b. $\int_0^1 \sinh(t)\,dt$,

c. $\int_0^1 \frac{5}{1+v^2}\,dv$,

d. $\int_1^{12} u^{-4}\,du$,

3.2 Basic Terms and Notations

Before we begin we need to introduce a simple classification of differential equations which will let us increase the complexity of the problems we consider in a systematic way.

3.2.1 Ordinary and Partial Differential Equations

The most significant distinction is between ordinary (ODE) and partial differential equations (PDE), and this depends on whether ordinary or partial derivatives occur.

Partial derivatives cannot occur when there is only one independent variable. The independent variables are usually the arguments of the function that we are trying to find, e.g. x in $f(x)$, t in $x(t)$, both x and y in $G(x, y)$. The most common independent variables we will use are x and t, and we will adopt a special shorthand notation for derivatives with respect to these variables: we will use a dot for d/dt, so that

$$\dot{z} = \frac{\mathrm{dz}}{\mathrm{dt}} \quad \text{and} \quad \ddot{z} = \frac{d^2 z}{\mathrm{dt}^2} \tag{3.10}$$

and a prime symbol for d/dx, so that

$$y' = \frac{\mathrm{dy}}{\mathrm{dx}} \quad \text{and} \quad y'' = \frac{d^2 z}{\mathrm{dx}^2}. \tag{3.11}$$

Usually we will prefer to use time as the independent variable.

In an ODE there is only one independent variable, for example the variable x in the equation

$$\frac{\mathrm{dy}}{\mathrm{dx}} = f(x) \tag{3.12}$$

specifying the slope of the graph of the function y or x. In

$$-\frac{\hbar^2}{2\,m}\frac{d^2\psi}{\mathrm{dx}^2} + V(x)\,\psi = \mathrm{E}\psi \tag{3.13}$$

where $\psi(x) = \alpha(x) + i\,\beta(x)$ is complex (this is the Schrödinger equation from quantum mechanics).

In a partial differential equation there is more than one independent variable and the derivatives are therefore partial derivatives, for example the temperature in a rod at position x and time t, $\theta(x, t)$, obeys the heat equation

$$\frac{\partial\theta(x, t)}{\partial t} = \kappa\,\frac{\partial^2\theta}{\partial x^2}. \tag{3.14}$$

Or the time dependent Schrödinger equation given by

$$-\frac{\hbar^2}{2\,m}\,\frac{\partial^2 \psi(x,\,t)}{\partial x^2} + V(x)\,\psi = i\,\hbar\,\frac{\partial \psi}{\partial t} \tag{3.15}$$

where the function ψ now becomes a field depending on two variables.

3.2.2 The Order of a Differential Equation

The **order** of a differential equation is the highest order derivative that occurs: the equation

$$\frac{\mathrm{dy}}{\mathrm{dx}} = f(x) \tag{3.16}$$

specifying the slope of a graph is first order, as is the following equation expressing energy conservation,

$$\frac{1}{2}\,m\,\dot{x}^2 + V(x) = E \tag{3.17}$$

($\frac{1}{2}\,m\,\dot{x}^2$ is the kinetic energy while $V(x)$ is the potential energy at a point x.
Newton's second law of motion

$$\frac{d^2\,x}{\mathrm{dt}^2} = F(x) \tag{3.18}$$

is second order; the equation

$$\frac{d^2}{\mathrm{dx}^2}\left(A(x)\,\frac{d^2}{\mathrm{dx}^2}\,u(x)\right) = f(x) \tag{3.19}$$

which occurs in theory of beam and in fact is the Euler-Bernoulli model for bending a beam representing a fourth order ODE.

To be more formal, an nth order ordinary differential equation for a function $y(x)$ is an equation of the form

$$F\left(\frac{d^n\,y}{\mathrm{dx}^n},\,\frac{d^{n-1}\,y}{\mathrm{dx}^{n-1}},\,\ldots,\,\frac{\mathrm{dy}}{\mathrm{dx}},\,y,\,x\right) = 0. \tag{3.20}$$

3.2.3 Linear and Nonlinear

Another important concept in the classification of differential equations is linearity. Generally, linear problems are relatively 'easy' (which means that we can find an explicit solution) and nonlinear problems are 'hard' (which means that we cannot solve them explicitly except in very particular cases).

An nth order ODE for $y(t)$ is said to be linear if it can be written in the form

$$a_n(t)\,\frac{d^n\,y}{dt^n} + a_{n-1}(t)\,\frac{d^{n-1}\,y}{dt^{n-1}} + \ldots + a_1(t)\,\frac{dy}{dt} + a_0(t)\,y = f(t) \tag{3.21}$$

i.e. only multiples of y and its derivatives occur. Such a linear equation is called homogeneous if $f(t) = 0$, and inhomogeneous if $f(t) \neq 0$.

3.2.4 Types of Solution

When we try to solve a differential equation we may obtain various possible types of solutions, depending on the equation. Ideally, perhaps, we would find a full explicit solution, in which the dependent variable is given explicitly as a combination of elementary functions of the independent variable, as in

$$y(t) = \sin(t) + t\,e^{2\,t}. \tag{3.22}$$

We can expect to be able to find such a fully explicit solution only for a very limited set of examples.

A little more likely is a solution in which y is still given directly as a function of t, but as an expression involving an integral, for example

$$y(t) = 1 + \int_0^t \sin(v)\,e^{-v^2}\,dv. \tag{3.23}$$

Here y is still an explicit function of t, but the integral cannot be evaluated in terms of elementary functions.

Sometimes, however, we will only be able to obtain an implicit form of the solution; this is when we obtain an equation that involves no derivatives and relates the dependent and independent variables. For example, the equation

$$\ln(y) + 6\ln(x) - e^{-y} - 7\,x + 4 = 0 \tag{3.24}$$

relates x and y, but cannot be solved explicitly for y as a function of x.

3.2.5 Tests and Exercises

The following two subsections serve to test your understanding of the last section. Work first on the test examples and then try to solve the exercises.

3.2.5.1 Test Problems

T1. What is a solution of a differential equation?

T2. What is the difference between linear and non linear differential equations?

T3. How is an initial value problem defined?

T4. What is the order of a differential equation?

T5. How do you recognize ordinary and partial differential equations?

3.2.5.2 Exercises

E1. Classify the following equations as ordinary or partial, give their order, and state whether they are linear or nonlinear. In each case identify the dependent and independent variables.

Bessel's equation (ν is a parameter)

$$x^2\, y'' + x\, y' + \left(x^2 - \nu^2\right) y = 0. \tag{1}$$

b. Burger's equation (ν is a parameter)

$$\frac{\partial u}{\partial t} - \nu\, \frac{\partial^2 u}{\partial x^2} + u\, \frac{\partial u}{\partial x} = 0. \tag{2}$$

c. The logistic population model (k is a parameter)

$$\frac{\mathrm{dp}}{\mathrm{dt}} = k\, p\, (1 - p). \tag{3}$$

d. The wave equation (c is a parameter)

$$\frac{\partial^2 u}{\partial t^2} = c^2\, \frac{\partial^2 u}{\partial x^2}. \tag{4}$$

3.3 Geometry of First Order Differential Equations

3.3.1 Terminology

A differential equation is an equation involving one or more derivatives of an unknown function. In this section we will denote the unknown function by $y = y(x)$ unless the differential equations arises from an application problem involving time, in which case we will denote it by $y = y(t)$. The order of a differential equation is the order of the highest derivative that it contains. Here are some examples:

Table 3.1. Different differential equations classified by their order.

Differential Equation	*Order*
$\frac{d}{\mathrm{dx}}\, y(x) = 3\, y(x)$	1
$\frac{d^2\, y(x)}{\mathrm{dx}^2} + y(x) = 2\, x^2$	2
$\frac{d^3\, y(x)}{\mathrm{dx}^3} + \frac{\mathrm{dy}(x)}{\mathrm{dx}} + \sin(y(x)) = 0$	3
$y' - x\, y(x) = 0$	1
$y'' + 2\, y' - y^2 = \sin(x)$	2

Examples for ordinary differential equations.

In the last two equations the derivatives of y are expressed in prime notation. You will usually be able to tell from the equation itself or the context in which it arises whether to interpret y' as dy/dx or as dy/dt.

Since the unknown function $y = y(x)$ depends only on one independent variable, we call the differential equation an ordinary differential equation. In case that the dependent variable depends on more than one independent variables and an equation contains a mixture of derivatives with respect to different independent variables, we call the differential equation a partial differential equation.

We begin by studying first order differential equations and we will assume that our equation is, or can be put, in the form

$$\frac{\mathrm{dy}}{\mathrm{dx}} = f(x,\ y). \tag{3.25}$$

where y is the dependent and x the independent variable. The problem her is: Given $f(x,\ y)$ find all functions $y(x)$ which satisfy the differential equation (3.25). We approach this problem in the following manner. A fundamental principle of mathematics is that the way to solve a new problem is to reduce it, in some manner, to a problem that we have already solved. In practice this usually entails successively simplifying the problem until it resembles one we have already solved. Since we are presently in the business of solving differential equations, it is advisable for us to take inventory and list all the differential equations we can solve. If we assume that our mathematical background consists of just elementary calculus then the very sad fact is that only first order differential equations of the following type can be solved at present

$$\frac{\mathrm{dy}}{\mathrm{dx}} = g(x) \tag{3.26}$$

where g is any integrable function of x. To solve Equation (3.26) simply integrate both sides with respect to x, which yields

$$y(x) = \int g(t)\,\mathrm{dt} + C. \tag{3.27}$$

Here C is an arbitrary constant of integration, and by $\int g(t)\,\mathrm{dt}$ we mean an anti derivative of g, that is, a function whose derivative is g. Thus, to solve any other differential equation we must somehow reduce it to the form (3.26). As we will see, this is impossible in most cases. Hence, we will not be able, without the aid of a computer, to solve most differential equations. It stands to reason, therefore, that to find those differential equations that we can solve, we should start with very simple equations. Experience has taught us that the simplest equations are those which are linear in the dependent variable y.

But before we go to solve these simple equation let us discuss two basic geometric principles helpful in the examination of first order ordinary differential equations. These two principles are related to the fact that the differential equation and the related function in (3.25) allow the interpretation that the equation exists in a space (manifold) which is generated by the coordinates x, y, and $y' = p$. This three dimensional manifold is also known as the skeleton of the differential equation. The triple $(x,\ y,\ p)$ defines the coordinates in the manifold. The related geometrical object to these coordinates are the direction field and the skeleton of the differential equation.

3.3.2 Functions of Two Variables

We will be concerned here with first order equations that are expected with the derivative by itself on one side of the equation. For example

$$y' = x^3 \tag{3.28}$$

and

$$y' = \sin(x\, y). \tag{3.29}$$

The first of these equations involves only x on the right hand side, so it has the form $y' = f(x)$. However, the second equation involves both x and y on the right side, so it has the form $y' = f(x, y)$, where the symbol $f(x, y)$ stands for a function of the two variables x and y. Later in this text we will study functions of two variables in more depth, but for now it will suffice to think of $f(x, y)$ as a formula that produces a unique output when values of x and y are given as input. For example if

$$f(x, y) = x^2 + 3\, y \tag{3.30}$$

and if the inputs are $x = 2$ and $y = -4$, then the output is

$$f(2, -4) = 2^2 + 3\,(-4) = 4 - 14 = -8. \tag{3.31}$$

Remark 3.2. In applied problems involving time, it is usual to use t as the independent variable, in which case we would be concerned with equations of the form $y' = f(t, y)$, where $y' = \mathrm{dy}/\mathrm{dt}$.

3.3.3 The Skeleton of an Ordinary Differential Equation

Let us consider the first order differential equation (3.25) in the implicit representation

$$F(x, y(x), y') = 0, \tag{3.32}$$

where $y' = \frac{dy}{dx}$ is the first order derivative. As discussed in the previous section there are two components of a given first order ordinary differential equation the solution and the geometric frame of the equation. The two components of an ordinary differential equation are defined as follows:

Definition 3.1. *The Skeleton*

The skeleton of a differential equation is defined as the surface

$$F(x, y, p) = 0 \tag{3.33}$$

in the space of the independent variables x, y, and p. y and p denote the sets of dependent variables and derivatives, respectively. The corresponding differential equation follows from the skeleton with the replacement of p by the derivative y'. ■

This definition is nothing more than the introduction of a 3 dimensional manifold $\mathfrak{M}$. This manifold is spanned by the independent and dependent variables (x, y) with the third direction denoting the first derivative $y' = p$. The second property of a differential equation is the class of solutions.

The class of solutions is defined as follows:

Definition 3.2. *Class of Solutions*

A solution is a continuously differentiable function $h(x)$ such that the curve $y = h(x)$, $y' = \frac{dh(x)}{dx}$ belongs to the skeleton, that is, $F\left(x, h(x), \frac{dh(x)}{dx}\right) = 0$ identically in x for some interval. ■

The combination of both components of a differential equation the skeleton and the solution allows us to solve a first-order equation. The crucial step in integrating differential equations is a simplification of the skeleton. This simplification can be gained by a suitable change of variables x and y. To this end, we use symmetries of differential equations, leaving the differential equation invariant. Provided a symmetry is known, a simplification of the skeleton can be carried out by introducing suitable variables, so called canonical variables. This kind of simplification is demonstrated by the following examples.

Example 3.1. Skeleton

Let us consider the following equation

$$\mathbf{riccati} = -\frac{2}{x^2} + y(x)^2 + \frac{\partial y(x)}{\partial x} = 0$$

$$-\frac{2}{x^2} + y'(x) + y(x)^2 = 0$$

Solution 3.1 The skeleton of this equation is defined by the algebraic equation

$$p - \frac{2}{x^2} + y^2 = 0$$

$$p - \frac{2}{x^2} + y^2 = 0$$

and its surface, a so-called hyperbolic paraboloid, can be displayed by defining the skeleton as a function in tow variables x and y by

$$f(\mathrm{x\_}, \mathrm{y\_}) := \frac{2}{x^2} - y^2$$

The corresponding surface $f(x, y)$ in three dimensions is shown in Figure 3.3.

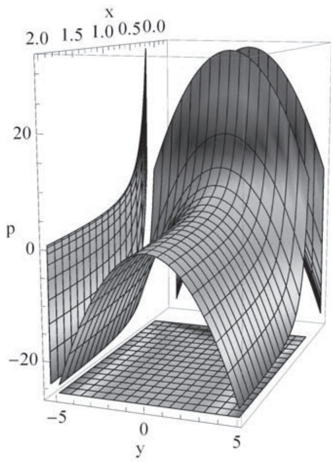


Figure 3.3. The skeleton of the first order ordinary differential equation $y' = \frac{2}{x^2} - y^2 = f(x, y) = p$.

Figure 3.3 shows that the skeleton in the coordinates x, y, and p has a singularity as $x \to 0$. We also observe the parabolic shape of the surface for large x-values. Thus, the surface is twisted in two directions, which obviously baffles the discovery of the solution. Our goal is to transform the twisted shape to a simpler non twisted representation. For the given equation, a one parameter symmetry is provided by the following scaling transformations representing a non homogeneous dilation

$$x = r\, e^{-a} \tag{3.34}$$

$$y = w(x\, e^{a})\, e^{a}. \tag{3.35}$$

Invariance of an ODE means that the original ODE subject to the transformation stays the same in the new coordinates. We can check the invariance of the equation by taking into account that the derivatives also need to be transformed in an invariance transformation. If we insert the transformation into the original equation we find

$$e^{2a}\left(-\frac{2}{r^2} + w(r)^2 + w'(r)\right) = 0 \tag{3.36}$$

which is the original first order equation now in terms of r and w. The corresponding canonical variables related to the transformation of the manifold are given by

$$x = e^{t} \tag{3.37}$$

$$y = \frac{w(\ln(x))}{x} \tag{3.38}$$

Application of this transformation to the first order ODE gives us the following result:

$$w' + w^2 = 2 + w \tag{3.39}$$

which if represented as a skeleton in (t, w, $w' = p$) simplifies in a nice way as shown in Figure 3.4. The simplification achieved is related to the curvature of the skeleton. It is obvious from Figure 3.4 that the twists in two directions disappeared and only the bending around the w' axis remains. The surprising feature is that the skeleton can be translated along the t-axis without changing the shape. This simplification is also obvious in Equation (3.39) where no explicit t dependence is present.

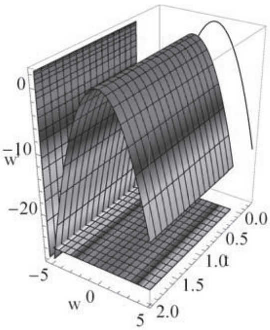


Figure 3.4. The skeleton of the first order ordinary differential equation $w' = -w^2 + 2 + w = f(t, w) = p$.▲

Example 3.2. Skeleton

Another example for a first-order ordinary differential equation which is now treated by means of *Mathematica* is

$$\textbf{example2} = \left(\frac{y(x)^2}{x^2} + \frac{y(x)^3}{x}\right) - \frac{\frac{\partial y(x)}{\partial x}}{x^2} = 0$$

$$-\frac{y'(x)}{x^2} + \frac{y(x)^2}{x^2} + \frac{y(x)^3}{x} = 0$$

Solution 3.2. This example is also invariant with respect to an inhomogeneous scaling transformation. We define this sort of transformation by a transformation rule like

scalingtrafo = {$x \to e^{-a}\, r$, $y \to$ Function[x, $w(x\, e^{a})\, e^{a}$]};
dtrafo = v_$^{(\text{n_})}$(a_. x_) :→ $a^{-n}\, v^{(n)}(a\, x)$;
scaling(x_) := x /. scalingtrafo /. dtrafo

These rules look somehow clumsy and strange. However, they serve to the job to transform the independent and dependent variables and their derivatives of an arbitrary order. The application of the function scaling[] to the second example shows the invariance of this equation

scaling(example2)

$$-\frac{e^{4a}\,w'(r)}{r^2}+\frac{e^{4a}\,w(r)^2}{r^2}+\frac{e^{4a}\,w(r)^3}{r}=0$$

The graphical representation of the skeleton in original variables (x, y, $y' = p$), for this equation is shown in Figure 3.5

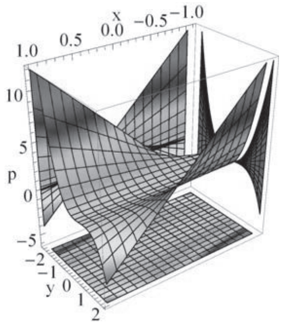


Figure 3.5. The skeleton of the first order ordinary differential equation $w' = -w^2 + 2 + w = f(t, w) = p$.▲

This three-dimensional representation of the skeleton looks like a stingray. The structure of this entangled surface is simplified if we apply the following canonical transformation:

$$\textbf{canonical} = \left\{x \to e^t,\ y \to \textbf{Function}\left[x,\ \frac{w(\log(x))}{x}\right]\right\};$$

canonicaltransform(x_) := Simplify[PowerExpand[*x* /. canonical]]

The transformation is carried out by

canonicaltransform(Thread[example2 exp(4 *t*), Equal])

$$w(t)^3 + w(t)^2 + w(t) = w'(t)$$

The related skeleton simplifies the surface to an invariant surface with respect to translations along the t-axis (Figure 3.6).

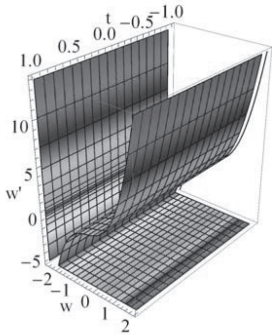


Figure 3.6. The skeleton of the first order ordinary differential equation $w' = -w^2 + 2 + w = f(t, w) = p$.

which is a third-order parabola translated along the t-axis. This example shows that the introduction of canonical coordinates radically simplifies the shape of the skeleton and thus allows access to the solution. ▲

3.3.4 The Direction Field

The direction filed uses the information available from the differential equation and is a representation of the line element in a projection of the manifold (x, y, p) to the plane (x, y). The slope in this plane is represented as a tangent line to the unknown function y. If we consider a certain point (x_0, y_0) in this projection, we know that this point is characterized by the differential equation, given as

$$y' = f(x, y). \tag{3.40}$$

The information on this point can be represented by the local line element given as the triple $(x_0, y_0, p_0) = (x_0, y_0, f(x_0, y_0))$. Where p_0 is the local slope of the curve passing through the point (x_0, y_0). This means that we can locally represent the relation for the unknown y by a linear approximation as

$$y(x) = y_0 + y'|_{x=x_0}(x - x_0) = y_0 + f(x_0, y_0)(x - x_0). \tag{3.41}$$

This kind of approximation was discussed in Section 2.2 when we discussed local approximations of functions. This local approximation represents in some way the function $y(x)$. Thus the direction field contains the complete information about the solution of a first order ordinary differential equation. This geometric interpretation is shown in Figure 3.7.

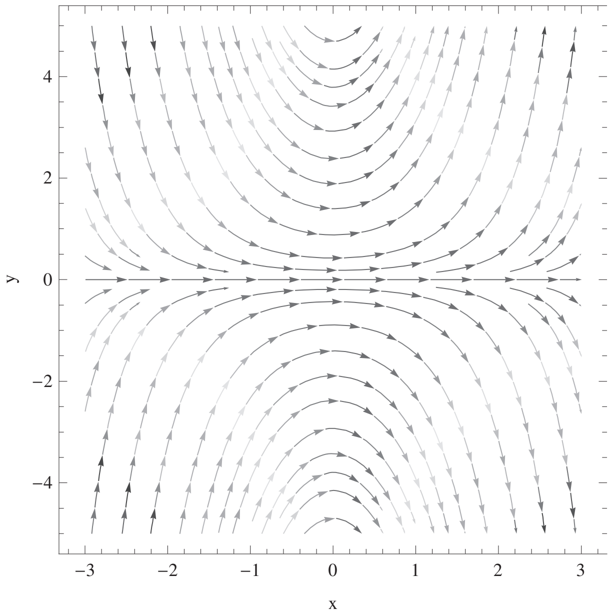


Figure 3.7. Direction filed of the first order ordinary differential equation $y' = x\, y$.

If we interpret dy/dx as the slope of a tangent line, then at a point (x, y) on an integral curve of the equation $dy/dx = f(x)$, the slope of the tangent line is $f(x)$. What is interesting about this is that the slope of the tangent lines to the integral curve can be obtained without actually solving the differential equation. For example, if

$$y' = \sqrt{x^2 + 1} \tag{3.42}$$

then we know without solving the equation that at the point where $x = 1$ the tangent line to the integral curve has slope $\sqrt{1^2 + 1} = \sqrt{2}$; and more generally, at a point where $x = a$, the tangent line to an integral curve has slope $\sqrt{a^2 + 1}$.

A geometric description of the integral curve of a differential equation $y' = f(x)$ can be obtained by choosing a rectangular grid of points in the x-y-plane, calculating the slopes of the tangent lines to the integral curves at the grid points, and drawing small portions of the tangent lines at those points. The resulting picture, which is called a direction field or slope field for the equation, shows the direction of the integral curves at the grid points. With sufficient many grid points it is often possible to visualize the integral curves themselves; for example Figure 3.8 shows a direction field for the differential equation $y' = x^2$, and Figure 3.9 shows that same field with the integral curves imposed on it—the more grid points that are used, the more completely the direction field reveals the shape of the integral curves. However, the amount of computation can be considerable, so computers are usually used when direction fields with many grid points are needed.

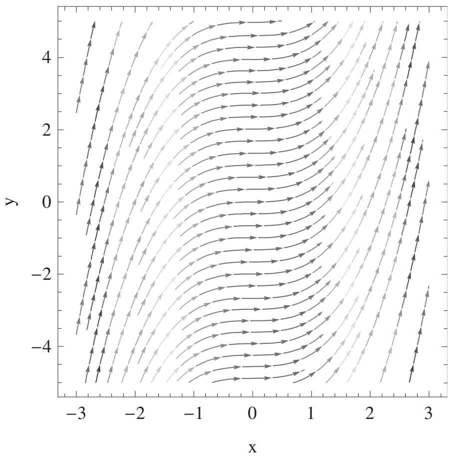


Figure 3.8. Direction filed of the first order ordinary differential equation $y' = x^2$.

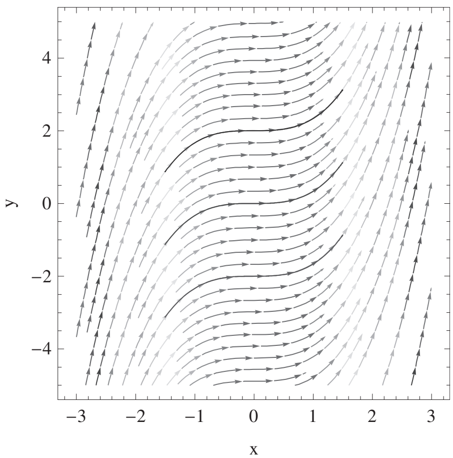


Figure 3.9. Direction filed of the first order ordinary differential equation $y' = x^2$.

The same principle not only holds for $y' = f(x)$ but also on ODEs like $y' = f(x, y)$. To see why this is so, let us review the basic idea. If we interpret y' as the slope of a tangent line, then the differential equation states that at each point (x, y) on an integral curve, the slope of the tangent line is equal to the value of f at that point (Figure 3.10). For example suppose that $f(x, y) = y - x$, in which case we have the differential equation

$y' = y - x.$

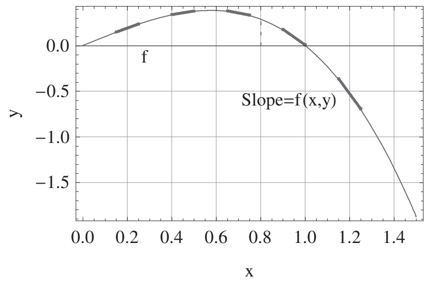


Figure 3.10. At each point (x, y) on an integral curve of $y' = f(x, y)$, the tangent line has slope $f(x, y)$.

It so happens that this equation can be solved exactly since it can be written as

$$y' - y = -x \tag{3.43}$$

which is a first-order linear equation. The solution of this equation is given as

$$y = x + 1 + c\, e^{x}. \tag{3.44}$$

Figure 3.11 shows some of the integral curves superimposed on the direction field.

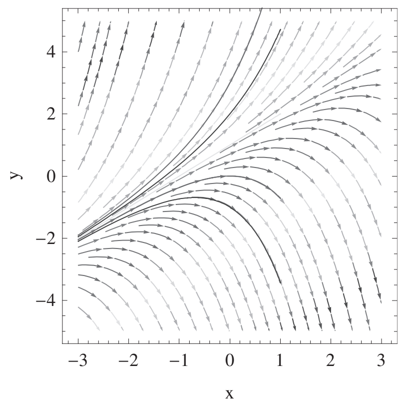


Figure 3.11. Direction filed of the first order ordinary differential equation $y' = y - x$.

Thus the direction field of a first order ordinary differential equation is a useful source to gain information on the solution of this equation. If we plot the direction field we get qualitative information on how the solution will look like at different locations (x_0, y_0) in the solution plane.

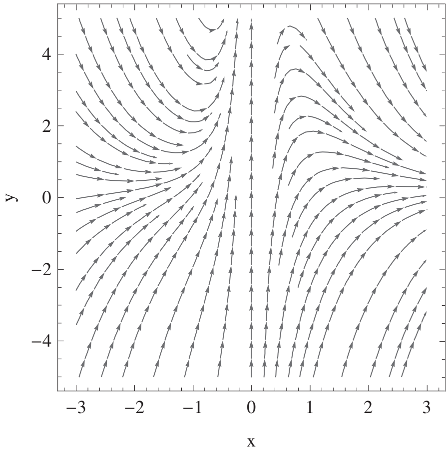


Figure 3.12. Direction filed of the first order ordinary differential equation $y' = \frac{2}{x^2} - y$.

Relation (3.41) for the local approximation allows another interpretation. If the points (x_0, y_0) are given points, so called initial conditions, then each point fixes a curve which includes this point. This suggests that each initial point (x_0, y_0) fixes one solution curve; e.g. given by the local approximation.

3.3.5 Solutions of Differential Equations

A function $y = y(x)$ is a solution of a differential equation on an open interval I if the equation is satisfied identically in I when y and its derivatives are substituted into the equation. For example

$$y = e^{2x} \tag{3.45}$$

is a solution of the differential equation

$$\frac{dy}{dx} - y = e^{2x} \tag{3.46}$$

on the interval $I = (-\infty, \infty)$, since substituting y and its derivative into the left side of this equation yields

$$\frac{dy}{dx} - y = \frac{d}{dx}\left(e^{2x}\right) - e^{2x} = 2\,e^{2x} - e^{2x} = e^{2x} \tag{3.47}$$

for all real values of x. The same calculation with a computer algebra system follows next. First define the solution as

solution = $y \to \left(x \mapsto e^{2x}\right)$

$y \to \left(x \mapsto e^{2x}\right)$

then the differential equation is written

oDE = $\dfrac{\partial y(x)}{\partial x} - y(x) = e^{2x}$

$y'(x) - y(x) = e^{2x}$

and the solution is inserted into the equation which yields on return a logical expression

oDE /. solution

True

telling us that the equation is satisfied. However, the given solution is not the only solution on I; for example, the function

$$y = C\, e^{x} + e^{2x} \tag{3.48}$$

is also a solution for every real value of the constant C, since

$$\frac{\mathrm{dy}}{\mathrm{dx}} - y = \frac{d}{\mathrm{dx}}\left(C\, e^{x} + e^{2x}\right) - \left(C\, e^{x} + e^{2x}\right) = \mathrm{C} e^{x} + 2\, e^{2x} - C\, e^{x} - e^{2x} = e^{2x}. \tag{3.49}$$

The solution in a CAS system reads

solution2 = $y \to \left(x \mapsto C\, e^{x} + e^{2x}\right)$

$y \to \left(x \mapsto C\, e^{x} + e^{2x}\right)$

inserting this solution into the ODE gives

oDE /. solution2

True

which demonstrates that the left hand side is equal the right hand side of the equation.

After developing some techniques for solving equations such as (3.46), we will be able to show that all solutions of (3.48) on $I = (-\infty, \infty)$ can be obtained by substituting values for the constant C in (3.48). On a given interval I, a solution of a differential equation from which all solutions in I can be derived by substituting values for arbitrary constants is called a general solution of the equation in I. Thus (3.48) is a general solution of (3.46) on the interval $I = (-\infty, \infty)$.

Remark 3.3. Usually, the general solution of an nth-order differential equation on an interval will contain n arbitrary constants. Although we will not prove this, it makes sense intuitively because n integrations are needed to recover a function from its nth derivative, and each integration introduces an arbitrary constant. For example, (3.48) has one arbitrary constant, which is consistent with the fact that it is the general solution of a first-order equation (3.46).

The graph of a solution of a differential equation is called an integral curve for the equation, so the general solution of a differential equation produces a family of integral curves corresponding to the different possible choices for the arbitrary constants. For example Figure 3.13 shows some integral curves for (3.46), which were obtained by assigning values to the arbitrary constant in (3.48).

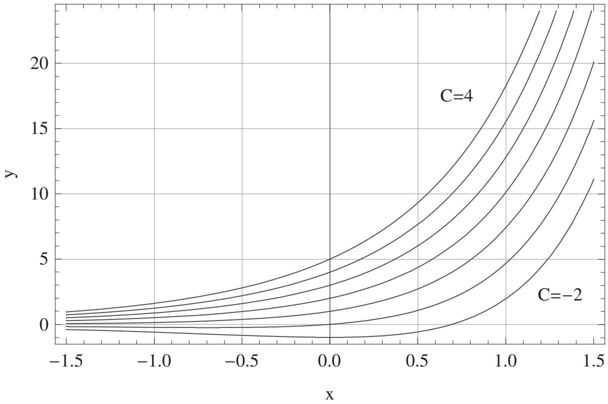


Figure 3.13. Solution family of the differential equation $y' - y = e^{2x}$ for different values of the integration constant $C = \{-2, -1, 0, 1, 2, 3, 4\}$.

3.3.6 Initial-Value Problem

When an applied problem leads to differential equation, there are usually conditions in the problem that determine specific values for the arbitrary constants. As a rule of thumb, it requires n conditions to determine values for all n arbitrary constants in the general solution of an nth order differential equation (one condition for each constant). For a first order equation, the single arbitrary constant can be determined by specifying the value of the unknown function $y(x)$ at an arbitrary x-value x_0, say $y(x_0) = y_0$. This is called an initial condition, and the problem of solving a first-order equation subject to an initial condition is called a first-order initial-value problem. Geometrically, the initial condition $y(x_0) = y_0$ has the effect of isolating the integral curve that passes through the point (x_0, y_0) from the complete family of integral curves.

Example 3.3. Initial Value Problem

The solution of the initial value problem

$$\frac{dy}{dx} - y = e^{2x}, \qquad y(0) = 3 \tag{3.50}$$

Solution 3.3. With the direction field shown in Figure 3.14

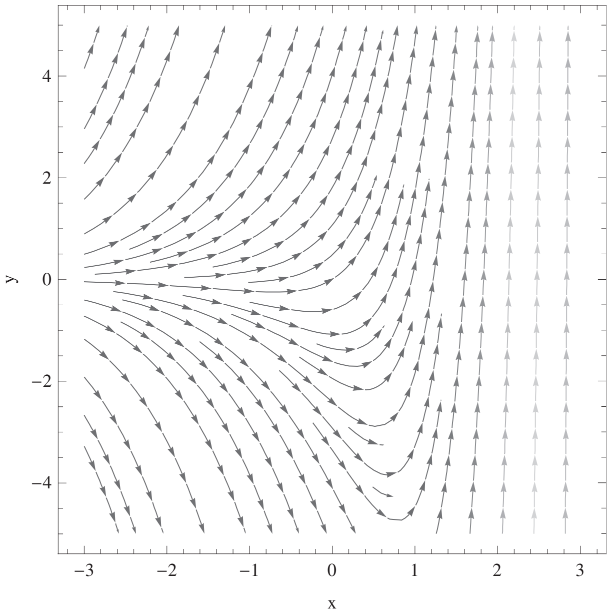


Figure 3.14. Direction filed of the first order ordinary differential equation $y' = y + e^{2x}$.

The following Figure 3.15 allows to interactively choose the initial point (⊕) so that the solution is started from your selection.

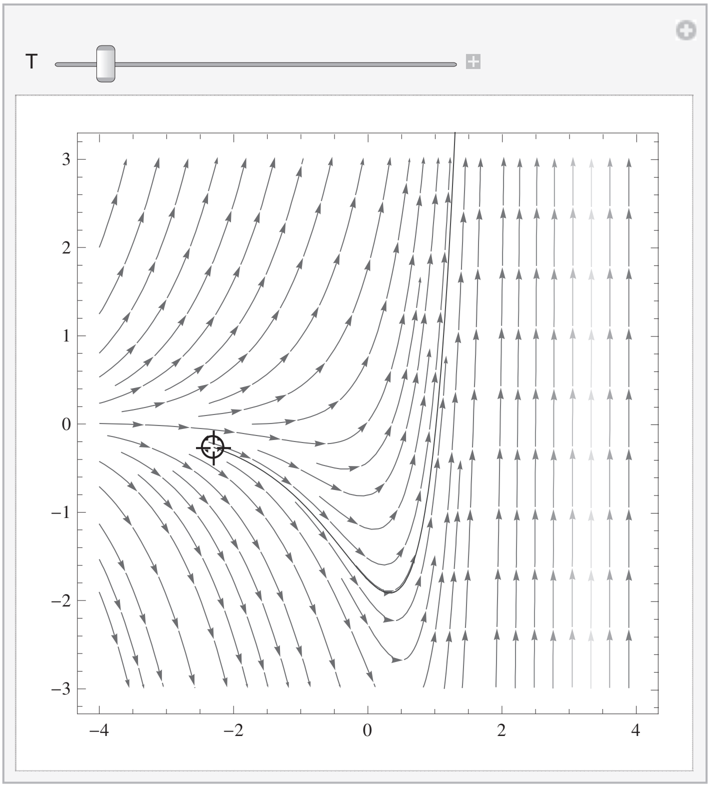


Figure 3.15. Direction filed of the first order ordinary differential equation $y' = y + e^{2x}$.

can be obtained by substituting the initial condition $x = 0$, $y = 3$ in the general solution (3.48) to find C. We obtain

$$3 = C e^0 + e^0 = C + 1. \tag{3.51}$$

Thus, $C = 2$, and the solution of the initial-value problem, which is obtained by substituting this value of C in (8), is

$$y = 2 e^x + e^{2x}. \tag{3.52}$$

Geometrically, this solution is realized as the integral curve in Figure 3.14 that passes though the point (0, 3).▲

3.3.7 Tests and Exercises

The following two subsections serve to test your understanding of the last section. Work first on the test examples and then try to solve the exercises.

3.3.7.1 Test Problems

T1. What type of geometric properties are useful in examining ODEs?

T2. How is a skeleton defined?

T3. What Information about the ODE is represented by a skeleton?

T4. What kind of information is included in a direction field?

T5. Which coordinates are used to represent a phase diagram?

T6. How is an initial value problem defined?

T7. What is a solution of a differential equation?

3.3.7.2 Exercises

E1. Show that $y = x - x^{-1}$ is a solution of the differential equation $x\, y' + y = 2\, x$.

E2. Show that $y = 2\, x \ln(x)$ is a solution of the differential equation $x\, y' - y = 2\, x$.

E3. Verify that $y = x - 2$ is a solution of the ODE $x\, y' - y = 2$.

E4. Show that every member of the family of functions $y = (\ln(x) + C)/x$ is a solution of the differential equation $x^2\, y' + x\, y = 1$. Graph several members of the family of solutions on a common screen.

E5. What can you say about a solution of the equation

$$y' + y^2 = 0 \tag{1}$$

just by looking at the differential equation?

a. Can you find a solution of the initial value problem (1) with $y(0) = 1/2$?

b. Verify that all members of the family $y = 1/(x + C)$ are solutions of equation (1).

c. Can you think of a solution of the differential equation (1) that is not a member of the family in part b)?

E6. What can you say about the graph of a solution of the equation $y' = x\, y^3$ when x is close to 0? What if x is large?

a. Verify that all members of the family are $y = \left(c - x^2\right)^{-1/2}$ are solutions of the differential equation $y' = x\, y^3$.

b. Graph several members of the family of solutions on a common screen. Do the graphs confirm what you predicted?

c. Find a solution of the initial-value problem $y(0) = 5$.

E7. Given the following differential equation

$$y' - y^2 = \frac{2}{x^2} \tag{2}$$

a. Generate a plot of the skeleton of the equation

b. Check if the equation is invariant with respect to the transformation $x = x\, e^{-\beta}$ and $y = y\, e^{\beta}$.

c. Use the canonical transformations $x = e^t$ and $y = w(\ln(x))/x$ to find another representation of (2). What is the difference between your result and (2).

d. Graph the skeleton of your derived equation.

E8. Graph the direction field for the following first order ODEs

a. $y' = 1 + 2\, x\, y$,

b. $y' = -2\, x\, y$,

c. $y' = 1 - 2\, x\, y$.

Sketch a direction field for the differential equation. Then use it to sketch three solution curves.

a. $y' = 1 + y$,

b. $y' = x^2 + y^2$,

c. $y' = y + 2\,x\,y$.

E10 Use a computer algebra system to draw a direction field for the given differential equation. Get a printout and sketch on it the solution curve that passes through $\left(0, \frac{3}{2}\right)$. Then use the CAS to draw the solution curve and compare it with your sketch.

a. $y' = y - 2\,x\,e^y$,

b. $y' = 1 - x\,y^2$,

c. $y' = x^2 \cos(y) + \frac{y}{\sqrt{x+1}}$.

3.4 Types of First Order Differential Equations

In this section we will discuss different types of first order differential equations. The different types are usually classified by their solution technique or the structure of the equation. In addition there are some special ODEs assigned to mathematicians who introduced the ODE or derived this equation the first time. Such kind of equations are the Riccati equation, Bernoulli equation, etc. To start with we will first discuss the simples first order ODE which is a linear ODE. Then we will examine separable and exact equations. After that we will switch to special types of equations like Riccati and others.

3.4.1 First-Order Linear Equations

The simplest first-order equations are those that can be written in the form

$$\frac{dy}{dx} = q(x) \tag{3.53}$$

Such equations can often be solved by integration. For example, if the function $q(x) = x^3$

$$\frac{dy}{dx} = x^3 \tag{3.54}$$

then

$$y = \int x^3\, dx = \frac{x^4}{4} + C \tag{3.55}$$

is the general solution of (3.53) on the interval $I = (-\infty, \infty)$. More generally, a first-order differential equation is called linear if it is expressible in the form

$$\frac{dy}{dx} + p(x)\,y = q(x). \tag{3.56}$$

Equation (3.53) is the special case of (3.56) that results when the function $p(x)$ is identically 0. Some other examples of first order linear differential equations are

$$\frac{dy}{dx} + x^2\, y = e^x \tag{3.57}$$

$$\frac{dy}{dx} + \sin(x)\, y + x^3 = 0 \tag{3.58}$$

$$\frac{dy}{dx} + 5\, y = 5 \tag{3.59}$$

Let us assume that the functions $p(x)$ and $q(x)$ are both continuous on some common open interval I. We will now describe a procedure for finding a general solution to (3.56) on I.

From the Fundamental Theorem of Calculus it follows that $p(x)$ has an anti derivative $P = P(x)$ on I. That is, there exists a differentiable function $P(x)$ on I such that $dP/dx = p(x)$. Consequently, the function $\mu = \mu(x)$ defined by $\mu = e^{P(x)}$ is differentiable on I with

$$\frac{d\mu}{dx} = \frac{d}{dx}\left(e^{P(x)}\right) = \frac{dP}{dx}\, e^{P(x)} = \mu\, p(x). \tag{3.60}$$

Suppose now that $y = y(x)$ is a solution of (3.56) on I. Then

$$\frac{d}{dx}(\mu\, y) = \mu\, \frac{dy}{dx} + y\, \frac{d\mu}{dx} = \mu\, \frac{dy}{dx} + \mu\, p(x)\, y = \mu\left(\frac{dy}{dx} + p(x)\, y\right) = \mu\, q(x). \tag{3.61}$$

That is the function $\mu\, y$ is an anti derivative (or integral) of the known function $\mu\, q(x)$. For this reason, the function $\mu = e^{P(x)}$ is known as an integrating factor for equation (3.56). On the other hand, the function $\mu\, q(x)$ is continuous on I and therefore possesses an anti derivative $H(x)$. It then follows that $\mu\, y = H(x) + C$ for some constant C or equivalently, that

$$y = \frac{1}{\mu}\,(H(x) + C). \tag{3.62}$$

Conversely, it is straight forward to check that for any choice of C, equation (3.62) defines a solution to (3.56) on I. We conclude that the general solution to (3.56) on I is given by (3.62). Since

$$\int \mu\, q(x)\, dx = H(x) + C. \tag{3.63}$$

This general solution can be expressed as

$$y = \frac{1}{\mu} \int \mu\, q(x)\, dx. \tag{3.64}$$

We will refer to this process for solving (3.56) as the method of integrating factors.

Example 3.4. Linear Equations

Solve the differential equation

$$\mathbf{deq} = \frac{\partial y(x)}{\partial x} - y(x) = e^{2x}$$

$$y'(x) - y(x) = e^{2x}$$

Solution 3.4. This is a first-order linear differential equation with the function $p(x) = -1$ and $q(x) = e^{2x}$ that are both continuous on the interval $I = (-\infty, \infty)$. Thus we can choose

$$P(x) = -x$$

$$-x$$

with

$$\mu = e^{-x}$$

$$e^{-x}$$

and

$$\mathbf{int} = e^{2x}\,\mu$$

$$e^{x}$$

so that the general solution to this equation on I is given by

$$\frac{C + \int \mathbf{int}\, dx}{\mu}$$

$$e^{x}\,(C + e^{x})$$

Note that this solution is in agreement with the calculation done by *Mathematica*

DSolve[deq, *y*, *x*]

$$\{\{y \to (\{x\} \longmapsto c_1\, e^{x} + e^{2x})\}\}$$

▲

It is not necessary to memorize Equation (3.64) to apply the method of integrating factors; you need only remember the integrating factor $\mu = e^{P(x)}$ and the steps used to derive Equation (3.64).

Example 3.5. Initial Value Problem

Solve the initial value problem

$$\mathbf{deq} = x\,\frac{\partial y(x)}{\partial x} - y(x) = x$$

$$x\,y'(x) - y(x) = x$$

inval = y(1) == 2

$y(1) = 2$

Solution 3.5. The differential equation can be rewritten in the standard form by dividing through by x. This yields

deq = Simplify[Thread[$\frac{\text{deq}}{x}$, Equal]]

$$y'(x) = \frac{y(x) + x}{x}$$

where now $q(x) = 1$ is continuous on $(-\infty, \infty)$ and $p(x) = -1/x$ is continuous on $(-\infty, 0)$ and $(0, \infty)$. Since we need $p(x)$ and $q(x)$ to be continuous on a common interval, and since our initial condition presumes a solution for $x = 1$, we will find the general solution of the equation on the interval $(0, \infty)$. On this interval

$$C + \int \frac{1}{x}\, dx$$

$$C + \log(x)$$

so we can take $P(x) = -\ln(x)$ with $\mu = e^{P(x)} = e^{-\ln(x)} = \frac{1}{x}$ the corresponding integrating factor. Multiplying both sides of the original equation by this integrating factor yields

Thread[$\frac{\text{deq}}{x}$, Equal]

$$\frac{y'(x)}{x} = \frac{y(x) + x}{x^2}$$

or the equivalent representation

$$\textbf{deq1} = \frac{\partial}{\partial x}\frac{y(x)}{x} == \frac{1}{x}$$

$$\frac{y'(x)}{x} - \frac{y(x)}{x^2} = \frac{1}{x}$$

The integration of both sides delivers

deq2 = Thread[$\int$ deq1 dx, Equal]

$$\frac{y(x)}{x} = \log(x)$$

Solving with respect to the unknown y

s1 = Flatten[Solve[deq2, $y(x)$]]

$\{y(x) \to x \log(x)\}$

gives us the solution

solution = $C\, x + y(x)$ /. s1

$C\, x + x \log(x)$

The initial condition $y(1) = 2$ requires

s2 = Flatten[Solve[(solution /. $x \to 1$) == 2, C]]

$\{C \to 2\}$

So that the final solution after inserting the result from the previous calculation is given as

solution /. s2

$2\, x + x \log(x)$

▲

The result of Example 3.5 illustrates an important property of first-order linear initial value problems: Given any x_0 in I and any value of y_0, there will always exist a solution $y = y(x)$ to the standard equation (3.56) on I with $y(x_0) = y_0$; furthermore, this solution will be unique. Such existence and uniqueness results need not hold for nonlinear equations.

3.4.2 First Order Separable Equations

Solving a first order linear differential equation involves only the integration of functions of x. We will now consider a collection of equations whose solutions require integration of functions of y as well. A first-order separable differential equation is one that can be written in the form

$$h(y)\, \frac{\mathrm{dy}}{\mathrm{dx}} = g(x). \tag{3.65}$$

For example, the equation

eq1 = $\dfrac{\partial y(x)}{\partial x}\,(4\, y(x) - \cos(y(x))) == 3\, x^2$

$y'(x)\,(4\, y(x) - \cos(y(x))) = 3\, x^2$

is a separable equation with

$$h(y) = 4\, y - \cos(y) \qquad \text{and } g(x) = 3\, x^2. \tag{3.66}$$

We will assume that the function $h(y)$ and $g(x)$ both possess anti derivatives in their respective variables y and x. That is, there exists a differentiable function $H(y)$ with $\mathrm{dH}/\mathrm{dy} = h(x)$ and there exists a differentiable function $G(x)$ with $\mathrm{dG}/\mathrm{dx} = g(x)$.

Suppose now that $y = y(x)$ is a solution to (3.65) on an open interval I. Then it follows from the chain rule that

$$\frac{d}{\mathrm{dx}}(H(y)) = \frac{\mathrm{dH}}{\mathrm{dy}}\frac{\mathrm{dy}}{\mathrm{dx}} = h(y)\frac{\mathrm{dy}}{\mathrm{dx}} = \frac{d}{\mathrm{dx}}(G(x)) = g(x). \tag{3.67}$$

In other words the function $H(y(x))$ is an anti derivative of $g(x)$ on the interval I and consequently there must exist a constant C such that $H(y(x)) = G(x) + C$ on I. Equivalently, the solution $y = y(x)$ to (3.65) is defined implicitly by Equation

$$H(y) = G(x) + C. \tag{3.68}$$

Consequently, suppose that for some choices of C a differentiable function $y = y(x)$ is defined implicitly by (3.68). Then $y(x)$ will be a solution to (3.65). We conclude that every solution to (3.65) will be given implicitly by equation (3.68) for some appropriate choice of C.

We can express Equation (3.68) symbolically by writing

$$\int h(y)\,dy = \int g(x)\,dx. \tag{3.69}$$

Informally, we first multiply both sides of Equation (3.65) by dx to separate the variables into the equation $h(y)\,dy = g(x)\,dx$. Integrating both sides of this equation then gives Equation (3.69). This process is called the method of separation of variables. Although separation of variables provides us with a convenient way of recovering Equation (3.68), it must be interpreted with care. For example, the constant C in Equation (3.68) is often not arbitrary; some choices of C may yield solutions, and other may not. Furthermore, even when solutions do exist, their domain can vary in unexpected ways with the choice of C. It is for reasons such as these that we will not refer to a general solution of a separable equation.

In some cases Equation (3.68) can be solved to yield explicit solutions to (3.65).

Example 3.6. Separable Differential Equations

Solve the differential equation

$$\mathbf{eq1} = \frac{\partial y(x)}{\partial x} == -4\,x\,y(x)^2$$

$$y'(x) = -4\,x\,y(x)^2$$

and determine the integration constant by the initial condition $y(0) = 1$.

Solution 3.6. For $y \neq 0$ we can write this equation in the form

$$\frac{1}{y^2}\frac{dy}{dx} = -4\,x. \tag{3.70}$$

The related direction field is shown in Figure 3.16

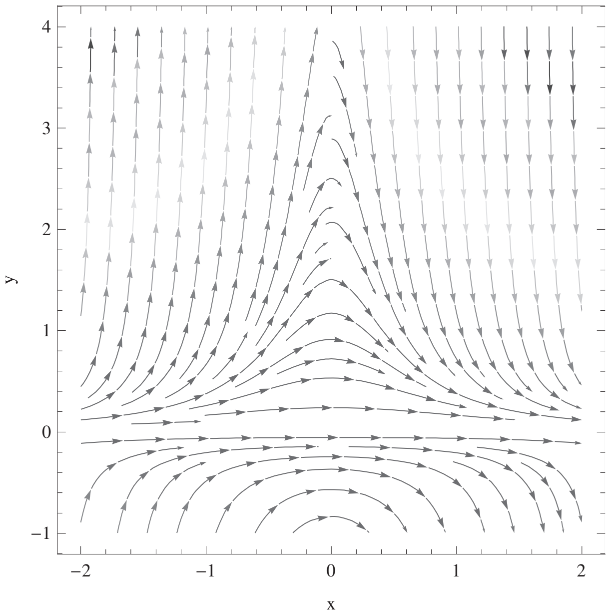


Figure 3.16. Direction filed of the first order ordinary differential equation $y' = -4\,x\,y^2$.

Separating variables and integrating yields

$$\frac{1}{y^2}\,dy = -4\,x\,dx \tag{3.71}$$

$$\int \frac{1}{y^2}\,dy = \int -4\,x\,dx \tag{3.72}$$

which is a symbolic expression of the equation

$$\frac{-1}{y} = -2\,x^2 + C. \tag{3.73}$$

Solving for y as a function of x, we obtain

$$y = \frac{1}{2\,x^2 - C}. \tag{3.74}$$

The initial condition $y(0) = 1$ requires that $y = 1$ when $x = 0$. Substituting these values into our solution yields $C = -1$. Thus a solution to this initial-value problem is

$$y = \frac{1}{2x^2 + 1}. \tag{3.75}$$

The solution of the initial value problem is automatically derived by *Mathematica* using the following line

sol1 = Flatten[DSolve[Join[{eq1}, {*y*(0) == 1}], *y*, *x*]]

$$\left\{y \to \left(\{x\} \longmapsto \frac{1}{2x^2 + 1}\right)\right\}$$

The solution derived can be used to generate a graphical representation (see Figure 3.17). It is important to know the steps used by the function DSolve[] to determine the solution. In fact the steps are the same as discussed above.

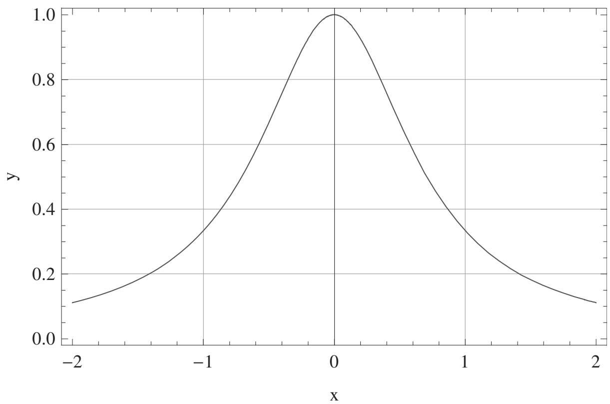


Figure 3.17. Solution of the initial vale problem $y' = -4\,x\,y^2$ with $y(0) = 1$.

If we are interested in the general solution of equation (3.70) then we will not specify the initial conditions in the function DSolve[]. Here is the general solution of this equation without initial conditions.

sol2 = Flatten[DSolve[eq1, *y*, *x*]]

$$\left\{y \to \left(\{x\} \longmapsto \frac{1}{2x^2 - c_1}\right)\right\}$$

The following plot shows the solution for different values of the integration constant C_1.

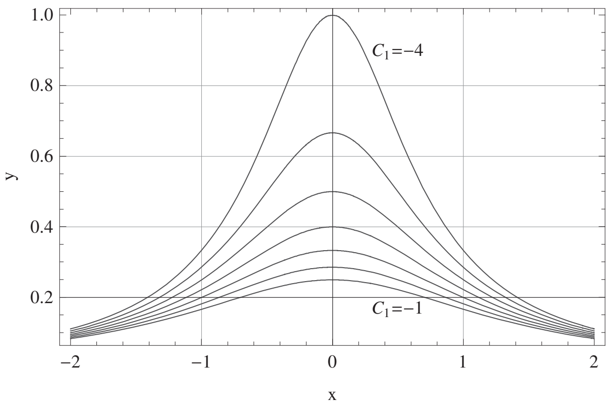


Figure 3.18. Family of solutions for the first order ordinary differential equation $y' = -4\,x\,y^2$ with $C_1 \in \{-4, -3.5, -3, -2.5, -2, -1.5, -1\}$.▲

If the solution of a differential equation is known symbolically, we have the total information on this equation available. Sometimes it is necessary to examine not only the solution itself but also the derivative. Since the solution is known as a function of the independent variable we are able to find an expression for the derivative with respect to the independent variable. Both quantities are known as parametric functions of the independent variable so we can form pairs of (y', y) to create a parametric representation of the solution. This kind of representation is known as phase space representation. The corresponding graphical plot is called a phase space diagram. The following Figure 3.19 shows the phase space diagram of the equation discussed in Example 3.6.

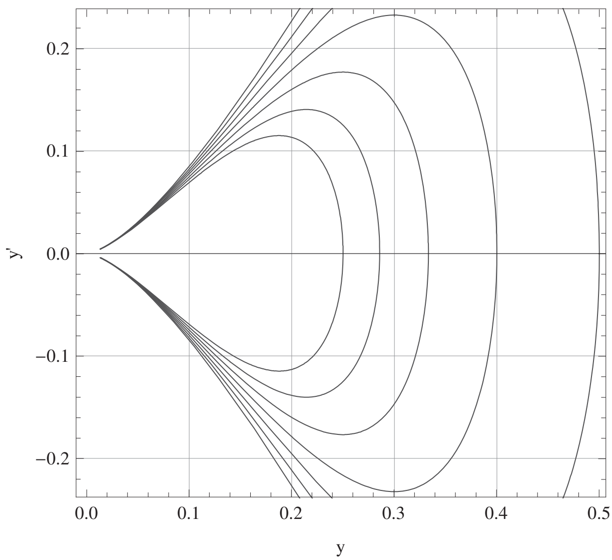


Figure 3.19. Phase space diagram of the first order ODE $y' = -4\,x\,y^2$. The different curves correspond to different values of $C_1 \in \{-4, -3.5, -3, -2.5, -2, -1.5, -1\}$.

In many engineering applications, it is often necessary to find the orthogonal trajectories of a given family of curves. (A curve which intersects each member of a family of curves at right angle is called an orthogonal trajectory of the given family.) For example, a charged particle moving under the influence of a magnetic field always travels on a curve which is perpendicular to each of the magnetic field lines. The problem of computing orthogonal trajectories of a family of curves can be solved in the following manner. Let the given family of curves be described by the relation

$$F(x,\ y,\ c) = 0. \tag{3.76}$$

Differentiating this equation with respect to x yields

$$\frac{d\,F}{\mathrm{dx}} + \frac{\mathrm{dy}}{\mathrm{dx}}\,\frac{\mathrm{dF}}{\mathrm{dy}} = 0 = F_x + y'\,F_y \tag{3.77}$$

which solved with respect to y' gives

$$y' = -\frac{F_x}{F_y}. \tag{3.78}$$

Next we solve for $c = c(x,\ y)$ from (3.76) and replace every c in (3.77) and (3.78) by this value $c(x,\ y)$. Finally, since the slope of curves which intersect orthogonally are negative reciprocals of each other, we see that the orthogonal trajectories of (3.76) are the solution curves of the equation

$$y' = \frac{F_y}{F_x}. \tag{3.79}$$

Example 3.7. Orthogonal Trajectories

As an application of these relations find the orthogonal trajectories of the family of parabolas

$$x = c\, y^2. \tag{3.80}$$

Solution 3.7. Differentiating the equation $x = c\, y^2$ gives

$$1 = 2\, c\, y\, y'. \tag{3.81}$$

Since $c = x \big/ y^2$, we see that

$$y' = \frac{y}{2\, x}. \tag{3.82}$$

Thus, the orthogonal trajectories of the family of parabolas $x = c\, y^2$ are the solution curves of the equation

$$y' = -\frac{2\, x}{y} \tag{3.83}$$

which represents a separable ODE of first order. The separation of variables and the integration delivers

$$\int y\, dy = -\int 2\, x\, dx \tag{3.84}$$

$$\frac{y^2}{2} = -x^2 + C \tag{3.85}$$

which simplifies to

$$y^2 + 2\, x^2 = k \tag{3.86}$$

which represents a family of ellipses shown in Figure 3.20.

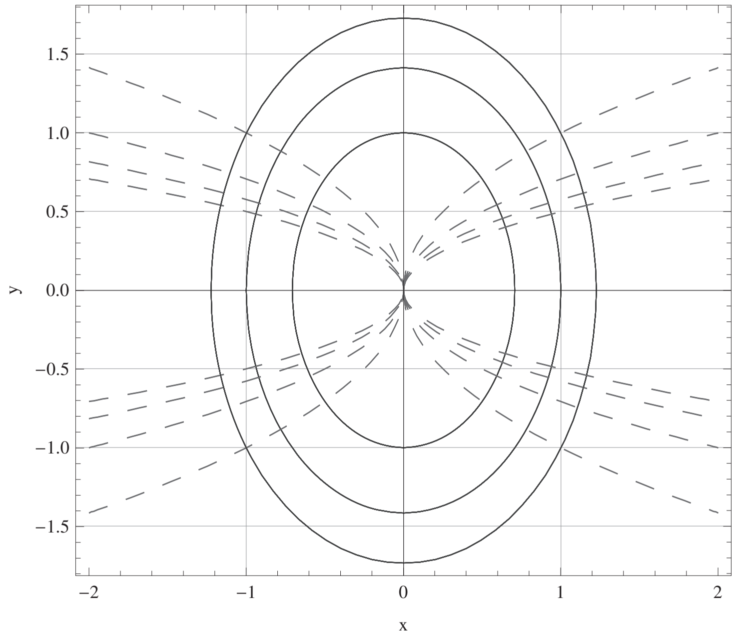


Figure 3.20. The parabolas $x = c\, y^2$ and their orthogonal trajectories $y^2 + 2\, x^2 = k$.

The solution of the same problem is gained by a CAS by the following lines. First define the parabolas

$$\mathbf{eq} = x - c\; y(x)^2$$

$$x - c\; y(x)^2$$

Then find the defining equations for the orthogonal trajectories

$$\mathbf{eq1} = \frac{\partial\, y(x)}{\partial x} == \left(-\frac{1}{\frac{\partial y(x)}{\partial x}} \;/.\; \mathbf{Flatten}\left[\mathbf{Solve}\left[\left(\frac{\partial\, \mathbf{eq}}{\partial x} \;/.\; \mathbf{Flatten}[\mathbf{Solve}[\mathbf{eq} == 0, c]]\right) == 0, \frac{\partial\, y(x)}{\partial x}\right]\right]\right)$$

$$y'(x) = -\frac{2\, x}{y(x)}$$

and solve the resulting ODE

DSolve[eq1, *y*, *x*]

$$\left\{\left\{y \to \left(\{x\} \mapsto -\sqrt{2}\, \sqrt{c_1 - x^2}\right)\right\}, \left\{y \to \left(\{x\} \mapsto \sqrt{2}\, \sqrt{c_1 - x^2}\right)\right\}\right\}$$

The solution is given by two branches of an ellipse.▲

3.4.3 Exact Equations

Here we will consider the firs order ordinary differential equation in the following representation. The most general form of a first order ODE is given by

$$y' = f(x, y) \tag{3.87}$$

which is an ODE solved with respect to the first derivative. Equivalently, this kind of equation can be written in a so called differential form

$$\omega = M(x, y)\,\mathrm{dx} + N(x, y)\,\mathrm{dy} = 0. \tag{3.88}$$

This representation is called exact; i.e. $\omega = \mathrm{d}\phi$ or

$$M(x, y)\,\mathrm{dx} + N(x, y)\,\mathrm{dy} = \frac{\partial\phi(x, y)}{\partial x}\,\mathrm{dx} + \frac{\partial\,\phi(x, y)}{\partial y}\,\mathrm{dy} \tag{3.89}$$

For a function $\phi = \phi(x, y)$ of two variables, if and only if it is closed; i.e.

$$\frac{\partial\,N(x, y)}{\partial x} = \frac{\partial M(x, y)}{\partial y}. \tag{3.90}$$

This in fact means that we have to compare the coefficients of the total differentials dx and dy which provides us with

$$\frac{\partial\phi(x, y)}{\partial x} = M(x, y) \tag{3.91}$$

$$\frac{\partial\phi(x, y)}{\partial y} = N(x, y). \tag{3.92}$$

Differentiating each of the both relations with respect to y and x, respectively delivers

$$\frac{\partial^2\phi(x, y)}{\partial x\,\partial y} = \frac{\partial M(x, y)}{\partial y} \tag{3.93}$$

$$\frac{\partial^2\phi(x, y)}{\partial y\,\partial x} = \frac{\partial N(x, y)}{\partial x}. \tag{3.94}$$

Since both expressions are the same

$$\frac{\partial^2\phi(x, y)}{\partial x\,\partial y} = \frac{\partial^2\phi(x, y)}{\partial y\,\partial x} \tag{3.95}$$

it follows that the relation

$$\frac{\partial N(x, y)}{\partial x} = \frac{\partial M(x, y)}{\partial y} \tag{3.96}$$

must hold.

The function ϕ is obtained by equation (3.89) rewritten as a system of differential equations for the unknown function ϕ:

$$\frac{\partial \phi(x, y)}{\partial x} = M(x, y) \tag{3.97}$$

$$\frac{\partial \phi(x, y)}{\partial y} = N(x, y). \tag{3.98}$$

The system (3.91) and (3.92) is overdetermined but it is integrable provided that the condition (3.90) is satisfied. Its solution is given by

$$\phi(x, y) = \int_{x_0}^{x} M(z, y)\, \mathrm{dz} + \int_{y_0}^{y} N(x_0, z)\, \mathrm{dz}. \tag{3.99}$$

Accordingly the first order differential equation (3.87) in the form

$$M(x, y)\, \mathrm{dx} + N(x, y)\, \mathrm{dy} = 0. \tag{3.100}$$

is said to be exact if its coefficients satisfy (3.90). The solution of the exact equation is given implicitly by

$\phi(x, y) = C$,

where ϕ is defined by (3.99) and C is an arbitrary constant.

Example 3.8. Exact ODE

Is the equation $1 + \cos(x + y) + \cos(x + y)\, y' = 0$ an exact equation and if so what is the solution.

Solution 3.8. This equation can be written as

$$\frac{d}{\mathrm{dx}}(x + \sin(x + y)) = 0 \tag{3.101}$$

Hence the function ϕ is

$$\phi(x, y) = x + \sin(x + y) = c \tag{3.102}$$

and thus

$$y = -x + \arcsin(c - x).\blacktriangle \tag{3.103}$$

3.4.4 The Riccati Equation

The general Riccati equation is an equation of first order with quadratic non linearity

$$y'(x) = P(x) + Q(x)\, y + R(x)\, y^2. \tag{3.104}$$

The distinctive property of the Riccati equation is that, unlike an arbitrary equation it admits a nonlinear superposition. Namely, the anharmonic ratio of any four solutions y_1, y_2, y_3, y_4 of the Riccati equation is constant:

$$\frac{\frac{y_4 - y_2}{y_4 - y_1}}{\frac{y_3 - y_2}{y_3 - y_1}} = C. \tag{3.105}$$

Consequently, given any three distinct particular solutions $y_1(x)$, $y_2(x)$, $y_3(x)$ of (3.104), its general solution y is expressed rationally in terms of y_1, y_2, y_3 and an arbitrary constant C upon solving the equation

$$\frac{\frac{y - y_2(x)}{y - y_1(x)}}{\frac{y_3(x) - y_2(x)}{y_3(x) - y_1(x)}} = C. \tag{3.106}$$

Definition 3.3. *Equivalence Transformation*

A change of variables $(x, y) \to (\overline{x}, \overline{y})$, is called an equivalence transformation of the Riccati equation if any equation of the form (3.104) is transformed into an equation of the same type with possibly different coefficients. Equations related by an equivalence transformation are said to be equivalent. ■

Theorem 3.2. *Equivalence*

The general equivalence transformation of the Riccati equation comprises of

a) changes of the independent variables

$$\overline{x} = \phi(x) \text{ and } \phi'(x) \neq 0, \tag{3.107}$$

b) linear-rational transformations of the dependent variable

$$\overline{y} = \frac{\alpha(x)\, y + \beta(x)}{\gamma(x)\, y + \delta(x)} \quad \text{with} \quad \alpha\delta - \beta\gamma \neq 0. \blacksquare \tag{3.108}$$

3.4.5 The Bernoulli Equation

The Bernoulli equation

$$y'(x) + P(x)\, y = Q(x)\, y^n \qquad \text{and } n \neq 0 \text{ and } n \neq 1 \tag{3.109}$$

Bernoulli = $\boldsymbol{P(x)\, y(x) + \frac{\partial y(x)}{\partial x} = Q(x)\, y(x)^n}$

$$P(x)\, y(x) + y'(x) = Q(x)\, y(x)^n$$

reduces to a linear equation by means of the substitution

substitution = $\boldsymbol{y \to \left(x \mapsto z(x)^{\frac{1}{1-n}}\right)}$

$$y \to \left(x \mapsto z(x)^{\frac{1}{1-n}}\right)$$

The resulting equation is

Simplify[Bernoulli /. substitution]

$$\frac{z(x)^{-\frac{n}{n-1}}\left((n-1)\, P(x)\, z(x) - z'(x)\right)}{n-1} = Q(x)\left(z(x)^{\frac{1}{1-n}}\right)^n$$

which reads

$$z'(x) + (1-n)\, P(x)\, z = (1-n)\, Q(x). \tag{3.110}$$

3.4.6 The Abel Equation

The Abel equation in its homogeneous form is given by the nonlinear ODE of first order

$$y' = a\, x^{(2n+1)} y^3 + b\, x^n\, y^2 + \frac{c}{x}\, y + d\, x^{-n-2} \tag{3.111}$$

where a, b, c, and d are constants and n represents the order of homogenity. The substitution $w = x^{(n+1)} y$ leads to a separable equation. This can be checked by transforming Able's equation

abelsEq = $\boldsymbol{\frac{\partial y(x)}{\partial x} = a\, x^{2n+1}\, y(x)^3 + b\, x^n\, y(x)^2 + \frac{c\, y(x)}{x} + d\, x^{-n-2}}$

$$y'(x) = a\, x^{2n+1}\, y(x)^3 + b\, x^n\, y(x)^2 + \frac{c\, y(x)}{x} + d\, x^{-n-2}$$

with the rule

transform = $\boldsymbol{y \to \left(x \mapsto x^{-n-1}\, w(x)\right)}$

$$y \to \left(x \mapsto x^{-n-1}\, w(x)\right)$$

which applied to Able's equation delivers

Simplify[abelsEq /. transform]

$$x^{-n-2}\left(x\, w'(x) - (n+1)\, w(x)\right) = x^{-n-2}\left(a\, w(x)^3 + b\, w(x)^2 + c\, w(x) + d\right)$$

a separable equation.

3.4.7 Tests and Exercises

The following two subsections serve to test your understanding of the last section. Work first on the test examples and then try to solve the exercises.

3.4.7.1 Test Problems

T1. How is a linear ODE defined. Give examples.

T2. What is a separable differential equation? Stet some examples.

T3. How is an exact ODE defined? State some of the equations.

T4. What is the difference between linear and non linear ODEs?

T5. How are Riccati, Bernoulli, and Abel equations defined?

3.4.7.2 Exercises

E1. Find the general solution of

$$u' = \frac{u - 3\,x \ln\left(x^2 + u^2\right)}{x + 3\,u \ln\left(x^2 + u^2\right)} \tag{1}$$

E2. Find the solution for

$$u' = \frac{1}{x^2\, e^{-u^2} - x\,u} \tag{2}$$

E3. Find the solution for

$$\frac{x\,u'}{u + x\,u'} = r\,(x\,u)^s \ln(u) + j\,(x\,u)^k \tag{3}$$

E4. Find the solution for

$$u' = \frac{1}{x^2\,u - x\,u^2} \tag{4}$$

E5. Find the solution to the following ODEs:

$$\frac{x\,u' - u}{x\,u' + u} = x^{n-1}\,u^{n+1} \tag{5}$$

$$u' = \frac{u}{x}\,\frac{1 + x^\alpha\,u^{\alpha+2}}{1 - x^\alpha\,u^{\alpha+2}} \tag{6}$$

$$\ln\left(\frac{x\,u' - u}{x\,u' + u}\right) = \left(\arcsin\left((x\,u)^{1/3}\right)\right)^2 + \ln\left(\frac{u}{x}\right) \tag{7}$$

E6. Find the solution to

$$\frac{1}{u'} + x\,(\ln(x))\,u^3 = x \tag{8}$$

E7. Find the solution to

$$\frac{u - a\,x^b\,u^n\,u'}{x^b\,u'} + \left(\frac{x^{1-b}}{1-b} - \frac{a}{n}\,y^n\right)\frac{1}{\ln(u)} = \ln(u) \tag{9}$$

where a, b, and n are constants.

E8. Find the solution to the equation

$$x\,u' = u + \frac{x^2}{1 + (u/x)^2\,e^{-x}} \tag{10}$$

E9. Find the solution to

$$u' = \frac{3\,u^2}{u^2 + 2\,u - 4\,e^{3\,x-u}} \tag{11}$$

E10 Find the solution to

$$u' = -\frac{a}{x} + b\,x^e\,e^{m\,u}, \tag{12}$$

where a, b, e, and m are constants, and $e \neq m\,a - 1$.

3.5 Numerical Approximations—Euler's Method

3.5.1 Introduction

In this section we will reexamine the concept of a direction field and we will discuss a method for approximating solutions of first-order equations numerically. Numerical approximations are important in cases where the differential equation cannot be solved exactly.

3.5.2 Euler's Method

Our next objective is to develop a method for approximating the solution of an initial-value problem of the form

$$y' = f(x,\ y) \quad \text{with} \quad y(x_0) = y_0. \tag{3.112}$$

We will not attempt to approximate $y(x)$ for all values of x; rather we will choose a some small increment Δx and focus on approximating the values of $y(x)$ at a succession of x-values spaced Δx units apart, starting from x_0. We will denote these x-values by

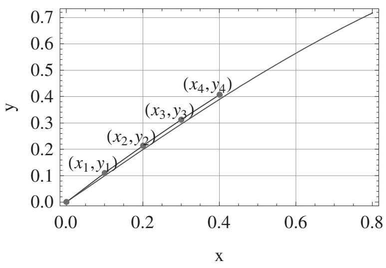
$$x_1 = x_0 + \Delta\text{x},\ \ x_2 = x_1 + \Delta\text{x} = x_0 + 2\,\Delta\text{x},\ \ x_3 = x_2 + \Delta\text{x} = x_0 + 3\,\Delta\text{x},\ \ \ldots \tag{3.113}$$

and we will denote the approximations of $y(x)$ at these points by

$$y_1 \approx y(x_1),\ \ y_2 \approx y(x_2),\ \ y_3 \approx y(x_3),\ \ \ldots. \tag{3.114}$$

The technique that we will describe for obtaining these approximations is called Euler's Method. Although there are better approximation methods available, many of them use Euler's method as a starting point, so the underlying concept are important to understand.

The basic idea behind Euler's method is to start at the known initial point (x_0, y_0) and draw a line segment in the direction determined by the direction field until we reach the point (x_1, y_1) with x-coordinate $x_1 = x_0 + \Delta x$ (see Figure 3.21). If Δx is small, then it is reasonable to expect that this line segment will not deviate much from the integral curve $y = y(x)$, and thus y_1 should closely approximate $y(x_1)$.



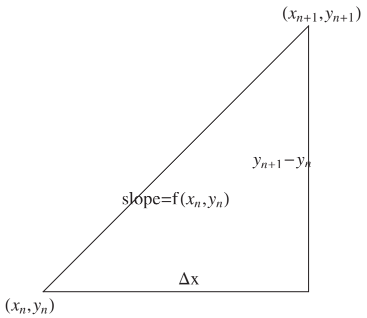


Figure 3.21. The first few steps of Euler's method to integrate a first order ODE by $y_{n+1} = y_n + \Delta x\, f(x_n, y_n)$ (top). The bottom part shows a single geometric element of Euler's approximation in step $n + 1$.

To obtain the subsequent approximation, we repeat the process using the direction field as a guide at each step. Starting at the endpoint (x_1, y_1), we draw a line segment determined by the direction field until we reach the point (x_2, y_2) with x-coordinate $x_2 = x_1 + \Delta x$, and from that point we draw a line segment determined by the direction field to the point (x_3, y_3) with x-coordinate $x_3 = x_2 + \Delta x$, and so forth. As indicated in Figure 3.21, this procedure produces a polygonal path that tends to follow the integral curve closely, so it is reasonable to expect that the y-values $y_2, y_3, y_4, \ldots$ will closely approximate $y(x_2), y(x_3), y(x_4), \ldots$.

To explain how the approximations $y_1, y_2, y_3, \ldots$ can be computed, let us focus on a typical line segment. As indicated in Figure 3.21 in the bottom part, assume that we have found the point (x_n, y_n), and we are trying to determine the next point $(x_{n+1},\ y_{n+1})$, where $x_{n+1} = x_n + \Delta x$. Since the slope of the line segment joining the points is determined by the direction field at the starting point, the slope is $f(x_n,\ y_n)$, and hence

$$\frac{y_{n+1} - y_n}{x_{n+1} - x_n} = \frac{y_{n+1} - y_n}{\Delta x} = f(x_n,\ y_n) \tag{3.115}$$

which we can rewrite as

$$y_{n+1} = y_n + \Delta x\, f(x_n,\ y_n). \tag{3.116}$$

This formula, which is the heart of Euler's method, tells us how to use each approximation to compute the next approximation.

Example 3.9. Euler's Method

Use Euler's Method with a step size of 0.1 to make a table of approximate values of the solution of the initial-value problem

$$y' = y - x \quad \text{with} \quad y(0) = 2 \tag{3.117}$$

over the interval $0 \le x \le 1$.

Solution 3.9. In this problem we have $f(x,\ y) = y - x$ and $x_0 = 0$, and $y_0 = 2$. Moreover, since the step size is 0.1, the x-values at which the approximation values will be obtained are

```
Table[0.1 i + 0, {i, 1, 10}]
```

```
{0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1.}
```

The approximation is generated by the following steps:

1. The right hand side of the ODE is defined as

```
f(x_, y_) := y - x
```

2. The spatial step is set to the value

```
Δx = 0.1
```

```
0.1
```

3. The initial point is set to the pair

```
{x, y} = {0, 2}
```

```
{0, 2}
```

and we set up a list collecting the results

```
lresult = ( x  y )
```

```
{{0, 2}}
```

Then the first iteration is carried out due to equation (3.116)

{xn1, yn1} = {x + Δx, Δx f(x, y) + y}

```
{0.1, 2.2}
```

In the next step we exchange and store the values in the a list and add them to the collection list

{x, y} = {xn1, yn1}

```
{0.1, 2.2}
```

AppendTo[lresult, {x, y}]

```
{{0, 2}, {0.1, 2.2}}
```

These steps should be repeated n- times. Therefore we repeat the iteration using the current values of (x, y) in the following loop

Do[{xn1, yn1} = {x + Δx, Δx f(x, y) + y}; {x, y} = {xn1, yn1}; AppendTo[lresult, {x, y}], {i, 1, 9}]

The results are collected in the list *lresult* which is tabulated in the following table

```
lresult // TableForm[#, TableHeadings → {{}, {"xn", "yn"}}] &
```

x_n	y_n
0	2
0.1	2.2
0.2	2.41
0.3	2.631
0.4	2.8641
0.5	3.11051
0.6	3.37156
0.7	3.64872
0.8	3.94359
0.9	4.25795
1.	4.59374

The graphical representation of these points is given in Figure 3.22

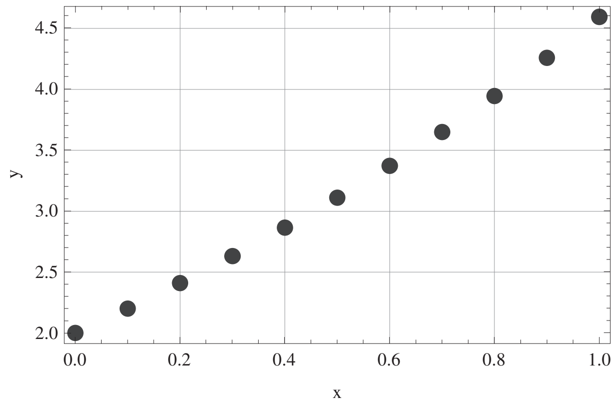


Figure 3.22. Numerical approximation of the initial-value problem $y' = y - x$ with $y(0) = 2$.

The total procedure can be implemented in a function called EulerMethod[] as follows

```
EulerMethod[rightHandside_, initialValues_List] :=
 Block[{Δx = 0.1, n = 10, lresult},
  (* --- assigne initial values--- *)
  {x, y} = initialValues;
  (* --- initialize the list for results --- *)
  lresult = {{x, y}};
  (* --- iterate Euler's equation --- *)
  Do[{xn1, yn1} = {x + Δx, Δx f[x, y] + y};
     {x, y} = {xn1, yn1};
   AppendTo[lresult, {x, y}],
   {i, 1, n}];
  (* --- return the results --- *)
  lresult
 ]
```

The application of this function will generate the numerical approximation of the initial value problem. The results are listed below.

```
nsol = EulerMethod[f[x, y], {0, 2}];
nsol // TableForm[#, TableHeadings → {{}, {"xn", "yn"}}] &
```

x_n	y_n
0	2
0.1	2.2
0.2	2.41
0.3	2.631
0.4	2.8641
0.5	3.11051
0.6	3.37156
0.7	3.64872
0.8	3.94359
0.9	4.25795
1.	4.59374

3.5.3 Tests and Exercises

The following two subsections serve to test your understanding of the last section. Work first on the test examples and then try to solve the exercises.

3.5.3.1 Test Problems

T1. What is Euler's method?

T2. Explain how Euler set up his integration method (the idea).

3.5.3.2 Exercises

E1. Solve the following initial value problem by Euler's method

$$u' = -\sqrt{3}\ u \text{ with } u(0) = 1. \tag{1}$$

E2. Solve the following initial value problem by Euler's method

$$u' = -250\, u \text{ with } u(0) = 1. \tag{2}$$

4

Elementary Linear Algebra

4.1 Vectors and Algebraic Operations

4.1.1 Introduction

Many physical quantities, such as area, length, mass, and temperature, are completely described once the magnitude of the quantity is known. Such quantities are called scalars. Other physical quantities such as the velocity, the position of an object in space, the angular velocity of a rotating disk, and the force acting in a certain direction determine both the magnitude of the quantity and a direction. These quantities are called vectors. For example, wind movement is usually described by giving the speed and direction, say 20km/h northeast. So the wind speed as the magnitude and the wind direction as the orientation form a vector called the wind velocity. As mentioned before other examples of vectors are force and displacement or positions. In this chapter our goal is to review some of the basic theory of vectors in two and three dimensions.

4.1.2 Geometry of Vectors

In this section vectors in 2D and 3D will be introduced geometrically, arithmetic operations on vectors will be defined, and some basic properties of these arithmetic operations will be established.

Vectors can be represented geometrically as directed line segments or arrows in 2D and 3D. The direction of the arrow specifies the direction of the vector, and the length of the arrow describes the magnitude. The tail of the arrow is called the initial point of the vector, and the tip of the arrow the terminal point. Symbolically, we shall denote vectors with lowercase letters with an arrow above the letter (for instance $\vec{a}$, $\vec{b}$, $\vec{c}$, and $\vec{x}$). When discussing vectors, we shall refer to numbers as scalars. For now, all our scalars will be real numbers and will be denoted with lowercase letters (for example a, b, c, and x).

If, as in Figure 4.1 the initial point of a vector $\vec{v}$ is A and the terminal point is B, we write

$$\vec{v} = \overrightarrow{AB}. \tag{4.1}$$

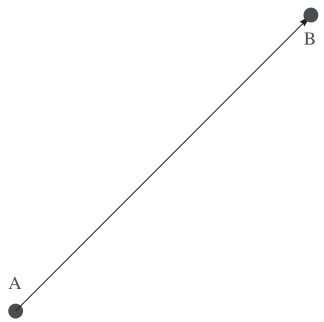


Figure 4.1. The vector $\overrightarrow{AB}$.

Vectors with the same length and same direction, such as those in Figure 4.2 are called equivalent. Since we want a vector to be determined solely by its length and direction, equivalent vectors are regarded as equal even though they may be located in different positions.

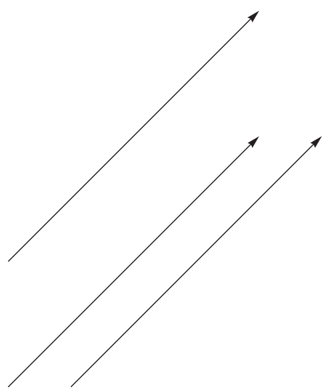


Figure 4.2. Equivalent vectors.

If $\vec{v}$ ad $\vec{w}$ are equivalent we write

$$\vec{v} = \vec{w}. \tag{4.2}$$

Definition 4.1. *Addition of Vectors*

If $\vec{v}$ and $\vec{w}$ are any two vectors, then the sum $\vec{v}+\vec{w}$ is a vector determined as follows: Position the vector $\vec{w}$ so that its initial point coincides with the terminal point of $\vec{v}$. The vector $\vec{v}+\vec{w}$ is represented by the arrow from the initial point of $\vec{v}$ to the terminal point of $\vec{w}$. ■

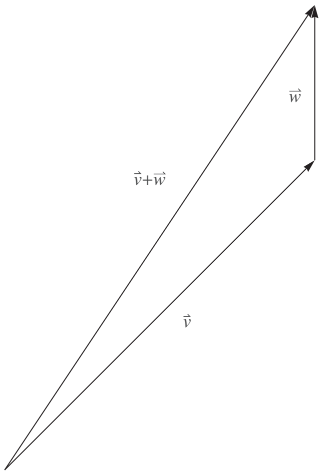


Figure 4.3. The sum of two vectors $\vec{v}+\vec{w}$.

In Figure 4.4 we have constructed the two sums $\vec{v}+\vec{w}$ and $\vec{w}+\vec{v}$. It is evident from this figure that

$$\vec{v}+\vec{w}=\vec{w}+\vec{v} \tag{4.3}$$

and that the sum coincided with the diagonal of the parallelogram determined by $\vec{v}$ and $\vec{w}$ when these vectors are positioned so that they have the same initial point.

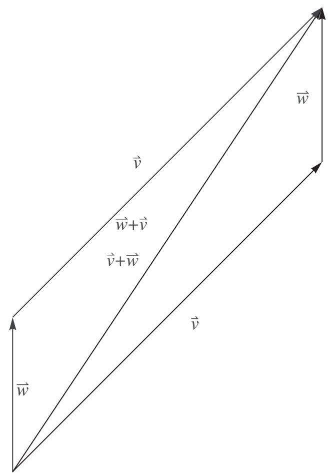


Figure 4.4. The sum of two vectors $\vec{v}+\vec{w}=\vec{w}+\vec{v}$.

The vector of length zero is called the zero vector and is denoted by $\vec{0}$. We define

$$\vec{0}+\vec{v}=\vec{v}+\vec{0}=\vec{v} \tag{4.4}$$

for every vector $\vec{v}$. Since there is no natural direction for the zero vector, we shall agree that it can be assigned any direction that is convenient for the problem being considered. If $\vec{v}$ is any nonzero vector, then $-\vec{v}$, the negative of $\vec{v}$, is defined to be the vector that has the same magnitude as $\vec{v}$ but is oppositely directed (see Figure 4.5)

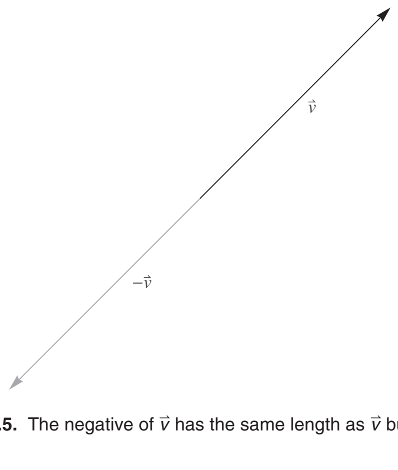


Figure 4.5. The negative of $\vec{v}$ has the same length as $\vec{v}$ but is oppositely directed.

This vector has the property

$$\vec{v} + (-\vec{v}) = \vec{0}. \tag{4.5}$$

In addition we define $\vec{0} = -\vec{0}$. Subtraction of vectors is defined as follows:

Definition 4.2. *Subtraction of Vectors*

If $\vec{v}$ ad $\vec{w}$ are any two vectors, then the difference of $\vec{w}$ from $\vec{v}$ is defined by

$$\vec{v} - \vec{w} = \vec{v} + (-\vec{w}).\blacksquare \tag{4.6}$$

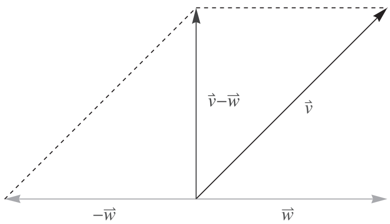


Figure 4.6. Difference of two vectors $\vec{v} - \vec{w}$.

To obtain the difference $\vec{v} - \vec{w}$ without constructing $-\vec{w}$, position $\vec{v}$ and $\vec{w}$ so that their initial points coincide; the vector from the terminal point of $\vec{w}$ to the terminal point of $\vec{v}$ is then the vector $\vec{v} - \vec{w}$ (see Figure 4.7).

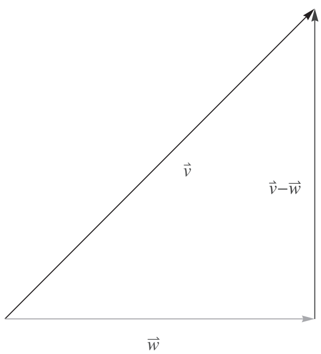


Figure 4.7. Difference of two vectors $\vec{v} - \vec{w}$.

Definition 4.3. *Product with a Scalar*

If $\vec{v}$ is a nonzero vector and k is a nonzero real number (scalar), then the product $k\,\vec{v}$ is defined to be the vector whose length is $|k|$ times the length of $\vec{v}$ and whose direction is the same as that of $\vec{v}$ if

$k > 0$ and opposite to that if $k < 0$. We define $k\vec{v} = \vec{0}$ if $k = 0$ or $\vec{v} = \vec{0} = 0$. ■

Remark 4.1. Note that the vector $(-1)\vec{v}$ has the same length as $\vec{v}$ but is oppositely directed. Thus $(-1)\vec{v}$ is just the negative of $\vec{v}$; that is, $(-1)\vec{v} = -\vec{v}$.

A vector of the form $k\vec{v}$ is called a scalar multiple of $\vec{v}$. Vectors that are scalar multiples of each other are parallel. Conversely, it can be shown that nonzero parallel vectors are scalar multiples of each other. We omit the proof here.

4.1.3 Vectors and Coordinate Systems

Problems involving vectors can often be simplified by introducing a rectangular coordinate system. For the moment we shall restrict to vectors in 2D (the plane). Let $\vec{v}$ be any vector in the plane, and assume, as shown in Figure 4.8 that $\vec{v}$ has been positioned so that its initial point is at the origin of a rectangular coordinate system. The coordinates (v_1, v_2) of the terminal point of $\vec{v}$ are called the components of $\vec{v}$, and we write for the vector

$$\vec{v} = (v_1, v_2). \tag{4.7}$$

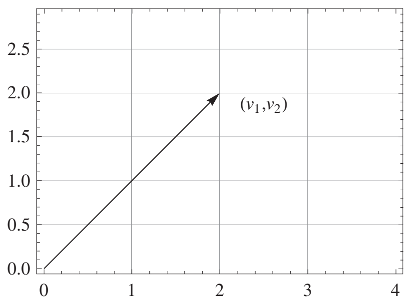


Figure 4.8. v_1 and v_2 are the components of the vector $\vec{v}$.

If equivalent vectors $\vec{v}$ and $\vec{w}$, are located so that their initial points fall at the origin, then it is obvious that their terminal points must coincide (since the vector have the same length and the same direction); thus the vectors have the same components. Conversely, vectors with the same components are equivalent since they have the same length an the same direction. In summary, two vectors

$$\vec{v} = (v_1, v_2) \qquad \text{and} \qquad \vec{w} = (w_1, w_2) \tag{4.8}$$

are equivalent if and only if

$$v_1 = w_1 \qquad \text{and} \qquad v_2 = w_2. \tag{4.9}$$

The operations of vector addition and multiplication by a scalar are easy to carry out in terms of components. As illustrated in Figure 4.9 if

$$\vec{v} = (v_1,\ v_2) \qquad \text{and} \qquad \vec{w} = (w_1,\ w_2) \tag{4.10}$$

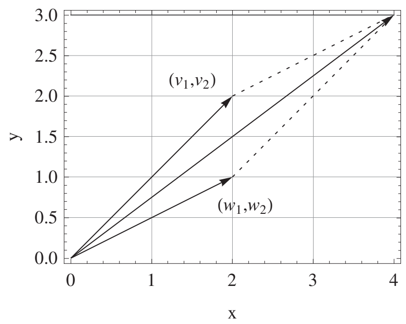


Figure 4.9. v_1 and v_2 are the components of the vector $\vec{v}$.

then the sum of the vectors is gained by adding the components of the vectors, so

$$\vec{v} + \vec{w} = (v_1,\ v_2) + (w_1,\ w_2) = (v_1 + w_1,\ v_2 + w_2). \tag{4.11}$$

If $\vec{v} = (v_1,\ v_2)$ and k is any scalar, then by using a geometric argument involving similar triangles, it can be shown that

$$k\,\vec{v} = k\,(v_1,\ v_2) = (k\,v_1,\ k\,v_2). \tag{4.12}$$

Thus for example, if $\vec{v} = (1,\ -5)$ and $\vec{w} = (6,\ 8)$, then

$$\vec{v} + \vec{w} = (1,\ -5) + (6,\ 8) = (7,\ 3) \tag{4.13}$$

and

$$6\,\vec{v} = 6\,(1,\ -5) = (6,\ -30). \tag{4.14}$$

Since $\vec{v} - \vec{w} = \vec{v} + (-1)\,\vec{w}$, it follows from (4.11) and (4.12) that

$$\vec{v} - \vec{w} = (v_1,\ v_2) - (w_1,\ w_2) = (v_1 - w_1,\ v_2 - w_2). \tag{4.15}$$

Just as vectors in the plane can be described by pairs of real numbers, vectors in 3D can be described by triples of real numbers by introducing a rectangular coordinate system. To construct such a coordinate system, select a point O, called the origin, and choose three mutually perpendicular lines, called coordinate axes, passing through the origin. Label these axes x, y, and z, and select a positive direction for each coordinate axis as well as a unit of length for measuring distances see Figure 4.10.

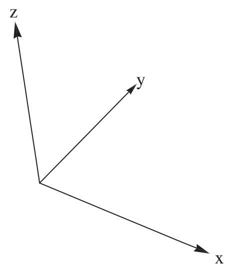


Figure 4.10. A 3D coordinate system with its mutually perpendicular axes.

Each pair of coordinate axes determines a plane called a coordinate plane. These are referred to as the (x, y)-plane, the (x, z)-plane, and the (y, z)-plane. To each point P in 3-space we assign a triple of numbers (x, y, z), called the coordinates of P, as follows: pass three planes through P parallel to the coordinate planes, and denote the points of intersection of these planes with the three coordinate axes by X, Y, and Z (see Figure 4.11).

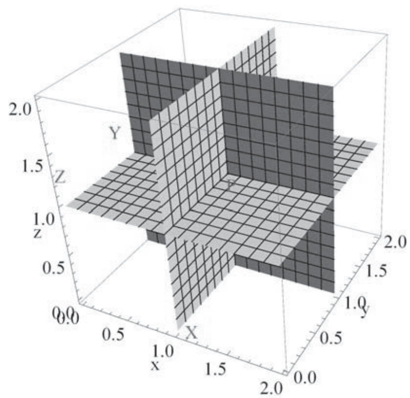


Figure 4.11. Representation of a point P by intersection of three mutually perpendicular coordinate planes.

The coordinates of P are defined to be the signed lengths

$$x = \overline{\mathrm{OX}}, \quad y = \overline{\mathrm{OY}}, \quad \text{and } z = \overline{\mathrm{OZ}}. \tag{4.16}$$

Rectangular coordinate systems in 3-space fall into two categories, left-handed and right-handed. A right-handed system has the property that an ordinary screw pointed in the positive direction on the z-

axis would be advanced if the positive x-axis were rotated 90° toward the positive y-axis; the system is left-handed if the screw would be retraced.

Remark 4.2. In this book we will use only right-handed coordinate systems.

If, as in Figure 4.11, a vector $\vec{v}$ in 3-space is positioned so its initial point is at the origin of a rectangular coordinate system, then the coordinates of the terminal point are called the components of $\vec{v}$, and we write

$$\vec{v} = (v_1, v_2, v_3). \tag{4.17}$$

If $\vec{v} = (v_1, v_2, v_3)$ and $\vec{w} = (w_1, w_2, w_3)$ are two vectors in 3-space, then arguments similar to those used for vectors in plane can be used to establish the following results.

$\vec{v}$ and $\vec{w}$ are equivalent if and only if $v_1 = w_1$, $v_2 = w_2$, and $v_3 = w_3$. The sum of two vectors is calculated by adding the components of the vectors $\vec{v} + \vec{w} = (v_1, v_2, v_3) + (w_1, w_2, w_3) = (v_1 + w_1, v_2 + w_2, v_3 + w_3)$ The product of a vector with a scalar k is defined by $k\,\vec{v} = k\,(v_1, v_2, v_3) = (k\,v_1, k\,v_2, k\,v_3)$.

Example 4.1. Vectors and its Components

Given a vector $\vec{v} = (2, 1, 5)$ and $\vec{w} = (4, 7, 8)$ find the sum the product with a scalar $k = 2$ and $k = -1$.

Solution 4.1. The sum of the two vectors is gained by adding the components of the two vectors

$$\vec{v} + \vec{w} = (2, 1, 5) + (4, 7, 8) = (2 + 4, 1 + 7, 5 + 8) = (6, 8, 13).$$

The product with the scalars follows by

$$k\,\vec{v} = 2\,(2, 1, 5) = (4, 2, 10)$$

and

$$k\,\vec{w} = -1\,(4, 7, 8) = (-4, -7, -8)$$

multiplying the vector with $k = -1$ reverses just the direction.▲

Example 4.2. Computer Color Models

Color on computer monitors are commonly based on what is called RGB color model.

Solution 4.2. Colors in this system are created by adding together percentages of the primary colors red (R), green (G), and blue (B). One way to do this is to identify the primary colors with the vectors

$\vec{r} = (1, 0, 0)$ pure red

$\vec{g} = (0, 1, 0)$ pure green

$\vec{b} = (0, 0, 1)$ pure blue

in $\mathbb{R}^3$ and to create all other colors by forming linear combinations of $\vec{r}$, $\vec{g}$, and $\vec{b}$ using coefficients between 0 and 1, inclusive. The so introduced coefficients represent the percentage of each pure color in the mix. The set of all such color vectors are called RGB space or the RGB color cube. Thus each color vector $\vec{c}$ in this cube is expressible as a linear combination of the form

$$\vec{c} = c_1\,\vec{r} + c_2\,\vec{g} + c_3\,\vec{b} \;=\; c_1(1,\,0,\,0) + c_2(0,\,1,\,0) + c_3(0,\,0,\,1) = (c_1,\,c_2,\,c_3)$$

where $0 \le c_i \le 1$. As indicated in Figure 4.12, the corners of the cube represent the pure primary colors together with the colors, black, withe, magenta, cyan, and yellow. The vectors along the diagonal running from black to white corresponds to shades of gray.

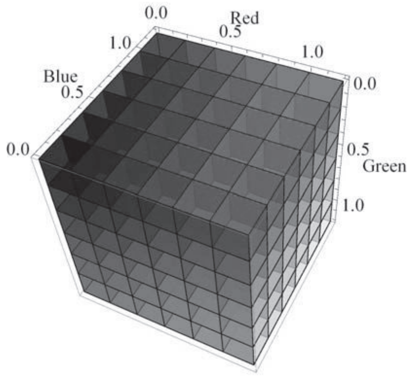


Figure 4.12. RGB cube. The surfaces of the cubes show the transition from pure colors to the mixed colors in the RGB color system as three dimensional vectors with coordinates (Red,Green, Blue).▲

Sometimes a vector is positioned so that its initial point is not at the origin. If the vector $\overrightarrow{P_1\,P_2}$ has initial point $P_1 = (x_1,\ y_1,\ z_1)$ and the terminal point $P_2 = (x_2,\ y_2,\ z_2)$, then

$$\overrightarrow{P_1\,P_2} = (x_2 - x_1,\ y_2 - y_1,\ z_2 - z_1). \tag{4.18}$$

That is, the components of $\overrightarrow{P_1\,P_2}$ are obtained by subtracting the coordinates of the initial point from the coordinates of the terminal point. This may be seen using Figure 4.13.

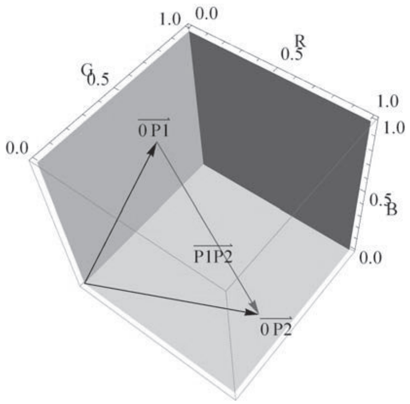


Figure 4.13. Vector with origin different from the origin.

Example 4.3. Components of Vectors

Find the components of the vector $\vec{v} = \overrightarrow{P_1 P_2}$ with the initial point $P_1 = (2, -1, 5)$ and the terminal point $P_2 = (3, 7, 6)$.

Solution 4.3. The components of the vector $\vec{v}$ defined by the two points P_1 and P_2 is given by

$\vec{v} = (2, -1, 5) - (3, 7, 6) = (-1, -8, -1)$.▲

4.1.4 Vector Arithmetic and Norm

The following Theorem collects the most important properties of vectors in 2D and 3D.

Theorem 4.1. *Vector Arithmetic*

If $\vec{u}$, $\vec{v}$, and $\vec{w}$ are vectors in 2D or 3D and k and l are scalars, then the following relationships hold

$a)\ \vec{u} + \vec{v} = \vec{v} + \vec{u}$ commutativity

$b)\ \vec{u} + \vec{0} = \vec{0} + \vec{u} = \vec{u}$ existence of the neutral element

$c)\ (\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$ associativity

$d)\ \vec{u} + (-\vec{u}) = 0$ existence of the inverse

$e)\ k\,(l\,\vec{u}) = (k\,l)\,\vec{u}$ scalar association

$f)\ k\,(\vec{u} + \vec{v}) = k\,\vec{u} + k\,\vec{u}$ distribution of scalars

$g)\ (k + l)\,\vec{u} = k\,\vec{u} + l\,\vec{u}$ distribution of vectors

$h)\ 1\,\vec{u} = \vec{u}$ existence of scalar unity.■

We note that we have developed two approaches to vectors: *geometric*, in which vectors are represented by arrows or directed line segments, and *analytic*, in which vectors are represented by pairs or triples of numbers called components. As a consequence the equations in Theorem 4.1 can be proved either geometrically or analytically. To illustrate the algebraic way for part c). The remaining proofs are left as exercises.

Proof 4.1. Part c) we shall give the proof for vectors in 3D, the proof in 2D is similar. If $\vec{u} = (u_1, u_2, u_3)$, $\vec{v} = (v_1, v_2, v_3)$, and $\vec{w} = (w_1, w_2, w_3)$, then

$$
\begin{aligned}
&(\vec{u}+\vec{v})+\vec{w} \\
=\ & ((u_1, u_2, u_3)+(v_1, v_2, v_3))+(w_1, w_2, w_3) \\
=\ & (u_1+v_1, u_2+v_2, u_3+v_3)+(w_1, w_2, w_3) \\
=\ & (u_1+v_1+w_1, u_2+v_2+w_2, u_3+v_3+w_3) \\
=\ & (u_1+(v_1+w_1), u_2+(v_2+w_2), u_3+(v_3+w_3)) \\
=\ & (u_1, u_2, u_3)+((v_1+w_1), (v_2+w_2), (v_3+w_3)) \\
=\ & (u_1, u_2, u_3)+((v_1, v_2, v_3)+(w_1, w_2, w_3)) \\
=\ & \vec{u}+(\vec{v}+\vec{w})
\end{aligned}
$$

The geometric proof is contained in Figure 4.14.

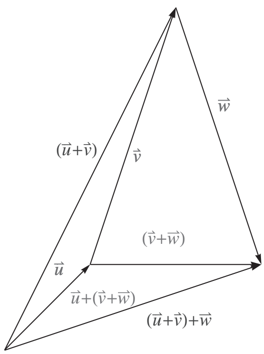


Figure 4.14. The geometric proof of associativity of vectors.

QED

Remark 4.3. In light of part c) of this theorem, the symbol $\vec{u}+\vec{v}+\vec{w}$ is unambiguous since the same sum is obtained no matter where parentheses are inserted. Moreover, if the vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$ are placed "tip to tail", then the sum $\vec{u}+\vec{v}+\vec{w}$ is the vector from initial point of $\vec{u}$ to the terminal point of $\vec{w}$ (see Figure 4.14).

The length of a vector $\vec{u}$ is often called the norm of $\vec{u}$ and is denoted by $\|\vec{u}\|$. It follows from the theorem of Pythagoras that the norm of a vector $\vec{u} = (u_1, u_2)$ in 2D is

$$\| \vec{u} \| = \sqrt{u_1^2 + u_2^2} \,. \tag{4.19}$$

This relation is show in Figure 4.15.

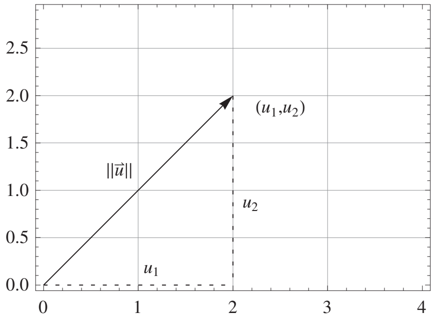


Figure 4.15. Norm of a vector $\vec{u} = (u_1, u_2)$ in 2D is $\| \vec{u} \| = \sqrt{u_1^2 + u_2^2}$.

Let $\vec{u} = (u_1, u_2, u_3)$ be a vector in 3D. The norm in 3D is defined by applying Pythagoras theorem for 3D as

$$\| \vec{u} \|^2 = u_1^2 + u_2^2 + u_3^2 \tag{4.20}$$

which gives

$$\| \vec{u} \| = \sqrt{u_1^2 + u_2^2 + u_3^2} \,. \tag{4.21}$$

A vector of norm 1 is called a unit vector.

If $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$ are two points in 3-space, then the distance d between them is the norm of the vector $\overrightarrow{P_1 P_2}$. Since

$$\overrightarrow{P_1 P_2} = (x_2 - x_1, y_2 - y_1, z_2 - z_1) \tag{4.22}$$

It follows from the definition of the norm that

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \,. \tag{4.23}$$

Similarly, this kind of relation holds in 2D and is given for two points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ by

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \,. \tag{4.24}$$

Example 4.4. Norm and Distance

Find the norm of the vector $\vec{u} = (1, 4, -7)$. Determine the distance between the two points

$P_1 = (1, 3, 6)$ and $P_2 = (4, 7, 9)$.

Solution 4.4. The norm of the vector is determined in terms of the formula

$$\|\vec{u}\| = \sqrt{x^2 + y^2 + z^2} = \sqrt{1 + 16 + 49} = \sqrt{66} = 8.12404$$

The distance between the two points is equal to the norm of the vector $\overrightarrow{P_1 P_2} = (1 - 4, 3 - 7, 6 - 9)$

$$\|\overrightarrow{P_1 P_2}\| = \sqrt{(1-4)^2 + (3-7)^2 + (6-9)^2} = \sqrt{34} = 5.83095.\blacktriangle$$

From the definition of the product $k\,\vec{u}$, the length of a vector $k\,\vec{u}$ is $|k|$ times the length of $\vec{u}$. Expressed as an equation, this statement says that

$$\|k\,\vec{u}\| = |k|\ \|\vec{u}\|. \tag{4.25}$$

This useful formula is applicable in both 2D and 3D.

4.1.5 Dot Product and Projection

In this section we shall discuss an important way of multiplying vectors in 2D and 3D. We shall then give some applications of this multiplication to geometry.

4.1.5.1 Dot Product of Vectors

Let $\vec{u}$ and $\vec{v}$ be two nonzero vectors in 2D and 3D, and assume these vectors have been positioned so that their initial points coincide. By the angle between $\vec{u}$ and $\vec{v}$, we shall mean the angle θ determined by $\vec{u}$ and $\vec{v}$ that satisfies $0 \le \theta \le \pi$

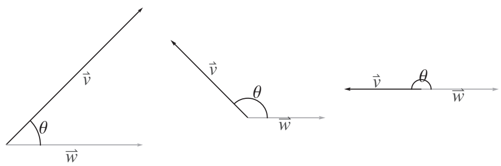


Figure 4.16. The angle θ between $\vec{u}$ and $\vec{v}$ satisfies $0 \le \theta \le \pi$.

Definition 4.4. *Dot Product, Inner Product*

If $\vec{u}$ and $\vec{v}$ are vectors in 2D or 3D and θ is the angle between $\vec{u}$ and $\vec{v}$ then the dot product or Euclidean inner product $\vec{u}.\vec{v}$ is defined by

$$\vec{u}.\vec{v} = \begin{cases} \|\vec{u}\|\ \|\vec{v}\|\cos(\theta) & \text{if } \vec{u} \neq 0 \text{ and } \vec{v} \neq 0 \\ 0 & \text{if } \vec{u} = 0 \text{ or } \vec{v} = 0 \text{ or } \vec{u} \text{ is orthogonal to } \vec{v}. \end{cases} \quad \blacksquare$$

Example 4.5. Dot Product

As shown in Figure 4.17 the angle between the vectors $\vec{u} = (0, 0, 1)$ and $\vec{v} = (0, 2, 2)$ is 45°. Show this fact by means of calculation.

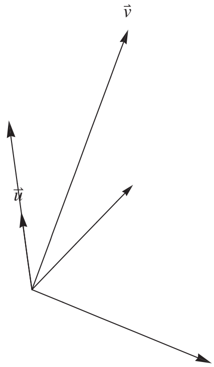


Figure 4.17. Dot product for two given vectors.

Solution 4.5. The dot product is defined by

$$\vec{u}.\vec{v} = \|\vec{u}\| \ \|\vec{v}\| \cos(\theta) = \left(\sqrt{0^2+0^2+1^2}\right)\left(\sqrt{0^2+2^2+2^2}\right)\left(\frac{1}{\sqrt{2}}\right) = 2. \blacktriangle$$

4.1.5.2 Component Form of the Dot Product

For purpose of computation, it is desirable to have a formula that expresses the dot product of two vectors in terms of the components of the vectors. We will derive such a formula for vectors in 3D; the derivation for vectors in 2D is similar.

Let $\vec{u} = (u_1, u_2, u_3)$ and $\vec{v} = (v_1, v_2, v_3)$ be two nonzero vectors. If, as shown in Figure 4.18, θ is the angle between $\vec{u}$ and $\vec{v}$, then the law of cosines yields the following relation.

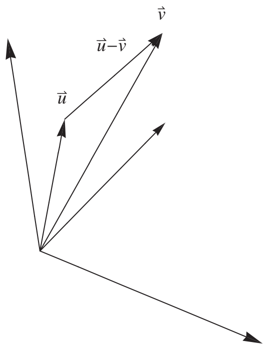


Figure 4.18. Components of two vectors $\vec{u}$ and $\vec{v}$ in connection with the dot product.

The vector between the endpoint of $\vec{u}$ and $\vec{v}$ is denoted as $\vec{d} = \vec{u} - \vec{v}$ which when applied the cosine law can be expressed as

$$\|\vec{d}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\, \|\vec{u}\|\, \|\vec{v}\| \cos(\theta) \tag{4.26}$$

Since $\vec{d} = \vec{u} - \vec{v}$, we can write this expression as

$$\|\vec{u}\|\, \|\vec{v}\| \cos(\theta) = \frac{1}{2}\left(\|\vec{u}\|^2 + \|\vec{v}\|^2 - \|\vec{u} - \vec{v}\|^2\right) \tag{4.27}$$

or

$$\vec{u}.\vec{v} = \frac{1}{2}\left(\|\vec{u}\|^2 + \|\vec{v}\|^2 - \|\vec{u} - \vec{v}\|^2\right). \tag{4.28}$$

Substituting

$$\|\vec{u}\| = u_1^2 + u_2^2 + u_3^2, \qquad \text{and} \qquad \|\vec{v}\| = v_1^2 + v_2^2 + v_3^2, \tag{4.29}$$

and

$$\|\vec{u} - \vec{v}\| = (u_1 - v_1)^2 + (u_2 - v_2)^2 + (u_3 - v_3)^2 \tag{4.30}$$

we obtain, after simplifying,

$$\vec{u}.\vec{v} = u_1\, v_1 + u_2\, v_2 + u_3\, v_3. \tag{4.31}$$

Although we derived this formula under the assumption that $\vec{u}$ and $\vec{v}$ are nonzero, the formula is also valid if $\vec{u} = 0$ or $\vec{v} = 0$ (verify this as an exercise).

If $\vec{u} = (u_1, u_2)$ and $\vec{v} = (v_1, v_2)$ are two vectors in 2D, then the formula corresponding to the above is

$$\vec{u}.\vec{v} = u_1\, v_1 + u_2\, v_2. \tag{4.32}$$

4.1.5.3 Angle Between Vectors

If $\vec{u}$ and $\vec{v}$ are nonzero vectors, then the definition of the scalar product can be used to rewrite the expressing as

$$\cos(\theta) = \frac{\vec{u}.\vec{v}}{\|\vec{u}\|\ \|\vec{v}\|}. \tag{4.33}$$

Example 4.6. Dot Product of Two Vectors

Consider the vectors $\vec{u} = (2, -1, 1)$ and $\vec{v} = (1, 1, 2)$. Find the dot product $\vec{u}.\vec{v}$ and determine the angle θ between $\vec{u}$ and $\vec{v}$.

Solution 4.6. Applying the formulas for the dot product we find

$$\vec{u}.\vec{v} = u_1\, v_1 + u_2\, v_2 + u_3\, v_3 = 2 * 1 + (-1) * 1 + 1 * 2 = 3$$

For the given vectors we have $\|\vec{u}\| = \|\vec{v}\| = \sqrt{6}$ so that we get from

$$\cos(\theta) = \frac{\vec{u}.\vec{v}}{\|\vec{u}\|\ \|\vec{v}\|} = \frac{3}{\sqrt{6}\ \sqrt{6}} = \frac{1}{2}$$

and thus $\theta = 60\,°$.

The same calculations can be carried out by using *Mathematica.* First define the vectors

u = {2, −1, 1}; v = {1, 1, 2};

Then evaluate the dot product

dotProduct = u.v

3

and finally find the angle

$$\boldsymbol{\theta = \cos^{-1}\left(\frac{u.v}{\sqrt{u.u}\ \sqrt{v.v}}\right)}$$

$$\frac{\pi}{3}$$

▲

Example 4.7. Vectors and Geometry

A common problem in geometry is to find the angle between a diagonal of a cube and one of its edges.

Solution 4.7. Let k be the length of the edge and introduce a coordinate system as shown in Figure 4.19

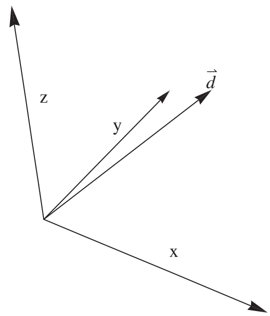


Figure 4.19. Diagonal in a cube with side length k.

If we let $\vec{u} = (k, 0, 0)$, $\vec{v} = (0, k, 0)$, and $\vec{w} = (0, 0, k)$, then the vector

$$\vec{d} = (k, k, k) = \vec{u} + \vec{v} + \vec{w}$$

is a diagonal of the cube. The angle θ between $\vec{d}$ and the edge $\vec{u}$ satisfies

$$\cos(\theta) = \frac{\vec{u}.\vec{d}}{\|\vec{u}\| \ \|\vec{d}\|} = \frac{k^2}{k\sqrt{3k^2}} = \frac{1}{\sqrt{3}}.$$

Thus $\theta = \arccos\left(\frac{1}{\sqrt{3}}\right) = 54.74°$

In *Mathematica* the calculation is as follows

$$\boldsymbol{d = \{k, k, k\}; u = \{k, 0, 0\};}$$

$$\boldsymbol{\theta = \cos^{-1}\left(\frac{d.u}{\sqrt{d.d}\ \sqrt{u.u}}\right)}$$

$$\cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

Note that the result is independent of k, as expected.▲

The following theorem shows how the dot product can be used to obtain information about the angle between two vectors; it also establishes an important relationship between the norm and the dot product.

Theorem 4.2. *Dot Product and Angle*

Let $\vec{u}$ and $\vec{v}$ be vectors in 2D and 3D.

a) $\vec{v}.\vec{v} = \|\vec{v}\|^2$, that is $\|\vec{v}\| = (\vec{v}.\vec{v})^{1/2}$.

b) If the vectors $\vec{u}$ and $\vec{v}$ are nonzero and θ is the angle between them, then

θ is acute *if and only if* $\vec{u}.\vec{v} > 0$
θ is obtuse *if and only if* $\vec{u}.\vec{v} < 0$
$\theta = \frac{\pi}{2}$ *if and only if* $\vec{u}.\vec{v} = 0$. ■

Proof 4.2. a) Since the angle θ between $\vec{v}$ is 0, we have

$\vec{v}.\vec{v} = \|\vec{v}\| \; \|\vec{v}\| \cos(\theta) = \|\vec{v}\|^2 \cos(0) = \|\vec{v}\|^2$.

b) Since θ satisfies $0 \leq \theta \leq \pi$, it follows that θ is acute if and only if $\cos(\theta) > 0$, that θ is obtuse if and only if $\cos(\theta) < 0$, and that $\theta = \pi/2$ if and only if $\cos(\theta) = 0$. But $\cos(\theta)$ has the same sign as $\vec{u}.\vec{v} = \|\vec{u}\| \; \|\vec{v}\| \cos(\theta) > 0$, and $\|\vec{v}\| > 0$. Thus the result follows.

QED

Example 4.8. Dot Products from Components

If $\vec{u} = (1, -2, 3)$, $\vec{v} = (-3, 4, 2)$, and $\vec{w} = (3, 6, 3)$ find the dot products of all combinations.

Solution 4.8. The calculation is straight forward

$$\vec{u}.\vec{v} = (1, -2, 3).(-3, 4, 2) = 1*(-3) + (-2)*4 + 3*2 = -5$$

$$\vec{v}.\vec{w} = (-3, 4, 2).(3, 6, 3) = (-3)*3 + 4*6 + 2*3 = 21$$

$$\vec{u}.\vec{w} = (1, -2, 3).(3, 6, 3) = 1*3 + (-2)*6 + 3*3 = 0$$

Therefore $\vec{u}$ and $\vec{v}$ make an obtuse angle, $\vec{v}$ and $\vec{w}$ make an acute angle, and $\vec{u}$ and $\vec{w}$ are perpendicular.▲

4.1.5.4 Orthogonal Vectors

Perpendicular vectors are also called orthogonal vectors. Due to Theorem 4.2 part b), two nonzero vectors are orthogonal if and only if their dot product is zero. If we agree to consider $\vec{u}$ and $\vec{v}$ to be perpendicular when either or both of these vectors are 0, then we can state without exception that two vectors $\vec{u}$ and $\vec{v}$ are orthogonal (perpendicular) if and only if $\vec{u}.\vec{v} = 0$. To indicate that $\vec{u}$ and $\vec{v}$ are orthogonal vectors, we write $\vec{u} \perp \vec{v}$.

Example 4.9. Vector Perpendicular to a Line

Show that in 2D the nonzero vector $\vec{n} = (a, b)$ is perpendicular to the line $a\,x + b\,y + c = 0$.

Solution 4.9. Let $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ be distinct points on the line, so that

$$a\,x_1 + b\,y_1 + c = 0$$

$$a\,x_2 + b\,y_2 + c = 0.$$

Since the following vector $\vec{d} = (x_2 - x_1,\ y_2 - y_1)$ runs along the line, we need only show that $\vec{n}$ and $\vec{d}$ are perpendicular. On the other hand if we subtract the two equations from each other, we find

$$a(x_2 - x_1) + b(y_2 - y_1) = 0$$

which can be expressed as

$$(a,\ b).(x_2 - x_1,\ y_2 - y_1) = 0 \qquad \text{or} \qquad \vec{n}.\vec{d} = 0.$$

Thus $\vec{n}$ and $\vec{d}$ are perpendicular.▲

The following theorem lists the most important properties of the dot product. They are useful in calculations involving vectors.

Theorem 4.3. *Algebraic Properties of a Dot Product*

If $\vec{u}$, $\vec{v}$ and $\vec{w}$ are vectors in 2D or 3D and k is a scalar, then the following holds

$a)\ \vec{u}.\vec{v} = \vec{v}.\vec{u}$ commutativity of the product

$b)\ \vec{u}.(\vec{v} + \vec{w}) = \vec{u}.\vec{v} + \vec{u}.\vec{w}$ distributive law

$c)\ k\,(\vec{u}.\vec{v}) = (k\,\vec{u}).\vec{v} = \vec{u}.(k\,\vec{v})$ associativity of scalars

$d)\ \vec{v}.\vec{v} > 0$ if $\vec{v} \neq 0$ and $\vec{v}.\vec{v} = 0$ if $\vec{v} = 0$.■

The Proof of these relations is left as an exercise.

4.1.5.5 Orthogonal Projection

In many applications it is of interest to decompose a vector $\vec{u}$ into a sum of two terms, one parallel to the specified nonzero vector $\vec{a}$ and the other perpendicular to $\vec{a}$. If $\vec{u}$ and $\vec{a}$ are positioned so that their initial point coincides at a point Q, we can decompose the vector $\vec{u}$ as follows (see Figure 4.20): Drop a perpendicular from the tip of $\vec{u}$ to the line through $\vec{a}$, and construct the vector $\vec{w}_1$ from Q to the foot of this perpendicular. Next form the difference

$$\vec{w}_2 = \vec{u} - \vec{w}_1. \tag{4.34}$$

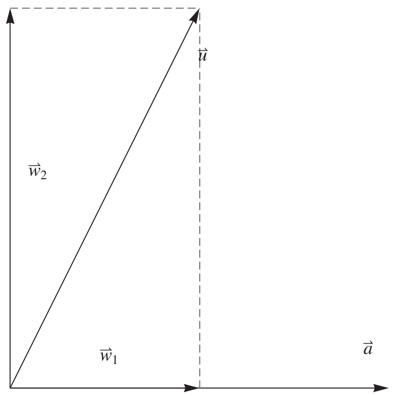


Figure 4.20. The vector $\vec{u}$ is the sum of $\vec{w}_1$ and $\vec{w}_2$, where $\vec{w}_1$ is parallel to $\vec{a}$ and $\vec{w}_2$ is perpendicular to $\vec{a}$.

As indicated in Figure 4.20 the vector $\vec{w}_1$ is parallel to $\vec{a}$, the vector $\vec{w}_2$ is perpendicular to $\vec{a}$, and

$$\vec{w}_1 + \vec{w}_2 = \vec{w}_1 + (\vec{u} - \vec{w}_1) = \vec{u}. \tag{4.35}$$

The vector $\vec{w}_1$ is called the orthogonal projection of $\vec{u}$ on $\vec{a}$ or sometimes the vector component of $\vec{u}$ along $\vec{a}$. It is denoted by

$$\mathrm{proj}_{\vec{a}}\, \vec{u} \tag{4.36}$$

Read: projection of $\vec{u}$ on $\vec{a}$. The vector $\vec{w}_2$ is called the vector component of $\vec{u}$ orthogonal to $\vec{a}$. Since we have $\vec{w}_2 = \vec{u} - \vec{w}_1$ this vector can be written as

$$\vec{w}_2 = \vec{u} - \mathrm{proj}_{\vec{a}}\, \vec{u}. \tag{4.37}$$

The following theorem gives formulas for calculating $\mathrm{proj}_{\vec{a}}\, \vec{u}$ and $\vec{u} - \mathrm{proj}_{\vec{a}}\, \vec{u}$.

Theorem 4.4. *Projection Theorem*

If $\vec{u}$ and $\vec{a}$ are vectors in 2D or 3D and if $\vec{a} \neq 0$, then

$$\mathrm{proj}_{\vec{a}}\, \vec{u} = \frac{\vec{u}.\vec{a}}{\|\vec{a}\|^2}\, \vec{a} \qquad \text{vector component of } \vec{u} \text{ along } \vec{a}.$$

$$\vec{u} - \mathrm{proj}_{\vec{a}}\, \vec{u} = \vec{u} - \frac{\vec{u}.\vec{a}}{\|\vec{a}\|^2}\, \vec{a} \qquad \text{vector component of } \vec{u} \text{ orthogonal to } \vec{a}. \blacksquare$$

The proof of this theorem is left as an exercise.

Example 4.10. Vector Component of $\vec{u}$ Along $\vec{a}$

Let $\vec{u} = (2, -1, 3)$ and $\vec{a} = (4, -1, 2)$. Find the vector component of $\vec{u}$ along $\vec{a}$ and the vector component of $\vec{u}$ orthogonal to $\vec{a}$.

Solution 4.10. First let us calculate the dot product

$$\vec{u}.\vec{a} = (2, -1, 3).(4, -1, 2) = 2*4 + (-1)*(-1) + 3*2 = 15$$

The value of the norm squared is

$$\| a \|^2 = 4^2 + (-1)^2 + 2^2 = 21.$$

Thus the vector component of $\vec{u}$ along $\vec{a}$ is

$$\text{proj}_{\vec{a}}\, \vec{u} = \frac{\vec{u}.\vec{a}}{\| \vec{a} \|^2} \vec{a} = \frac{15}{21} (4, -1, 2) = \left(\frac{20}{7}, \frac{-5}{7}, \frac{10}{7} \right)$$

and the vector component of $\vec{u}$ orthogonal to $\vec{a}$ is

$$\vec{u} - \text{proj}_{\vec{a}}\, \vec{u} = \vec{u} - \frac{\vec{u}.\vec{a}}{\| \vec{a} \|^2} \vec{a} = (2, 01, 3) - \left(\frac{20}{7}, \frac{-5}{7}, \frac{10}{7} \right) = \left(\frac{-6}{7}, \frac{-2}{7}, \frac{11}{7} \right).$$

As a check, you may wish to verify that the vectors $\vec{u} - \text{proj}_{\vec{a}}\, \vec{u}$ and $\vec{a}$ are perpendicular by showing that their dot product is zero.▲

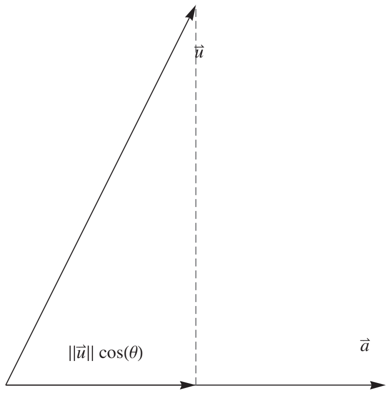


Figure 4.21. Projection of $\vec{u}$ on $\vec{a}$ by using the relation $\| \text{proj}_{\vec{a}}\, \vec{u} \| = \| \vec{u} \| \cos(\theta)$. In the figure it is assumed that $0 \le \theta \le \pi/2$.

A formula for the length of the vector component of $\vec{u}$ along $\vec{a}$ can be obtained by writing

$$\| \text{proj}_{\vec{a}}\, \vec{u} \| = \left\| \frac{\vec{u}.\vec{a}}{\| \vec{a} \|^2} \vec{a} \right\| = \left| \frac{\vec{u}.\vec{a}}{\| \vec{a} \|^2} \right| \| \vec{a} \| = \frac{|\vec{u}.\vec{a}|}{\| \vec{a} \|^2} \| \vec{a} \| \tag{4.38}$$

which yields

$$\| \text{proj}_{\vec{a}} \vec{u} \| = \frac{|\vec{u}.\vec{a}|}{\|\vec{a}\|}. \tag{4.39}$$

If θ denotes the angle between $\vec{u}$ and $\vec{a}$, then $\vec{u}.\vec{a} = \|\vec{u}\| \, \|\vec{a}\| \cos(\theta)$ so the expression above can be simplified by

$$\| \text{proj}_{\vec{a}} \vec{u} \| = \|\vec{u}\| \cos(\theta) \tag{4.40}$$

verify this as an exercise! A geometric interpretation of this result is given in Figure 4.21.

As an example, we will use vector methods to derive a formula for the distance from a point in the plane to a line.

Example 4.11. Distance Between a Point and a Line

Find the formula for the distance D between a point $P_0(x_0, \ y_0)$ and a line $a\,x + b\,y + c = 0$.

Solution 4.11. Let $Q(x_1, \ y_1)$ be any point on the line, and position the vector $\vec{n} = (a, \ b)$ so that its initial point is at Q. By virtue of Example 4.9, the vector $\vec{n}$ is perpendicular to the line.

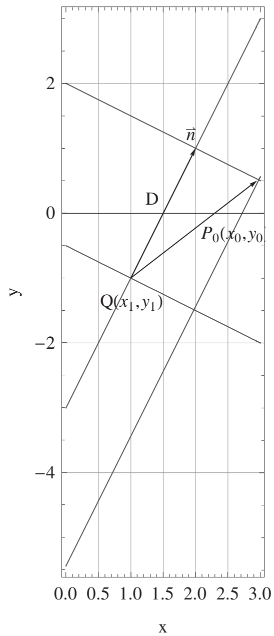


Figure 4.22. Distance of a point P_0 to a line $a\,x + b\,y + c = 0$.

As indicated in the figure above, the distance D is equal to the length of the orthogonal projection of $\overrightarrow{\mathrm{QP}_0}$ on $\vec{n}$; thus we have

$$D = \| \mathrm{proj}_{\vec{n}} \overrightarrow{\mathrm{QP}_0} \| = \frac{\left| \overrightarrow{\mathrm{QP}_0} . \vec{n} \right|}{\| \vec{n} \|}.$$

In detail we have

$$\overrightarrow{\mathrm{QP}_0} = (x_0 - x_1,\ y_0 - y_1)$$

$$\overrightarrow{\mathrm{QP}_0} . \vec{n} = a(x_0 - x_1) + b(y_0 - y_1)$$

$$\| \vec{n} \| = \sqrt{a^2 + b^2}$$

so

$$D = \frac{a(x_0 - x_1) + b(y_0 - y_1)}{\sqrt{a^2 + b^2}}. \blacktriangle$$

Example 4.12. Distance Formula

Use the derived formula for the distance to determine the distance between the Point $(1, -2)$ to the line $3\,x + 4\,y - 6 = 0$.

Solution 4.12. The insertion of the values into the formula give

$$D = \frac{|3 * 1 + 4 * (-2) - 6|}{\sqrt{3^2 + 4^2}} = \frac{|-11|}{\sqrt{25}} = \frac{11}{5}. \blacktriangle$$

4.1.5.6 Cross Product of Vectors

In many applications of vectors to problems in geometry, physics and engineering, it is of great interest to construct a vector in 3D that is perpendicular to two given vectors. In this section we show how to calculate this perpendicular vector.

Recall from Section 4.1.5.1 that the dot product of two vectors in 2D or 3D produce a scalar. We will now define a type of vector multiplication that produces a vector as the product but this calculation is only applicable in 3D.

Definition 4.5. *Cross Product*

If $\vec{u} = (u_1, u_2, u_3)$ and $\vec{v} = (v_1, v_2, v_3)$ are vectors in 3-space, then the cross product $\vec{u} \times \vec{v}$ is the vector defined by

$$\vec{u} \times \vec{v} = (u_2\, v_3 - u_3\, v_2,\ u_3\, v_1 - u_1\, v_3,\ u_1\, v_2 - u_2\, v_1). \blacksquare$$

Remark 4.4. Instead of memorizing the formula from above you can obtain the components of $\vec{u} \times \vec{v}$ as follows:

From the array of 3×2 symbols $\begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \\ u_3 & v_3 \end{pmatrix}$ whose first column contains the components of $\vec{u}$ and whose second row contains the components of $\vec{v}$. To find the components of $\vec{u}\times\vec{v}$ delete the first row and take the diagonal products first from upper left to lower right and subtract the second product of the diagonal lower left times upper right from the previous product $u_2\, v_3 - u_3\, v_2$ which represents the first component of the cross product. The second component follows from with the same procedure by deleting the second row and multiplying with -1, the result is $-(u_1\, v_3 - u_3\, v_1) = u_3\, v_1 - u_1\, v_3$. The last component follows by deleting the last row in the array and applying the same procedure as for the first component $u_1\, v_2 - u_2\, v_1$.

Example 4.13. Cross Product

Find the cross product $\vec{u}\times\vec{v}$ where $\vec{u} = (1, 4, 7)$ and $\vec{v} = (2, 0, 4)$.

Solution 4.13. From either the formula or the mnemonic in the preceding remark we find

$$\vec{u}\times\vec{v} = \begin{pmatrix} 1 \\ 4 \\ 7 \end{pmatrix} \times \begin{pmatrix} 2 \\ 0 \\ 4 \end{pmatrix} = \begin{pmatrix} 4*4-0 \\ 7*2-1*4 \\ 1*0-4*2 \end{pmatrix} = \begin{pmatrix} 16 \\ 10 \\ -8 \end{pmatrix}$$

In *Mathematica* this product is carried out by first defining the vectors

```
u = {1, 4, 7}; v = {2, 0, 4};
```

and applying the cross product function Cross[] to the two vectors

```
u×v
```

{16, 10, −8}

Obviously the result is the same.▲

There is an important difference between the dot product and the cross product of two vectors—the dot product is a scalar and the cross product is a vector. The following theorem gives some important relationships between the dot product and the cross product and also shows that $\vec{u}\times\vec{v}$ is orthogonal to both $\vec{u}$ and $\vec{v}$.

Theorem 4.5. *Relations Between Cross and Dot Products*

If $\vec{u}$, $\vec{v}$, and $\vec{w}$ are vectors in 3-space, then

a) $\vec{u}.(\vec{u}\times\vec{v}) = 0$ — $(\vec{u}\times\vec{v})$ is orthogonal to $\vec{u}$

b) $\vec{v}.(\vec{u}\times\vec{v}) = 0$ — $(\vec{u}\times\vec{v})$ is orthogonal to $\vec{v}$

c) $\|\vec{u}\times\vec{v}\|^2 = \|\vec{u}\|^2\,\|\vec{v}\|^2 - (\vec{u}.\vec{v})^2$ — Lagrange's identity

d) $\vec{u}\times(\vec{v}\times\vec{w}) = (\vec{u}.\vec{w})\,\vec{v} - (\vec{u}.\vec{v})\,\vec{w}$ — relationship between cross and dot product

$e)\ (\vec{u}\times\vec{v})\times\vec{w} = (\vec{u}.\vec{w})\,\vec{v} - (\vec{v}.\vec{w})\,\vec{u}$ relationship between cross and dot product.■

Proof 4.5. Let the vectors be represented in general symbolic terms $\vec{u} = (u_1, u_2, u_3)$, $\vec{v} = (v_1, v_2, v_3)$, and $\vec{w} = (w_1, w_2, w_3)$. Then the relations can be calculated as follows

u = {u1, u2, u3}; v = {v1, v2, v3}; w = {w1, w2, w3};

$a = u.u\times v$

u3 (u1 v2 – u2 v1) + u2 (u3 v1 – u1 v3) + u1 (u2 v3 – u3 v2)

which simplifies to

Simplify[a]

0

demonstrating the cross product of $\vec{u}$ and $\vec{v}$ is perpendicular to $\vec{u}$. The proof for b) follows by

$b = v.u\times v$

v3 (u1 v2 – u2 v1) + v2 (u3 v1 – u1 v3) + v1 (u2 v3 – u3 v2)

and the simplification gives

Simplify[b]

0

Again $\vec{v}$ is perpendicular to the cross product $\vec{u}\times\vec{v}$.
Lagrange's identity can be proved by

l1 = $u\times v.u\times v$

$(\text{u1 v2} - \text{u2 v1})^2 + (\text{u3 v1} - \text{u1 v3})^2 + (\text{u2 v3} - \text{u3 v2})^2$

l2 = $u.u\ v.v - (u.v)^2$

$(\text{u1}^2 + \text{u2}^2 + \text{u3}^2)(\text{v1}^2 + \text{v2}^2 + \text{v3}^2) - (\text{u1 v1} + \text{u2 v2} + \text{u3 v3})^2$

the difference of both expressions is

Simplify[l1 – l2]

0

which demonstrates the identity.

QED

The proofs of d) and e) are left as an exercise.

Example 4.14. Perpendicular Vectors

Consider the vectors $\vec{u} = (1, 2, 4)$ and $\vec{v} = (3, -7, 8)$ construct the perpendicular vector to $\vec{u}$ and $\vec{v}$.

Solution 4.14. Due to Theorem 4.5 we have the identity

$\vec{u}.(\vec{u}\times\vec{v}) = 0.$

The application of the dot and cross product on the vectors shows this identity

u = {1, 2, 4}; v = {3, −7, 8};

u.u×v

0

where the cross product is given by

x = u×v

{44, 4, −13}

The result demonstrates that the vector $\vec{x}$ is perpendicular to $\vec{u}$

u.x

0

In addition $\vec{x}$ is also perpendicular to $\vec{v}$

v.x

0

which again shows the validity of the relations a) and b) of Theorem 4.5.▲

The main arithmetic properties of the cross product are listed in the next theorem.

Theorem 4.6. *Algebraic Properties of the Cross Product*

If $\vec{u}$, $\vec{v}$, and $\vec{w}$ are any vectors in 3-space and k is any scalar, then

a) $\vec{u}\times\vec{v} = -(\vec{v}\times\vec{u})$

b) $\vec{u}\times(\vec{v}+\vec{w}) = (\vec{u}\times\vec{v})+(\vec{u}\times\vec{w})$

c) $(\vec{u}+\vec{v})\times\vec{w} = (\vec{u}\times\vec{w})+(\vec{v}\times\vec{w})$

d) $k\,(\vec{u}\times\vec{v}) = (k\,\vec{u})\times\vec{v} = \vec{u}\times(k\,\vec{v})$

e) $\vec{u}\times\vec{0} = \vec{0}\times\vec{u} = 0$

f) $\vec{u}\times\vec{u} = 0.$■

The proof follows immediately from Definition 4.5; for example a) can be proved as follows:

Proof 4.6. a) Interchanging $\vec{u}$ and $\vec{v}$ in the definition of the cross product interchanges the two columns in the array and thus the components in the product are interchanged and the sign changes too. Thus $\vec{u}\times\vec{v} = -\vec{v}\times\vec{u}$.

QED

The proof of the remaining parts are left as an exercise.

Example 4.15. Unit Vectors

Consider the vectors

i = {1, 0, 0};
j = {0, 1, 0};
k = {0, 0, 1};

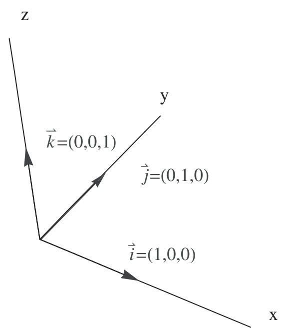


Figure 4.23. The standard unit vectors.

These vectors each have length 1 and lie along the coordinate axes (see Figure 4.23). They are called the standard unit vectors in 3-space.

Solution 4.15. Every vector $\vec{v} = (v_1, v_2, v_3)$ in 3D is expressible in terms of $\vec{i}$, $\vec{j}$, and $\vec{k}$ since we can write

$$\vec{v} = (v_1, v_2, v_3) = v_1(1, 0, 0) + v_2(0, 1, 0) + v_3(0, 0, 1) = v_1\,\vec{i} + v_2\,\vec{j} + v_3\,\vec{k}.$$

For example

{2, 4, 8} == 2 i + 4 j + 8 k

True

In addition we can use Theorem 4.5 to show the following properties

$i \times j$

{0, 0, 1}

j×k

{1, 0, 0}

k×i

{0, 1, 0}

the following relations are generally valid

i×i

{0, 0, 0}

j×j

{0, 0, 0}

and

k×k

{0, 0, 0}

The reader should have no trouble to obtain from the calculations above the following results:

$$\vec{i}\times\vec{i} = 0 \quad \vec{j}\times\vec{j} = 0 \quad \vec{k}\times\vec{k} = 0$$
$$\vec{i}\times\vec{j} = \vec{k} \quad \vec{j}\times\vec{k} = \vec{i} \quad \vec{k}\times\vec{i} = \vec{j}$$
$$\vec{j}\times\vec{i} = -\vec{k} \quad \vec{k}\times\vec{j} = -\vec{i} \quad \vec{i}\times\vec{k} = -\vec{j}.\blacktriangle$$

We know from Theorem 4.5 that $\vec{u}\times\vec{v}$ is orthogonal to both $\vec{u}$ and $\vec{v}$. If $\vec{u}$ and $\vec{v}$ are nonzero vectors, it can be shown that the direction of $\vec{u}\times\vec{v}$ can be determined using the following "right-hand rule": Let θ be the angle between $\vec{u}$ and $\vec{v}$, and suppose $\vec{u}$ is rotated counter clockwise through the angle θ until $\vec{u}$ coincides with $\vec{v}$. If the fingers of the right hand are cupped so that they point in the direction of the rotation, then the thumb indicates the direction of $\vec{u}\times\vec{v}$.

The reader may find it instructive in connection with Figure 4.23 to practice this rule with the products

$$\vec{i}\times\vec{j} = \vec{k}, \qquad \vec{j}\times\vec{k} = \vec{i} \qquad \vec{k}\times\vec{i} = \vec{j}. \tag{4.41}$$

If $\vec{u}$ and $\vec{v}$ are vectors in 3-space, then the norm of $\vec{u}\times\vec{v}$ has a useful geometric interpretation. Lagrange's identity, given in Theorem 4.5 states that

$$\|\vec{u}\times\vec{v}\|^2 = \|\vec{u}\|^2 \|\vec{v}\|^2 - (\vec{u}.\vec{v})^2. \tag{4.42}$$

If θ denotes the angle between $\vec{u}$ and $\vec{v}$, then $\vec{u}.\vec{v} = \|\vec{u}\| \|\vec{v}\| \cos(\theta)$, so Lagrange's identity can be rewritten as

$$\|\vec{u}\times\vec{v}\|^2 = \|\vec{u}\|^2 \|\vec{v}\|^2 - \|\vec{u}\|^2 \|\vec{v}\|^2 \cos^2(\theta) \tag{4.43}$$

$$\Longleftrightarrow \quad \|\vec{u}\times\vec{v}\|^2 = \|\vec{u}\|^2 \|\vec{v}\|^2 \left(1-\cos^2(\theta)\right) \tag{4.44}$$

$$\Longleftrightarrow \quad \|\vec{u}\times\vec{v}\|^2 = \|\vec{u}\|^2 \, \|\vec{v}\|^2 \sin^2(\theta). \tag{4.45}$$

Since $0 \leq \theta \leq \pi$, it follows that $\sin(\theta) \geq 0$, so this formula can be rewritten as

$$\|\vec{u}\times\vec{v}\| = \|\vec{u}\| \, \|\vec{v}\| \sin(\theta). \tag{4.46}$$

But $\|\vec{v}\| \sin(\theta)$ is the altitude of the parallelogram determined by $\vec{u}$ and $\vec{v}$ see Figure 4.24. Thus the relation given is equivalent to the area of the related parallelogram given by

$$A = (\text{base})\,(\text{altitude}) = \|\vec{u}\| \, \|\vec{v}\| \sin(\theta) = \|\vec{u}\times\vec{v}\|. \tag{4.47}$$

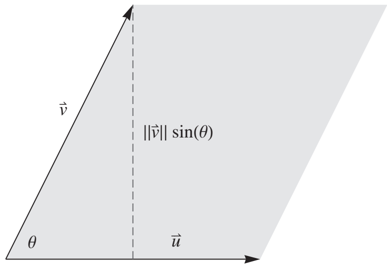


Figure 4.24. Geometric interpretation of the norm of the cross product $\|\vec{u}\times\vec{v}\|$ as area.

This result is even correct if $\vec{u}$ and $\vec{v}$ are collinear, since the parallelogram determined by $\vec{u}$ and $\vec{v}$ has zero area and from the area formula we have $\vec{u}\times\vec{v} = 0$ because $\theta = 0$ in this case. Thus the following Theorem holds.

Theorem 4.7. *Area of a Parallelogram*

If $\vec{u}$ and $\vec{v}$ are vectors in 3D, then $\|\vec{u}\times\vec{v}\|$ is equal to the area of the parallelogram determined by $\vec{u}$ and $\vec{v}$. ■

Example 4.16. Area of a Triangle

Find the area of a triangle determined by the points $P_1(1, 3, 5)$, $P_2(-3, 4, 7)$, and $P_3(4, 6, 8)$.

Solution 4.16. The area if a triangle is $1/2$ of the are of a parallelogram determined by the vectors $\overrightarrow{P_1 P_2}$ and $\overrightarrow{P_1 P_3}$. Using the method discussed above to introduce the vectors it follows from the points:

u = {1, 3, 5} − {−3, 4, 7};
v = {1, 3, 5} − {4, 6, 8};

The cross product of the two vectors is given by

u×v

{−3, 18, −15}

Then the area of the parallelogram is

$$\mathbf{Apar} = \sqrt{u \times v . u \times v}$$

$$3\sqrt{62}$$

and the triangle area follows by

$$\mathbf{Atri} = \frac{\mathbf{Apar}}{\mathbf{2}}$$

$$3\sqrt{\frac{31}{2}}$$

▲

Somehow related to this example is the following definition of a product of three vectors.

Definition 4.6. *Scalar Triple Product*

If $\vec{u}$, $\vec{v}$, and $\vec{w}$ are vectors in 3-space, then

$$\vec{u}.(\vec{v}\times\vec{w})$$

is called the scalar triple product of $\vec{u}$, $\vec{v}$, and $\vec{w}$.■

The scalar triple product of $\vec{u} = (u_1, u_2, u_3)$, $\vec{v} = (v_1, v_2, v_3)$, and $\vec{w} = (w_1, w_2, w_3)$ can be calculated by

```
u = {u1, u2, u3};
v = {v1, v2, v3};
w = {w1, w2, w3};
```

```
u.v×w
```

u1 (v2 w3 − v3 w2) + u2 (v3 w1 − v1 w3) + u3 (v1 w2 − v2 w1)

we will see later on that this product is related to the so called determinant of a matrix which is generated by the column vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$.

Example 4.17. Scalar Triple Product

Calculate the scalar triple product $\vec{u}.(\vec{v}\times\vec{w})$ of the vectors

$$u = 3\,i - 4\,j + 8\,k;$$
$$v = i + 3\,j - 9\,k;$$
$$w = 2\,i + 4\,k;$$

Solution 4.17. Using our formulas for the unit vectors in the cross product and the rules for scalars and the dot product delivers

u.v×w

76

▲

Remark 4.5. The symbol $(\vec{u}.\vec{v})\times\vec{w}$ makes no sense because we cannot form the cross product of a scalar and a vector. Thus no ambiguity arises if we write $\vec{u}.\vec{v}\times\vec{w}$ rather than $\vec{u}.(\vec{v}\times\vec{w})$. However for clarity we shall usually keep the parentheses.

Applying the Theorems 4.5 and 4.6 we can show that the scalar triple product satisfies the following relation

$$\vec{u}.(\vec{v}\times\vec{w}) = \vec{w}.(\vec{u}\times\vec{v}) = \vec{v}.(\vec{w}\times\vec{u}) \tag{4.48}$$

so the scalar triple product is cyclic with respect to the exchange of $\vec{w} \to \vec{u} \to \vec{v}$ etc.

4.1.6 Lines and Planes

In this section we will use vectors to derive equations of lines and planes in 3D. We shall then use these equations to solve some basic geometric problems.

4.1.6.1 Lines in Space

A line in 2D or 3D can be determined uniquely by specifying a point on the line and a nonzero vector parallel to the line. The following theorem gives parametric equations of a line through a point P_0 and parallel to a nonzero direction vector $\vec{v}$.

Theorem 4.8. *Parametric Line*

If $\overrightarrow{0P_0}$ *and* $\vec{v}$ *are two nonzero vectors in 2D or 3D then a parametric line is represented by*

$$\vec{x}(t) = \overrightarrow{0P_0} + t\,\vec{v}.$$

In components these equations are given by the equations

$$x(t) = x_0 + t\,a, \qquad y(t) = y_0 + t\,b$$

where $\overrightarrow{0P_0} = (x_0,\ y_0)$ *and* $\vec{v} = (a,\ b)$ *are the point on the line and the direction in 2D, respectively.*

For the 3D case we have $\overrightarrow{0P_0} = (x_0,\ y_0,\ z_0)$ *and* $\vec{v} = (a,\ b,\ c)$, *so that*

$$x(t) = x_0 + t\,a, \qquad y(t) = y_0 + t\,b, \text{ and } \quad z(t) = z_0 + t\,c.\blacksquare$$

The validity of this theorem follows from the geometric meaning of the relations shown in Figure 4.25.

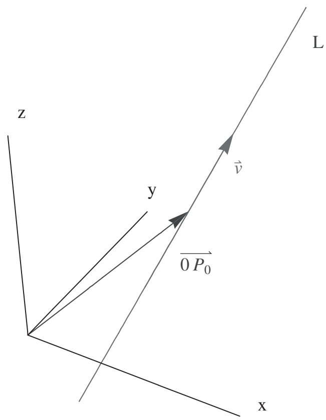


Figure 4.25. A unique line L passes through P_0 and is parallel to $\vec{v}$.

Example 4.18. Line in 3D

Find the parametric equations of the line L passing through the points $P_1(2, 4, -2)$ and $P_2(7, 0, 5)$. Where does the line intersect the xy-plane?

Solution 4.18. The line $\overrightarrow{P_1 P_2}$ is parallel to the line L and the point P_1 lies on the line L, so it follows that L has the parametric equation

P1 = {2, 4, −2};
P2 = {7, 0, 5};

P1P2 = P1 − P2

$\{-5, 4, -7\}$

$\boldsymbol{L} = \mathbf{P1} + \mathbf{P1P2}\,\boldsymbol{t}$

$\{2 - 5\,t, 4\,t + 4, -7\,t - 2\}$

which is in components

$$x = 2 - 5\,t, \quad y = 4 + 4\,t, \quad \text{and}\; z = -2 - 7\,t$$

The graph of this line is shown in the following Figure 4.26

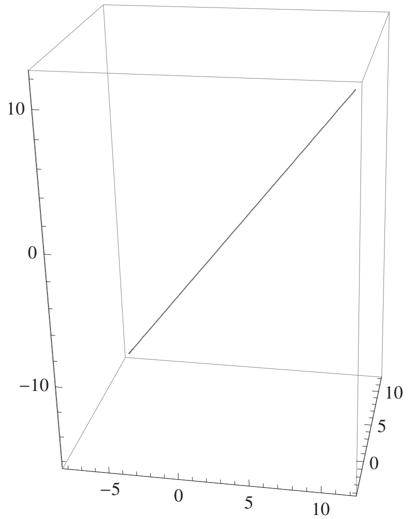


Figure 4.26. Parametric representation of a line in space.

It follows from the parametric equations of the line that the intersection is located in xy-plane if the z component of the three parametric equations vanishes. This means that we are looking for the solution of the equation

$$z = -2 - 7\,t = 0$$

which is given by $t = -2/7$. Which can be verified by

solution = Solve[L⟦3⟧ == 0, t]

$$\left\{\left\{t \to -\frac{2}{7}\right\}\right\}$$

inserting this special value of the parameter into the line equation we get

L **/. solution**

$$\left(\frac{24}{7} \quad \frac{20}{7} \quad 0\right)$$

which represents the point in the xy-plane.▲

4.1.6.2 Planes in 3D

In analytic geometry a line in 2D can be specified by giving its slope and one of its points. Similarly, one can specify a plane in 3D by giving its inclination and specifying one of its points. A convenient method for describing the inclination of a plane is to specify a nonzero vector, called normal, that is perpendicular to the plane.

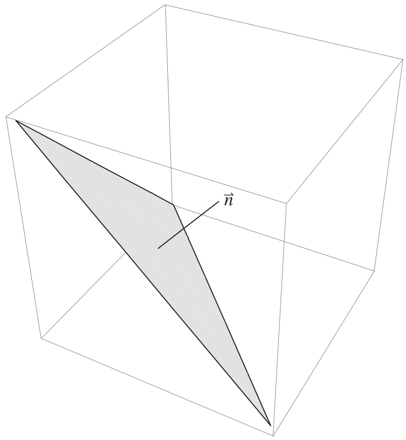


Figure 4.27. Geometric definition of a plane with its normal.

Suppose that we want to find the equation of the plane passing through the point $P_0(x_0, y_0, z_0)$ and having the nonzero vector $\vec{n} = (a, b, c)$ as a normal, that is perpendicular to the plane. It is evident from Figure 4.27 that the plane consists precisely of those points $P(x, y, z)$ for which the vector $\overrightarrow{P_0 P_1}$ is orthogonal to $\vec{n}$; that is

$$\vec{n}.\overrightarrow{P_0 P_1} = 0. \tag{4.49}$$

Since $\overrightarrow{P_0 P_1} = (x - x_0, y - y_0, z - z_0)$, this equation can be written as

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0. \tag{4.50}$$

We call this the point-normal form of the equation of a plane.

Example 4.19. Point-Normal Form of a Plane

Find the equation of a plane passing through the point $\vec{r}_0 = (3, -1, 7)$ and perpendicular to the vector $\vec{n} = (4, 2, -5)$.

Solution 4.19. The point-normal form of the equation is

$$\vec{n}.(\vec{r} - \vec{r}_0) = 0$$

which is in components

$$(4, 2, -5).(x - 3, y - 2, z + 5) = 0 = 4(x - 3) + 2(y - 2) - 5(z + 5).\blacktriangle$$

The following theorem shows that every equation of the form $a\,x + b\,y + c\,z + d = 0$ represents a plane in 3D.

Theorem 4.9. *Point-Normal Form of a Plane*

If a, b, and c are not all zero, then the graph of the equation

$$a\,x + b\,y + c\,z + d = 0 \tag{4.51}$$

is a plane that has the vectors $\vec{n} = (a, b, c)$ *as a normal.*■

Equation (4.51) is called the general form of the equation of a plane.

Example 4.20. Parallel Planes

Determine whether the planes

$$3\,x - 4\,y + 5\,z = 0$$

and

$$-6\,x + 8\,y - 10\,z - 4 = 0$$

are parallel.

Solution 4.20. It is clear geometrically that two planes are parallel if and only if their normals are parallel vectors. A normal to the first plane is

$$\vec{n}_1 = (3, -4, 5)$$

and a normal to the second plane is

$$\vec{n}_2 = (-6, 8, -10).$$

Since $\vec{n}_2$ is a scalar multiple of $\vec{n}_1$, the normals are parallel, and hence so are the planes.

Another way to show that the two normals are parallel is to calculate the cross product of the two vectors

{3, −4, 5}×{−6, 8, −10}

{0, 0, 0}

The result shows that the two vectors have the same direction and thus only the null vector can be perpendicular to them.▲

We have seen that a unique plane is determined by a point in the plane and a nonzero vector normal to the plane. In contrast, a unique plane is not determined by a point in the plane and a nonzero vector parallel to the plane. However, a unique plane is determined by a point in the plane and two nonparallel vectors that are parallel to the plane. A unique plane is also determined by three non colinear points that lie in the plane.

Example 4.21. Parallel Lines

Determine whether the line

$$x = 3 + 8\,t, \qquad y = 4 + 5\,t, \qquad z = -3 - t$$

is parallel to the plane $x - 3\,y + 5\,z = 12$.

Solution 4.21. The vector along the line is given by $\vec{v} = (8, 5, -1)$ the vector normal to the plane is $\vec{n} = (1, -3, 5)$. If the line is parallel to the plane then the two vectors must be perpendicular to each other. We test orthogonality by the dot product.

```
v = {8, 5, -1};
n = {1, -3, 5};
```

The dot product is given by

```
v.n
```

-12

which is finite and thus the line is not parallel to the plane.▲

Example 4.22. Intersection Point

Find the intersection of the line and the plane in Example 4.21.

Solution 4.22. If we let (x_0, y_0, z_0) be the point of intersection, then the coordinates of this point satisfy both the equation of the plane and the parametric equation of the line. Thus,

$$x_0 - 3\,y_0 + 5\,z_0 = 12$$

and for some value of t the parametric equations of the line become

$$x_0 = 3 + 8\,t, \qquad y_0 = 4 + 5\,t, \qquad z_0 = -3 - t.$$

Substituting the two equation into each other yields

```
tEquation = x0 - 3 y0 + 5 z0 == 12 /. {x0 → 8 t + 3, y0 → 5 t + 4, z0 → -t - 3}
```

$5(-t-3) + 8\,t - 3(5\,t+4) + 3 = 12$

which in return delivers an equation for determining t which can be solved with respect to t

```
solT = Flatten[Solve[tEquation, t]]
```

$\{t \to -3\}$

This in turn determines the intersection point to be

```
{x0 → 8 t + 3, y0 → 5 t + 4, z0 → -t - 3} /. solT
```

{x0 → −21, y0 → −11, z0 → 0}

▲

Two distinct intersecting planes determine two positive angles of intersection—an acute angle θ that satisfies the condition $0 \leq \theta \leq \pi/2$ and the supplement of that angle. If $\vec{n}_1$ and $\vec{n}_2$ are normal to the planes, then depending on the directions of $\vec{n}_1$ and $\vec{n}_2$, the angle θ is either the angle between $\vec{n}_1$ and $\vec{n}_2$ or the angle between $\vec{n}_1$ and $-\vec{n}_2$. In both cases, the angle is determined by

$$\cos(\theta) = \frac{|\vec{n}_1.\vec{n}_2|}{\|\vec{n}_1\|\ \|\vec{n}_2\|}. \tag{4.52}$$

Example 4.23. Angle Between two Planes

Find the acute angle of intersection between the two planes

$$2\,x - 4\,y + 4\,z = 7 \qquad \text{and} \qquad 6\,x + 2\,y - 3\,z = 2.$$

Solution 4.23. The given equations yield the normals

n1 = {2, −4, 4};
n2 = {6, 2, −3};

Thus the angle is determined by

$$N\left[\frac{\cos^{-1}\left(\frac{|\mathbf{n1.n2}|}{\sqrt{\mathbf{n1.n1}}\;\sqrt{\mathbf{n2.n2}}}\right)}{\circ}\right]$$

79.0194

The angle between the two planes is thus approximately 79°.▲

Next we will consider three basic distance problems in 3D.

- Find the distance between a point and a plane.
- Find the distance between two parallel planes.
- Find the distance between tow skew lines

The three problems are related to each other. If we can find the distance between a point and a plane, then we can find the distance between parallel planes by computing the distance between one of the planes and an arbitrary point P_0 in the other plane. Moreover, we can find the distance between tow skew lines by computing the distance between parallel planes containing them.

Theorem 4.10. *Distance to a Plane*

The distance D between a point $P_0(x_0,\ y_0,\ z_0)$ and the plane $a\,x + b\,y + c\,z + d = 0$ is

$$D = \frac{|\,a\,x_0 + b\,y_0 + c\,z_0 + d\,|}{\sqrt{a^2 + b^2 + c^2}}.\blacksquare$$

Example 4.24. Distance of a Point to a Plane

Fine the distance D between the point $(1, -4, -3)$ and the plane $2\,x - 3\,y + 6\,z = -1$.

Solution 4.24. The formula given in the theorem requires the plane to be rewritten in the form $a\,x + b\,y + c\,z + d = 0$. Thus we rewrite the equation of the given plane as

$$2\,x - 3\,y + 6\,z + 1 = 0$$

from which we obtain $a = 2$, $b = -3$, $c = 6$, and $d = 1$. Substituting these values and the coordinates of the given point, we gain

$$D = \frac{|2 \times 1 + (-3)(-4) + 6(-3) + 1|}{\sqrt{2^2 + (-3)^2 + 6^2}} = \frac{|-3|}{7} = \frac{3}{7}. \blacktriangle$$

4.1.7 Tests and Exercises

The following two subsections serve to test your understanding of the last section. Work first on the test examples and then try to solve the exercises.

4.1.7.1 Test Problems

T1. What is a norm and how can you define a norm?

T2. What is the difference between the coordinates of a vector and a coordinate system?

T3. Give a geometric interpretation of the projection of a vector.

T4. What kind of products can we define for vectors?

T5. What is the length of a vector?

T6. How can vectors be used to represent a line?

T7. Give examples to represent a plane by using vectors.

T8. How is the sum and difference of vectors defined?

4.1.7.2 Exercises

E1. Show that the unit vectors $\vec{e}_i$, in an n-dimensional space are similar to Cartesian unit vectors in three-dimensional Euclidean space.

E2. Calculate the scalar or inner product $\vec{y}.\vec{x}$ and the lengths $\|x\|$ and $\|y\|$ where

$$\vec{x} = (1, 2, 1, 2), \;\; \vec{y} = (i, 2, 3, 2\,i). \tag{1}$$

and $i = \sqrt{-1}$.

E3. Determine which of the following points $(2, 0, 4)$, $(1, 3, -2)$, and $(-1, 9, -15)$ lie on a single line.

E4. Let L be the line with parametric equations $x = 3\,t + 5$, $y = -2\,t + 1$. Write L in vector form $\vec{p} + t\,\vec{v}$, and draw the line together with the vectors $\vec{p}$ and $\vec{v}$.

E5. Let L be the line in $\mathbb{R}^3$ that consists of all the points $(2\,t + 1, t + 3, t)$. Find a position vector and a direction vector for this line, and write te line in the form $\vec{p} + t\,\vec{v}$.

E6. Determine which, if any, of the points listed below are on the line that has vector form $\vec{p} + t\,\vec{v}$, where $p = (7, 2, 4)$ and $\vec{v} = (2, 1, 3)$.

$$\vec{A} = (3, 0, -2), \;\; \vec{B} = (6, 1, 1), \;\; \vec{C} = (9, 4, 0). \tag{2}$$

E7. Let $\vec{p} + t\,\vec{v}$ describe a line L in $\mathbb{R}^3$. Let $\vec{q}$ and $\vec{r}$ be the vectors defined by $\vec{q} = \vec{p} + 2\,\vec{v}$ and $\vec{w} = \vec{p} - \vec{v}$. Explain why $\vec{q} + t\,\vec{v}$ and $\vec{r} + t\,\vec{v}$ also represent the line L.

E8. Find a vector form $\vec{p} + t\,\vec{v}$ of the line $2\,x - y = 4$. Also sketch the line together with the vectors $\vec{p}$ and $\vec{v}$.

E9. Let $\vec{p} = (2, 3)$ and $\vec{v} = (4, -1)$. Draw the following two sets of points on one set of axes:

$$\{t\,\vec{v} \mid -1 \le t \le 2\}, \;\; \{\vec{p} + t\,\vec{v} \mid -1 \le t \le 2\}. \tag{3}$$

E10 Let $\vec{u} = (2, -1)$ and $\vec{v} = (-1, 3)$. Draw the following set of points, and describe it geometrically in words:

$$\{a\,\vec{u} + b\,\vec{v} \mid -1 \le a \le 2 \text{ and } 0 \le b \le 1\}. \tag{4}$$

4.2 Systems of Linear Equations

4.2.1 Introduction

Information in engineering and mathematics is often organized into rows and columns to form regular arrays, called matrices (plural of matrix). Matrices are often tables of numerical data that arise from observations, but they also occur in various mathematical contexts. For example, we shall see in this chapter that to solve a system of equations such as

$$5x + y = 3 \tag{4.53}$$

$$2x - y = 4 \tag{4.54}$$

all of the information required for the solution is contained in the matrix

$$\begin{pmatrix} 5 & 1 & 3 \\ 2 & -1 & 4 \end{pmatrix} \tag{4.55}$$

and that the solution can be obtained by performing appropriate operations on this matrix. This is particularly important in developing computer programs to solve systems of linear equations because computers are well suited for manipulating arrays of numerical information (see Vol. IV). However, matrices are not simply a notational tool for solving systems of equations; they can be viewed as mathematical objects in their own right, and there is a rich and important theory associated with them that has a wide variety of applications. In this chapter we will begin the studies of matrices.

4.2.2 Linear Equations

Systems of linear algebraic equations and their solutions constitute one of the major topics studied in the course as linear algebra. In this section we shall introduce some basic terminology and discuss a method for solving such systems.

Any straight line in the xy-plane can be represented algebraically by an equation of the form

$$a_1 x + a_2 y = b \tag{4.56}$$

where a_1, a_2, and b are constants and a_1 and a_2 are not both zero. An equation of this form is called a linear equation in the variables x and y because the power of both variables is one. More generally, we define a linear equation in the n variables $x_1, x_2, \ldots, x_n$ to be one that can be expressed in the form

$$a_1 x_1 + a_2 x_2 + a_3 x_3 + \ldots + a_n x_n = b \tag{4.57}$$

where $a_1, a_2, \ldots, a_n$ and b are real constants. The variables in a linear equation are sometimes called unknowns.

Example 4.25. Linear Equations

The following equations

$$x + 9\, y = 8, \quad y = \frac{1}{3}\, x + z - 5, \;\text{ and } x_1 - 2\, x_2 - 6\, x_3 = 8$$

are linear. Observe that a linear equation does not involve any product or roots of variables.

Solution 4.25. All variables occur only to the first power and do not appear as arguments for functions like trigonometric, logarithmic, or exponential. Thus the following equations are obviously

$$x + 7\sqrt{y} = 10, \quad 7\, x + 2\, y - x\, z = 2, \;\text{ and } y = \cos(z)$$

nonlinear.▲

A solution of a linear equation $a_1\, x_1 + a_2\, x_2 + \ldots + a_n\, x_n = b$ is a sequence of n numbers $s_1, s_2, s_3, \ldots, s_n$ such that the equation is satisfied when we substitute $x_1 = s_1,\ x_2 = s_2,\ \ldots,\ x_n = s_n$ into the original equation. The set of all solutions of the equation is called its solution set or sometimes the general solution of the equation.

A finite set of linear equations in the variables $x_1, x_2, \ldots, x_n$ is called a system of linear equations or a linear system. The sequence of numbers $s_1, s_2, s_3, \ldots, s_n$ is called a solution of the system if $x_1 = s_1$, $x_2 = s_2, \ldots, x_n = s_n$ is a solution of every equation in the system.

A system of equations that has no solutions is said to be inconsistent; if there is at least one solution of the system, it is called consistent. To illustrate the possibilities that can occur in solving systems of linear equations, consider a general system of two linear equations in the unknowns $x_1 = x$ and $x_2 = y$:

$$a_1\, x + b_1\, y = c_1 \qquad \text{with } a_1,\ b_1 \text{ not both zero} \tag{4.58}$$

$$a_2\, x + b_2\, y = c_2 \qquad \text{with } a_2,\ b_2 \text{ not both zero.} \tag{4.59}$$

The graphs of these equations are lines. Since a point (x, y) lies on a line if and only if the numbers x and y satisfy the equation of the line, the solutions of the system of equations correspond to points of intersection of line l_1 and line l_2. There are three possibilities, illustrated in Figure 4.28

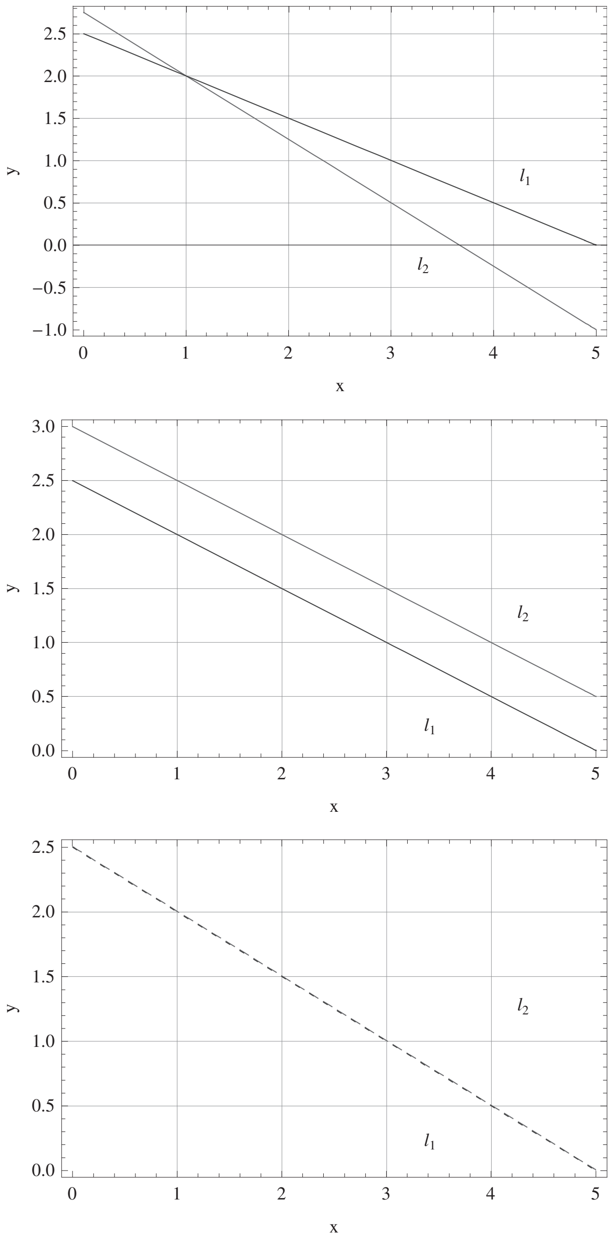


Figure 4.28. The three possible scenarios to solve a system of linear equations. Top: a single and unique solution exists, middle: no solution exists, and bottom: an infinite number of solutions exists.

- The line l_1 and l_2 may intersect at only one point, in which the system has exactly one solution
- The lines l_1 and l_2 may be parallel, in which case there is no intersection and consequently no solution to the system.
- The lines l_1 and l_2 may coincide, in which case there are infinitely many points of intersection and consequently infinitely many solutions to the system.

Although we have considered only two equations with two unknowns here, we will show that the same three possibilities hold for arbitrary linear systems:

Remark 4.6. Every system of linear equations has no solutions, or has exactly one solution, or has infinitely many solutions.

So the question is how can we decide whether a given system of linear equations allows a solution. We will begin with a few remarks concerning the last of the three questions posed. Our first task is to generalize the ideas of a single linear equation to a system of equations and to write down this system of m linear equations with n unknown $x_1, \ldots, x_n$ in a systematic way. The system of m equations in n unknown is given as follows

$$a_{11}\, x_1 + a_{12}\, x_2 + \ldots + a_{1\,n}\, x_n = b_1 \tag{4.60}$$

$$a_{21}\, x_1 + a_{22}\, x_2 + \ldots + a_{1\,n}\, x_n = b_2 \tag{4.61}$$

$$\vdots$$

$$a_{\mathrm{m}1}\, x_1 + a_{\mathrm{m}2}\, x_2 + \ldots + a_{\mathrm{mn}}\, x_n = b_m. \tag{4.62}$$

Here the a_{ij} are $m \times n$ given real numbers. The double sub scripting of the coefficients of the unknowns is a usefulI device that is used to specify the location of the coefficients in the system. The first subscript on the coefficient a_{ij} indicates the equation in which the coefficient occurs, and the second subscript indicates which unknown it multiplies. Thus, a_{12} is in the first equation and multiplies unknown x_2. In addition to the a_{ij} the numbers of the "right hand side" $b_1, \ldots, b_m$ are given. Numbers $s_1, \ldots, s_n$ that satisfy these equations are called solutions. Note that a solution $(x_1, x_2, \ldots, x_n) = (s_1, \ldots, s_n)$ may be considered as a point in $\mathbb{R}^n$. This leads immediately to a geometric interpretation of systems of linear equations. To make things clear, let us consider the case for a 2×2 system of equations. For this kind of system, we have $m = n = 2$

$$\begin{aligned} a\,x + b\,y &= s \\ c\,x + d\,y &= t. \end{aligned} \tag{4.63}$$

The solutions (x, y) of the first equation are the points of the line

$$y = \frac{1}{b}\,(s - a\,x) \qquad \text{with } b \neq 0. \tag{4.64}$$

Similarly, the solutions of the second equation lie on some other line.

$$y = \frac{1}{d}(t - c\,x) \qquad \text{with } d \neq 0. \tag{4.65}$$

The solutions for both equations must lie on both lines, that is, on their intersection.

In general two lines in $\mathbb{R}^2$ intersect in a single point. This means that there exists a unique solution of the system of equations. However, there are exceptions from this uniqueness if the two lines are parallel and different from each other or parallel and lay on each other. This means that they do not intersect, and thus the system has no solution. Or the two lines may be identical, and all points on this single line are solutions. So there are an infinite number of solutions. The actual behavior of a finite or infinite set of solutions depends basically on the coefficients of the system of equations.

The solution of the two equations can be derived by solving each equation with respect to one of the unknowns such as solving the first with respect to y

Solve[*a x* + *b y* == *s*, *y*]

$$\left\{\left\{y \to \frac{s - a\,x}{b}\right\}\right\}$$

The second one also with respect to y

Solve[*c x* + *d y* == *t*, *y*]

$$\left\{\left\{y \to \frac{t - c\,x}{d}\right\}\right\}$$

These solutions show that there is a dependence on x which can be eliminated such that the two unknowns are determined by

Solve[{*a x* + *b y* == *s*, *c x* + *d y* == *t*}, {*x*, *y*}]

$$\left\{\left\{x \to -\frac{b\,t - d\,s}{a\,d - b\,c},\ y \to -\frac{a\,t - c\,s}{b\,c - a\,d}\right\}\right\}$$

The solution is well defined if the quotients are defined.▲

Example 4.26. Intersection of Lines and Solutions

In the previous example let: $a = 1$, $b = 2$, $c = 3$, $d = 4$, $s = 5$, $t = 11$, then we have the system $x + 2\,y = 5$, $3\,x + 4\,y = 11$.

Solution 4.26. The first equation will have the solution $y = \frac{5-x}{2}$ which has the graph

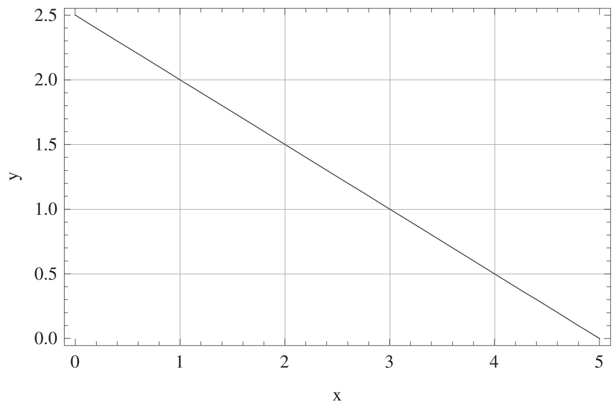


Figure 4.29. Solution $y = (5 - x)/2$.

The second equation has the solution $y = \frac{11-3x}{4}$ which graphically is represented in the following Figure 4.30

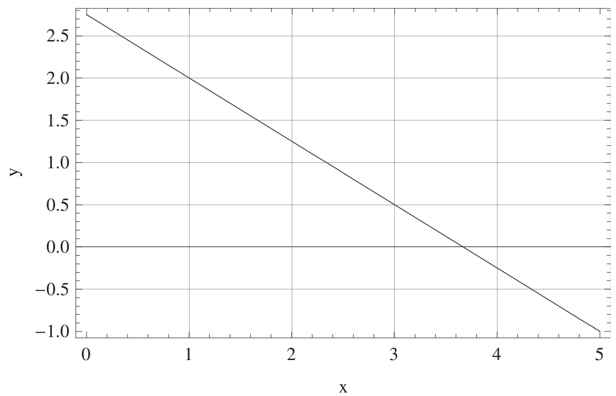


Figure 4.30. Solution $y = (11 - 3x)/4$.

If we plot the two solutions together in a single plot, we can locate the intersection point in the plane.

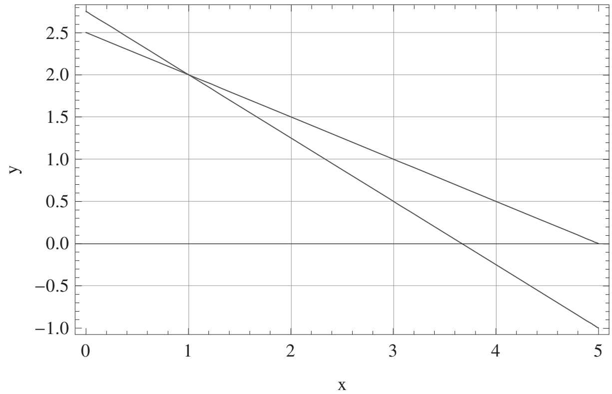


Figure 4.31. Both solutions $y = (5 - x)/2$ and $y = (11 - 3x)/4$.

We realize that the two lines intersect at the point (1, 2), which is indeed the solution of that system. This can be shown be solving the system of equations directly by

```
Solve[{x + 2 y == 5, 3 x + 4 y == 11}, {x, y}]
```

$\{\{x \to 1, y \to 2\}\}$

representing the x and y value of the solution.▲

For cases when $n = 3$, a single equation

$$a x + b y + c z = d \tag{4.66}$$

describes a plane in 3-space. Two planes in 3-space intersect in general in a line. The intersection of a line with a plane is in general a single point. So, in general, for 3 equations, there is a unique solution.

This argument generalizes to arbitrary n −space. A single equation

$$a_1 x_1 + a_2 x_2 + \ldots + a_n x_n = b \tag{4.67}$$

describes an hyperplane in $\mathbb{R}^n$ (an $(n-1)$-dimensional affine linear subspace). Two such hyperplanes intersect in an $(n-2)$ −dimensional subspace, etc. So, in general, for $(m \leq n)$ the solutions of a system of m equations form an $(n-m)$ −dimensional subspace. (Note that a 0-dimensional subspace is just a point).

4.2.3 Augmented Matrices

If we refer to equations (4.60-62) and mentally keep track of the location of the +'s, the x's, and the ='s, a system of m linear equations in n unknowns can be abbreviated by writing only the regular array of numbers:

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1\,n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2\,n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{\mathrm{m}1} & a_{\mathrm{m}2} & \dots & a_{\mathrm{mn}} & b_m \end{pmatrix} \tag{4.68}$$

This is called the augmented matrix for the system. The term matrix is used in mathematics to denote a rectangular array of numbers. Matrices arise in many contexts, which we will consider in more detail in later sections. For example the augmented matrix for the system of equations

$$x_1 + 3\,x_2 + 7\,x_3 = 8 \tag{4.69}$$

$$2\,x_1 + 4\,x_2 + 9\,x_3 = 1 \tag{4.70}$$

$$5\,x_1 + 2\,x_2 + 6\,x_3 = 4 \tag{4.71}$$

is

$$\begin{pmatrix} 1 & 3 & 7 & 8 \\ 2 & 4 & 9 & 1 \\ 5 & 2 & 6 & 4 \end{pmatrix} \tag{4.72}$$

Remark 4.7. When constructing an augmented matrix, we must write the unknowns in the same order in each equation, and the constants must be on the right.

The basic method for solving a system of linear equations is to replace the given system by a new system that has the same solution set but is easier to solve. This new system is generally obtained in a series of steps by applying the following three types of operations to eliminate unknowns symbolically:

1. Multiply an equation by a nonzero constant.

2. Interchange two equations.

3. Add a multiple of one equation to another.

Since the rows (horizontal lines) of an augmented matrix correspond to the equations in the associated system, these three operations correspond to the following operations on the rows of the augmented matrix:

1. Multiply a row by a nonzero constant.

2. Interchange two rows.

3. Add a multiple of one row to another row.

These are called elementary row operations. The following example illustrates how these operations can be used to solve systems of linear equations. Since a systematic procedure for finding solutions will be derived in the next section, it is not necessary to worry about how the steps in this example were selected. The main effort at this time should be devoted to understanding the computations and the discussion.

Example 4.27. Using Elementary Row Operations

In the first line below we solve a system of linear equations by operating on the equations in the system, and in the same line we solve the same system by operating on the rows of the augmented matrix.

$$\begin{aligned} x+y+2z&=9 \\ 2x+4y-3z&=1 \\ 3x+6y-5z&=0 \end{aligned} \qquad \begin{pmatrix} 1 & 1 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{pmatrix}$$

Add -2 times the first equation (row) to the second to obtain which

$$\begin{aligned} x+y+2z&=9 \\ 2y-7z&=-17 \\ 3x+6y-5z&=0 \end{aligned} \qquad \begin{pmatrix} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 3 & 6 & -5 & 0 \end{pmatrix}$$

Add -3 times the first equation (row) to the third to obtain

$$\begin{aligned} x+y+2z&=9 \\ 2y-7z&=-17 \\ 3y-11z&=-27 \end{aligned} \qquad \begin{pmatrix} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 0 & 3 & -11 & -27 \end{pmatrix}$$

Multiply the second equation (row) by $1/2$ to obtain

$$\begin{aligned} x+y+2z&=9 \\ y-\tfrac{7}{2}z&=\tfrac{-17}{2} \\ 3y-11z&=-27 \end{aligned} \qquad \begin{pmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & \frac{-7}{2} & \frac{-17}{2} \\ 0 & 3 & -11 & -27 \end{pmatrix}$$

Add -3 times the second equation (row) to the third to obtain

$$\begin{aligned} x+y+2z&=9 \\ y-\tfrac{7}{2}z&=\tfrac{-17}{2} \\ \tfrac{-1}{2}z&=\tfrac{-3}{2} \end{aligned} \qquad \begin{pmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & \frac{-7}{2} & \frac{-17}{2} \\ 0 & 0 & \frac{-1}{2} & \frac{-3}{2} \end{pmatrix}$$

Multiply the third equation (row) by -2 to obtain

$$\begin{aligned} x+y+2z&=9 \\ y-\tfrac{7}{2}z&=\tfrac{-17}{2} \\ z&=3 \end{aligned} \qquad \begin{pmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & \frac{-7}{2} & \frac{-17}{2} \\ 0 & 0 & 1 & 3 \end{pmatrix}$$

Add -1 times the second equation (row) to the first to obtain

$$\begin{aligned} x+\tfrac{11}{2}z&=\tfrac{35}{2} \\ y-\tfrac{7}{2}z&=\tfrac{-17}{2} \\ z&=3 \end{aligned} \qquad \begin{pmatrix} 1 & 0 & \frac{11}{2} & \frac{35}{2} \\ 0 & 1 & \frac{-7}{2} & \frac{-17}{2} \\ 0 & 0 & 1 & 3 \end{pmatrix}$$

Add $\frac{-11}{2}$ times the third equation (row) to the first and $\frac{7}{2}$ times the third equation to the second to obtain

$$\begin{array}{l} x = 1 \\ y = 2 \\ z = 3 \end{array} \qquad \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{pmatrix}$$

The solution thus is $x = 1$, $y = 2$, $z = 3$.▲

4.2.4 Gaussian Elimination

In this section we shall develop a systematic procedure for solving systems of linear equations. The procedure is based on the idea of reducing the augmented matrix of a system to another augmented matrix that is simple enough that the solution of the system can be found by inspection.

4.2.4.1 Echelon Forms

In example of 4.27, we solved a linear system in the unknowns x, y, and z by reducing the augmented matrix to the form

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{pmatrix}$$

from which the solution $x = 1$, $y = 2$, and $z = 3$ became evident. This is an example of a matrix that is in reduced row-echelon form. To be of this form, a matrix must have the following properties:

1. If a row does not consist entirely of zeros, then the first nonzero number in the row is a 1. We call this a leading 1.
2. If there are any rows that consist entirely of zeros, then they are grouped together at the bottom of the matrix.
3. In any two successive rows that do not consist entirely of zeros, the leading 1 in the lower row occurs farther to the right than the leading 1 in the higher row.
4. Each column that contains a leading 1 has zeros everywhere else in that column.

A matrix that has the first three properties is said to be in row-echelon form. Thus, a matrix in reduced row-echelon form is of necessity in row-echelon form, but not conversely.

Example 4.28. Row-Echelon and Reduced Row-Echelon Form

The following matrices are in reduced row-echelon form.

$$\begin{pmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

The following matrices are in row-echelon form.

$$\begin{pmatrix} 1 & 4 & -3 & 7 \\ 0 & 1 & 6 & 2 \\ 0 & 0 & 1 & 5 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 2 & 6 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

We leave it for you to confirm that each of the matrices in this example satisfies all of the requirements for its stated form.▲

Example 4.29. More on Row-Echelon and Reduced Row-Echelon Form

As the last example illustrates, a matrix in row-echelon form has zeros below each leading 1, whereas a matrix in reduced row-echelon form has zeros below and above each leading 1. Thus with any real numbers substituted for the *'s, all matrices of the following types are in row-echelon form:

$$\begin{pmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & 1 & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 1 & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \end{pmatrix}.$$

Moreover, all matrices of the following types are in reduced row-echelon form:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & * & 0 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 1 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 1 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \end{pmatrix}. \blacktriangle$$

If by a sequence of elementary row operations, the augmented matrix for a system of linear equations is put in reduced row-echelon form, then the solution set of the system will be evident by inspection or after a few simple steps. The next example illustrates this situation.

Example 4.30. Solution of Four Linear Systems

Suppose that the augmented matrix for a system of linear equations has been reduced by row operations to the given reduced row-echelon form. Solve the system.

a) $$\begin{pmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 4 \end{pmatrix}$$

b) $$\begin{pmatrix} 1 & 0 & 0 & 4 & -1 \\ 0 & 1 & 0 & 2 & 6 \\ 0 & 0 & 1 & 3 & 2 \end{pmatrix}$$

c) $$\begin{pmatrix} 1 & 6 & 0 & 0 & 4 & -2 \\ 0 & 0 & 1 & 0 & 3 & 1 \\ 0 & 0 & 0 & 1 & 5 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

d) $$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Solution 4.30. For part a) we can rewrite the augmented matrix to the following system of equations

$$x_1 = 5$$
$$x_2 = -2$$
$$x_3 = 4$$

It is obvious from this representation that the solution is given by $x_1 = 5$, $x_2 = -2$, and $x_3 = 4$.

For part b) the matrix represents the following system of equations

$$x_1 + 4\,x_4 = -1$$
$$x_2 + 2\,x_4 = 6$$
$$x_3 + 3\,x_4 = 2$$

Since x_1, x_2, and x_3 correspond to the leading 1's in the augmented matrix, we call them leading variables or pivots. The non leading variables, in this case x_4, are called free variables. Solving for the leading variables in terms of the free variables gives

$$x_1 = -1 - 4\,x_4$$
$$x_2 = 6 - 2\,x_4$$
$$x_3 = 2 - 3\,x_4$$

From this representation of the equations we see that the free variable x_4 can be assigned an arbitrary value, say $\xi \in \mathbb{R}$, which then determines the values of the leading variables x_1, x_2, and x_3. Thus there are infinitely many solutions, and the general solution is given by the formulas

$$x_1 = -1 - 4\,\xi$$
$$x_2 = 6 - 2\,\xi$$
$$x_3 = 2 - 3\,\xi$$
$$x_4 = \xi$$

The matrix for part c) is

$$\begin{pmatrix} 1 & 6 & 0 & 0 & 4 & -2 \\ 0 & 0 & 1 & 0 & 3 & 1 \\ 0 & 0 & 0 & 1 & 5 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

which contains a row containing only zeros. This row lead to the equation $0\,x_1 + 0\,x_2 + \ 0\,x_3 + 0\,x_4 + 0\,x_5 = 0$, which places no restriction on the solution. Thus we actually can omit this equation and write the corresponding system as

$$\begin{aligned} x_1 + 6\,x_2 + 4\,x_5 &= -2 \\ x_3 + 3\,x_5 &= 1 \\ x_4 + 5\,x_5 &= 2 \end{aligned}$$

Here the leading variables are x_1, x_3, and x_4 and the free variables are x_2 and x_5. Solving for the leading variables in terms of the free variables gives

$$\begin{aligned} x_1 &= -2 - 6\,x_2 - 4\,x_5 \\ x_3 &= 1 - 3\,x_5 \\ x_4 &= 2 - 5\,x_5 \end{aligned}$$

Since x_5 can be assigned an arbitrary value ξ and x_2 can be also assigned an arbitrary value τ, there are infinitely many solutions. The general solution is given by the formula

$$\begin{aligned} x_1 &= -2 - 6\,\tau - 4\,\xi \\ x_3 &= 1 - 3\,\xi \\ x_4 &= 2 - 5\,\xi \end{aligned}$$

For part d) we have the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where the last row of the augmented matrix is given as

$$0\,x_1 + 0\,x_2 + 0\,x_3 = 1.$$

Since this equation cannot be satisfied, there is no solution to the system.▲

4.2.4.2 Elimination Methods

We have just seen how easy it is to solve a system of linear equations once its augmented matrix is in reduced row-echelon form. Now we shall give a step-by-step elimination procedure that can be used to reduce any matrix to reduced row-echelon form. As we state each step in the procedure, we shall illustrate the idea by reducing the following matrix to reduced row-echelon form. To demonstrate the procedure let us consider the following augmented matrix

$$\begin{pmatrix} 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -10 & 6 & 12 & 28 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{pmatrix}$$

1. **Step**: Locate the leftmost column that does not consist entirely of zeros.

$$\begin{pmatrix} \mathbf{0} & 0 & -2 & 0 & 7 & 12 \\ \mathbf{2} & 4 & -10 & 6 & 12 & 28 \\ \mathbf{2} & 4 & -5 & 6 & -5 & -1 \end{pmatrix}$$

2. Step: Interchange the top row with another row, if necessary, to bring a nonzero entry to the top of the column found in step 1.

$$\begin{pmatrix} \mathbf{2} & \mathbf{4} & \mathbf{-10} & \mathbf{6} & \mathbf{12} & \mathbf{28} \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{pmatrix}$$

3. Step: If the entry that is now at the top of the column found in step 1 is a, multiply the first row by $1/a$ in order to introduce a leading 1.

$$\begin{pmatrix} \mathbf{1} & \mathbf{2} & \mathbf{-5} & \mathbf{3} & \mathbf{6} & \mathbf{14} \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{pmatrix}$$

4. Step: Add suitable multiples of the top row to the rows below so that all entrees below the leading 1 become zeros.

$$\begin{pmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ \mathbf{0} & \mathbf{0} & \mathbf{5} & \mathbf{0} & \mathbf{-17} & \mathbf{-29} \end{pmatrix}$$

5. Step: Now cover the top row in the matrix and begin again with step 1 applied to the submatrix that remains. Continue in this way until the entire matrix is in row-echelon form.

$$\begin{pmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & \mathbf{1} & \mathbf{0} & \mathbf{\frac{-7}{2}} & \mathbf{-6} \\ 0 & 0 & \mathbf{5} & \mathbf{0} & \mathbf{-17} & \mathbf{-29} \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & \frac{-7}{2} & -6 \\ 0 & 0 & \mathbf{0} & \mathbf{0} & \mathbf{\frac{1}{2}} & \mathbf{1} \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & \frac{-7}{2} & -6 \\ 0 & 0 & 0 & 0 & \mathbf{1} & \mathbf{2} \end{pmatrix}$$

The entire matrix is now in row-echelon form. To find the reduced row echelon form we need the following additional step.

6. Step: Beginning with the last nonzero row and working upward, add suitable multiples of each row to the rows above to introduce zeros above the leading 1's.

$$\begin{pmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & \frac{-7}{2} & -6 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & \mathbf{0} & \mathbf{1} \\ 0 & 0 & 0 & 0 & 1 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & -5 & 3 & \mathbf{0} & \mathbf{2} \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & \mathbf{0} & \mathbf{3} & \mathbf{0} & \mathbf{7} \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{pmatrix}$$

The last matrix is in reduced row-echelon form.

If we use only the first five steps, the above procedure produces a row-echelon form and is called Gaussian elimination. Carrying out step 6 in addition to the first five steps which generates the reduced row-echelon form is called Gauss-Jordan elimination.

Remark 4.8. It can be shown that every matrix has a unique reduced-echelon form; that is, one will arrive at the same reduced row-echelon form for a given matrix no matter how the row operations are varied. In contrast, a row-echelon form of a given matrix is not unique; different sequences of row operations can produce different row-echelon forms.

In *Mathematica* there exists a function which generates the reduced-echelon form of a given matrix. For the example above the calculation is done by

$$\mathbf{MatrixForm}\left[\mathbf{RowReduce}\left[\begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{-2} & \mathbf{0} & \mathbf{7} & \mathbf{12} \\ \mathbf{2} & \mathbf{4} & \mathbf{-10} & \mathbf{6} & \mathbf{12} & \mathbf{28} \\ \mathbf{2} & \mathbf{4} & \mathbf{-5} & \mathbf{6} & \mathbf{-5} & \mathbf{-1} \end{pmatrix}\right]\right]$$

$$\begin{pmatrix} 1 & 2 & 0 & 3 & 0 & 7 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{pmatrix}$$

Example 4.31. Gauss-Jordan Elimination

Solve by Gauss-Jordan elimination the following system of linear equations

$$\begin{array}{lllllll} x_1 & +5\,x_2 & -3\,x_3 & & +5\,x_5 & & = 1 \\ 3\,x_1 & +7\,x_2 & -4\,x_3 & -x_4 & +3\,x_5 & +2\,x_6 & = 3 \\ 2\,x_1 & +9\,x_2 & & +9\,x_4 & +3\,x_5 & -12\,x_6 & = 7 \end{array}$$

Solution 4.31. The augmented matrix of this system is

$$\mathbf{am} = \begin{pmatrix} 1 & 5 & -3 & 0 & 5 & 0 & 1 \\ 3 & 7 & -4 & -1 & 3 & 2 & 3 \\ 2 & 9 & 0 & 9 & 3 & -12 & 7 \end{pmatrix};$$

The reduced-echelon form is gained by adding -3 times the first row to the second row

$$\mathbf{am} = \begin{pmatrix} 1 & 5 & -3 & 0 & 5 & 0 & 1 \\ 0 & -8 & 5 & -1 & -12 & 2 & 0 \\ 2 & 9 & 0 & 9 & 3 & -12 & 7 \end{pmatrix};$$

Adding in addition -2 times the first row to the third one gives

$$\mathbf{am} = \begin{pmatrix} 1 & 5 & -3 & 0 & 5 & 0 & 1 \\ 0 & -8 & 5 & -1 & -12 & 2 & 0 \\ 0 & -1 & 6 & 9 & -7 & -12 & 5 \end{pmatrix};$$

Interchanging the second with the third row and multiplying the third by -1 will give

$$\mathbf{am} = \begin{pmatrix} 1 & 5 & -3 & 0 & 5 & 0 & 1 \\ 0 & 1 & -6 & -9 & 7 & 12 & -5 \\ 0 & -8 & 5 & -1 & -12 & 2 & 0 \end{pmatrix};$$

A multiple of the second row of 8 will give

$$\mathbf{am} = \begin{pmatrix} 1 & 5 & -3 & 0 & 5 & 0 & 1 \\ 0 & 1 & -6 & -9 & 7 & 12 & -5 \\ 0 & 0 & -43 & -73 & 44 & 98 & -40 \end{pmatrix};$$

Division of the last row by -43 gives

$$\mathbf{am} = \begin{pmatrix} 1 & 5 & -3 & 0 & 5 & 0 & 1 \\ 0 & 1 & -6 & -9 & 7 & 12 & -5 \\ 0 & 0 & 1 & \frac{73}{43} & \frac{-44}{43} & \frac{-98}{43} & \frac{40}{43} \end{pmatrix};$$

Adding a multiple of 6 of the third row to the second row produces

$$\mathbf{am} = \begin{pmatrix} 1 & 5 & -3 & 0 & 5 & 0 & 1 \\ 0 & 1 & 0 & \frac{51}{43} & \frac{37}{43} & \frac{-72}{43} & \frac{25}{43} \\ 0 & 0 & 1 & \frac{73}{43} & \frac{-44}{43} & \frac{-98}{43} & \frac{40}{43} \end{pmatrix};$$

Three time the last row added to the first row generates

$$\mathbf{am} = \begin{pmatrix} 1 & 5 & 0 & \frac{219}{43} & \frac{83}{43} & \frac{-294}{43} & \frac{163}{43} \\ 0 & 1 & 0 & \frac{51}{43} & \frac{37}{43} & \frac{-72}{43} & \frac{25}{43} \\ 0 & 0 & 1 & \frac{73}{43} & \frac{-44}{43} & \frac{-98}{43} & \frac{40}{43} \end{pmatrix};$$

-5 times the second row added to the first row gives

$$\textbf{am} = \begin{pmatrix} 1 & 0 & 0 & -\frac{36}{43} & -\frac{102}{43} & \frac{66}{43} & \frac{38}{43} \\ 0 & 1 & 0 & \frac{51}{43} & \frac{37}{43} & \frac{-72}{43} & \frac{25}{43} \\ 0 & 0 & 1 & \frac{73}{43} & \frac{-44}{43} & \frac{-98}{43} & \frac{40}{43} \end{pmatrix};$$

The same result is generated by

MatrixForm[RowReduce[am]]

$$\begin{pmatrix} 1 & 0 & 0 & -\frac{36}{43} & -\frac{102}{43} & \frac{66}{43} & \frac{38}{43} \\ 0 & 1 & 0 & \frac{51}{43} & \frac{37}{43} & -\frac{72}{43} & \frac{25}{43} \\ 0 & 0 & 1 & \frac{73}{43} & -\frac{44}{43} & -\frac{98}{43} & \frac{40}{43} \end{pmatrix}$$

in a single step. The result is that there are three leading variables x_1, x_2, and x_3 which are determined by x_4, x_5, and x_6▲

4.2.5 Preliminary Notations and Rules for Matrices

To simplify the notation and introduce a compact representation of equations we introduce the following notation. The right hand side of a system of linear equations of m equations and n unknowns

$$a_{11}\, x_1 + a_{12}\, x_2 + \ldots + a_{1\,n}\, x_n = b_1 \tag{4.73}$$

$$a_{21}\, x_1 + a_{22}\, x_2 + \ldots + a_{1\,n}\, x_n = b_2$$

$$\vdots$$

$$a_{\text{m}1}\, x_1 + a_{\text{m}2}\, x_2 + \ldots + a_{\text{mn}}\, x_n = b_m.$$

are abbreviated by the vector $\vec{b} = (b_1, b_2, b_3, \ldots, b_m)$. The vector of the n unknowns $\vec{x}$ is similarly abbreviated by $\vec{x} = (x_1, x_2, \ldots, x_n)$. The coefficients in front of the n unknowns are collected in the matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1\,n} \\ a_{21} & a_{22} & \ldots & a_{2\,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{\text{m}1} & a_{\text{m}2} & \ldots & a_{\text{mn}} \end{pmatrix} \tag{4.74}$$

The system of equations can now be reduced to the expression

$$A.\vec{x} = \vec{b} \tag{4.75}$$

where we agree on an extended use of the dot product between vectors and a matrix. The meaning here is that the column vector $\vec{x}$ is multiplied by each row of A in the usual way by using the dot product of ordinary vectors

$$A.\vec{x} = \begin{pmatrix} \vec{a}_1 \\ \vec{a}_2 \\ \vdots \\ \vec{a}_m \end{pmatrix} . \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} a_{11}\, x_1 + a_{12}\, x_2 + \ldots + a_{1\,n}\, x_n \\ a_{21}\, x_1 + a_{22}\, x_2 + \ldots + a_{2\,n}\, x_n \\ \vdots \\ a_{\mathrm{m}1}\, x_1 + a_{\mathrm{m}2}\, x_2 + \ldots + a_{\mathrm{mn}}\, x_n \end{pmatrix} \tag{4.76}$$

Using this notation we can simplify and reduce our system of equations to

$$A.\vec{x} = \vec{b}. \tag{4.77}$$

4.2.6 Solution Spaces of Homogeneous Systems

If $A\,\vec{x} = \vec{b}$ is a system of linear equations, then each vector $\vec{x}$ that satisfies this equation $\vec{x} = \vec{s}$ is called a solution vector of the system. The following theorem shows that the solution vector of a homogeneous linear system form a vector space, which we shall call the solution space of the system.

Theorem 4.11. *Solution Space*

If $A\,\vec{x} = 0$ is a homogeneous linear system of m equations in n unknowns, then the set of solution vectors is a subspace of $\mathbb{R}^n$. ■

Applications of this theorem follow next.

Example 4.32. Solution Space

Consider the linear system of equations

a) $$\begin{pmatrix} 1 & -2 & 3 \\ 2 & -4 & 6 \\ 3 & -6 & 9 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

b) $$\begin{pmatrix} 1 & -2 & 3 \\ -3 & 7 & -8 \\ -2 & 4 & -6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

c) $$\begin{pmatrix} 1 & -2 & 3 \\ -3 & 7 & -8 \\ 4 & 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

d) $$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Each of these systems had three unknowns, so that the solution form a subspace of $\mathbb{R}^3$. Geometrically, this means that each solution space must be the origin only, a line through the origin, a plane through the origin, or the total of $\mathbb{R}^3$. We shall now verify that this is so (leaving it to the reader to solve the system).

Solution 4.32. The solutions of the four different systems can be derived by using *Mathematica* for example.

The solution for case a) are

$$\textbf{caseA = Thread}\left[\begin{pmatrix} 1 & -2 & 3 \\ 2 & -4 & 6 \\ 3 & -6 & 9 \end{pmatrix}.\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} == \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\right]$$

$\{\{x_1 - 2x_2 + 3x_3\} = \{0\}, \{2x_1 - 4x_2 + 6x_3\} = \{0\}, \{3x_1 - 6x_2 + 9x_3\} = \{0\}\}$

solA = Solve[caseA, {x_1, x_2, x_3}]

```
Solve::svars :
 Equations may not give solutions for all "solve" variables. More…
```

$\{\{x_1 \to 2x_2 - 3x_3\}\}$

from which it follows that

$$x_1 - 2x_2 + 3x_3 = 0.$$

But this is an equation of the plane through the origin with the normal $\vec{n} = (1, -2, 3)$.

In case b) we find the solution from

$$\textbf{caseB = Thread}\left[\begin{pmatrix} 1 & -2 & 3 \\ -3 & 7 & -8 \\ -2 & 4 & -6 \end{pmatrix}.\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} == \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\right]$$

$\{\{x_1 - 2x_2 + 3x_3\} = \{0\}, \{-3x_1 + 7x_2 - 8x_3\} = \{0\}, \{-2x_1 + 4x_2 - 6x_3\} = \{0\}\}$

solB = Solve[caseB, {x_1, x_2, x_3}]

```
Solve::svars :
 Equations may not give solutions for all "solve" variables. More…
```

$\{\{x_1 \to -5x_3, x_2 \to -x_3\}\}$

which means that we have the following relations

$$x_1 = -5t, \;\; x_2 = -t, \;\; \text{and } x_3 = t$$

which are parametric equations for a line through the origin parallel to the vector $\vec{v} = (-5, -1, 1)$.

In case c) we find from the equations the following solution

$$\textbf{caseC = Thread}\left[\begin{pmatrix} 1 & -2 & 3 \\ -3 & 7 & -8 \\ 4 & 1 & 2 \end{pmatrix}.\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} == \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\right]$$

$\{\{x_1 - 2x_2 + 3x_3\} = \{0\}, \{-3x_1 + 7x_2 - 8x_3\} = \{0\}, \{4x_1 + x_2 + 2x_3\} = \{0\}\}$

solC = Solve[caseC, {x_1, x_2, x_3}]

$\{\{x_1 \to 0, x_2 \to 0, x_3 \to 0\}\}$

So the solution space is the origin only.

The solution in case d) are

$$\textbf{caseD} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} == \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

True

The equations are satisfied for any symbol used in x_1, x_2, and x_3. This means that we can insert any number from $\mathbb{R}^3$. So we are free to choose

$x_1 = r, \qquad x_2 = s,$ and $x_3 = t$

where r, s, and t have arbitrary values.▲

In subsection 4.1.5.6 we introduced the linear combination of column vectors. The following definition extends this idea to more general vectors.

Definition 4.7. *Linear Combinations*

A vector $\vec{w}$ is called a linear combination of the vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r$ if it can be expressed in the form

$\vec{w} = k_1 \vec{v}_1 + k_2 \vec{v}_2 + \ldots + k_r \vec{v}_r = \sum_{j=1}^{r} k_j \vec{v}_j$

where the k_j's $\{j \mid 1, 2, \ldots, r\}$ are scalars.■

Remark 4.9. If $r = 1$, then the equation in the preceding definition reduces to $\vec{w} = k_1 \vec{v}_1$; that is, $\vec{w}$ is a linear combination of a single vector $\vec{v}_1$ if it is a scalar multiple of $\vec{v}_1$.

Example 4.33. Vectors in $\mathbb{R}^3$

Every vector $\vec{v} = (a, b, c)$ in $\mathbb{R}^3$ is expressible as a linear combination of the standard basis vectors

$\vec{i} = (1, 0, 0)$

$\vec{j} = (0, 1, 0)$

$\vec{k} = (0, 0, 1)$.

Solution 4.33. Applying the definition we can write

$$\vec{v} = (a, b, c) = a(1, 0, 0) + b(0, 1, 0) + c(0, 0, 1) = a\vec{i} + b\vec{j} + c\vec{k}.\blacktriangle$$

Example 4.34. Checking Linear Combinations

Consider the vectors $\vec{u} = (1, 2, -1)$ and $\vec{v} = (6, 4, 2)$ in $\mathbb{R}^3$. Show that $\vec{w} = (9, 2, 7)$ is a linear combination of $\vec{u}$ and $\vec{v}$ and that $\vec{\omega} = (4, -1, 8)$ is not a linear combination of $\vec{u}$ and $\vec{v}$.

Solution 4.34. In order for $\vec{w}$ to be a linear combination of $\vec{u}$ and $\vec{v}$, there must be scalars k_1 and k_2 such that $\vec{w} = k_1\,\vec{u} + k_2\,\vec{v}$; that is

eqs = Thread[{9, 2, 7} == k_1 {1, 2, −1} + k_2 {6, 4, 2}]; TableForm[eqs]

$$9 = k_1 + 6\,k_2$$
$$2 = 2\,k_1 + 4\,k_2$$
$$7 = 2\,k_2 - k_1$$

Solving this system using the Gauss-Jordan elimination yields

gaussElim = RowReduce[$\begin{pmatrix} 1 & 6 & 9 \\ 2 & 4 & 2 \\ -1 & 2 & 7 \end{pmatrix}$]; MatrixForm[gaussElim]

$$\begin{pmatrix} 1 & 0 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

or in closed form

sol = Solve[eqs, {k_1, k_2}]

$\{\{k_1 \to -3,\ k_2 \to 2\}\}$

so that $\vec{w} = -3\,\vec{u} + 2\,\vec{v}$ which we verify by inserting the solutions into the equations

eqs /. sol

(True True True)

Similarly for $\vec{\omega}$ to be a linear combination of $\vec{u}$ and $\vec{v}$, there must be scalars k_1 and k_2 such that $\vec{\omega} = k_1\,\vec{u} + k_2\,\vec{v}$; that is

eqsω = Thread[{4, −1, 8} == k_1 {1, 2, −1} + k_2 {6, 4, 2}]; TableForm[eqsω]

$$4 = k_1 + 6\,k_2$$
$$-1 = 2\,k_1 + 4\,k_2$$
$$8 = 2\,k_2 - k_1$$

The Gauss-Jordan elimination

$$\textbf{gaussElim = RowReduce}\left[\begin{pmatrix} 1 & 6 & 4 \\ 2 & 4 & -1 \\ -1 & 2 & 8 \end{pmatrix}\right]\textbf{; MatrixForm[gaussElim]}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

shows that the system is inconsistent, so that no such scalars k_1 and k_2 exist. Consequently, $\vec{\omega}$ is not a linear combination of $\vec{u}$ and $\vec{v}$.▲

4.2.7 Tests and Exercises

The following two subsections serve to test your understanding of the last section. Work first on the test examples and then try to solve the exercises.

4.2.7.1 Test Problems

T1. What is a linear system of equations? Give examples.

T2. How is the coefficient matrix derived from a linear system of equations?

T3. What is an augmented matrix?

T4. What is a solution of a linear system of equations?

T5. What kind of solution exist for linear systems of equations?

T6. Does a solution change if you interchange the equations in a linear system of equations?

T7. What are echelon forms?

T8. Recall the steps of Gauß elimination.

4.2.7.2 Exercises

E1. Determine the matrix of coefficients and augmented matrix of each of the following systems of equations.

a. $\begin{array}{rcrcr} x_1 & + & 3x_2 & = & 7 \\ 2x_1 & - & 5x_2 & = & -3 \end{array}$,

b. $\begin{array}{rcrcrcr} 5x_1 & + & +2x_2 & - & 3x_3 & = & 8 \\ x_1 & + & 3x_2 & + & 6x_3 & = & 4 \\ 4x_1 & + & 6x_2 & - & 9x_3 & = & 7 \end{array}$,

c. $\begin{array}{rcrcrcr} -x_1 & + & 3x_2 & + & 5x_3 & = & -3 \\ 2x_1 & - & 2x_2 & + & 5x_3 & = & 8 \\ x_1 & + & 3x_2 & & & = & 6 \end{array}$,

d. $\begin{array}{rrrcr} x_1 & & & = & -3 \\ & x_2 & & = & 12 \\ & & x_3 & = & 8 \end{array}$,

E2. Interpret the following matrices as augmented matrices of systems of equations. Write down each system of equations.

a. $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$,

b. $\begin{pmatrix} 1 & 9 & -3 \\ 6 & 0 & 5 \end{pmatrix}$,

c. $\begin{pmatrix} 8 & 9 & 6 & 5 \\ 2 & 0 & 5 & 4 \\ 12 & 6 & 5 & 9 \end{pmatrix}$,

$$\begin{pmatrix} 2 & -6 & 3 & 8 \\ 7 & -1 & -6 & 12 \\ -1 & 0 & 7 & -4 \end{pmatrix},$$

e. $\begin{pmatrix} 1 & 3 & 5 & 8 \\ 0 & 2 & 9 & 6 \\ 0 & 0 & 7 & 8 \end{pmatrix}$,

f. $\begin{pmatrix} 1 & 0 & 0 & 6 \\ 7 & 1 & 0 & 3 \\ 8 & -2 & 1 & 12 \end{pmatrix}$,

E3. The following systems of equations all have unique solutions. Solve these systems using the method of Gauß-Jordan elimination with matrices.

a. $\begin{array}{rcrcr} x_1 & - & 2x_2 & = & -8 \\ 2x_1 & - & 3x_2 & = & -11 \end{array}$,

b. $\begin{array}{rcrcr} 2x_1 & + & 2x_2 & = & 4 \\ 3x_1 & + & 2x_2 & = & 3 \end{array}$,

c. $\begin{array}{rcrcrcr} x_1 & & & + & x_3 & = & 3 \\ & & 2x_2 & - & 2x_3 & = & -4 \\ & & x_2 & - & 2x_3 & = & 5 \end{array}$,

d. $\begin{array}{rcrcrcr} -x_1 & + & x_2 & - & x_3 & = & -2 \\ 3x_1 & + & x_2 & + & x_3 & = & 10 \\ 4x_1 & + & 2x_2 & + & 3x_3 & = & 14 \end{array}$.

E4. The following systems of equations all have unique solutions. Solve these systems using the method of Gauß-Jordan elimination with matrices.

a. $\begin{array}{rcrcrcr} & & 2x_2 & + & 4x_3 & = & 8 \\ 2x_1 & + & 2x_2 & & & = & 6 \\ x_1 & + & x_2 & + & x_3 & = & 5 \end{array}$,

b. $\begin{array}{rcrcrcr} x_1 & & & - & x_3 & = & 3 \\ -x_1 & & & + & 2x_3 & = & -8 \\ 3x_1 & + & x_2 & - & x_3 & = & 0 \end{array}$,

c. $\begin{array}{rcrcrcr} 2x_1 & + & 2x_2 & - & 4x_3 & = & 14 \\ 3x_1 & + & x_2 & + & x_3 & = & 8 \\ 2x_1 & - & x_2 & + & 2x_3 & = & -1 \end{array}$,

d. $\begin{array}{rcrcrcr} -3x_1 & - & 6x_2 & - & 15x_3 & = & -3 \\ 2x_1 & + & 3x_2 & + & 9x_3 & = & 1 \\ -4x_1 & - & 7x_2 & - & 17x_3 & = & -4 \end{array}$.

E5. Determine whether the following matrices are in reduced echelon form. If a matrix is not in reduced echelon form, give a reason.

a. $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$,

b. $\begin{pmatrix} 1 & 2 & 0 & 4 \\ 0 & 0 & 1 & 7 \end{pmatrix}$,

c. $\begin{pmatrix} 1 & 4 & 0 & 5 \\ 0 & 0 & 2 & 9 \end{pmatrix}$,

$$\begin{pmatrix} 1 & 0 & 0 & 3 & 2 \\ 0 & 2 & 0 & 6 & 1 \\ 0 & 0 & 1 & 2 & 3 \end{pmatrix},$$

e. $\begin{pmatrix} 1 & 3 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 & 5 \end{pmatrix}$,

f. $\begin{pmatrix} 1 & 0 & 5 & 0 \\ 0 & 1 & -7 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$,

E6. The following systems with 3 equations have all unique solutions. Solve these systems using the method of Gauß-Jordan elimination with matrices.

a. $\begin{array}{rcrcrcr} \frac{3}{2}x_1 & & & + & 3x_3 & = & 8 \\ -x_1 & + & 7x_2 & - & 9x_3 & = & -45 \\ 2x_1 & & & + & 5x_3 & = & 22 \end{array},$

b. $\begin{array}{rcrcrcr} x_1 & - & x_2 & + & 3x_3 & = & 3 \\ 2x_1 & - & x_2 & + & 2x_3 & = & 2 \\ 3x_1 & + & x_2 & - & 2x_3 & = & 3 \end{array},$

c. $\begin{array}{rcrcrcr} x_1 & + & 2x_2 & + & 3x_3 & = & 14 \\ 2x_1 & + & 5x_2 & + & 8x_3 & = & 36 \\ x_1 & - & x_2 & & & = & -4 \end{array},$

d. $\begin{array}{rcrcrcr} x_1 & + & x_2 & + & 3x_3 & = & 6 \\ x_1 & + & 2x_2 & + & 4x_3 & = & 9 \\ 2x_1 & + & x_2 & + & 6x_3 & = & 11 \end{array}.$

E7. Each of the following matrices is the reduced echelon form of the augmented matrix of a system of linear equations. Give the solution (if it exists) to each system of equations.

a. $\begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -3 \end{pmatrix}$,

b. $\begin{pmatrix} 1 & 2 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 & 6 \\ 0 & 0 & 0 & 1 & 5 \end{pmatrix}$,

c. $\begin{pmatrix} 1 & 0 & 5 & 0 \\ 0 & 1 & -7 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$,

d. $\begin{pmatrix} 1 & 0 & 0 & 5 & 3 \\ 0 & 1 & 0 & 6 & -2 \\ 0 & 0 & 0 & 1 & 5 \end{pmatrix}$,

e. $\begin{pmatrix} 1 & 3 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 & 5 \end{pmatrix}$,

f. $\begin{pmatrix} 1 & 0 & -3 & 4 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & 0 & 0 \end{pmatrix}$,

E8. Solve (if possible) each of the following systems of three equations in three variables using the method of Gauß-Jordan elimination.

$$\begin{array}{rcrcrcr} 3x_1 & - & 3x_2 & + & 9x_3 & = & 24 \\ 2x_1 & - & 2x_2 & + & 7x_3 & = & 17\,, \\ -x_1 & + & 2x_2 & - & 4x_3 & = & -11 \end{array}$$

b.
$$\begin{array}{rcrcrcr} x_1 & + & x_2 & + & x_3 & = & 7 \\ 2x_1 & + & 3x_2 & + & x_3 & = & 18, \\ -x_1 & + & x_2 & - & 3x_3 & = & 1 \end{array}$$

c.
$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & + & 4x_3 & = & 15 \\ 2x_1 & + & 4x_2 & + & 9x_3 & = & 33, \\ x_1 & + & 3x_2 & + & 5x_3 & = & 20 \end{array}$$

d.
$$\begin{array}{rcrcrcr} x_1 & + & 4x_2 & + & 3x_3 & = & 1 \\ 2x_1 & + & 8x_2 & + & 11x_3 & = & 7. \\ x_1 & + & 6x_2 & + & 7x_3 & = & 3 \end{array}$$

e.
$$\begin{array}{rcrcrcr} x_1 & - & x_2 & + & x_3 & = & 3 \\ 2x_1 & - & x_2 & + & 4x_3 & = & 7, \\ 3x_1 & - & 5x_2 & - & x_3 & = & 7 \end{array}$$

E9. Use row reduction and back substitution to solve each linear system listed below. Start by writing the augmented matrix for the system.

a.
$$\begin{array}{rcrcrcr} x_1 & - & 3x_2 & + & x_3 & = & 4 \\ 2x_1 & + & 3x_2 & + & x_3 & = & 1, \\ 3x_1 & + & x_2 & + & 4x_3 & = & 0 \end{array}$$

b.
$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & + & x_3 & = & 0 \\ 2x_1 & + & 3x_2 & + & x_3 & = & 1' \end{array}$$

c.
$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & + & x_3 & = & 0 \\ 2x_1 & + & 3x_2 & + & x_3 & = & 1, \\ 3x_1 & + & 5x_2 & + & 2x_3 & = & 1 \end{array}$$

d.
$$\begin{array}{rcrcrcr} x_1 & - & 3x_2 & + & x_3 & = & 4 \\ 2x_1 & + & 3x_2 & + & x_3 & = & 1. \\ 3x_1 & & & + & 2x_3 & = & 4 \end{array}$$

E10 Find the reduced row echelon form for the matrix below.

$$\begin{pmatrix} 1 & 2 & 0 & 5 \\ 0 & 4 & 6 & 10 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 2 & 4 \end{pmatrix} \tag{1}$$

4.3 Vector Spaces

4.3.1 Introduction

In the last sections we introduced vectors in 2D and 3D and mentioned that these kind of objects also exist in an *n*-dimensional space. In this section we shall generalize the concept of a vector still further. We will state a set of axioms that, if satisfied by a class of objects, will entitle those objects to be called *vectors*. These generalized vectors will include, among other things, various kinds of matrices and functions. Our work in this section is not an idle exercise in mathematics; it will provide a powerful tool for extending our geometric visualization to a wide variety of important mathematical problems where geometric intuition would not otherwise be available. We can visualize vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$ as arrows, which enable us to draw or form mental pictures to solve problems. Because the axioms we give to define our new kinds of vectors will be based on properties of vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$, the new vectors will have many familiar properties. Consequently, when we went to solve a problem involving our new kinds of vectors, say matrices or functions, we may be able to get a foothold on the problem by visualizing what the corresponding problem would be alike in $\mathbb{R}^2$ or $\mathbb{R}^3$.

4.3.2 Real Vector Spaces

In this subsection we shall extend the concept of a vector by extending the most important properties of familiar vectors and turning them into axioms (definitions). Thus when a set of objects satisfies these axioms, they will automatically have the most important properties of familiar vectors, thereby making it reasonable to regard these objects as new kinds of vectors.

4.3.2.1 Vector Space Axioms

The following definition consists of ten axioms. As you read each axiom, keep in mind that you have already seen each of them as part of various definitions and theorems in the preceding sections. Remember, too, that you do not prove axioms; they are simply the *rules of the game*.

Definition 4.8. *Vector Space*

Let V be an arbitrary nonempty set of objects on which two operations are defined: addition and multiplication by scalars. By addition we mean a rule for associating with each pair of objects $\vec{u}$ and $\vec{v}$ in V an object $\vec{u}+\vec{v}$, called the sum of $\vec{u}$ and $\vec{v}$; by scalar multiplication we mean a rule for associating with each scalar k and each object $\vec{u}$ in V an object $k\,\vec{v}$, called the scalar multiple of $\vec{u}$ by k. If the following axioms are satisfied by all objects $\vec{u}$, $\vec{v}$, and $\vec{w}$ in V and all scalars k and m, then we call V a vector space and we call the objects in V vectors.

1. *If $\vec{u}$ and $\vec{v}$ are objects in V, then $\vec{u}+\vec{v}$ is in V.* *(closure condition)*
2. $\vec{u}+\vec{v}=\vec{v}+\vec{u}$ *(commutativity of the sum)*
3. $\vec{u}+(\vec{v}+\vec{w})=(\vec{u}+\vec{v})+\vec{w}$ *(associativity of the sum)*
4. *There is an object $\vec{0}=0$ in V, called the zero vector for V, such that $0+\vec{u}=\vec{u}+0=\vec{u}$ for all $\vec{u}$ in*

V.

5. *For each $\vec{u}$ in V, there is an object $-\vec{u}$ in V, called the inverse (negative) of $\vec{u}$, such that $\vec{u}+(-\vec{u})=(-\vec{u})+\vec{u}=0$.*
6. *If k is any scalar and $\vec{u}$ is any object in V, then $k\,\vec{u}$ is in V. (closure condition by scalar multiplication)*
7. *$k\,(\vec{u}+\vec{v}) = k\,\vec{u}+k\,\vec{v}$. (distribution law of scalars)*
8. *$(k+m)\,\vec{u}=k\,\vec{u}+m\,\vec{u}$. (distribution law of vectors)*
9. *$k\,(m\,\vec{u})=(k\,m)\,\vec{u}$. (associativity of scalars and vectors in a product)*
10. *$1\,\vec{u}=\vec{u}$.■ (existence of the unit)*

Remark 4.10. Depending on the application, scalars may be real numbers or complex numbers. Vector spaces in which the scalars are complex numbers are called complex vector spaces, and those in which the scalar must be real are called real vector spaces.

The reader should keep in mind that the definition of a vector space specifies neither the nature of the vectors nor the operations. Any kind of objects can be a vector, and the operations of addition and scalar multiplication may not have any relationship or similarity to the standard vector operations on $\mathbb{R}^n$. The only requirement is that the ten vector space axioms be satisfied. Some authors use the notation $\oplus$ and $\odot$ for vector addition and scalar multiplication to distinguish these operations from addition and multiplication of real numbers; we will not use this convention, however.

4.3.2.2 Vector Space Examples

The following examples will illustrate the variety of possible vector spaces. In each example we will specify a nonempty set V and two operations, addition and scalar multiplication; then we shall verify that the ten vector space axioms are satisfied, thereby entailing V, with the specified operations, to be called a vector space.

Example 4.35. $\mathbb{R}^n$ is a Vector Space

The set $V=\mathbb{R}^n$ with the standard operations of addition and scalar multiplication defined in Section 4.1.4 is a vector space. Axiom 1 and 6 follows from the definition of the standard operations on $\mathbb{R}^n$; the remaining axioms follow from Theorem 4.1 generalized to $\mathbb{R}^n$.▲

The three most important special cases of $\mathbb{R}^n$ are $\mathbb{R}$, the real numbers, $\mathbb{R}^2$, the vectors in a plane, and $\mathbb{R}^3$ the vectors in 3D space.

Example 4.36. A Vector Space of Real-Valued Functions

Let V be a set of real-valued functions defined on the entire real line $(-\infty, +\infty)$. If $\vec{f}=f(x)$ and $\vec{g}=g(x)$ are two such functions and k is any real number.

Solution 4.36. Define the sum of functions $\vec{f}+\vec{g}$ and the scalar multiple $k\,\vec{f}$, respectively by

$$\left(\vec{f}+\vec{g}\right)(x)=f(x)+g(x) \qquad \text{and}$$

$$\left(k\,\vec{f}\right)(x)=k\,f(x).$$

In words, the value of the function $\vec{f} + \vec{g}$ at x is obtained by adding together the values of $\vec{f}$ and $\vec{g}$ at x. Similarly, the value of $k\,\vec{f}$ at x is k times the value of $\vec{f}$ at x. It is obvious that the algebraic properties of the functions satisfying the vector space axioms (proof them as an exercise), so V is a vector space. This vector space is denoted by $F(-\infty, +\infty)$. Consequences and examples of the vector space axioms are the following. If $\vec{f}$ and $\vec{g}$ are vectors in this space, then to say that $\vec{f} = \vec{g}$ is equivalent to say that $f(x) = g(x)$ for all x in the interval $(-\infty, \infty)$.

The vector $\vec{0}$ in $F(-\infty, +\infty)$ is the constant function that is identically zero for all values of x. The graph of this function is the line that coincides with the x-axis. The negative of a vector $\vec{f}$ is the function $-\vec{f} = -f(x)$. Geometrically, the graph of $-\vec{f}$ is the reflection of the graph of $\vec{f}$ across the x-axis.▲

Remark 4.11. In the preceding example we focused on the interval $(-\infty, +\infty)$. Had we restricted our attention to some closed interval $[a, b]$ ore some open interval (a, b), the functions defined on those intervals with the operations stated in the example would also have produced a vector space. Those vector spaces are denoted by $F[a, b]$ and $F(a, b)$.

Example 4.37. A Set That is Not a Vector Space

Let $V = \mathbb{R}^2$ and define addition and scalar multiplication operations as follows: If $\vec{u} = (u_1, u_2)$ and $\vec{v} = (v_1, v_2)$, then we define

$$\vec{u} + \vec{v} = (u_1 + v_1,\ u_2 + v_2)$$

and if k is any real number, then define

$k\,\vec{u} = (k\,u_1, 0)$. Show by checking the ten axioms that one of them is violated and thus the set as defined is not a vector space.

Solution 4.37. For example, if $\vec{u} = (2, 6)$ and $\vec{v} = (1, 4)$, and $k = 3$, then

$$\vec{u} + \vec{v} = (2, 6) + (1, 4) = (2 + 1,\ 6 + 4) = (3,\ 10)$$

$$k\,\vec{u} = 3\,(2, 6) = (3 * 2, 0) = (6, 0).$$

The addition operation is the standard addition operation in $\mathbb{R}^2$, but the scalar multiplication operation is not the standard scalar multiplication. It is easy to verify the first nine axioms for this set (try it). However the last axiom 10 fails to hold. For example, if $\vec{u} = (u_1, u_2)$ is such that $u_2 \neq 0$, then

$$1\,\vec{u} = 1\,(u_1, u_2) = (1\,u_1, 0) = (u_1, 0) \neq \vec{u}.$$

Thus V is not a vector space with the standard operations.▲

Example 4.38. Every Plane Through the Origin is a Vector Space

Let V be any plane through the origin in $\mathbb{R}^3$. We shall show that the points in V form a vector space under the standard addition and scalar multiplication operations for vectors in $\mathbb{R}^3$. From Example 4.35 we know that $\mathbb{R}^3$ itself is a vector space under these operations. Thus Axiom 2, 3, 7, 8, 9, and 10 hold for all points in $\mathbb{R}^3$ and consequently for all points in the plane V. We therefore need only show that Axiom 1, 4, 5, and 6 are satisfied.

Solution 4.38. Since the plane V passes through the origin, it has an equation of the form

$$a\,x + b\,y + c\,z = 0.$$

Thus if $\vec{u} = (u_1, u_2, u_3)$ and $\vec{v} = (v_1, v_2, v_3)$ are points in V, then $a\,u_1 + b\,u_2 + c\,u_3 = 0$ and $a\,v_1 + b\,v_2 + c\,v_3 = 0$. Adding these equations gives

$$a\,(u_1 = v_1) + b\,(u_2 + v_2) + c\,(u_3 + v_3) = 0\,.$$

The equality tells us that the coordinates of the point

$$\vec{u} + \vec{v} = (u_1 + v_1,\ u_2 + v_2,\ u_3 + v_3)$$

satisfies the original equation of the plane; thus $\vec{u} + \vec{v}$ lies in the plane V. This proves that Axiom 1 is satisfied. The verification of Axiom 4 and 6 are left as an exercise; however, we will prove that Axiom 5 is satisfied. Multiplying $a\,u_1 + b\,u_2 + c\,u_3 = 0$ through by -1 gives

$$a\,(-u_1) + \ b\,(-u_2) + c\,(-u_3) = 0.$$

Thus $-\vec{u} = (-u_1, -u_2, -u_3)$ lies in V. This establishes Axiom 5.▲

Example 4.39. Zero Vector Space

Let V consist of a single object, which we denote by $\vec{0}$, and define

$$\vec{0} + \vec{0} = \vec{0} \qquad \text{and}$$

$$k\,\vec{0} = \vec{0}$$

for all scalars k.

Solution 4.39. It is easy to check that all vector space axioms are satisfied. We call this the zero vector space.▲

4.3.3 Subspaces

It is possible for one vector space to be contained within another vector space. For example, we showed in the preceding section that planes through the origin are vector spaces that are contained in the vector space $\mathbb{R}^3$. In this section we will study this important concept in detail.

A subset of a vector space V that is itself a vector space with respect to the operations of vector addition and scalar multiplication defined in V is given a special name.

Definition 4.9. *Subspaces*

A subset W of a vector space V is called a subspace of V if W is itself a vector space under the addition and scalar multiplication defined on V.■

In general, one must verify the ten vector space axioms to show that the set W with addition and scalar multiplication forms a vector space. However, if W is part of a larger set V that is already known to be a vector space, then certain axioms need not be verified for W because they are inherited from V. For example, there is no need to check that $\vec{u} + \vec{v} = \vec{v} + \vec{u}$ (Axiom 2) for W because this holds for all vectors in V and consequently for all vectors in W. Other axioms inherited by W from V are 3, 7, 8, 9, and 10. Thus to show that a set W is a subspace of a vector space V, we need only verify Axiom 1, 4, 5, and 6. The following theorem shows that even Axiom 4 and 5 can be omitted.

Theorem 4.12. *Addition and Multiplication for Subspaces*

If W is a set of one or more vectors from a vector space V, then W is a subspace of V if and only if the following conditions hold.

a) If $\vec{u}$ and $\vec{v}$ are vectors in W, then $\vec{u} + \vec{v}$ is in W.

b) If k is any scalar and $\vec{u}$ is any vector in W, then $k\,\vec{u}$ is in W. ■

Proof 4.12. If W is a subspace of V, then all the vector space axioms are satisfied; in particular, Axiom 1 and 6 hold. But these are precisely condition a) and b) of the theorem.

Conversely, assume condition a) and b) hold. Since these conditions are vector space Axioms 1 and 6, we need only show that W satisfies the remaining eight axioms. Axioms 2, 3, 7, 8, 9, and 10 are automatically satisfied by the vectors in W since they are satisfied by all vectors in V. Therefore, to complete the proof, we need only verify that Axioms 4 and 5 are satisfied by vectors in W.

Let $\vec{u}$ be a vector in W. By condition b), $k\,\vec{u}$ is in W for every scalar k. Setting $k = 0$, it follows that $0\,\vec{u} = 0$ is in W, and setting $k = -1$, it follows that $(-k)\,\vec{u} = -\vec{u}$ is in W.

QED

Remark 4.12. A set W of one or more vectors from a vector space V is said to be closed under addition if condition a) in Theorem 4.12 holds and closed under scalar multiplication if condition b) holds. Thus Theorem 4.12 states that W is a subspace of V if and only if W is closed under addition and closed under scalar multiplication.

Example 4.40. Polynomials of Degree Smaller or Equal to n

Let n be an non negative integer, and let W consist of all functions expressible in the form

$$p(x) = a_0 + a_1\,x + a_2\,x^2 + \ldots + a_n\,x^n = \sum_{j=0}^{n} a_j\,x^j$$

where a_0, a_1, $\ldots$, a_n are real numbers. Thus W consists of all real polynomials of degree n or less.

Solution 4.40. The set W is a subspace of the vector space of all real-valued functions discussed in Example 4.36 of the preceding section. To see this, let $\vec{p}$ and $\vec{q}$ be the polynomials

$$\vec{p} = p(x) = a_0 + a_1\,x + a_2\,x^2 + \ldots + a_n\,x^n = \sum_{j=0}^{n} a_j\,x^j$$

and

$$\vec{q} = q(x) = b_0 + b_1\, x + b_2\, x^2 + \ldots + b_n\, x^n = \sum_{j=0}^{n} b_j\, x^j$$

Then

$$(\vec{p} + \vec{q})\,(x) = p(x) + q(x) = \sum_{j=0}^{n} a_j\, x^j + \sum_{j=0}^{n} b_j\, x^j = \sum_{j=0}^{n} \left(a_j + b_j\right) x^j$$

and

$$(k\,\vec{p})\,(x) = k\; p(x) = k \sum_{j=0}^{n} a_j\, x^j = \sum_{j=0}^{n} k\, a_j\, x^j.$$

These functions have the original form, so $\vec{p} + \vec{q}$ and $k\,\vec{p}$ lie in W. This vector space W of polynomials is usually denoted by the symbol P_n.▲

Example 4.41. Subspaces of Functions

Recall from calculus that if $\vec{f}$ and $\vec{g}$ are continuous functions on the interval $(-\infty, +\infty)$ and k is a constant, then $\vec{f} + \vec{g}$ and $k\,\vec{f}$ are also continuous. Thus the continuous functions on the interval $(-\infty, +\infty)$ form a subspace of $F(-\infty, +\infty)$, since they are closed under addition and scalar multiplication. We denote this subspace by $C(-\infty, +\infty)$. Similarly, if $\vec{f}$ and $\vec{g}$ have continuous first order derivatives on $(-\infty, +\infty)$, then so do $\vec{f} + \vec{g}$ and $k\,\vec{f}$. Thus the functions with continuous first derivative on $(-\infty, +\infty)$ form a subspace of $F(-\infty, +\infty)$. We denote this subspace by $C^1(-\infty, +\infty)$, where the superscript 1 is used to emphasize the first derivative. However, there is a theorem in calculus that every differentiable function is continuous, so $C^1(-\infty, \infty)$ is actually a subspace of $C(-\infty, +\infty)$.

To take this a step further, for each positive integer m, the functions with continuous mth derivatives on $(-\infty, +\infty)$ form a subspace of $C^1(-\infty, +\infty)$ as do the functions with continuous derivatives of all order. We denote the subspace of functions with continuous mth order derivatives on $(-\infty, +\infty)$ by $C^m(-\infty, +\infty)$, and we denote the subspace of functions that have continuous derivatives of all order on $(-\infty, +\infty)$ by $C^\infty(-\infty, +\infty)$. Finally, there is a theorem in calculus that polynomials have continuous derivatives of all order, so P_n is a subspace of $C^\infty(-\infty, +\infty)$. The hierarchy of subspaces discussed in this example is shown in Figure 4.32.

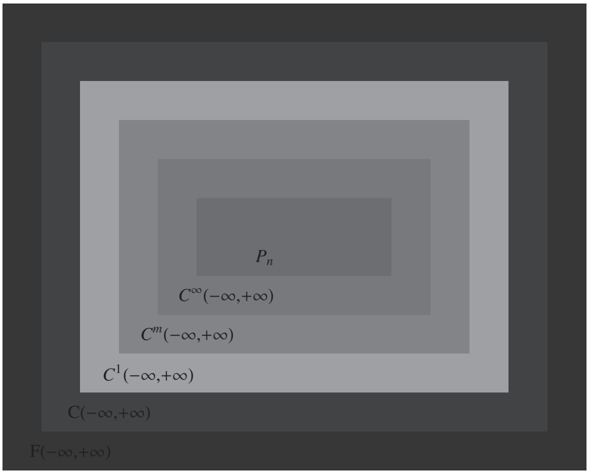


Figure 4.32. Vector space structure of functions as subspaces.▲

4.3.4 Spanning

If $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r$ are vectors in a vector space V, then generally some vectors in V may be linear combinations of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r$ and others may not. The following theorem shows that if we construct a set W consisting of all those vectors that are expressible as linear combinations of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r$, then W forms a subspace of V.

Theorem 4.13. *Subspace*

If $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r$ are vectors in a vector space V, then

a) The set W of all linear combinations of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r$ is a subspace of V.

b) W is the smallest subspace of V that contains $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r$ in the sense that every other subspace of V that contains $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r$ must contain W.■

Proof 4.13. a) To show that W is a subspace of V, we must prove that it is closed under addition and scalar multiplication. There is at least one vector in W—namely 0, since $0 = 0\,\vec{v}_1 + 0\,\vec{v}_2 + \ldots + 0\,\vec{v}_r$. If $\vec{u}$ and $\vec{v}$ are vectors in W, then

$$\vec{u} = c_1\,\vec{v}_1 + c_2\,\vec{v}_2 + \ldots + c_r\,\vec{v}_r$$

and

$$\vec{v} = k_1\,\vec{v}_1 + k_2\,\vec{v}_2 + \ldots + k_r\,\vec{v}_r$$

where $c_1, c_2, \ldots, c_r, k_1, k_2, \ldots, k_r$ are scalars. Therefore,

$$\vec{u} + \vec{v} = (c_1 + k_1)\,\vec{v}_1 + (c_2 + k_2)\,\vec{v}_2 + \ldots + (c_r + k_r)\,\vec{v}_r$$

and for any scalar k,

$$k\,\vec{u} = (k\,c_1)\,\vec{v}_1 + (k\,c_2)\,\vec{v}_2 + \ldots + (k\,c_r)\,\vec{v}_r$$

Thus $\vec{u} + \vec{v}$ and $k\,\vec{u}$ are linear combinations of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r$, and consequently lie in W. Therefore, W is closed under addition and scalar multiplication.

b) Each vector $\vec{v}_i$ is a linear combination of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r$ since we can write

$$\vec{v}_i = 0\,\vec{v}_1 + 0\,\vec{v}_2 + \ldots + 1\,\vec{v}_i + \ldots + 0\,\vec{v}_r$$

Therefore, the subspace W contains each of the vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r$. Let W' be any other subspace that contains $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r$. Since W' is closed under addition and scalar multiplication, it must contain all linear combinations of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r$. Thus W' contains each vector of W.

QED

Based on these observations we make the following definition

Definition 4.10. *Span*

If $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r\}$ is a set of vectors in a vector space V, then the subspace W of V containing of all linear combinations of the vectors in S is called the space spanned by $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r$, and we say that the vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r$ span W. To indicate that W is the space spanned by the vectors in the set $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r\}$, we write

$$W = \text{span}(S) \qquad \text{or} \qquad W = \text{span}\,\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r\}.\blacksquare$$

Example 4.42. Spaces Spanned by One or Two Vectors

If $\vec{v}_1$ and $\vec{v}_2$ are non colinear vectors in $\mathbb{R}^3$ with their initial points at the origin, then span$\{\vec{v}_1, \vec{v}_2\}$, which consists of all linear combinations $k_1\,\vec{v}_1 + k_2\,\vec{v}_2$, is the plane determined by $\vec{v}_1$ and $\vec{v}_2$. Similarly, if $\vec{v}$ is a nonzero vector in $\mathbb{R}^3$ or $\mathbb{R}^2$ the span$\{\vec{v}\}$, which is the set of all scalar multiples $k\,\vec{v}$, is the line determined by $\vec{v}$.▲

Example 4.43. Spanning Set for P_n

The polynomials $1, x, x^2, x^3, \ldots, x^n$ span the vector space P_n. Since each polynomial $\vec{p}$ in P_n can be written as

$$\vec{p} = p(x) = a_0 + a_1\,x + a_2\,x^2 + \ldots + a_n\,x^n = \sum_{j=0}^{n} a_j\,x^j$$

which is a linear combination of $1, x, x^2, x^3, \ldots, x^n$. We can denote this by writing

$$P_n = \text{span}\left\{1, x, x^2, x^3, \ldots, x^n\right\}.\blacktriangle$$

Example 4.44. Three Vectors that do not Span $\mathbb{R}^3$

Determine whether $\vec{v}_1 = (1, 1, 2)$, $\vec{v}_2 = (1, 0, 1)$, and $\vec{v}_3 = (2, 1, 3)$ span the vector space $\mathbb{R}^3$.

Solution 4.44. We must determine whether an arbitrary vector $\vec{b} = (b_1, b_2, b_3)$ in $\mathbb{R}^3$ can be represented as a linear combination

$$\vec{b} = k_1\,\vec{v}_1 + k_2\,\vec{v}_2 + k_3\,\vec{v}_3$$

of the vectors $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$. Expressing this equation in terms of components gives

eqs = Thread[{b_1, b_2, b_3} = k_1 {1, 1, 2} + k_2 {1, 0, 1} + k_3 {2, 1, 3}]; TableForm[eqs]

$$b_1 = k_1 + k_2 + 2\,k_3$$
$$b_2 = k_1 + k_3$$
$$b_3 = 2\,k_1 + k_2 + 3\,k_3$$

The problem thus reduces to determine whether this system is consistent for all values of b_1, b_2, and b_3. The application of the Gauss-Jordan elimination shows that

$$\textbf{red = RowReduce}\left[\begin{pmatrix} \mathbf{1} & \mathbf{1} & \mathbf{2} & \mathbf{b1} \\ \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{b2} \\ \mathbf{2} & \mathbf{1} & \mathbf{3} & \mathbf{b3} \end{pmatrix}\right]\textbf{; MatrixForm[red]}$$

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

the system is inconsistent and thus the three vectors $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$ do not span $\mathbb{R}^3$.▲

Spanning sets are not unique. For example, any two non colinear vectors that lie in a plane will span the same plane, and any nonzero vector on a line will span the same line. We leave the proof of the following useful theorem as an exercise.

Theorem 4.14. *Equivalence of Spans*

If $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r\}$ and $S' = \{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_r\}$ are two sets of vectors in a vector space V, then

$$\text{span}\,\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r\} = \text{span}\,\{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_r\}$$

if and only if each vector in S is a linear combination of those in S' and each vector in S' is a linear combination of those in S.■

4.3.5 Linear Independence

In the preceding section we learned that a set of vectors $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r\}$ spans a given vector space V if every vector in V is expressible as a linear combination of the vectors in S. In general, there may be more than one way to express a vector in V as a linear combination of vectors in a spanning set. In this section we will study conditions under which each vector in V is expressible as a linear combination of the spanning vectors in exactly one way. Spanning sets with this property play a fundamental role in the study of vector spaces.

Definition 4.11. *Linearly Independent and Dependent Vectors*

If $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r\}$ *is a nonempty set of vectors, then the vector equation*

$$k_1 \vec{v}_1 + k_2 \vec{v}_2 + \ldots + k_r \vec{v}_r = 0$$

has at least one solution, namely

$$k_1 = 0, \;\; k_2 = 0, \;\; \ldots, \;\; k_r = 0.$$

If this is the only solution, then S is called a linearly independent set. If there are other solutions, then S is called a linearly dependent set.■

Example 4.45. A Linearly Dependent Set

Let $\vec{v}_1 = (2, -1, 0, 3)$, $\vec{v}_2 = (1, 2, 5, -1)$, and $\vec{v} = (7, -1, 5, 8)$. Determine whether the given vectors are linearly independent or dependent.

Solution 4.45. According to the definition we should find scalars k_1, k_2, and k_3 which are unequal to zero to have linear dependence. So we write

```
syst = Thread[k1 {2, -1, 0, 3} + k2 {1, 2, 5, -1} + k3 {7, -1, 5, 8} == 0];
TableForm[syst, TableAlignments → {Right}]
```

$$\begin{aligned} 2k_1 + k_2 + 7k_3 &= 0 \\ -k_1 + 2k_2 - k_3 &= 0 \\ 5k_2 + 5k_3 &= 0 \\ 3k_1 - k_2 + 8k_3 &= 0 \end{aligned}$$

To find the coefficients k_i we apply the Gauss-Jordan elimination to the augmented matrix to find

$$\textbf{red = RowReduce}\left[\begin{pmatrix} 2 & 1 & 7 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & 5 & 5 & 0 \\ 3 & -1 & 8 & 0 \end{pmatrix}\right]; \textbf{MatrixForm[red]}$$

$$\begin{pmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

showing that k_1 depends on k_3, and k_2 depends on k_3 in the following form

$$k_1 = -3\,t, \;\; k_2 = -t, \;\; \text{and}\, k_3 = t$$

where t is an arbitrary number. Thus an infinite set of choices for k_i will result to a vanishing linear combination. In conclusion the three given vectors are not linearly independent.▲

Example 4.46. A Linearly Dependent Set of Polynomials

Given are the polynomials

$$\vec{p}_1 = 1 - x,\ \vec{p}_2 = 5 + 3\,x - 2\,x^2,\ \text{and}\ \vec{p}_3 = 1 + 3\,x - x^2.$$

Determine if the three vectors are linearly independent from each other

Solution 4.46. Since the maximal order of the polynomials is 2 we can represent the vectors by the three coefficients of the vector $\{x^0, x^1, x^2\}$. The related vectors in the linear combinations now read

```
syst2 = Thread[k1 {1, -1, 0} + k2 {5, 3, -2} + k3 {1, 3, -1} == {0, 0, 0}];
TableForm[syst2, TableAlignments → {Right}]
```

$$\begin{aligned} k_1 + 5\,k_2 + k_3 &= 0 \\ -k_1 + 3\,k_2 + 3\,k_3 &= 0 \\ -2\,k_2 - k_3 &= 0 \end{aligned}$$

The related augmented matrix is used in a Gauss-Jordan elimination

$$\textbf{red = RowReduce}\left[\begin{pmatrix} 1 & 5 & 1 & 0 \\ -1 & 3 & 3 & 0 \\ 0 & -2 & -1 & 0 \end{pmatrix}\right]\textbf{; MatrixForm[red]}$$

$$\begin{pmatrix} 1 & 0 & -\frac{3}{2} & 0 \\ 0 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

which shows that $k_1 = 3/2\,k_3$ and $k_2 = -1/2\,k_3$. Thus there exists an infinite set of combinations to represent linear combinations of $\vec{p}_1$, $\vec{p}_2$, and $\vec{p}_3$ and thus the vectors are not linearly independent.▲

Example 4.47. A Linearly Independent Set

Determine whether the vectors

$$\vec{v}_1 = (1, -2, 3),\ \vec{v}_2 = (5, 6, -1),\ \text{and}\ \vec{v}_3 = (3, 2, 0)$$

form a linearly dependent set or a linearly independent set.

Solution 4.47. Applying the definition we have

```
sys3 = Thread[k1 {1, -2, 3} + k2 {5, 6, -1} + k3 {3, 2, 0} == {0, 0, 0}];
TableForm[sys3, TableAlignments → {Right}]
```

$$\begin{aligned} k_1 + 5\,k_2 + 3\,k_3 &= 0 \\ -2\,k_1 + 6\,k_2 + 2\,k_3 &= 0 \\ 3\,k_1 - k_2 &= 0 \end{aligned}$$

The related augmented matrix is given by

$$\mathbf{red = RowReduce}\left[\begin{pmatrix} \mathbf{1} & \mathbf{5} & \mathbf{3} & \mathbf{0} \\ \mathbf{-2} & \mathbf{6} & \mathbf{2} & \mathbf{0} \\ \mathbf{3} & \mathbf{-1} & \mathbf{0} & \mathbf{0} \end{pmatrix}\right]\mathbf{; MatrixForm[red]}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

The only solution for this system of equations is $k_1 = k_2 = k_3 = 0$. Thus the vectors are linearly independent from each other.▲

The term linearly dependent suggests that the vectors depend on each other in some way. The following theorem shows that this is in fact the case.

Theorem 4.15. *Linearly Dependent Vectors*

A set S with two or more vectors is

a) Linearly dependent if and only if at least one of the vectors in S is expressible as a linear combination of the other vectors in S.

b) Linearly independent if and only if no vector in S is expressible as a linear combination of the other vectors in S.■

We shall prove part a) and leave the proof of part b) as an exercise.

Proof 4.15. a) Let $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r\}$ be a set with two ore more vectors. If we assume that S is linearly dependent, then there are scalars $k_1, k_2, \ldots, k_r$, not all zero, such that

$$k_1 \vec{v}_1 + k_2 \vec{v}_2 + \ldots + k_r \vec{v}_r = 0$$

To be specific, suppose that $k_1 \neq 0$. Then we can write this relation as

$$\vec{v}_1 = -\frac{k_2}{k_1} \vec{v}_2 - \ldots - \frac{k_r}{k_1} \vec{v}_r$$

which expresses $\vec{v}_1$ as a linear combination of the other vectors in S. Similarly, if $k_j \neq 0$ in this relation for some $j = 2, 3, \ldots, r$, then $\vec{v}_j$ is expressible as a linear combination of the other vectors in S.

Conversely let us assume that at least one of the vectors in S is expressible as a linear combination of the other vectors. To be specific, suppose that

$$\vec{v}_1 = c_2 \vec{v}_2 + \ldots + c_r \vec{v}_r$$

so

$$\vec{v}_1 - c_2 \vec{v}_2 - \ldots - c_r \vec{v}_r = 0.$$

It follows that S is linearly dependent since the equation

$$k_1 \vec{v}_1 + k_2 \vec{v}_2 + \ldots + k_r \vec{v}_r = 0$$

is satisfied by

$$k_1 = 1, k_2 = -c_2, \ldots, k_r = -c_r$$

which are not all zero. The proof in the case where some vector other than $\vec{v}_1$ is expressible as a linear combination of the other vectors in S is similar.

QED

The following theorem gives two simple facts about linear independence that are important to know.

Theorem 4.16. *Linearly Dependent Vectors*

a) A finite set of vectors that contains the zero vector is linearly dependent.

b) A set with exactly two vectors is linearly independent if and only if neither vector is a scalar multiple of the other. ■

We shall prove part a) and leave the proof of part b) as an exercise.

Proof 4.16. a) For any vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r$ the set $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r, 0\}$ is linearly dependent since the equation

$$0\,\vec{v}_1 + 0\,\vec{v}_2 + \ldots + 0\,\vec{v}_r + 1 \times 0 = 0$$

expresses 0 as a linear combination of the vectors in S with coefficients that are not all zero.

QED

Linear independence has some useful geometric interpretations in $\mathbb{R}^2$ and $\mathbb{R}^3$:

- In $\mathbb{R}^2$ or $\mathbb{R}^3$, a set of two vectors is linearly independent if and only if the vectors do not lie on the same line when they are placed with their initial points at the origin.
- In $\mathbb{R}^3$, a set of three vectors is linearly independent if and only if the vectors do not lie in the same plane when they are placed with their initial points at the origin.

The first result follows from the fact that two vectors are linearly independent if and only if neither vector is a scalar multiple of the other. Geometrically, this is equivalent to stating that the vectors do not lie on the same line when they are positioned with their initial points at the origin.

The second result follows from the fact that three vectors are linearly independent if and only if none of the vectors is a linear combination of the other two. Geometrically, this is equivalent to stating that none of the vectors lies in the same plane as the other two, or, alternatively, that the three vectors do not lie in a common plane when they are positioned with their initial points at the origin.

The next theorem shows that a linearly independent set in $\mathbb{R}^n$ can contain at most n vectors.

Theorem 4.17. *Maximum Number of Independent Vectors*

Let $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r\}$ be a set of vectors in $\mathbb{R}^n$. If $r > n$, then S is linearly dependent. ■

Remark 4.13. This theorem tells us that a set in $\mathbb{R}^2$ with more than two vectors is linearly dependent and a set in $\mathbb{R}^3$ with more than three vectors is linearly dependent.

4.3.6 Tests and Exercises

The following two subsections serve to test your understanding of the last section. Work first on the test examples and then try to solve the exercises.

4.3.6.1 Test Problems

T1. What is a vector space? Give examples.

T2. How is the Span defined?

T3. How is a linear combination of vectors defined?

T4. What are linearly dependent vectors?

T5. What are linearly independent vectors?

T6. Which properties are defined in e linear vector space?

T7. What are subspaces of a linear vector space?

4.3.6.2 Exercises

E1. Prove that the set of n-dimensional vectors C^n with the operations of addition and scalar multiplication defined as follows is a vector space.

$$(u_1,\ \ldots,\ u_n) + (v_1,\ \ldots,\ v_n) = (u_1 + v_1,\ \ldots,\ u_n + v_n) \tag{1}$$

$$c\,(u_1,\ \ldots,\ u_n) = (c\,u_1,\ \ldots,\ c\,u_n) \tag{2}$$

Determine $\vec{u} + \vec{v}$ and $c\,\vec{u}$ for the following vectors and scalars in C^2.

a. $\vec{u} = (2 - i,\ 3 + 4\,i)$, $\vec{v} = (5,\ 1 + 3\,i)$, $c = 3 - 2\,i$,

b. $\vec{u} = (1 + 5\,i,\ -2 - 3\,i)$, $\vec{v} = (4\,i,\ 3 - 7\,i)$, $c = 4 + 2\,i$.

E2. Let U be the set of all vectors in $\mathbb{R}^3$ that are perpendicular to a vector $\vec{u}$. Prove that U is a vector space.

E3. Is the set of 3×3 symmetric matrices a vector space?

E4. Consider the following sets with operations of point wise addition and scalar multiplication, having the real numbers as their domain. Are they vector spaces?

a. The set of even functions. $f(-x) = f(x)$.

b. The set of odd functions. $f(-x) = -f(x)$.

E5. Consider the sets of vectors of the following form. Prove that the sets are subspaces of $\mathbb{R}^2$. Give the geometrical interpretation of each subspace.

a. $(a,\ 0)$,

b. $(a,\ a)$,

c. $(a,\ 2\,a)$,

d. $(a,\ a + 3\,b)$.

E6. Consider the set of vectors of the following form. Determine which of the sets are subspaces of $\mathbb{R}^3$.

a. $(a,\ b,\ a - 4\,b)$,

b. $(a,\ 1,\ 1)$,

c. $(a,\ b,\ 2\,a + 3\,b + 6)$,

d. $\left(a,\ a^2,\ 6\,a\right)$.

E7. Which of the following subsets of $\mathbb{R}^3$ are subspaces? The set of all vectors of the form $(a,\ b,\ c)$ where a, b, and c are

a. integers,

b. non negative real numbers,

c. rational numbers.

E8. Show that the following sets of vectors span $\mathbb{R}^2$. Express the vector $(3,\ 5)$ in terms of each spanning set.

$(1, 1), (1, -1)$,

b. $(1, 4), (-2, 0)$,

c. $(1, 3), (3, 10)$,

d. $(1, 0), (0, 1)$.

E9. Determine whether the following vectors span $\mathbb{R}^2$.

a. $(1, 1), (-2, 1)$,

b. $(3, 2), (1, 1), (1, 0)$,

c. $(-3, 1), (3, -1)$,

d. $(4, -1), (2, 3), (6, 5)$.

E10 Let U be the subspace of $\mathbb{R}^2$ generated by the vector $(-1, 3)$. Let V be the subspace of $\mathbb{R}^2$ generated by the vector $(-2, 6)$. Show that $U = V$.

4.4 Matrices

Rectangular arrays of real numbers arise in many contexts other than as augmented matrices for systems of linear equations. In this section we begin to introduce matrix theory by giving some of the fundamental definitions of the subject. We will see how matrices can be combined through the arithmetic operations of addition, subtraction, and multiplication.

4.4.1 Matrix Notation and Terminology

In Section 4.1.2 we used rectangular arrays, called augmented matrices, to abbreviate systems of linear equations. However, rectangular arrays of numbers occur in other contexts as well. For example, the following rectangular array with three rows and seven columns describes the number of hours a student spent studying three different subjects during a certain week:

	Mon.	Tues.	Wed.	Thurs.	Fri.	Sat.	Sun.
Math	4	3	2	2	1	5	4
Phys	2	1	1	4	6	2	1
Eng	1	4	3	5	2	1	1

If we suppress the headings, we are left with the following rectangular array of numbers with three rows and seven columns, called a matrix.

$$\begin{pmatrix} 4 & 3 & 2 & 2 & 1 & 5 & 4 \\ 2 & 1 & 1 & 4 & 6 & 2 & 1 \\ 1 & 4 & 3 & 5 & 2 & 1 & 1 \end{pmatrix}$$

More generally we make the following definition

Definition 4.12. *Matrix*

A matrix is a rectangular array of numbers. The numbers in the array are called the entries in the matrix. ■

Some examples of matrices are:

$\begin{pmatrix} 2 & 4 \\ 0 & 1 \\ 5 & 7 \end{pmatrix}$ *a* matrix with three rows and two columns.

$\begin{pmatrix} 1 & 5 & 0 & 8 \end{pmatrix}$ *a* matrix with *a* single row and four columns *a* row vector.

$\begin{pmatrix} 1 & \sqrt{2} & 6 & e \\ 0 & 1 & \pi & -3 \\ 7 & 3 & 1 & -12 \\ 9 & -1 & 4 & 1 \end{pmatrix}$ *a* four times four matrix.

The size of a matrix is described in terms of the numbers of rows (horizontal lines) and columns (vertical lines) it contains. For example the first matrix above has three rows and two columns, so its size is 3 by 2, written as 3×2. In a size description, the first number always denotes the number of rows, and the second denotes the number of columns. The remaining matrices from above have size 1×4 and 4×4, respectively. A matrix with only one column is called a column matrix (or a column vector), and a matrix with only one row is called a row matrix (or a row vector).

Remark 4.14. It is common practice to omit the brackets for a 1×1 matrix. Thus we might write 4 rather than (4) for a scalar. Although this makes it impossible to tell whether 4 denotes the number 4 or a 1×1 matrix whose entry is 4, however this rarely causes problems, since it is usually possible to tell which is meant from the context in which the symbol is used.

We shall use capital letters to denote matrices and lower letters to denote numbers; thus we might write

$$A = \begin{pmatrix} 2 & 3 & 9 \\ 6 & 4 & -1 \\ 3 & 6 & 8 \end{pmatrix} \qquad \text{or} \qquad C = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

When distinguishing matrices, it is common to refer to numerical quantities as scalars. Unless stated otherwise, scalars will be real numbers.

The entry that occurs in row i and column j of matrix A will be denoted by a_{ij}. Thus a general 2×3 matrix might be written as

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} \tag{4.78}$$

and a general $m\times n$ matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \tag{4.79}$$

When compactness of notation is desired, the preceding matrix can be written as

$$\left(a_{ij}\right)_{m\times n} \qquad \text{or} \qquad \left(a_{ij}\right) \tag{4.80}$$

the first notation being used when it is important in the discussion to know the size, and the second being used when the size need not be emphasized. Usually, we shall match the letter denoting a matrix with the letter denoting its entries; thus, for a matrix B we would generally use b_{ij} for the entry in row i and column j, and for a matrix C we would use the notation c_{ij}.

The entry in row i and column j of a matrix A is also commonly denoted by the symbol $(A)_{ij}$. Thus for the general matrix A we write

$$(A)_{ij} = a_{ij}. \tag{4.81}$$

Row and column matrices are of special importance, and it is common practice to denote them by lowercase vector symbols. For such matrices, double sub scripting of the entries is unnecessary. Thus the general $1\times n$ row matrix $\vec{a}$ and the general $m\times 1$ matrix $\vec{b}$ would be written as

$$\vec{a} = (a_1,\ a_2,\ \ldots,\ a_n) \qquad \text{and} \qquad \vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix} \tag{4.82}$$

A matrix A with n rows and n columns is called a square matrix of order n, and the entries a_{ii} are called the main diagonal of A.

$$A = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{pmatrix} \tag{4.83}$$

4.4.2 Operations with Matrices

So far we have used matrices to abbreviate the work in solving systems of linear equations. For other applications, however, it is desirable to develop the arithmetic of matrices in which matrices can be added, subtracted, and multiplied in a useful way.

Definition 4.13. *Equal Matrices*

Two matrices A and B are defined to be equal if they have the same size and their corresponding entries are equal

$$a_{ij} = b_{ij}.\blacksquare$$

In matrix notation, if $A = \left(a_{ij}\right)$ and $B = \left(b_{ij}\right)$ have the same size, then $A = B$ if and only if $(A)_{ij} = (B)_{ij}$ for all i and j. Examples for this kind of equality are demonstrated by the following matrices

$$A = \begin{pmatrix} 1 & 3 \\ 7 & x \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 3 \\ 7 & 6 \end{pmatrix}, \qquad C = \begin{pmatrix} 1 & 2 & 8 \\ 7 & 6 & 2 \end{pmatrix}$$

If $x = 6$, then $A = B$, but for all other values of x the matrices A and B are not equal, since not all of their corresponding entries are equal. There is no value of x for which $A = C$ since A and C have different size.

Definition 4.14. *Sum of Matrices*

If A and B are matrices of the same size, then the sum $A + B$ is the matrix obtained by adding the entries of B to the corresponding entries of A, and the difference $A - B$ is the matrix obtained by subtracting the entries of B from the corresponding entries of A. Matrices of different size cannot be added or subtracted.■

In matrix notation, if $A = (a_{ij})$ and $B = (b_{ij})$ have the same size, then

$$(A + B)_{ij} = (A)_{ij} + (B)_{ij} = a_{ij} + b_{ij} \tag{4.84}$$

and

$$(A - B)_{ij} = (A)_{ij} - (B)_{ij} = a_{ij} - b_{ij} \tag{4.85}$$

Example 4.48. Addition and Subtraction

Consider the matrices

$$\mathbf{A} = \begin{pmatrix} 1 & 4 & 6 \\ 8 & 3 & -2 \\ 5 & 6 & 4 \end{pmatrix}; \mathbf{B} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ 3 & 4 & 1 \end{pmatrix}; \mathbf{F} = \begin{pmatrix} 1 & 4 & 7 \\ 7 & 4 & 1 \end{pmatrix};$$

Add and subtract the different matrices if possible.

Solution 4.48. Then

$\mathbf{A + B}$

$$\begin{pmatrix} 2 & 6 & 9 \\ 10 & 4 & 2 \\ 8 & 10 & 5 \end{pmatrix}$$

and

$\mathbf{A - B}$

$$\begin{pmatrix} 0 & 2 & 3 \\ 6 & 2 & -6 \\ 2 & 2 & 3 \end{pmatrix}$$

The expressions $A + F$, $B + F$, $A - F$, and $B - F$ are undefined. This behavior is notified by *Mathematica* with an error message

A* + *F

```
Thread::tdlen : Objects of unequal length in
    {{1, 4, 6}, {8, 3, -2}, {5, 6, 4}} + {{1, 4, 7}, {7, 4, 1}}
    cannot be combined. More…
```

$$\begin{pmatrix} 1 & 4 & 7 \\ 7 & 4 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 4 & 6 \\ 8 & 3 & -2 \\ 5 & 6 & 4 \end{pmatrix}$$

B* + *F

```
Thread::tdlen : Objects of unequal length in
    {{1, 2, 3}, {2, 1, 4}, {3, 4, 1}} + {{1, 4, 7}, {7, 4, 1}}
    cannot be combined. More…
```

$$\begin{pmatrix} 1 & 4 & 7 \\ 7 & 4 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ 3 & 4 & 1 \end{pmatrix}$$

A* − *F

```
Thread::tdlen : Objects of unequal length in
    {{1, 4, 6}, {8, 3, -2}, {5, 6, 4}} + {{-1, -4, -7}, {-7, -4, -1}}
    cannot be combined. More…
```

$$\begin{pmatrix} -1 & -4 & -7 \\ -7 & -4 & -1 \end{pmatrix} + \begin{pmatrix} 1 & 4 & 6 \\ 8 & 3 & -2 \\ 5 & 6 & 4 \end{pmatrix}$$

The return value is the undefined sum or difference as a symbolic expression.▲

Definition 4.15. *Product with a Scalar*

If A is any matrix and c is any scalar, then the product c A is the matrix obtained by multiplying each entry of the matrix A by c. The matrix c A is called to be a scalar multiple of A.■

In matrix notation, if $A = (a_{ij})$, then

$$c\,A = c\,(A)_{ij} = c\,a_{ij} \tag{4.86}$$

Example 4.49. Scalar Multiples

Find the scalar multiples by $c = \sqrt{2}$, $b = -1$ of the following matrices

$$\mathbf{A} = \begin{pmatrix} \mathbf{1} & \mathbf{4} & \mathbf{6} \\ \mathbf{8} & \mathbf{3} & \mathbf{-2} \\ \mathbf{5} & \mathbf{6} & \mathbf{4} \end{pmatrix}; \mathbf{B} = \begin{pmatrix} \mathbf{1} & \mathbf{2} & \mathbf{3} \\ \mathbf{2} & \mathbf{1} & \mathbf{4} \\ \mathbf{3} & \mathbf{4} & \mathbf{1} \end{pmatrix}; \mathbf{F} = \begin{pmatrix} \mathbf{1} & \mathbf{4} & \mathbf{7} \\ \mathbf{7} & \mathbf{4} & \mathbf{1} \end{pmatrix};$$

Solution 4.49. The multiplication of each entry with c or b gives

$\sqrt{2}\ \boldsymbol{A}$

$$\begin{pmatrix} \sqrt{2} & 4\sqrt{2} & 6\sqrt{2} \\ 8\sqrt{2} & 3\sqrt{2} & -2\sqrt{2} \\ 5\sqrt{2} & 6\sqrt{2} & 4\sqrt{2} \end{pmatrix}$$

$\sqrt{2}\ \boldsymbol{B}$

$$\begin{pmatrix} \sqrt{2} & 2\sqrt{2} & 3\sqrt{2} \\ 2\sqrt{2} & \sqrt{2} & 4\sqrt{2} \\ 3\sqrt{2} & 4\sqrt{2} & \sqrt{2} \end{pmatrix}$$

$\sqrt{2}\ \boldsymbol{F}$

$$\begin{pmatrix} \sqrt{2} & 4\sqrt{2} & 7\sqrt{2} \\ 7\sqrt{2} & 4\sqrt{2} & \sqrt{2} \end{pmatrix}$$

$-\boldsymbol{A}$

$$\begin{pmatrix} -1 & -4 & -6 \\ -8 & -3 & 2 \\ -5 & -6 & -4 \end{pmatrix}$$

$-\boldsymbol{B}$

$$\begin{pmatrix} -1 & -2 & -3 \\ -2 & -1 & -4 \\ -3 & -4 & -1 \end{pmatrix}$$

$-\boldsymbol{F}$

$$\begin{pmatrix} -1 & -4 & -7 \\ -7 & -4 & -1 \end{pmatrix}$$

It is common practice to denote $(-1)\,A$ by $-A$, etc.▲

If $A_1, A_2, \ldots, A_n$ are matrices of the same size and $c_1, c_2, c_3, \ldots, c_n$ are scalars, then an expression of the form

$$c_1 A_1 + c_2 A_2 + \ldots + c_n A_n = \sum_{j=1}^{n} c_j A_j \tag{4.87}$$

is called a linear combination of $A_1, A_2, \ldots, A_n$ with coefficients $c_1, c_2, c_3, \ldots, c_n$. For example, if A, B, and F are the matrices

$$A = \begin{pmatrix} 1 & 4 & 6 \\ 8 & 3 & -2 \\ 5 & 6 & 4 \end{pmatrix}; B = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ 3 & 4 & 1 \end{pmatrix}; F = \begin{pmatrix} -2 & 6 & 9 \\ 1 & 6 & 0 \\ 8 & -5 & 3 \end{pmatrix};$$

then a linear combination with $c_1 = 2$, $c_2 = -4$, and $c_3 = 1$ is

$$\mathbf{2\,A - 4\,B + F}$$

$$\begin{pmatrix} -4 & 6 & 9 \\ 9 & 8 & -20 \\ 6 & -9 & 7 \end{pmatrix}$$

Thus far we have defined multiplication of a matrix by a scalar but not the multiplication of two matrices. Since matrices are added by adding corresponding entries and subtracted by subtracting corresponding entries, it would seem natural to define multiplication of matrices by multiplying corresponding entries. However, it turns out that such a definition would not be very useful for most problems. Experience had led mathematicians to the following more useful definition of matrix multiplication.

Definition 4.16. *Matrix Multiplication*

If A is an $m\times r$ matrix and B is an $r\times m$ matrix, then the product $A\,B = A.B$ is the $m\times n$ matrix whose entries are determined as follows. To find the entry in row i and column j of $A\,B$, single out row i from matrix A and column j from matrix B. Multiply the corresponding entries from the row and column together, and then add up the resulting products. ■

Example 4.50. Matrix Product

Consider the following two matrices

$$A = \begin{pmatrix} 1 & 4 & 6 \\ 2 & 5 & 9 \end{pmatrix}; B = \begin{pmatrix} -1 & 3 & 8 & 4 \\ 0 & 3 & 6 & 1 \\ 1 & 1 & 3 & 2 \end{pmatrix};$$

and calculate the product $A.B$.

Solution 4.50. Since A is a 2×3 matrix and B is a 3×4 matrix, the product $A.B$ is a 2×4 matrix. The product is given by

$$\mathbf{A.B}$$

$$\begin{pmatrix} 5 & 21 & 50 & 20 \\ 7 & 30 & 73 & 31 \end{pmatrix}$$

The detailed calculation is based on the rule given in Definition 4.16. In detail we extract the row and column vectors and apply the dot product for each of these vectors to gain the entry at $(A.B)_{\text{ij}}$. This procedure is give in the following line in a step by step procedure in *Mathematica*.

```
{{TakeRows[A, {1}].TakeColumns[B, {1}],
  TakeRows[A, {1}].TakeColumns[B, {2}], TakeRows[A, {1}].
   TakeColumns[B, {3}], TakeRows[A, {1}].TakeColumns[B, {4}]},
 {TakeRows[A, {2}].TakeColumns[B, {1}], TakeRows[A, {2}].
   TakeColumns[B, {2}], TakeRows[A, {2}].TakeColumns[B, {3}],
  TakeRows[A, {2}].TakeColumns[B, {4}]}}
```

```
{{{{5}}, {{21}}, {{50}}, {{20}}}, {{{7}}, {{30}}, {{73}}, {{31}}}}
```

The result except the brackets is the same as given above.▲

The definition of matrix multiplication requires that the number of columns of the first factor A be the same as the numbers of rows of the second factor B in order to form the product $A.B$. If this condition is not satisfied, the product is undefined. A convenient way to determine whether a product of two matrices is defined is to write down the size of the first factor and, to the right of it, write down the size of the second factor. If the inside numbers are the same then the product is defined.

$$\begin{array}{ccc} A & B & A.B \\ m\times r & r\times n & m\times n \end{array} \tag{4.88}$$

The outside numbers give the size of the product.

In general, if $A = (a_{\text{ij}})$ is a $m\times r$ matrix and $B = (b_{\text{ij}})$ is a $r\times n$ matrix, then, as illustrated by the color in the following equation

$$A.B = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1r} \\ a_{21} & a_{22} & \ldots & a_{2r} \\ \vdots & \vdots & & \vdots \\ a_{\text{i}1} & a_{\text{i}2} & \ldots & a_{\text{ir}} \\ \vdots & \vdots & & \vdots \\ a_{\text{m}1} & a_{\text{m}2} & \ldots & a_{\text{mr}} \end{pmatrix} . \begin{pmatrix} b_{11} & b_{12} & \ldots & b_{1j} & \ldots & b_{1n} \\ b_{21} & b_{22} & \ldots & b_{2j} & \ldots & b_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ b_{\text{r}1} & b_{\text{r}2} & \ldots & b_{\text{rj}} & \ldots & b_{\text{rn}} \end{pmatrix} \tag{4.89}$$

the entry $(A.B)_{\text{ij}}$ in row i and column j of $A.B$ is given by

$$(A.B)_{\text{ij}} = a_{\text{i}1}\, b_{1j} + a_{\text{i}2}\, b_{2j} + a_{\text{i}3}\, b_{3j} + \ldots + a_{\text{ir}}\, b_{\text{rj}} = \sum_{\rho=1}^{r} a_{\text{i}\rho}\, b_{\rho\text{j}} = a_{\text{i}\rho}\, b_{\rho\text{j}} \tag{4.90}$$

where the last expression is known as Einstein's sum convention.

In addition to this standard matrix product there exists another product frequently used in finite element methods and control theory. This product is called Kronecker product. The following definition which also introduces the Kronecker sum and difference fixes the notation and its meaning.

Definition 4.17. *Kronecker Product*

The Kronecker product of an $n\times m$ matrix A and an $r\times s$ matrix B generates an $n\,r\times m\,s$ matrix, as follows

$$A \otimes B = \begin{pmatrix} a_{11} B & a_{12} B & \dots & a_{1m} B \\ a_{21} B & & \dots & a_{2m} B \\ \vdots & & \ddots & \vdots \\ a_{n1} B & & \dots & a_{nm} B \end{pmatrix}$$

In addition to the Kronecker product the Kronecker sum and difference is defined as

$$A \oplus B = A \otimes I_n + I_n \otimes B$$

and

$$A \ominus B = A \otimes I_n - I_n \otimes B$$

where I is an $n \times n$ *unit matrix.*■

The following example shows an application of the Kronecker product. Lets assume we have two matrices given by A and B as

$$\boldsymbol{A = \begin{pmatrix} 2 & -1 \\ -1 & 3 \end{pmatrix}}$$

$$\begin{pmatrix} 2 & -1 \\ -1 & 3 \end{pmatrix}$$

and

$$\boldsymbol{B = \begin{pmatrix} \frac{3}{2} & 1 \\ 1 & \frac{5}{2} \end{pmatrix}}$$

$$\begin{pmatrix} \frac{3}{2} & 1 \\ 1 & \frac{5}{2} \end{pmatrix}$$

Then the Kronecker product is

KroneckerProduct[*A*, *B*]

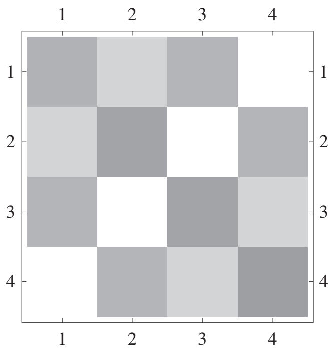
$$\begin{pmatrix} 3 & 2 & -\frac{3}{2} & -1 \\ 2 & 5 & -1 & -\frac{5}{2} \\ -\frac{3}{2} & -1 & \frac{9}{2} & 3 \\ -1 & -\frac{5}{2} & 3 & \frac{15}{2} \end{pmatrix}$$

The Kronecker sum of these matrices follows

kSum = KroneckerProduct[*A*, IdentityMatrix[2]] + KroneckerProduct[IdentityMatrix[2], *B*]

$$\begin{pmatrix} \frac{7}{2} & 1 & -1 & 0 \\ 1 & \frac{9}{2} & 0 & -1 \\ -1 & 0 & \frac{9}{2} & 1 \\ 0 & -1 & 1 & \frac{11}{2} \end{pmatrix}$$

These relations can be checked my simple manual calculations, try it as an exercise. If we look at a graphical representation of the Kronecker sum, we observe that there is some structure inside the matrix



Based on this observation a matrix can be subdivided or partitioned into smaller matrices by inserting horizontal and vertical rules between selected rows and columns. For example, the following are three possible partitions of a general 3×4 matrix A — the first is a partition of A into four submatrices A_{11}, A_{12}, A_{21}, and A_{22}.

$$\boldsymbol{A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{pmatrix};}$$

A_{11} = TakeMatrix(*A*, {1, 1}, {2, 3})

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}$$

A_{12} = TakeMatrix(*A*, {1, 4}, {2, 4})

$$\begin{pmatrix} a_{14} \\ a_{24} \end{pmatrix}$$

A_{21} = TakeMatrix(*A*, {3, 1}, {3, 3})

$$\begin{pmatrix} a_{31} & a_{32} & a_{33} \end{pmatrix}$$

A_{22} = TakeMatrix(A, {3, 4}, {3, 4})

(a_{34})

The second is a partition of A into its row matrices $\vec{r}_1$, $\vec{r}_2$, and $\vec{r}_3$

r_1 = TakeRows(A, {1})

$(a_{11} \quad a_{12} \quad a_{13} \quad a_{14})$

r_2 = TakeRows(A, {2})

$(a_{21} \quad a_{22} \quad a_{23} \quad a_{24})$

r_3 = TakeRows(A, {3})

$(a_{31} \quad a_{32} \quad a_{33} \quad a_{34})$

The third kind of a partition is to take the column vectors $\vec{c}_1$, $\vec{c}_2$, $\vec{c}_3$ and $\vec{c}_4$ as submatrices

c_1 = TakeColumns(A, {1})

$$\begin{pmatrix} a_{11} \\ a_{21} \\ a_{31} \end{pmatrix}$$

c_2 = TakeColumns(A, {2})

$$\begin{pmatrix} a_{12} \\ a_{22} \\ a_{32} \end{pmatrix}$$

c_3 = TakeColumns(A, {3})

$$\begin{pmatrix} a_{13} \\ a_{23} \\ a_{33} \end{pmatrix}$$

c_4 = TakeColumns(A, {4})

$$\begin{pmatrix} a_{14} \\ a_{24} \\ a_{34} \end{pmatrix}$$

4.4.2.1 Functions defined by Matrices

The equation $A.\vec{x} = \vec{b}$ with A and $\vec{b}$ given defines a linear system of equations to be solved for $\vec{x}$. But we could also write this equation as $\vec{y} = A.\vec{x}$, where A and $\vec{x}$ are given. In this case, we want to compute $\vec{y}$. If A is a $m \times n$ matrix, then this is a function that associates with every $n \times 1$ column vector $\vec{x}$ an $m \times 1$ column vector $\vec{y}$, and we may view A as defining a rule that shows how a given $\vec{x}$ is mapped into a corresponding $\vec{y}$. This idea is discussed in more detail in a later section where we discuss linear transforms. An example may demonstrate what we mean by a matrix function.

Example 4.51. Matrix Function

Consider the following matrices

$$\boldsymbol{A} = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{-1} \end{pmatrix};$$

and a point in space $\vec{x}$ defined by its coordinates

$$\boldsymbol{x} = \begin{pmatrix} \boldsymbol{a} \\ \boldsymbol{b} \end{pmatrix};$$

What is the result of the product $A.\vec{x}$? In addition let B be another matrix given by

$$\boldsymbol{B} = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{-1} & \mathbf{0} \end{pmatrix};$$

what is the action of this matrix if a vector $\vec{x}$ is multiplied from the right?

Solution 4.51. The product $\vec{y} = A.\vec{x}$ is

$$\boldsymbol{y = A.x}$$

$$\begin{pmatrix} a \\ -b \end{pmatrix}$$

so the effect of multiplying A by a column vector is to change the sign of the second entry of the column vector which means a reflection on the x-axis.

For the second matrix B we find

$$\boldsymbol{y = B.x}$$

$$\begin{pmatrix} b \\ -a \end{pmatrix}$$

So the effect of multiplying B by a column vector is to interchange the first and second entries of the column vector, also changing the sign of the first entry.

If we assume that the column vector $\vec{x}$ is locating a point (a, b) in the plane, then the effect of A is to reflect the point about the x-axis whereas the effect of B is to rotate the line segment from the origin to the point through a right angle; 90° rotation.

To use these ideas in a parametric representation of a function as $\vec{f}(t) = (x(t),\ y(t))$ we can use these matrices to generate different representations of these functions. For example assume that $\vec{f} = (t,\ t^2)$ then we find with

$$\boldsymbol{f = \{t, t^2\}}$$

$$\{t, t^2\}$$

the graphical representation as

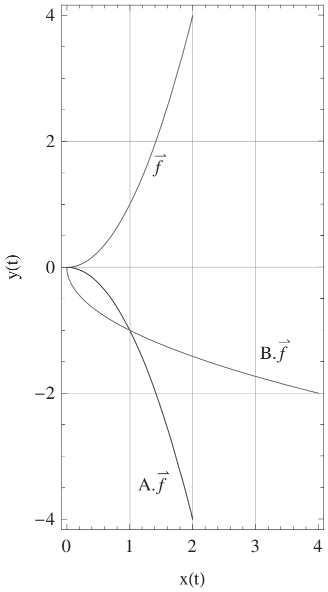


Figure 4.33. Vector space structure of functions as subspaces.

This examples shows that matrix operations are efficient in transformations.▲

4.4.2.2 Transposition and Trace of a Matrix

We conclude this section by defining two matrix operations that have no analogues in the calculations with real numbers. The first property is the so called transposition of a matrix. Since a matrix is a two dimensional array of numbers we can write it down in a row column based way or in a column row based way. This means that there are in principle two different representations of a matrix. This two ways of writing down a matrix is summarized in the following definition.

Definition 4.18. *Transposition of a Matrix*

If A is any $m \times n$ matrix, then the transposition of A, denoted by A^T, is defined to be the $n \times m$ matrix that results from interchanging the rows and columns of A; that is, the first column of A^T is the first row of A, the second column of A^T is the second row of A, and so forth.■

Example 4.52. Transposed Matrices

The following matrices serve as examples to show the transposition of matrices

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{pmatrix};$$

Solution 4.52. The transpose of A is generated by interchanging the rows and the columns

$$A^T$$

$$\begin{pmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \\ a_{14} & a_{24} & a_{34} \end{pmatrix}$$

▲

A matrix B given by

$$B = \begin{pmatrix} 1 & 4 \\ 3 & 5 \end{pmatrix};$$

has the transpose representation

$$B^T$$

$$\begin{pmatrix} 1 & 3 \\ 4 & 5 \end{pmatrix}$$

A column vector transposed

$$F = \begin{pmatrix} 1 \\ a \end{pmatrix};$$

results to a row vector

$$F^T$$

$$\begin{pmatrix} 1 & a \end{pmatrix}$$

Observe that not only are the columns of A^T the rows of A, but the rows of A^T are the columns of A. Thus the entry in row i and column j of A^T is the entry in row j and column i of A; that is

$$\left(A^T\right)_{\mathrm{ij}} = (A)_{\mathrm{ji}} \tag{4.91}$$

Note the reversal of the subscripts.

In the special case where A is a square matrix; that is a $n\times n$ matrix, the transpose of A can be obtained by interchanging entries that are symmetrically positioned about the main diagonal. This is shown in the following 4×4 matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix};$$

$\boldsymbol{A}^{\mathrm{T}}$

$$\begin{pmatrix} a_{11} & a_{21} & a_{31} & a_{41} \\ a_{12} & a_{22} & a_{32} & a_{42} \\ a_{13} & a_{23} & a_{33} & a_{43} \\ a_{14} & a_{24} & a_{34} & a_{44} \end{pmatrix}$$

The trace of a matrix is the second property which is not known in arithmetic of real numbers. The trace is a property which is an invariant of a matrix under similarity transformation. The trace of a square matrix is defined in the following definition.

Definition 4.19. *Trace of a Matrix*

If A is a square matrix, then the trace of A, denoted tr(A), is defined to be the sum of the entries on the main diagonal of A. The trace of A is undefined if A is not a square matrix. ■

Example 4.53. Trace of a Matrix

The following two matrices are examples for square matrices. Find their traces.

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}; \; \boldsymbol{B} = \begin{pmatrix} 1 & 4 \\ 3 & 5 \end{pmatrix};$$

Solution 4.53. The trace of A is

$$\operatorname{tr}(A) = \sum_{i=1}^{4} a_{\mathrm{ii}} = a_{11} + a_{22} + a_{33} + a_{44}$$

Tr[*A*]

$a_{11} + a_{22} + a_{33} + a_{44}$

The numerical example is given by matrix B

$$\boldsymbol{B} = \begin{pmatrix} 1 & 4 \\ 3 & 5 \end{pmatrix};$$

Tr[*B*]

6

here $\operatorname{tr}(B) = 1 + 5 = 6$.▲

4.4.3 Matrix Arithmetic

In this section we will discuss some properties of the arithmetic operations on matrices. We shall see that many of the basic rules of arithmetic for real numbers also hold for matrices, but few do not.

4.4.3.1 Operations for Matrices

For real numbers a and b, we always have $a\,b = b\,a$, which is called the commutativity law for multiplication. For matrices, however, $A.B$ and $B.A$ need not be equal. Equality can fail to hold for three reasons: It can happen that the product $A.B$ is defined but $B.A$ is undefined. For example , this is the case if A is a 2×3 matrix and B is a 3×4 matrix. Also, it can happen that $A.B$ and $B.A$ are both defined but have different sizes. This is the situation if A is a 2×3 matrix and B is a 3×2 matrix. Finally, it is possible to have $A.B \neq B.A$ even if both $A.B$ and $B.A$ are defined and have the same size.

Example 4.54. $A.B \neq B.A$

Consider the two square matrices

$$\mathbf{A = \begin{pmatrix} -1 & 0 \\ 2 & 3 \end{pmatrix}; B = \begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix};}$$

Find the products $A.B$ and $B.A$ and compare both with each other.

Solution 4.54. Multiplication of $A.B$ gives

$$\mathbf{AB = A.B}$$

$$\begin{pmatrix} -1 & -2 \\ 11 & 4 \end{pmatrix}$$

and

$$\mathbf{BA = B.A}$$

$$\begin{pmatrix} 3 & 6 \\ -3 & 0 \end{pmatrix}$$

The observation is that $A.B \neq B.A$.▲

Although the commutative law for multiplication is not valid in matrix arithmetic, many familiar laws of arithmetic are valid for matrices. Some of the most important ones and their names are summarized in the following theorem.

Theorem 4.18. *Arithmetic Laws for Matrices*

Assuming that the sizes of the matrices are such that the indicated operations can be performed, the following rules of matrix arithmetic are valid.

1. $A + B = B + A$ *(Commutative law for addition)*
2. $A + (B + C) = (A + B) + C$ *(Associative law for addition)*
3. $A.(B.C) = (A.B).C$ *(Associative law for multiplication)*
4. $A.(B + C) = A.B + A.C$ *(Left distribution law)*
5. $(B + C).A = B.A + C.A$ *(Right distribution law)*
6. $A.(B - C) = A.B - A.C$
7. $(B - C).A = B.A - C.A$

8. $a\,(B+C) = a\,B + a\,C$
9. $a\,(B-C) = a\,B - a\,C$
10. $(a+b)\,C = a\,C + b\,C$
11. $(a-b)\,C = a\,C - b\,C$
12. $a\,(b\,C) = (a\,b)\,C$
13. $a\,(B.C) = (a\,B).C = B.(a\,C)$.■

To prove the equalities in this theorem, we must show that the matrix of the left side has the same size as the matrix on the right side and that corresponding entries on the two sides are equal. With the exception of the associative law in part 3), the proofs all follow the same pattern. We shall prove part 4) as an illustration.

Proof 4.18. 4) We must show that $A.(B+C)$ and $A.B+A.C$ have the same size and that corresponding entries are equal. To form $A.(B+C)$, the matrices B and C must have the same size, say $m\times n$, and matrix A must have m columns, so its size must be of the form $r\times m$. This makes $A.(B+C)$ an $r\times n$ matrix. It follows that $A.B+A.C$ is also an $r\times n$ matrix and, consequently $A.(B+C)$ and $A.B+A.C$ have the same size.

Suppose that $A=(a_{\text{ij}})$, $B=(b_{\text{ij}})$, and $C=(c_{\text{ij}})$. We want to show that the corresponding entries of $A.(B+C)$ and $A.B+A.C$ are equal; that is

$$(A.(B+C))_{\text{ij}} = (A.B+A.C)_{\text{ij}} \tag{4.92}$$

for all values of i and j. But from the definition of matrix addition and matrix multiplication, we have

$$(A.(B+C))_{\text{ij}} = \sum_{\rho=1}^{m} a_{\text{i}\rho}\left(b_{\rho\text{j}}+c_{\rho\text{j}}\right) = \sum_{\rho=1}^{m} a_{\text{i}\rho}\,b_{\rho\text{j}} + \sum_{\rho=1}^{m} a_{\text{i}\rho}\,c_{\rho\text{j}} = (A.B)_{\text{ij}} + (A.C)_{\text{ij}}. \tag{4.93}$$

QED

Remark 4.15. Although the operations of matrix addition and matrix multiplication were defined for pairs of matrices, associative laws 2) and 3) enable us to denote sums and products of three matrices as $A+B+C$ and $A.B.C$ without inserting any parenthesis. This is justified by the fact that no matter how parentheses are inserted, the associative laws guarantee that the same end result will be obtained. In general, given any sum or any product of matrices, pairs of parentheses can be inserted or deleted anywhere within the expression without affecting the end result.

Example 4.55. Associativity of Multiplication

To show the validity of associativity in a multiplication consider the three matrices.

$$\boldsymbol{A} = \begin{pmatrix} \mathbf{1} & \mathbf{2} \\ \mathbf{4} & \mathbf{1} \\ \mathbf{7} & \mathbf{8} \end{pmatrix}; \boldsymbol{B} = \begin{pmatrix} \mathbf{5} & \mathbf{2} \\ \mathbf{1} & \mathbf{6} \end{pmatrix}; \boldsymbol{F} = \begin{pmatrix} \mathbf{4} & \mathbf{8} \\ \mathbf{7} & \mathbf{9} \end{pmatrix};$$

Solution 4.55. To show the validity for the product we calculate $A.B$ and $B.F$ first

AB = *A*.*B*

$$\begin{pmatrix} 7 & 14 \\ 21 & 14 \\ 43 & 62 \end{pmatrix}$$

and

BF = *B*.*F*

$$\begin{pmatrix} 34 & 58 \\ 46 & 62 \end{pmatrix}$$

Thus the products $(A.B).F$ should be equal to $A.(B.F)$

AB.*F*

$$\begin{pmatrix} 126 & 182 \\ 182 & 294 \\ 606 & 902 \end{pmatrix}$$

***A*.BF**

$$\begin{pmatrix} 126 & 182 \\ 182 & 294 \\ 606 & 902 \end{pmatrix}$$

Comparing the results demonstrates the validity of statement 3) in Theorem 4.18.▲

4.4.3.2 Zero Matrices

A matrix, all of whose entries are zero, such as

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 \end{pmatrix} \tag{4.94}$$

is called a zero matrix. A zero matrix will be denoted by 0; if it is important to emphasize the size, we will write $0_{n\times m}$ for an $n\times m$ zero matrix.

If A is any matrix and 0 is the zero matrix with the same size, it is obvious that $A+0=0+A=A$. The matrix 0 plays much the same role in these matrix equations as the 0 play in the numerical equations $a+0=0+a=a$.

Since we already know that some of the rules of arithmetic for real numbers do not carry over to matrix arithmetic, it would be foolhardy to assume that all the properties of the real number zero carry over to zero matrices. For example, consider the following two standard results in the arithmetic of real numbers.

- If $a\,b=a\,c$ and $a\neq 0$, then $b=c$. (This is called the cancellation law.)
- If $a\,d=0$, then at least one of the factors on the left is 0.

As the next example shows, the corresponding results are not generally true in matrix arithmetic.

Example 4.56. Invalid Cancellation Law

Given the four matrices

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix}; B = \begin{pmatrix} 1 & 1 \\ 3 & 4 \end{pmatrix}; F = \begin{pmatrix} 2 & 5 \\ 3 & 4 \end{pmatrix}; G = \begin{pmatrix} 3 & 7 \\ 0 & 0 \end{pmatrix};$$

Demonstrate the cancellation law for scalar arithmetic is not valid for matrices! Show also that the product of $A.G$ vanishes but do not draw the wrong conclusion from that.

Solution 4.56. First let us determine $A.B$ and $A.F$

AB = *A.B*

$$\begin{pmatrix} 3 & 4 \\ 6 & 8 \end{pmatrix}$$

AF = *A.F*

$$\begin{pmatrix} 3 & 4 \\ 6 & 8 \end{pmatrix}$$

Although $A.B = A.F$ we cannot cancel out A since $B \neq F$.

For the product of $A.G$ we find

AG = *A.G*

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

This result does not mean that one of the matrices vanishes in fact both matrices are unequal to the zero matrix.▲

In spite of the above example, there are a number of familiar properties of the real number 0 that do apply to the zero matrices. Some of the more important ones are summarized in the next theorem. The proofs are left as exercises.

Theorem 4.19. *Zero Matrices*

Assuming that the size of the matrices are such that the indicated operations can be performed, the following rules of matrix arithmetic are valid.

1. $A + 0 = 0 + A = A$

2. $A - A = 0$

3. $0 - A = -A$

4. $A.0 = 0$ *and* $0.A = 0$.■

4.4.3.3 Identity Matrices

Of special interest are square matrices with 1's on the main diagonal and 0's off the main diagonal, such as

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \text{and so on.} \tag{4.95}$$

A matrix of this form is called an identity matrix and is denoted by I. If it is important to emphasize the size, we shall write I_n for an $n \times n$ identity matrix.

If A is an $m \times n$ matrix, then as illustrated in the next example

$$A.I_n = A \quad \text{and} \quad I_n.A = A. \tag{4.96}$$

Thus, an identity matrix plays much the same role in matrix arithmetic that the number 1 plays in the numerical relationship $a\,1 = 1\,a = a$.

Example 4.57. Identity Matrix

Consider the matrix

$$\mathbf{A} = \begin{pmatrix} \boldsymbol{a_{11}} & \boldsymbol{a_{12}} & \boldsymbol{a_{13}} \\ \boldsymbol{a_{21}} & \boldsymbol{a_{22}} & \boldsymbol{a_{23}} \end{pmatrix};$$

Solution 4.57. We start with I_2 and examine the products with appropriate identity matrices.

$$\begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{pmatrix} . \boldsymbol{A}$$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}$$

and check the next product with I_3

$$\boldsymbol{A}. \begin{pmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \end{pmatrix}$$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}$$

Thus the relation $I_2.A = A = A.I_3$ holds.▲

As the next theorem shows, identity matrices arise naturally in studying reduced row-echelon forms of square matrices.

Theorem 4.20. *Identity Matrix and Reduced Rowechelon Form*

If R is the reduced row-echelon form of a $n \times n$ matrix A, then either R has a row of zeros or R is the *identity matrix I_n.* ■

Proof 4.20. Suppose the row echelon form of A is

$$R = \begin{pmatrix} r_{11} & r_{12} & \cdots & r_{1n} \\ r_{21} & r_{22} & \cdots & r_{2n} \\ \vdots & \vdots & & \vdots \\ r_{n1} & r_{n2} & \cdots & r_{nn} \end{pmatrix}$$

Either the last row in this matrix consists entirely of zeros or it does not. If not, the matrix contains no zero rows, and consequently each of the n rows has a leading entry of 1. Since these leading 1's must occur progressively farther to the right as we move down the matrix, each of these 1's must occur on the main diagonal. Since the other entries in the same column as one of these 1's are zero, R must be I_n. Thus either R has a row of zeros or $R = I_n$.

QED

Definition 4.20. *Invertible Matrix*

If A is a square matrix, and if a matrix B of the same size can be found such that A.B = B.A = I, then A is said to be invertible and B is called an inverse of A. If no such matrix B can be found, Then A is said to be singular. ■

Example 4.58. Requirements for Inverse Matrices

The matrices A and B are given as follows

$$A = \begin{pmatrix} 2 & -5 \\ -1 & 3 \end{pmatrix}; \; B = \begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix};$$

Test if B is the inverse of A.

Solution 4.58. Due to the definition given we have to check the products $A.B$ and $B.A$ and verify that $A.B = B.A = I$.

AB = *A.B*

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

BA = *B.A*

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

In both cases the identity matrix is derived and thus B is the inverse of A.▲

If a matrix does not have an inverse then we call this matrix singular.

Example 4.59. Singular Matrix

Consider the matrices

$$A = \begin{pmatrix} 1 & 4 & 0 \\ 2 & 5 & 0 \\ 3 & 6 & 0 \end{pmatrix}; B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix};$$

and decide if A is singular.

Solution 4.59. To show that A is singular consider the products $A.B$ and $B.A$

A.B

$$\begin{pmatrix} b_{11}+4\,b_{21} & b_{12}+4\,b_{22} & b_{13}+4\,b_{23} \\ 2\,b_{11}+5\,b_{21} & 2\,b_{12}+5\,b_{22} & 2\,b_{13}+5\,b_{23} \\ 3\,b_{11}+6\,b_{21} & 3\,b_{12}+6\,b_{22} & 3\,b_{13}+6\,b_{23} \end{pmatrix}$$

B.A

$$\begin{pmatrix} b_{11}+2\,b_{12}+3\,b_{13} & 4\,b_{11}+5\,b_{12}+6\,b_{13} & 0 \\ b_{21}+2\,b_{22}+3\,b_{23} & 4\,b_{21}+5\,b_{22}+6\,b_{23} & 0 \\ b_{31}+2\,b_{32}+3\,b_{33} & 4\,b_{31}+5\,b_{32}+6\,b_{33} & 0 \end{pmatrix}$$

The product contains as last column 0's. This is an indication that A does not have an inverse and thus is singular.▲

4.4.3.4 Properties of the Inverse

It is reasonable to ask whether an invertible matrix can have more than one inverse. The next theorem shows that the answer is no—an invertible matrix has exactly one inverse.

Theorem 4.21. *Uniqueness of the Inverse*

If B and C are both inverses of the matrix A, then B = C.■

Proof 4.21. Since B is an inverse of A, we have $B.A = I$. Multiplying both sides on the right by C gives $(B.A).C = 1.\,C = C$. But

$$(B.A).C = B.(A.C) = B.I = B,$$

so

$$C = B.$$

QED

As a consequence of this important result, we can now speak of the inverse of an invertible matrix. If A is invertible, then its inverse will be denoted by the symbol A^{-1}. Thus

$$A.A^{-1} = I \text{ and } A^{-1}.A = I. \tag{4.97}$$

The inverse of A plays much the same role in matrix arithmetic that the reciprocal a^{-1} plays in the numerical relationships $a\,a^{-1} = 1$ and $a^{-1}\,a = 1$.

In the next section we will develop a method for finding inverses of invertible matrices of any size; however, the following theorem gives conditions under which a 2×2 matrix is invertible and provides a simple formula for the inverse.

Theorem 4.22. *Inverse of a 2×2 Matrix*

The matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is invertible if $a\,d - b\,c \neq 0$, in which case the inverse is given by the formula

$$A^{-1} = \frac{1}{a\,d - b\,c}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} \frac{d}{a\,d-b\,c} & -\frac{b}{a\,d-b\,c} \\ -\frac{c}{a\,d-b\,c} & \frac{a}{a\,d-b\,c} \end{pmatrix}. \blacksquare$$

Proof 4.22. We have to show that $A^{-1}.A = I$ and $A.A^{-1} = I$

$$\boldsymbol{A = \begin{pmatrix} a & b \\ c & d \end{pmatrix};\ \mathrm{A1} = \begin{pmatrix} \frac{d}{a\,d-b\,c} & -\frac{b}{a\,d-b\,c} \\ -\frac{c}{a\,d-b\,c} & \frac{a}{a\,d-b\,c} \end{pmatrix};}$$

Simplify[*A*.A1]

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Simplify[A1.*A*]

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

which shows the validity of the formula.

QED

Theorem 4.23. *Product of Invertible*

If A and B are invertible matrices of the same size, then A.B is invertible and

$$(A.B)^{-1} = B^{-1}.A^{-1}. \blacksquare$$

Proof 4.23. If we can show that $(A.B).\left(B^{-1}\,A^{-1}\right) = \left(B^{-1}\,A^{-1}\right).(A.B) = I$, then we will have simultaneously shown that the matrix $A.B$ is invertible and that

$$(A.B)^{-1} = B^{-1}.A^{-1}.$$

But

$$(A.B).\left(B^{-1}\,A^{-1}\right) = A.\left(B.B^{-1}\right).A^{-1} = A.I.A^{-1} = A.A^{-1} = I.$$

A similar argument shows that

$$\left(B^{-1}.A^{-1}\right).(A.B) = I.$$

QED

Although we will not prove it, this result can be extended to include three or more factors; that is

A product of any number of invertible matrices is invertible, and the inverse of the product is the product of the inverse in the reverse order.

Example 4.60. Inverse of a Product

Consider the matrices

$$\boldsymbol{A} = \begin{pmatrix} \mathbf{1} & \mathbf{2} \\ \mathbf{1} & \mathbf{3} \end{pmatrix}; \boldsymbol{B} = \begin{pmatrix} \mathbf{3} & \mathbf{2} \\ \mathbf{2} & \mathbf{2} \end{pmatrix}; \mathbf{AB} = \boldsymbol{A}.\boldsymbol{B}$$

$$\begin{pmatrix} 7 & 6 \\ 9 & 8 \end{pmatrix}$$

Solution 4.60. Applying the formula from Theorem 4.22, we obtain

$$\mathbf{A1} = \begin{pmatrix} \mathbf{3} & \mathbf{-2} \\ \mathbf{-1} & \mathbf{1} \end{pmatrix}; \mathbf{B1} = \begin{pmatrix} \mathbf{1} & \mathbf{-1} \\ \mathbf{-1} & \frac{\mathbf{3}}{\mathbf{2}} \end{pmatrix}; \mathbf{AB1} = \begin{pmatrix} \mathbf{4} & \mathbf{-3} \\ -\frac{\mathbf{9}}{\mathbf{2}} & \frac{\mathbf{7}}{\mathbf{2}} \end{pmatrix};$$

Also,

B1.A1

$$\begin{pmatrix} 4 & -3 \\ -\frac{9}{2} & \frac{7}{2} \end{pmatrix}$$

Therefore $(A.B)^{-1} = B^{-1}.A^{-1}$ as guaranteed by Theorem 4.23.▲

4.4.3.5 Powers of a Matrix

Next we shall define powers of a square matrix and discuss their properties.

Definition 4.21. *Powers of a Matrix*

If A is a square matrix, then we define the non negative integer power of A to be

$$A^0 = I$$

$$A^n = A.A.A\,....A$$

as a product consisting of n factors with n > 0.

Moreover, if A is invertible, then we define the negative integer powers to be

$$A^{-n} = \left(A^{-1}\right)^n = A^{-1}.A^{-1}.....A^{-1}$$

as a product of n factors of A^{-1}. ■

Because this definition parallels that for real numbers, the usual laws of exponentiation hold.

Theorem 4.24. *Laws of Exponents*

If A is a square matrix and r and s are integers, then

$$A^r.A^s = A^{r+s}$$

and

$$(A^r)^s = A^{r\,s}. \blacksquare$$

The next theorem provides some useful properties of negative exponents

Theorem 4.25. *Laws of Negative Exponents*

If A is an invertible matrix, then:

1. A^{-1} *is invertible and* $\left(A^{-1}\right)^{-1} = A$
2. A^n *is invertible and* $(A^n)^{-1} = \left(A^{-1}\right)^n$ *for* $n = 0, 1, 2, 3, \ldots$
3. *For any nonzero scalar k, the matrix k A is invertible and* $(k\,A)^{-1} = \frac{1}{k} A^{-1}$. ■

Proof 4.25. a) Since $A.A^{-1} = A^{-1}.A = I$, the matrix A^{-1} is invertible and $\left(A^{-1}\right)^{-1} = A$.

The proof of part b) is left as an exercise.

c) If k is any nonzero scalar, then we can write

$$(k\,A).\left(\frac{1}{k}\,A^{-1}\right) = \frac{1}{k}\,(k\,A).A^{-1} = \frac{1}{k}\,k\,A.A^{-1} = 1\,I = I.$$

Similarly,

$$\left(\frac{1}{k}\,A^{-1}\right).(k\,A) = I$$

so that $k\,A$ is invertible and

$$(k\,A)^{-1} = \frac{1}{k}\,A^{-1}.$$

QED

Example 4.61. Powers of a Matrix

Let A and A^{-1} be as in Example 4.60; that is

$$\mathbf{A} = \begin{pmatrix} \mathbf{1} & \mathbf{2} \\ \mathbf{1} & \mathbf{3} \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}$$

and

$$\mathbf{A1} = \begin{pmatrix} \mathbf{3} & \mathbf{-2} \\ \mathbf{-1} & \mathbf{1} \end{pmatrix}$$

$$\begin{pmatrix} 3 & -2 \\ -1 & 1 \end{pmatrix}$$

Then

A3 = *A.A.A*

$$\begin{pmatrix} 11 & 30 \\ 15 & 41 \end{pmatrix}$$

Am3 = A1.A1.A1

$$\begin{pmatrix} 41 & -30 \\ -15 & 11 \end{pmatrix}$$

Here some caution is needed if *Mathematica* is used unprejudiced. The direct exponentiation will give faulty results because the arithmetic operation in *Mathematica* is applied directly to all list elements.▲

4.4.3.6 Matrix Polynomials

If A is a square matrix, say a $m \times m$, and if

$$p(x) = a_0 + a_1\, x + \ldots + a_n\, x^n \tag{4.98}$$

is a polynomial, then we define

$$p(A) = a_0\, I\; + a_1\, A + a_2\, A^2 + \ldots + a_n\, A^n \tag{4.99}$$

where I_m is the identity matrix. In words, $p(A)$ is the $m \times m$ matrix that results when A is substitute for x in the polynomial and a_0 is replaced by $a_0\, I$.

Example 4.62. Matrix Polynomial

Create the second order matrix polynomial by using

$p(x) = 2\, x^2 - 3\, x + 4$ with the matrix

$$A = \begin{pmatrix} -1 & 2 \\ 0 & 3 \end{pmatrix};$$

Solution 4.62. The polynomial is generated by

$$\mathbf{pol} = -3\,A + 2\,A.A + 4\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 9 & 2 \\ 0 & 13 \end{pmatrix}$$

representing a 2×2 matrix.▲

4.4.3.7 Properties of the Transpose

The next theorem lists the main properties of the transpose operation.

Theorem 4.26. *Properties of the Transpose*

If the size of the matrices are such that the standard operations can be performed, then

1. $((A)^T)^T = A$
2. $(A+B)^T = A^T + B^T$ *and* $(A-B)^T = A^T - B^T$
3. $(k\,A)^T = k\,A^T$, *where k is any real scalar*
4. $(A.B)^T = B^T.A^T$.■

If we keep in mind that transposing a matrix interchanges its rows and columns, parts 1), 2), and 3) should be self-evident. For example, part 1) states that interchanging rows and columns twice leaves a matrix unchanged; part 2) asserts that adding and then interchanging rows and columns yields the same result as first interchanging rows and columns and then adding, and part 3) asserts that multiplying by a scalar and then interchanging rows and columns yields the same result as first interchanging rows and columns and then multiplying by the scalar. Part 4) is left as an exercise.

Although we shall not prove it, part 4) of this theorem can be extended to include three or more factors; that is: The transpose of a product of any number of matrices is equal to the product of their transposes in the reverse order.

Remark 4.16. Note the similarity between this result and the result and the result about the inverse of a product of matrices.

4.4.3.8 Invertibility of the Transpose

The following theorem establishes a relationship between the inverse of an invertible matrix and the inverse of its transpose.

Theorem 4.27. *Inverse of the Transpose*

If A is an invertible matrix, then A^T *is also invertible and*

$(A^T)^{-1} = (A^{-1})^T$.■

Proof 4.27. We can prove the invertibility of A^T and obtain the relation in the theorem by showing that

$$A^T.(A^{-1})^T = (A^{-1})^T.A^T = I.$$

But from part 4) of Theorem 4.26 and the fact that $I^T = I$, we have

$$A^T.(A^{-1})^T = (A^{-1}.A)^T = I^T = I$$

$$(A^{-1})^T.A^T = (A.A^{-1})^T = I^T = I \tag{4.100}$$

which completes the proof.

QED

Example 4.63. Inverse of the Transpose

Given a matrix A show the property of the theorem above.

$$\mathbf{A = \begin{pmatrix} -5 & -3 \\ 2 & 1 \end{pmatrix};}$$

Solution 4.63. The transpose and the inverse are determined by the formulas and the application of the theorems

$$\mathbf{AT} = A^{\mathrm{T}}$$

$$\begin{pmatrix} -5 & 2 \\ -3 & 1 \end{pmatrix}$$

The inverse of A is

$$\mathbf{AI} = A^{-1}$$

$$\begin{pmatrix} 1 & 3 \\ -2 & -5 \end{pmatrix}$$

The inverse of the transposed matrix is

$$\mathbf{ATI = AT^{-1}}$$

$$\begin{pmatrix} 1 & -2 \\ 3 & -5 \end{pmatrix}$$

An the transpose of this matrix is

ATI^{T}

$$\begin{pmatrix} 1 & 3 \\ -2 & -5 \end{pmatrix}$$

which agrees with the inverse of the matrix A.

AT^{-1}

$$\begin{pmatrix} 1 & -2 \\ 3 & -5 \end{pmatrix}$$

▲

4.4.4 Calculating the Inverse

In this section we will develop an algorithm for finding the inverse of an invertible matrix. We will also discuss some of the basic properties of invertible matrices.

We begin with the definition of a special type of matrix than can be used to array out an elementary row operation by matrix multiplication.

Definition 4.22. *Elementary Matrix*

An $n \times n$ matrix is called an elementary matrix if it can be obtained from the $n \times n$ identity matrix I_n by performing a single elementary row operation.■

Examples of this kind of matrices are listed below

$$\begin{pmatrix} 1 & 0 \\ 0 & -3 \end{pmatrix}$$

an elementary matrix results if we multiply the second row of I_2 by -3.

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

an elementary matrix results by interchanging the fourth with the second row.

$$\begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

an elementary matrix is generated if we multiply the third row of I_3 by 3 and add it to the first row.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

the identity matrix is already an elementary matrix.

When a matrix A is multiplied on the left by an elementary matrix E, the effect is to perform an elementary row operation on A. This is the content of the following theorem, the proof of which is left for an exercise.

Theorem 4.28. *Elementary Matrix*

If the elementary matrix E results from performing a certain row operation on I_m and if A is an $m \times m$ matrix, then the product E.A is the matrix that results when this same row operation is performed on A.■

Example 4.64. Elementary Matrices

Given the matrix A and the elementary matrix E

$$A = \begin{pmatrix} 1 & 0 & 2 & 3 \\ 2 & -1 & 3 & 6 \\ 1 & 4 & 4 & 0 \end{pmatrix}; \text{El} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix};$$

The elementary matrix results from adding three times the first row of I_3 to the third row. What is the result of the product?

Solution 4.64. The product of $E.A$ is

El.*A*

$$\begin{pmatrix} 1 & 0 & 2 & 3 \\ 2 & -1 & 3 & 6 \\ 4 & 4 & 10 & 9 \end{pmatrix}$$

which is precisely the same matrix if we add 3 times the first row of A to the third row.▲

Remark 4.17. Theorem 4.28 is primarily of theoretical interest and will be used for developing some results about matrices and systems of linear equations. Computationally, it is preferable to perform now operations directly rather than multiplying on the left by an elementary matrix.

If an elementary operation is applied to an identity matrix I to produce an elementary matrix E, then there is a second row operation that, when applied to E, produces I back again. For example, if E is multiplied by a nonzero constant c, then I can be reversed if the ith row of E is multiplied by $1/c$. The various possibilities are listed in Table 4.1. The operations on the right side of this table are called the inverse operations of the corresponding operations on the left.

Table 4.1. Row operations to generate an elementary matrix and its inverse operations.

Row operation on I that produces E	*Row operation on E that reproduces I*
Multiply row i by $c \neq 0$	Multiply row i by 1/c
Interchange rows i and j	Interchange rows i and j
Add c times row i to row j	Add - c times row i to row j

Elementary matrix

The next theorem gives an important property of elementary matrices.

Theorem 4.29. *Property of Elementary Matrices*

Every elementary matrix is invertible, and the inverse is also an elementary matrix. ■

Proof 4.29. If E is an elementary matrix, then E results from performing some row operation on I. Let E_0 be the matrix that results when the inverse of this operation is performed on I. Applying Theorem 4.28 and using the fact that inverse row operations cancel the effect of each other, it follows that

$$E_0.E = I \qquad \text{and} \qquad E.E_0 = I.$$

Thus, the elementary matrix E_0 is the inverse of E.

QED

The next theorem establishes some fundamental relationships among invertibility, homogeneous linear systems, reduced row-echelon forms, and elementary matrices. These results are extremely important and will be used many times in later sections.

Theorem 4.30. *Equivalent Statements*

If A is an $n \times n$ matrix, then the following statements are equivalent, that is, all true or all false.

1. *A is invertible.*

2. *$A.\vec{x} = 0$ has only the trivial solution.*

3. *The reduced row-echelon form of A is I_n.*

4. *A is expressible as a product of elementary matrices.* ■

Proof 4.29. We shall prove the equivalence by establishing the chain of implications: 1) ⇒ 2) ⇒ 3) ⇒ 4) ⇒ 1).

1) ⇒ 2): Assume A is invertible and let $\vec{x}_0$ be a solution of $A.\vec{x} = 0$; thus $A.\vec{x}_0 = 0$. Multiplying both sides of this equation by the matrix A^{-1} gives $A^{-1}.(A.\vec{x}_0) = A^{-1}.0$, or $\left(A^{-1}.A\right).\vec{x}_0 = 0$ or $I.\vec{x}_0 = 0$. Thus, $A.\vec{x} = 0$ has only the trivial solution.

2) ⇒ 3): Let $A.\vec{x} = 0$ be the matrix form of the system

$$\begin{aligned}
a_{11}\,x_1 + a_{12}\,x_2 + \ldots + a_{1\,n}\,x_n &= 0 \\
a_{21}\,x_1 + a_{22}\,x_2 + \ldots + a_{2\,n}\,x_n &= 0 \\
&\vdots \\
a_{\text{n}1}\,x_1 + a_{\text{n}2}\,x_2 + \ldots + a_{\text{nn}}\,x_n &= 0
\end{aligned}$$

and assume that the system has only the trivial solution. If we solve by Gauss-Jordan elimination, then the system of equations corresponding to the reduced row-echelon form of the augmented matrix will be

$$\begin{aligned}
x_1 &= 0 \\
x_2 &= 0 \\
&\vdots \\
x_n &= 0
\end{aligned}$$

The augmented matrix

$$\begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1\,n} & 0 \\ a_{21} & a_{22} & \ldots & a_{2\,n} & 0 \\ \vdots & \vdots & & \vdots & 0 \\ a_{\text{n}1} & a_{\text{n}2} & \ldots & a_{\text{nn}} & 0 \end{pmatrix}$$

for the system of equations the augmented matrix can be reduced to

$$\begin{pmatrix} 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & & \vdots & 0 \\ 0 & 0 & \ldots & 1 & 0 \end{pmatrix}$$

by a sequence of elementary row operations. If we disregard the last column (of zeros) in each of these matrices, we can conclude that the reduced row-echelon form of A is I_n.

3) ⇒ 4): Assume that the reduced row-echelon form of A is I_n, so that A can be reduced to I_n by a finite sequence of elementary row operations. By Theorem 4.28, each of these operations can be accomplished by multiplying on the left by an appropriate elementary matrix. Thus we can find elementary matrices $E_1, E_2, \ldots, E_k$ such that

$$E_k.E_{k-1}.\ldots.E_2.E_1.A = I_n \tag{4.101}$$

By Theorem 4.29 $E_1, E_2, \ldots, E_k$ are invertible. Multiplying both sides of this equation on the left successively by $E_k^{-1}, E_{k-1}^{-1}, \ldots, E_1^{-1}$ we obtain

$$A \;=\; E_1^{-1}\,.E_2^{-1} \ldots E_k^{-1}\, I_n = E_1^{-1}\,.E_2^{-1} \ldots E_k^{-1} \tag{4.102}$$

By Theorem 4.29, this equation expresses A as a product of elementary matrices.

4) ⇒ 1): If A is a product of elementary matrices, then from Theorem 4.23 and 4.29, the matrix A is a product of invertible matrices and hence is invertible.

QED

4.4.4.1 Row Equivalence

If a matrix B can be obtain from a matrix A by performing a finite sequence of elementary row operations, then obviously we can get from B back to A by performing the inverse of these elementary row operations in reverse order. Matrices that can be obtained from one another by a finite sequence of elementary row operations are said to be row equivalent. With this terminology, it follows from 1) and 3) of Theorem 4.30 that an $n\times n$ matrix A is invertible if and only if it is row equivalent to the $n\times n$ identity matrix.

4.4.4.2 Inverting Matrices

As our first application of Theorem 4.30, we establish a method for determining the inverse of an invertible matrix. Multiplying (4.101) on the right by A^{-1} yields

$$E_k.E_{k-1}....E_2.E_1.I_n = A^{-1} \tag{4.103}$$

which tells us that A^{-1} can be obtained by multiplying I_n successively on the left by the elementary matrices E_1, E_2, …, E_k. Since multiplication on the left by one of these elementary matrices performs a row operation, it follows, by comparing Equation (4.101) with (4.103), that the sequence of row operations that reduce A to I_n will reduce I_n to A^{-1}. Thus we have the following result: To find the inverse of an invertible matrix A, we must find a sequence of elementary row operations that reduces A to the identity and then perform this same sequence of operations on I_n to obtain A^{-1}.

A simple method for carrying out this procedure is given in the following example.

Example 4.65. Row Operations and A^{-1}

Find the inverse of

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{pmatrix}.$$

Solution 4.65. We want to reduce A to the identity matrix by row operations and simultaneously apply these operations to I to produce A^{-1}. To accomplish this we shall adjoin the identity matrix to the right side of A, thereby producing a matrix of the form

$$(A\,|\,I).$$

Then we shall apply row operations to this matrix until the left side is reduced to I; these operations will convert the right side to A^{-1}, so the final Matrix will have the form

$$(I\,|\,A).$$

The computation are as follows:

Apply the Gauss-Jordan elimination on the following augmented matrix

$$\mathbf{A} = \begin{pmatrix} \mathbf{1} & \mathbf{-2} & \mathbf{5} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{2} & \mathbf{6} & \mathbf{3} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{1} & \mathbf{7} & \mathbf{8} & \mathbf{0} & \mathbf{0} & \mathbf{1} \end{pmatrix};$$

A1 = RowReduce[A]

$$\begin{pmatrix} 1 & 0 & 0 & \frac{9}{31} & \frac{17}{31} & -\frac{12}{31} \\ 0 & 1 & 0 & -\frac{13}{93} & \frac{1}{31} & \frac{7}{93} \\ 0 & 0 & 1 & \frac{8}{93} & -\frac{3}{31} & \frac{10}{93} \end{pmatrix}$$

The result is an augmented matrix containing in the first part the identity matrix and in the second part the inverse matrix A^{-1}.

TakeMatrix(A1, {1, 4}, {3, 6})

$$\begin{pmatrix} \frac{9}{31} & \frac{17}{31} & -\frac{12}{31} \\ -\frac{13}{93} & \frac{1}{31} & \frac{7}{93} \\ \frac{8}{93} & -\frac{3}{31} & \frac{10}{93} \end{pmatrix}$$

This is a simple way to calculate an inverse matrix.▲

Often it will not be known in advance whether a given matrix is invertible. If an $n\times n$ matrix A is not invertible, then it cannot be reduced to I_n by elementary row operations. Stated another way, the reduced row-echelon form of A has at least one row of zeros. Thus if the procedure in the last example is attempted on a matrix that is not invertible, then at some point in the computation a row of zeros will occur on the left side. It can then be concluded that the given matrix is not invertible, and the computations can be stopped.

Example 4.66. A Matrix Which is not Invertible

Check if the following matrix is invertible.

$$A = \begin{pmatrix} 1 & 6 & 4 \\ 2 & 4 & -1 \\ -1 & 2 & 5 \end{pmatrix};$$

Solution 4.66. Applying the Gauss-Jordan elimination on the augmented matrix we find

$$\text{AE} = \begin{pmatrix} 1 & 6 & 4 & 1 & 0 & 0 \\ 2 & 4 & -1 & 0 & 1 & 0 \\ -1 & 2 & 5 & 0 & 0 & 1 \end{pmatrix};$$

RowReduce[AE]

$$\begin{pmatrix} 1 & 0 & -\frac{11}{4} & 0 & \frac{1}{4} & -\frac{1}{2} \\ 0 & 1 & \frac{9}{8} & 0 & \frac{1}{8} & \frac{1}{4} \\ 0 & 0 & 0 & 1 & -1 & -1 \end{pmatrix}$$

The vanishing of the last row of A indicates that this matrix is not invertible.▲

4.4.5 System of Equations and Invertibility

In this section we will establish more results on systems of linear equations and invertibility of matrices. Our work will lead to a new method for solving n equations in n unknowns.

In Section 4.2 we mentioned that every linear system has no solutions, or has one solution, or has infinitely many solutions. We are now in a position to prove this fundamental result.

Theorem 4.31. *Solutions of a Linear System of Equations*

Every system of linear equations has no solutions, or has exactly one solution, or has infinitely man solutions. ■

Proof 4.31. If $A.\vec{x} = \vec{b}$ is a system of linear equations, exactly one of the following is true: a) the system has no solutions, b) the system has exactly one solution, or c) the system has more than one solution. The proof will be complete if we can show that the system has infinitely many solutions in case c).

Assume that $A.\vec{x} = \vec{b}$ has more than one solution, and let $\vec{x}_0 = \vec{x}_1 - \vec{x}_2$, where $\vec{x}_1$ and $\vec{x}_2$ are any two distinct solutions. Because $\vec{x}_1$ and $\vec{x}_2$ are distinct, the matrix $\vec{x}_0$ is nonzero; moreover

$$A.\vec{x}_0 = A.(\vec{x}_1 - \vec{x}_2) = A.\vec{x}_1 - A.\vec{x}_2 = \vec{b} - \vec{b} = 0.$$

If we now let k be a scalar, then

$$A.(\vec{x}_1 + k\,\vec{x}_0) = A.\vec{x}_1 + k\,A.\vec{x}_0 = \vec{b} + k\,0 = \vec{b} + 0 = \vec{b}.$$

But this says that $\vec{x}_1 + k\,\vec{x}_0$ is a solution of $A.\vec{x} = \vec{b}$. Since $\vec{x}_0$ is nonzero and there are infinitely many choices for k, the system $A.\vec{x} = \vec{b}$ has infinitely many solutions.

QED

Thus far, we have studied two methods for solving linear systems: Gaussian elimination and Gauss-Jordan elimination. The following theorem provides a new method for solving certain linear systems.

Theorem 4.32. *Solutions of a Linear System of Equations by A^{-1}*

If A is an invertible $n \times n$ matrix, then for each $n \times 1$ matrix $\vec{b}$, the system of equations $A.\vec{x} = \vec{b}$ has exactly one solution, namely, $\vec{x} = A^{-1}\,\vec{b}$. ■

Proof 4.32. Since $A.\left(A^{-1}\,\vec{b}\right) = \vec{b}$, it follows that $\vec{x} = A^{-1}.\vec{b}$ is a solution of $A.\vec{x} = \vec{b}$. To show that this is the only solution, we will assume that $\vec{x}_0$ is an arbitrary solution and then show that $\vec{x}_0$ must be the solution $A^{-1}\,\vec{b}$.

If $\vec{x}_0$ is any solution, then $A.\vec{x}_0 = \vec{b}$. Multiplying both sides by A^{-1}, we obtain $\vec{x}_0 = A^{-1}\,\vec{b}$.

QED

Example 4.67. Solutions of a Linear System of Equations by A^{-1}

Let us look for a solution of the linear equations

$$\begin{array}{rrrcr} x_1 & +2\,x_2 & +3\,x_3 & = & 5 \\ 2\,x_1 & +5\,x_2 & +3\,x_3 & = & 3 \\ x_1 & & +8\,x_3 & = & 17 \end{array}$$

In matrix form this system can be written as $A.\vec{x} = \vec{b}$ where

$$\mathbf{A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{pmatrix}; x = \begin{pmatrix} x1 \\ x2 \\ x3 \end{pmatrix}; b = \begin{pmatrix} 5 \\ 3 \\ 17 \end{pmatrix};}$$

The inverse of A is give by

$$\mathbf{Ai = A^{-1}}$$

$$\begin{pmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{pmatrix}$$

Solution 4.67. Applying the ideas from above we can write down the solution of the linear system of equations by the product

$$\mathbf{xs = Ai.b}$$

$$\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

which gives the solution as a vector for the components $x_1 = 1$, $x_2 = -1$, and $x_3 = 2$.▲

Remark 4.18. Note that this method only applies when the system has as many equations as unknowns and the coefficient matrix is invertible. This method is less efficient, computationally, than the Gauß elimination, but it is important in the analysis of equations involving matrices.

Frequently one is concerned with solving a sequence of systems

$$A.\vec{x} = \vec{b}_i \qquad \text{with } i = 1, 2, \ldots, k. \tag{4.104}$$

Each of the systems has the same square coefficient matrix A. If A is invertible, then the solutions

$$\vec{x}_i = A^{-1}\,\vec{b}_i \qquad \text{with } i = 1, 2, \ldots, k \tag{4.105}$$

can be obtained with one matrix inversion and k matrix multiplications. Once again, however, a more efficient method is to form the matrix

$$\left(A \,\middle|\, \vec{b}_1 \ldots \vec{b}_k\right) \tag{4.106}$$

in which the coefficient matrix A is augmented by all k of the matrices $\vec{b}_i$, and then reduce this equation to a reduced row-echelon form by Gauss-Jordan elimination. In this way we can solve all k systems at once. This method has the added value that it applies even when A is not invertible.

Example 4.68. Electrical Networks

Two networks with two loops and two knots allow the following equations for currents I_1, I_2, and I_3.

$$\begin{array}{rrrcr} I_1 & +2\,I_2 & +3\,I_3 & = & 5 \\ 2\,I_1 & +5\,I_2 & +3\,I_3 & = & 3 \\ I_1 & & +8\,I_3 & = & 17 \end{array}$$

In the second network the relations are changed in such a way that the right hand side if the equation changes

$$\begin{array}{rrrcr} I_1 & +2\,I_2 & +3\,I_3 & = & 2 \\ 2\,I_1 & +5\,I_2 & +3\,I_3 & = & 1 \\ I_1 & & +8\,I_3 & = & 15 \end{array}$$

The question is which currents I_i result from the different configurations.

Solution 4.68. Here we can apply both methods of solution either the A^{-1} or Gauss-Jordan method. Let us start with Gauss-Jordan. The augmented matrix is given by

$$\mathbf{Ag} = \begin{pmatrix} \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{5} & \mathbf{2} \\ \mathbf{2} & \mathbf{5} & \mathbf{3} & \mathbf{3} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} & \mathbf{8} & \mathbf{17} & \mathbf{15} \end{pmatrix};$$

the reduced matrix in echelon form follows from

RowReduce[Ag]

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 71 \\ 0 & 1 & 0 & -1 & -24 \\ 0 & 0 & 1 & 2 & -7 \end{pmatrix}$$

The result tells us that for the first network the solutions are $I_1 = 1$, $I_2 = -1$ and $I_3 = 2$. For the second configuration we get $I_1 = 71$, $I_2 = -24$, and $I_3 = -7$.

The second approach is based on the inverse matrix A^{-1}. The inverse matrix is gained by

$$\mathbf{Ai} = \begin{pmatrix} \mathbf{1} & \mathbf{2} & \mathbf{3} \\ \mathbf{2} & \mathbf{5} & \mathbf{3} \\ \mathbf{1} & \mathbf{0} & \mathbf{8} \end{pmatrix}^{-1}$$

$$\begin{pmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{pmatrix}$$

Now the two solutions follow by

s1 = Ai.{5, 3, 17}

{1, −1, 2}

and

s2 = Ai.{2, 1, 15}

{71, −24, −7}

which agrees with the Gauss-Jordan method.▲

Up to now, to show that an $n \times n$ matrix A is invertible, it has been necessary to find an $n \times n$ matrix B such that

$$A.B = I = B.A. \tag{4.107}$$

The next theorem shows that if we produce an $n \times n$ matrix B satisfying either condition, then the other condition holds automatically.

Theorem 4.33. *Existence of* A^{-1}

Let A be a square matrix.

a) If B is a square matrix satisfying $B.A = I$, *then* $B = A^{-1}$.

b) If B is a square matrix satisfying $A.B = I$, *then* $B = A^{-1}$.■

We shall prove a) and leave part b) as an exercise.

Proof 4.33. a) Assume that $B.A = I$. If we can show that A is invertible, the proof can be completed by multiplying $B.A = I$ on both sides by A^{-1} to obtain

$$B.A.A^{-1} = I.A^{-1} \qquad \text{or} \qquad B.I = I.A^{-1} \qquad \text{or} \qquad B = A^{-1}.$$

To show that A is invertible, it suffices to show that the system $A.\vec{x} = 0$ has only the trivial solution. Let $\vec{x}_0$ be any solution of this system. If we multiply both sides of $A.\vec{x}_0 = 0$ on the left by B, we obtain $B.A.\vec{x}_0 = B.0$ or $I.\vec{x}_0 = 0$ or $\vec{x}_0 = 0$. Thus, the system of equations $A.\vec{x} = 0$ has only the trivial solution.

QED

We are now in a position to sum up our knowledge and formulate equivalent statements.

Theorem 4.34. *Equivalent Statement*

If A is an $n \times n$ *matrix, then the following are equivalent.*

a) A is invertible.

b) $A.\vec{x} = 0$ *has only the trivial solution.*

c) The reduced row-echelon form of A is I_n.

d) A is expressible as a product of elementary matrices.

e) $A.\vec{x} = \vec{b}$ is consistent for every $n \times 1$ matrix $\vec{b}$.

f) $A.\vec{x} = \vec{b}$ has exactly one solution for every $n \times 1$ matrix $\vec{b}$. ■

Proof 4.34. Since we extended Theorem 4.30, we can base our proof on the same sequence of arguments and extend the loop by two additional steps. So here it is sufficient to prove a) ⇒ f) ⇒ e) ⇒ a).

The step a) ⇒ f) was already proved in Theorem 4.32. f) ⇒ e) is self-evident: If $A.\vec{x} = \vec{b}$ has exactly one solution for every $\vec{b}$, then $A.\vec{x} = \vec{b}$ is consistent for every $\vec{b}$.

The last step e) ⇒a) is based on; if the system $A.\vec{x} = \vec{b}$ is consistent for every $\vec{b}$, then in particular, the systems

$$A.\vec{x} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, A.\vec{x} = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \ldots, A.\vec{x} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

are consistent. Let $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n$ be solutions of the respective systems, and let us form an $n \times n$ matrix C having these solutions as columns. Thus C has the form

$$C = (\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n).$$

As discussed in Section 4.4.2, the successive columns of the product $A.C$ will be

$$A.\vec{x}_1, A.\vec{x}_2, \ldots, A.\vec{x}_n.$$

Thus

$$A.C = (A.\vec{x}_1, A.\vec{x}_2, \ldots, A.\vec{x}_n) = \begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & 1 \end{pmatrix} = I.$$

By part b) of Theorem 4.33, it follows that $C = A^{-1}$. Thus A is invertible.

QED

We know from earlier work that invertible matrix factors produce an invertible product. The following theorem, which will be stated at the moment without proof, looks at the converse: It shows that if the product of square matrices is invertible, then the factors themselves must be invertible.

Theorem 4.35. *Product of Inverse*

Let A and B be square matrices of the same size. If A.B is invertible, then A and B must also be invertible.■

In our future work the following fundamental problem will occur frequently in various contexts.

Let A be a fixed $n \times n$ matrix, Find all matrices $\vec{b}$ such that the system of equations $A.\vec{x} = \vec{b}$ is consistent.

If A is an invertible matrix, Theorem 4.32 completely solves this problem by asserting that for every vector $\vec{b}$ the linear system $A.\vec{x} = \vec{b}$ has a unique solution $\vec{x} = A^{-1}.\vec{b}$. If A is not square, or if A is square but not invertible, then Theorem 4.32 does not apply. In these cases the vector $\vec{b}$ must satisfy certain conditions in order for $A.\vec{x} = \vec{b}$ to be consistent. The following example illustrates how the elimination method can be used to determine such conditions.

Example 4.69. Consistency of Equations

What conditions must b_1, b_2, and b_3 satisfy in order for the system of equations

$$\begin{aligned} x_1 \quad + x_2 \quad + 2\,x_3 &= b_1 \\ x_1 \qquad\qquad + x_3 &= b_2 \\ 2\,x_1 \quad + x_2 \quad + 3\,x_3 &= b_3 \end{aligned}$$

to be consistent?

Solution 4.69. The related augmented matrix is

$$\mathbf{A} = \begin{pmatrix} \mathbf{1} & \mathbf{1} & \mathbf{2} & \boldsymbol{b_1} \\ \mathbf{1} & \mathbf{0} & \mathbf{1} & \boldsymbol{b_2} \\ \mathbf{2} & \mathbf{1} & \mathbf{3} & \boldsymbol{b_3} \end{pmatrix};$$

which can be reduced to

$$\begin{pmatrix} 1 & 1 & 2 & b_1 \\ 0 & -1 & -1 & b_2 - b_1 \\ 0 & -1 & -1 & b_3 - 2\,b_1 \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & 1 & 2 & b_1 \\ 0 & 1 & 1 & b_1 - b_2 \\ 0 & 0 & 0 & b_3 - b_2 - b_1 \end{pmatrix}$$

It is now evident from the third row in the matrix that the system has a solution if and only if b_1, b_2, and b_3 satisfy the condition

$$b_3 - b_2 - b_1 = 0 \qquad \text{or} \qquad b_3 = b_1 + b_2.$$

To express this condition another way, $A.\vec{x} = \vec{b}$ is consistent if and only if $\vec{b}$ is a vector of the form

$$\vec{b} = (b_1, b_2, b_1 + b_2)$$

where b_1 and b_2 are arbitrary.▲

Example 4.70. Consistency of Equations

What conditions must b_1, b_2, and b_3 satisfy in order for the system of equations

$$\begin{array}{cccc} x_1 & +2x_2 & +3x_3 & = & b_1 \\ 2x_1 & +5x_2 & +3x_3 & = & b_2 \\ x_1 & & +8x_3 & = & b_3 \end{array}$$

to be consistent?

Solution 4.70. The related augmented matrix is

$$\mathbf{A} = \begin{pmatrix} \mathbf{1} & \mathbf{2} & \mathbf{3} & \boldsymbol{b_1} \\ \mathbf{2} & \mathbf{5} & \mathbf{3} & \boldsymbol{b_2} \\ \mathbf{1} & \mathbf{0} & \mathbf{8} & \boldsymbol{b_3} \end{pmatrix};$$

which can be reduced by

RowReduce[*A*]

$$\begin{pmatrix} 1 & 0 & 0 & -40\,b_1 + 16\,b_2 + 9\,b_3 \\ 0 & 1 & 0 & 13\,b_1 - 5\,b_2 - 3\,b_3 \\ 0 & 0 & 1 & 5\,b_1 - 2\,b_2 - b_3 \end{pmatrix}$$

For this case there is no restriction on the b_i's, the given system $A.\vec{x} = \vec{b}$ has a unique solution.▲

4.4.6 Tests and Exercises

The following two subsections serve to test your understanding of the last section. Work first on the test examples and then try to solve the exercises.

4.4.6.1 Test Problems

T1. How is a matrix defined? Give examples.

T2. State some examples where matrices are used to formulate equations.

T3. What is a symmetric matrix?

T4. What is the trace of a matrix?

T5. How is the inverse of a matrix calculated?

T6. What is an identity matrix?

T7. What is the meaning of a matrix function?

T8. How are powers of a matrix defined?

4.4.6.2 Exercises

E1. Let the following matrices A, B, C, and D be given by

$$A = \begin{pmatrix} 5 & 3 \\ 2 & -1 \\ 5 & 7 \end{pmatrix}, \; B = \begin{pmatrix} 2 & 7 \\ 3 & -6 \\ -2 & 5 \end{pmatrix}, \; C = \begin{pmatrix} 1 & 1 \\ 4 & 5 \end{pmatrix}, \text{ and } D = \begin{pmatrix} 2 & 4 \\ -4 & 7 \end{pmatrix} \qquad (1)$$

Compute the following algebraic expressions (if they exist).

a. $C + D$,
b. $A + D$,
c. $-C$,
d. $2\,A + B$,
e. $A - B$,
f. $2\,B$,
g. $A.C$,
h. $C.A$

E2. Given the following matrices

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ B = \begin{pmatrix} 0 & 1 \\ 3 & 7 \end{pmatrix},\ C = \begin{pmatrix} 3 \\ 4 \end{pmatrix},\ \text{and } D = \begin{pmatrix} 2 & 4 & 3 \\ -4 & 7 & 2 \end{pmatrix} \tag{2}$$

Compute the following expressions (if they exist).

a. $A\,B$,
b. $B\,A$,
c. $C\,A$,
d. $A\,C$,
e. $A^2 - B$,
f. $D\,B$,
g. $B\,C$,
h. $B\,A\,D$

E3. Given the following matrices

$$A = \begin{pmatrix} 1 & 0 & 8 \\ 0 & -1 & 6 \\ 2 & 9 & -1 \end{pmatrix} \text{and } B = \begin{pmatrix} 0 & 1 & 9 \\ 3 & 7 & -4 \\ 5 & 3 & -1 \end{pmatrix} \tag{3}$$

Let O_3 and I_3 be the 3×3 zero and identity matrices. Show that

a. $A + O_3 = O_3 + A = A$,
b. $B\,O_3 = O_3\,B = O_3$,
c. $B\,I_3 = I_3\,B = B$,
d. $A\,I_3 - I_3\,A = O_3$.
e. $A\,I_3\,B - B\,I_3\,A = O_3$,

E4. Let A be a 2×2 matrix, B a 2×2 matrix, C a 3×4 matrix, D a 4×2 matrix, and E a 4×5 matrix. Determine which of the following matrix expressions exist and give the size of the resulting matrices when they do exist

a. $A\,B$,
b. $A\,C$,
c. $3\,(E\,B) + 7\,D$,
d. $2\,(E\,B) + D\,A$.
e. $C\,D - 4\,(C\,E)\,B$.

E5. Let $R = P\,Q$ and $S = Q\,P$, where

$$P = \begin{pmatrix} 1 & 0 \\ 0 & -1 \\ 2 & 9 \end{pmatrix} \text{and } Q = \begin{pmatrix} 0 & 1 & 9 \\ 3 & 7 & -4 \end{pmatrix} \tag{4}$$

Determine the following elements (if they exist) of R and S, without computing the complete matrices.

a. r_{21},

r_{31},

c. s_{21},

d. s_{11},

e. s_{23},

E6. Let A be an $m\times r$ matrix, B be an $r\times n$ matrix and $C = A\,B$. Let the column submatrices of B be $B_1, B_2, \ldots, B_n$ and of C be $C_1, C_2, \ldots, C_n$. We can write B in the form $B = (B_1, B_2, \ldots, B_n)$ and $C = (C_1, C_2, \ldots, C_n)$. Prove that $C_j = A\,B_j$.

E7. Let D be a matrix whose second column is all zero. Let C be any matrix such that $C\,D$ exists. Prove that the second column of $C\,D$ is all zero.

E8. Given the matrix equation $A\,\vec{x} = \vec{b}$ where $\vec{x}$ is the vector of unknowns. Find the solution for this system of equations by using the inverse matrix A^{-1}. The specific matrix A is given by

$$A = \begin{pmatrix} -1 & 8 & 6 & -9 & -1 \\ 6 & 5 & 2 & 5 & -4 \\ -3 & 2 & -6 & 2 & 1 \\ 1 & 9 & 6 & -8 & 11 \\ 2 & 4 & -6 & 9 & -12 \end{pmatrix}. \tag{5}$$

Use the following right hand sides $\vec{b}$ to determine the solutions $\vec{x}$ for

a. $\vec{b} = (1, 3, 5, 7, 9)$,

b. $\vec{b} = (2, 4, 6, 8, 10)$,

c. $\vec{b} = (-1, 1, -1, 1, -1)$.

E9. Determine the trace of the following matrix A

$$A = \begin{pmatrix} 2 & 8 & 6 & -9 & -1 \\ 6 & 3 & 2 & 1 & -4 \\ -3 & 2 & 5 & 2 & 1 \\ 1 & 1 & 6 & 9 & 11 \\ 2 & 4 & -6 & 9 & -12 \end{pmatrix}. \tag{6}$$

E10 Given the following matrix A

$$A = \begin{pmatrix} 2 & 8 & 6 & -9 & -1 \\ 6 & 3 & 2 & 1 & -4 \\ -3 & 2 & 5 & 2 & 1 \\ 1 & 1 & 6 & 9 & 11 \end{pmatrix} \tag{7}$$

a. Use matrix multiplication to find the sum of the elements in each row of the matrix A.

b. Use matrix multiplication to find the sum of elements in each column of the matrix A.

c. Extend these results. How can matrix multiplication be used to add the elements in each row of a matrix? How can matrix multiplication be used to add the elements in each column of a matrix?

4.5 Determinants

A determinant is a certain kind of function that associates a real number with a square matrix. In this section we will define this function. As a consequence of our work here, we will obtain a formula for the inverse of an invertible matrix as well as a formula for the solution to certain systems of linear equations in terms of determinants.

4.5.1 Cofactor Expansion

Recall from Theorem 4.22 that the 2×2 matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{4.108}$$

is invertible if $a\,d - c\,b \neq 0$. The expression $a\,d - c\,b$ occurs frequently in mathematics that it has a name; it is called the determinant of the 2×2 matrix A and is denoted by the symbol $\det(A)$ or $|A|$. With this notation, the formula for A^{-1} given in Theorem 4.22 is

$$A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}. \tag{4.109}$$

One of the goals of this chapter is to obtain analogs of this formula to square matrices of higher order. This will require that we extend the concept of a determinant to square matrices of all order.

4.5.1.1 Minors and Cofactors

There are several ways in which we might proceed. The approach in this section is a recursive approach: It defines the determinant of an $n\times n$ matrix in terms of the determinants of the $(n-1)\times(n-1)$ matrices. The $(n-1)\times(n-1)$ matrices that will appear in this definition are submatrices of the original matrix. These submatrices are given a special name.

Definition 4.23. *Minors and Cofactors*

If A is a square matrix, then the minor of entry a_{ij} is denoted by M_{ij} and is defined to be the determinant of the submatrix that remains after the ith and jth column are deleted from A. The number $(-1)^{i+j}\, M_{ij}$ is denoted by C_{ij} and is called the cofactor of entry a_{ij}. ∎

Example 4.71. Minors and Cofactors

Given the matrix

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix};$$

Find the minors and cofactors of this matrix.

Solution 4.71. The minors in *Mathematica* are defined in slightly different way using the lower right corner of the matrix as the starting point. However, we can correct this by reversing the order of the derived matrix in the following way

minors = Map[Reverse, Minors[*A*], {0, 1}]

$$\begin{pmatrix} a_{22}\,a_{33} - a_{23}\,a_{32} & a_{21}\,a_{33} - a_{23}\,a_{31} & a_{21}\,a_{32} - a_{22}\,a_{31} \\ a_{12}\,a_{33} - a_{13}\,a_{32} & a_{11}\,a_{33} - a_{13}\,a_{31} & a_{11}\,a_{32} - a_{12}\,a_{31} \\ a_{12}\,a_{23} - a_{13}\,a_{22} & a_{11}\,a_{23} - a_{13}\,a_{21} & a_{11}\,a_{22} - a_{12}\,a_{21} \end{pmatrix}$$

which actually are the 2×2 determinants of the matrix A if the entries $(i,\ j)$ of the matrix are selected. The cofactors follow from the minors by changing the sign due to our definition $C_{ij} = (-1)^{i+j} M_{ij}$

cofactors = Table$\left[(-1)^{i+j}\right.$ **Map[Reverse, Minors[*A*], {0, 1}]⟦*i*, *j*⟧, {*i*, 1, Length[*A*]}, {*j*, 1, Length[*A*]}**$\left.\right]$

$$\begin{pmatrix} a_{22}\,a_{33} - a_{23}\,a_{32} & a_{23}\,a_{31} - a_{21}\,a_{33} & a_{21}\,a_{32} - a_{22}\,a_{31} \\ a_{13}\,a_{32} - a_{12}\,a_{33} & a_{11}\,a_{33} - a_{13}\,a_{31} & a_{12}\,a_{31} - a_{11}\,a_{32} \\ a_{12}\,a_{23} - a_{13}\,a_{22} & a_{13}\,a_{21} - a_{11}\,a_{23} & a_{11}\,a_{22} - a_{12}\,a_{21} \end{pmatrix}$$

So finally the cofactors are the sign changed minors. To avoid mistakes in our future calculations we introduce our own definitions for minors and cofactors based on Definition 4.23. We define

```
MatrixMinors[A_] := Map[Reverse, Minors[A], {0, 1}]
```

and for the cofactors

```
Cofactors[A_] := Block[{minors}, minors = MatrixMinors[A];
  Table[(-1)^(i+j) minors[[i, j]], {i, 1, Length[A]}, {j, 1, Length[A]}]]
```

These two little functions will be used in the following to extract minors and cofactors. ▲

Note that the cofactor and the minor of an element a_{ij} differ only in sign; that is, $C_{ij} = \pm M_{ij}$. A quick way to determine whether to use + or − is to use the fact that the sign relating C_{ij} and M_{ij} is in the ith row and jth column of the checkerboard array

$$\begin{pmatrix} + & - & + & - & + & \cdots \\ - & + & - & + & - & \cdots \\ + & - & + & - & + & \cdots \\ - & + & - & + & - & \cdots \\ + & - & + & - & + & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

For example $C_{11} = M_{11}$, $C_{12} = -M_{12}$, $C_{22} = M_{22}$ and so on.

Strictly speaking the determinant of a matrix is a number. However, it is common practice to abuse the terminology slightly and use the term determinant to refer to the matrix whose determinant is being computed. Thus we might refer to

$$\begin{vmatrix} 3 & 1 \\ 4 & -2 \end{vmatrix}$$

as a 2×2 determinant and call 3 the entry in the first row and first column of the determinant.

4.5.1.2 Cofactor Expansion

The definition of a 3×3 determinant in terms of minors and cofactors is

$$\det(A) = a_{11}\,M_{11} + a_{12}(-M_{12}) + a_{13}\,M_{13} = a_{11}\,C_{11} + a_{12}\,C_{12} + a_{13}\,C_{13} \tag{4.110}$$

Equation (4.110) shows that the determinant of A can be computed by multiplying the entries in the first row of A by their corresponding cofactors and adding the resulting products. More generally, we define the determinant of an $n \times n$ matrix to be

$$\det(A) = a_{11}\, C_{11} + a_{12}\, C_{12} + \ldots + a_{1\,n}\, C_{1\,n}. \tag{4.111}$$

The method of evaluating $\det(A)$ is called cofactor expansion along the first row of A.

Example 4.72. Cofactor Expansion

Let

$$A = \begin{pmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{pmatrix};$$

Evaluate $\det(A)$ by cofactor expansion along the first row of A.

Solution 4.72. The minors of the matrix A are given by

minors = MatrixMinors(*A*)

$$\begin{pmatrix} -4 & -11 & 12 \\ -2 & -6 & 7 \\ 3 & 9 & -10 \end{pmatrix}$$

and the cofactors follow with

coFactors = Cofactors(*A*)

$$\begin{pmatrix} -4 & 11 & 12 \\ 2 & -6 & -7 \\ 3 & -9 & -10 \end{pmatrix}$$

The determinant of matrix A is then derived as the dot product of the first row of A and the first row of the cofactor matrix as

```
A⟦1⟧.coFactors⟦1⟧
```

```
-1
```

We can check this result by using the *Mathematica* function Det for determining the determinant

```
Det[A]
```

```
-1
```

which shows that the determinant derived by the cofactor method is consistent with the method used by *Mathematica.*▲

If A is a 3×3 matrix, then its determinant is formally determined by the cofactor method as

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} +$$

$$a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}(a_{22}\,a_{33} - a_{32}\,a_{23}) - a_{12}(a_{21}\,a_{33} - a_{31}\,a_{23}) + a_{13}(a_{21}\,a_{32} - a_{31}\,a_{22}) =$$
$$-a_{13}\,a_{22}\,a_{31} + a_{12}\,a_{23}\,a_{31} + a_{13}\,a_{21}\,a_{32} - a_{11}\,a_{23}\,a_{32} - a_{12}\,a_{21}\,a_{33} + a_{11}\,a_{22}\,a_{33}$$

By rearranging the terms in various ways, it is possible to obtain other formulas. There should be no trouble checking that all of the following are correct.

$$\begin{aligned} \det(A) &= a_{11}\,C_{11} + a_{12}\,C_{12} + a_{13}\,C_{13} \\ &= a_{11}\,C_{11} + a_{21}\,C_{21} + a_{31}\,C_{31} \\ &= a_{21}\,C_{21} + a_{22}\,C_{22} + a_{32}\,C_{32} \\ &= a_{12}\,C_{12} + a_{22}\,C_{22} + a_{32}\,C_{32} \\ &= a_{31}\,C_{31} + a_{32}\,C_{32} + a_{33}\,C_{33} \\ &= a_{13}\,C_{13} + a_{23}\,C_{23} + a_{33}\,C_{33}. \end{aligned} \tag{4.113}$$

Note that in each equation, the entries and cofactors all come from the same row or column. These equations are called the cofactor expansion of $\det(A)$.

The results we have given for 3×3 matrices form a special case of the following general theorem, which states without proof the following.

Theorem 4.36. *Cofactor Expansion*

The determinant of an $n\times n$ matrix A can be computed by multiplying the entries in any row (or column) by their cofactors and adding the resulting products; that is for each $1 \le i \le n$ and $1 \le j \le n$,

$\det(A) =$

$$a_{1\,j}\,C_{1\,j} + a_{2\,j}\,C_{2\,j} + \ldots + a_{n\,j}\,C_{n\,j} = \sum_{k=1}^{n} a_{kj}\,C_{kj} \qquad \textit{cofactor expansion along the jth column}$$

and

$$\det(A) = a_{i1}\,C_{i1} + a_{i2}\,C_{i2} + \ldots + a_{in}\,C_{in} = \sum_{k=1}^{n} a_{ik}\,C_{ik} \qquad \textit{cofactor expansion along the ith row}.\blacksquare$$

Note that we may choose any row or any column.

Example 4.73. Cofactor Expansion

Let again be the matrix as in Example 4.72

$$\boldsymbol{A} = \begin{pmatrix} \mathbf{3} & \mathbf{1} & \mathbf{0} \\ \mathbf{-2} & \mathbf{-4} & \mathbf{3} \\ \mathbf{5} & \mathbf{4} & \mathbf{-2} \end{pmatrix};$$

Evaluate det(A) by cofactor expansion along any row of A.

Solution 4.73. The cofactors of the matrix A are given by

coFactors = Cofactors(A)

$$\begin{pmatrix} -4 & 11 & 12 \\ 2 & -6 & -7 \\ 3 & -9 & -10 \end{pmatrix}$$

Selecting the second row of the matrix to expand det(A) into cofactors we find

A⟦2⟧.coFactors⟦2⟧

-1

the third row delivers

A⟦3⟧.coFactors⟦3⟧

-1

The result is the same as expected but the number of terms used to evaluate this result is larger than the number of non vanishing terms in the previous one.▲

Remark 4.19. In this example we had to compute three cofactors, but in Example 4.72 we only had to compute two of them, since the third was multiplied by a zero. In general, the best strategy for evaluating a determinant by cofactor expansion is to expand along a row or column having the largest number of zeros.

4.5.1.3 Adjoint of a Matrix

In a cofactor expansion we compute det(A) by multiplying the entries in a row or column by their cofactors and adding the resulting products. It turns out that if one multiplies the entries of A in any row by the corresponding cofactors from a different row, the sum of the products is always zero. This result also holds for columns. Although we omit the general proof, the next example illustrates the idea of the proof in a special case.

Example 4.74. Cofactor Multiplication

Let the matrix A be given as a 3×3 matrix with general entries.

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix};$$

What is the result of the multiplication of one of the rows of A with one of the rows of the cofactor matrix?

Solution 4.74. First the cofactor matrix is generated by the function Cofactors. The cofactors are

Ac = Cofactors(*A*)

$$\begin{pmatrix} a_{22}\,a_{33}-a_{23}\,a_{32} & a_{23}\,a_{31}-a_{21}\,a_{33} & a_{21}\,a_{32}-a_{22}\,a_{31} \\ a_{13}\,a_{32}-a_{12}\,a_{33} & a_{11}\,a_{33}-a_{13}\,a_{31} & a_{12}\,a_{31}-a_{11}\,a_{32} \\ a_{12}\,a_{23}-a_{13}\,a_{22} & a_{13}\,a_{21}-a_{11}\,a_{23} & a_{11}\,a_{22}-a_{12}\,a_{21} \end{pmatrix}$$

If we multiply the first row of A with the first row of the cofactor matrix we observe that

Simplify[*A*⟦1⟧.Ac⟦1⟧]

$$a_{13}\,(a_{21}\,a_{32}-a_{22}\,a_{31})+a_{11}\,(a_{22}\,a_{33}-a_{23}\,a_{32})+a_{12}\,(a_{23}\,a_{31}-a_{21}\,a_{33})$$

corresponds to the determinant of the matrix A. If we now change the row of the cofactor matrix we gain

Simplify[*A*⟦1⟧.Ac⟦2⟧]

0

and

Simplify[*A*⟦1⟧.Ac⟦3⟧]

0

This means that the cofactor expansion of a determinant is uniquely defined.▲

Now we will use this fact to get a formula for the inverse matrix A^{-1}.

Definition 4.24. *Adjoint Matrix*

If A is any $n\times n$ matrix and C_{ij} is the cofactor of a_{ij}, then the matrix

$$C=\begin{pmatrix} C_{11} & C_{12} & \ldots & C_{1\,n} \\ C_{21} & C_{22} & \ldots & C_{2\,n} \\ \vdots & \vdots & & \vdots \\ C_{\text{n}1} & C_{\text{n}2} & \ldots & C_{\text{nn}} \end{pmatrix}$$

is called the matrix of cofactors from A. The transpose of this matrix is called the adjoint of A and is denoted by adj(A).■

Example 4.75. Adjoint Matrix

Let a matrix A be given by

$$\mathbf{A}=\begin{pmatrix} \mathbf{2} & \mathbf{4} & \mathbf{1} \\ \mathbf{0} & \mathbf{-1} & \mathbf{3} \\ \mathbf{6} & \mathbf{7} & \mathbf{2} \end{pmatrix};$$

Determine the cofactors and the adjoint of A.

Solution 4.75. The cofactors of A are

Ac = Cofactors(*A*)

$$\begin{pmatrix} -23 & 18 & 6 \\ -1 & -2 & 10 \\ 13 & -6 & -2 \end{pmatrix}$$

The adjoint representation of the matrix A follows by transposition

$\mathbf{Ac}^{T}$

$$\begin{pmatrix} -23 & -1 & 13 \\ 18 & -2 & -6 \\ 6 & 10 & -2 \end{pmatrix}$$

▲

We are now in a position to derive a formula for the inverse of an invertible matrix. We need to use an important fact that will be proved later on, that is a square matrix A is invertible if and only if det(A) is not zero.

Theorem 4.37. *Inverse Matrix*

If A is an invertible matrix, then

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A). \blacksquare$$

Proof 4.37. Since det(A) is a scalar we have the relation

$$\det(A)\, A^{-1} = \operatorname{adj}(A)$$

multiply this relation by A from the left then we find

$$\det(A)\, A.A^{-1} = A.\operatorname{adj}(A)$$

which is

$$\det(A)\, I = A.\operatorname{adj}(A).$$

Consider now the right hand side which is given by

$$A.\operatorname{adj}(A) = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \ldots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{pmatrix} . \begin{pmatrix} C_{11} & C_{21} & \ldots & C_{j1} & \ldots & C_{n1} \\ C_{12} & C_{22} & \ldots & C_{j2} & \ldots & C_{n2} \\ \vdots & \vdots & & \vdots & & \vdots \\ C_{1n} & C_{2n} & \ldots & C_{jn} & \ldots & C_{nn} \end{pmatrix}$$

The entries in the ith row and jth column of product $A.\operatorname{adj}(A)$ is

$a_{i1}\,C_{j1} + a_{i2}\,C_{j2} + \ldots + a_{in}\,C_{jn}$

If $i = j$ then this is the cofactor expansion of det(A), and if $i \neq j$, then the a's and the cofactors come from different rows of A, so for this case the product will vanish. Therefore

$$A.\mathrm{adj}(A) = \begin{pmatrix} \det(A) & 0 & \ldots & 0 \\ 0 & \det(A) & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & \det(A) \end{pmatrix} = \det(A).I.$$

Since A is invertible, $\det(A) \neq 0$. Therefore we can write

$$\frac{1}{\det(A)} A\,\mathrm{adj}(A) = I \qquad \text{or} \qquad A.\left(\frac{1}{\det(A)}\,\mathrm{adj}(A)\right) = I$$

And finally

$$\left(\frac{1}{\det(A)}\,\mathrm{adj}(A)\right) = A^{-1}$$

This can be verified in the 3×3 case as follows

$$\boldsymbol{A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix};}$$

find the cofactors

Ac = Cofactors(*A*)

$$\begin{pmatrix} a_{22}\,a_{33} - a_{23}\,a_{32} & a_{23}\,a_{31} - a_{21}\,a_{33} & a_{21}\,a_{32} - a_{22}\,a_{31} \\ a_{13}\,a_{32} - a_{12}\,a_{33} & a_{11}\,a_{33} - a_{13}\,a_{31} & a_{12}\,a_{31} - a_{11}\,a_{32} \\ a_{12}\,a_{23} - a_{13}\,a_{22} & a_{13}\,a_{21} - a_{11}\,a_{23} & a_{11}\,a_{22} - a_{12}\,a_{21} \end{pmatrix}$$

and multiply A by the adj(A) we find

Simplify[*A*.adj(*A*)]

$\{\{a_{13}\,(-a_{22}\,a_{31} + a_{21}\,a_{32}) + a_{12}\,(a_{23}\,a_{31} - a_{21}\,a_{33}) + a_{11}\,(-a_{23}\,a_{32} + a_{22}\,a_{33}),\ 0,\ 0\},$
$\{0,\ a_{13}\,(-a_{22}\,a_{31} + a_{21}\,a_{32}) + a_{12}\,(a_{23}\,a_{31} - a_{21}\,a_{33}) + a_{11}\,(-a_{23}\,a_{32} + a_{22}\,a_{33}),\ 0\},$
$\{0,\ 0,\ a_{13}\,(-a_{22}\,a_{31} + a_{21}\,a_{32}) + a_{12}\,(a_{23}\,a_{31} - a_{21}\,a_{33}) + a_{11}\,(-a_{23}\,a_{32} + a_{22}\,a_{33})\}\}$

which is $\det(A).I_3$.

QED

Example 4.76. Finding the Inverse A^{-1} by adj(A)

Given the matrix A find the inverse A^{-1} of a and verify that $A.A^{-1} = I$.

$$\mathbf{A} = \begin{pmatrix} \mathbf{1} & \mathbf{5} & \mathbf{4} \\ \mathbf{3} & \mathbf{2} & \mathbf{9} \\ \mathbf{-1} & \mathbf{1} & \mathbf{0} \end{pmatrix};$$

Solution 4.76. The determinant of this matrix is

det = |*A*|

-34

which is not equal to zero and thus A is invertible by

$$\frac{\mathbf{adj}(A)}{|A|}$$

$$\begin{pmatrix} \frac{9}{34} & -\frac{2}{17} & -\frac{37}{34} \\ \frac{9}{34} & -\frac{2}{17} & -\frac{3}{34} \\ -\frac{5}{34} & \frac{3}{17} & \frac{13}{34} \end{pmatrix}$$

▲

4.5.1.4 Cramer's Rule

The next theorem provides a formula for the solution of certain linear systems of n equations in n unknowns. This formula, known as Cramer's rule, is nowadays of marginal interest for computational purposes, but it is a useful tool for studying the mathematical properties of a solution without the need for solving the system. Compared with Gauss-Jordan elimination it is straight forward.

Theorem 4.38. *Cramer's Rule*

If $A.\vec{x} = \vec{b}$ is a system of n linear equations in n unknowns such that det(A) ≠ 0, then the system has a unique solution. This solution is

$$x_1 = \frac{\det(A_1)}{\det(A)},\ x_2 = \frac{\det(A_2)}{\det(A)},\ \ldots,\ x_n = \frac{\det(A_n)}{\det(A)}$$

where A_j is the matrix obtained by replacing the entries in the jth column of A by the entries in the vector $\vec{b} = (b_1, b_2, \ldots, b_n)$. ■

Proof 4.38. If $\det(A) \neq 0$, then A is invertible, and $\vec{x} = A^{-1}.\vec{b}$ is the unique solution of $A.\vec{x} = \vec{b}$. Therefore we have

$$\vec{x} = A^{-1}\,\vec{b} = \frac{1}{\det(A)}\operatorname{adj}(A).\vec{b} = \frac{1}{\det(A)} \begin{pmatrix} C_{11} & C_{21} & \ldots & C_{n1} \\ C_{12} & C_{22} & \ldots & C_{n2} \\ \vdots & \vdots & & \vdots \\ C_{1n} & C_{2n} & \ldots & C_{nn} \end{pmatrix} . \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

The entry in the jth row of $\vec{x}$ is therefore

$$x_j = \frac{b_1 C_{1j} + b_2 C_{2j} + \ldots + b_n C_{nj}}{\det(A)}$$

Now let

$$A_j = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1\,j-1} & b_1 & a_{1\,j+1} & \ldots & a_{1\,n} \\ a_{21} & a_{22} & \ldots & a_{2\,j-1} & b_2 & a_{2\,j+1} & \ldots & a_{2\,n} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{n\,j-1} & b_n & a_{n\,j+1} & \ldots & a_{nn} \end{pmatrix}$$

Since A_j differs from A only on the jth column, it follows that the cofactors of entries $b_1, b_2, \ldots, b_n$ in A_j are the same as the cofactors of the corresponding entries in the jth column of A. The cofactor expansion of $\det(A)$ along the jth column is therefore

$$\det(A_j) = b_1 C_{1j} + b_2 C_{2j} + \ldots + b_n C_{nj}$$

This leads to

$$x_j = \frac{\det(A_j)}{\det(A)}.$$

QED

Example 4.77. Cramer's Rule

Use Cramer's Rule to solve the following system of linear equations

$$\begin{array}{cccccc} x_1 & + & +2\,x_3 & = & 6 \\ -3\,x_1 & +4\,x_2 & +6\,x_3 & = & 30 \\ -x_1 & -2\,x_2 & 3\,x_3 & = & 8 \end{array}$$

Solution 4.77. The four different matrices we need are

$$\mathbf{A} = \begin{pmatrix} \mathbf{1} & \mathbf{0} & \mathbf{2} \\ \mathbf{-3} & \mathbf{4} & \mathbf{6} \\ \mathbf{-1} & \mathbf{-2} & \mathbf{3} \end{pmatrix};$$

The matrix where the first column is replaced by $\vec{b}$

$$\mathbf{A1} = \begin{pmatrix} \mathbf{6} & \mathbf{0} & \mathbf{2} \\ \mathbf{30} & \mathbf{4} & \mathbf{6} \\ \mathbf{8} & \mathbf{-2} & \mathbf{3} \end{pmatrix};$$

The next is the second with the second column replaced by $\vec{b}$

$$\mathbf{A2} = \begin{pmatrix} \mathbf{1} & \mathbf{6} & \mathbf{2} \\ -\mathbf{3} & \mathbf{30} & \mathbf{6} \\ -\mathbf{1} & \mathbf{8} & \mathbf{3} \end{pmatrix};$$

and finally the third with the third column replaced by $\vec{b}$

$$\mathbf{A3} = \begin{pmatrix} \mathbf{1} & \mathbf{0} & \mathbf{6} \\ -\mathbf{3} & \mathbf{4} & \mathbf{30} \\ -\mathbf{1} & -\mathbf{2} & \mathbf{8} \end{pmatrix};$$

Therefore using the different matrices in the ratios with the determinant of the original matrix delivers

$$\mathbf{x1} = \frac{|\mathbf{A1}|}{|A|}$$

$$-\frac{10}{11}$$

$$\mathbf{x2} = \frac{|\mathbf{A2}|}{|A|}$$

$$\frac{18}{11}$$

$$\mathbf{x3} = \frac{|\mathbf{A3}|}{|A|}$$

$$\frac{38}{11}$$

finally the result is $x_1 = -10/11$, $x_2 = 18/11$, and $x_3 = 38/11$ representing the solution.▲

Remark 4.20. To solve a system of n equations in n unknown by using Cramer's rule, it is necessary to evaluate $n+1$ determinants of $n \times n$ matrices. For systems with more than three equations, Gaussian elimination is far more efficient. However, Cramer's rule does give a formula for the solution if the determinant of the coefficient matrix is nonzero.

In the following sections we will show that the determinant of a square matrix can be evaluated by reducing the matrix to row-echelon form. This method is important since it is the most computationally efficient way to find the determinant of a general matrix.

4.5.1.5 Basic Theorems

We begin with a fundamental theorem that will lead us to an efficient procedure for evaluating the determinant of a matrix of any order n.

Theorem 4.39. *Zero Determinant*

Let A be a square matrix. If A has a row of zeros or columns of zeros, then det(A) = 0.■

Proof 4.39. The determinant of A found by cofactor expansion along the row or column of all zeros is

$$\det(A) = 0\,C_1 + 0\,C_2 + \ldots + 0\,C_n$$

where $C_1,\ C_2,\ \ldots,\ C_n$ are the cofactors for that row or column. Hence $\det(A) = 0$.

QED

Here is another useful theorem.

Theorem 4.40. *Determinant of A^T*

Let A be a square matrix. Then $det(A) = det\left(A^T\right)$. ■

Proof 4.40. The determinant of A found by cofactor expansion along its first row is the same as the determinant of A^T found by cofactor expansion along its first column.

QED

Remark 4.21. Because of Theorem 4.40, nearly every theorem about determinants that contains the word row in its statement is also true when the word column is substituted for row. To prove a column statement, one need only transpose the matrix in question, to convert the column statement to a row statement, and then apply the corresponding known result for rows.

4.5.1.6 Elementary Row Operations

The next theorem shows how an elementary row operation on a matrix affects the value of its determinant.

Theorem 4.41. *Row Operation*

Let A be an $n \times n$ matrix.

a) If B is the matrix that results when a single row or single column of A is multiplied by a scalar k, then

$$\det(B) = k\,\det(A). \tag{4.114}$$

b) If B is the matrix that results when two rows or two columns of A are interchanged, then

$$\det(B) = -\det(A). \tag{4.115}$$

c) If B is the matrix that results when a multiple of one row of A is added to another row or when a multiple of one column is added to another column, then

$$\det(B) = \det(A).\blacksquare \tag{4.116}$$

We omit the proof but give the following example that illustrates the theorem for 3×3 determinants.

Example 4.78. Determinants

A general 3×3 matrix is given by

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix};$$

The determinant of the matrix B which results from A by multiplying one row with a constant k is given by

$$\boldsymbol{B} = \begin{pmatrix} a_{11}\,k & a_{12}\,k & a_{13}\,k \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix};$$

$$|\boldsymbol{A}|$$

$$-a_{13}\,a_{22}\,a_{31} + a_{12}\,a_{23}\,a_{31} + a_{13}\,a_{21}\,a_{32} - a_{11}\,a_{23}\,a_{32} - a_{12}\,a_{21}\,a_{33} + a_{11}\,a_{22}\,a_{33}$$

$$|\boldsymbol{B}|$$

$$a_{13}\,a_{22}\,a_{31}\,(-k) + a_{12}\,a_{23}\,a_{31}\,k + a_{13}\,a_{21}\,a_{32}\,k - a_{11}\,a_{23}\,a_{32}\,k - a_{12}\,a_{21}\,a_{33}\,k + a_{11}\,a_{22}\,a_{33}\,k$$

which is a result of the cofactor expansion with respect to the first row. If we interchange two rows in A then

$$\boldsymbol{B} = \begin{pmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{pmatrix};$$

$$|\boldsymbol{A}| + |\boldsymbol{B}|$$

$$0$$

This indicated that for two rows interchanged $\det(B) = -\det(A)$.

$$\boldsymbol{B} = \begin{pmatrix} a_{21}\,k + a_{11} & a_{22}\,k + a_{12} & a_{23}\,k + a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix};$$

$$|\boldsymbol{B}| - |\boldsymbol{A}|$$

$$0$$

The result shows that the two determinants have the same value so that $\det(B) = \det(A)$.▲

4.5.1.7 Elementary Matrices

Recall that an elementary matrix results from performing a single elementary row operation on an identity matrix; thus, if we let $A = I_n$ in Theorem 4.41, so that we have $\det(A) = \det(I_n) = 1$, then the matrix B is an elementary matrix, and the theorem yields the following result about determinants of elementary matrices.

Theorem 4.42. *Row Operation*

Let E be an $n \times n$ elementary matrix.

a) If E results from multiplying a row of I_n by k, then det(E) = k.

b) If E results from interchanging two rows of I_n, then det(E) = −1.

c) If E results from adding a multiple of one row of I_n to another, then det(E) = 1.■

Example 4.79. Determinants of Elementary Matrices

The following determinants of elementary matrices illustrate Theorem 4.42.

$$\mathbf{A} = \begin{pmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{3} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \end{pmatrix}; |\mathbf{A}|$$

3

A constant in a single row occurs which is not equal to 1 then the result is det(A) = k det(I).

$$\mathbf{A} = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix}; |\mathbf{A}|$$

−1

The interchange of two rows changes the sign of the determinant.

$$\mathbf{A} = \begin{pmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{6} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \end{pmatrix}; |\mathbf{A}|$$

1

Any additional off diagonal element does not affect the determinant.▲

4.5.1.8 Matrices with Proportional Rows or Columns

If a square matrix A has two proportional rows, then a row of zeros can be introduced by adding a suitable multiple of one of the rows to the other. Similarly for columns. But adding a multiple of one row or column to another does not change the determinant, so from Theorem 4.39, we must have det(A) = 0. This proves the following theorem.

Theorem 4.43. *Row Operation*

If A is a square matrix with two proportional rows or two proportional columns, then det(A) = 0.■

Example 4.80. Zero Rows

The following computations illustrates the content of the theorem

$$\mathbf{A} = \begin{pmatrix} \mathbf{1} & \mathbf{-4} & \mathbf{3} & \mathbf{6} \\ \mathbf{16} & \mathbf{1} & \mathbf{5} & \mathbf{2} \\ \mathbf{4} & \mathbf{-16} & \mathbf{1} & \mathbf{24} \\ \mathbf{2} & \mathbf{-8} & \mathbf{6} & \mathbf{12} \end{pmatrix}; |\mathbf{A}|$$

0

If a matrix has two rows which are proportional then the determinant is zero.

$$\mathbf{A} = \begin{pmatrix} \mathbf{3} & \mathbf{12} \\ \mathbf{-6} & \mathbf{-24} \end{pmatrix}; |\mathbf{A}|$$

0

▲

4.5.1.9 Evaluating Matrices by Row Reduction

We shall now give a method for evaluating determinants that involves substantially less computation than the cofactor expansion method. The idea of the method is to reduce the given matrix to upper triangular form by elementary row operations, then compute the determinant of the upper triangular matrix which is an easy computation, and then relate that determinant to that of the original matrix. Here is an example:

Example 4.81. Triangle Diagonalization (Tridiagonalization)

Evaluate det(A) where

$$\mathbf{A} = \begin{pmatrix} \mathbf{0} & \mathbf{1} & \mathbf{5} \\ \mathbf{3} & \mathbf{-6} & \mathbf{9} \\ \mathbf{2} & \mathbf{6} & \mathbf{1} \end{pmatrix};$$

Solution 4.81. We will reduce A to row-echelon form which is upper triangular and apply the different theorems.

$$\det(A) = \begin{vmatrix} 0 & 1 & 5 \\ 3 & -6 & 9 \\ 2 & 6 & 1 \end{vmatrix} = -\begin{vmatrix} 3 & -6 & 9 \\ 0 & 1 & 5 \\ 2 & 6 & 1 \end{vmatrix} = -3\begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 2 & 6 & 1 \end{vmatrix}$$

$$= -3\begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 10 & -5 \end{vmatrix} = -3\begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & -55 \end{vmatrix} = -3\,(-55)\,(1) = 165. \blacktriangle$$

Remark 4.22. The method of row reduction is well suited for computer evaluation of determinants because it is computationally efficient and easily programmed. However, cofactor expansion is often easier for paper and pencil calculations.

4.5.2 Properties of Determinants

In this section we will develop some of the fundamental properties of the determinant function. Our work here will give us some further insight into the relationship between a square matrix and its determinant. One of the immediate consequences of this material will be the determinant test for the invertibility of a matrix.

4.5.2.1 Basic Properties of Determinants

Suppose that A and B are $n \times n$ matrices and k is any scalar. We begin by considering possible relationships between $\det(A)$, $\det(B)$, and $\det(k\,A)$, $\det(A+B)$, and $\det(A.B)$.

Since a common factor of any row of a matrix can be moved through the determinant sign, and since each of the n rows in $k\,A$ has a common factor of k, we obtain

$$\det(k\,A) = k^n \det(A). \tag{4.117}$$

For example,

$$\begin{vmatrix} k\,a_{11} & k\,a_{12} & k\,a_{13} \\ k\,a_{21} & k\,a_{22} & k\,a_{23} \\ k\,a_{31} & k\,a_{32} & k\,a_{33} \end{vmatrix} = k^3 \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}. \tag{4.118}$$

Unfortunately, no simple relationship exists among $\det(A)$, $\det(B)$, and $\det(A+B)$. In particular, we emphasize that $\det(A+B)$ will usually not be equal to $\det(A)+\det(B)$. The following example illustrates this fact.

Example 4.82. $\det(A + B) \neq \det(A) + \det(B)$

Given the two matrices

$$\boldsymbol{A} = \begin{pmatrix} \mathbf{2} & \mathbf{6} \\ \mathbf{1} & \mathbf{9} \end{pmatrix}; \boldsymbol{B} = \begin{pmatrix} \mathbf{-1} & \mathbf{5} \\ \mathbf{-7} & \mathbf{3} \end{pmatrix};$$

Find the determinant of the sum of the matrices and the sum of the determinants.

Solution 4.82. First let us calculate the sum of the determinant by adding the two matrices

AB = *A* + *B*

$$\begin{pmatrix} 1 & 11 \\ -6 & 12 \end{pmatrix}$$

Then we determine the determinant by

|AB|

78

and the sum of the determinants is

$$|A| + |B|$$

44

The result demonstrates that the relation $\det(A+B) \neq \det(A) + \det(B)$ for this specific example.▲

However there is one important exception from this relation concerning sums of determinants. To obtain the general relation let us first examine the case with two 2×2 matrices that differ only in the second row:

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}; B = \begin{pmatrix} a_{11} & a_{12} \\ b_{21} & b_{22} \end{pmatrix};$$

If we now calculate the determinant of the sum of $A + B$ we find

$$|A| + |B|$$

$$-a_{12} a_{21} + a_{11} a_{22} - a_{12} b_{21} + a_{11} b_{22}$$

which is equal to

$$\left| \begin{pmatrix} a_{11} & a_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \end{pmatrix} \right|$$

$$-a_{12} a_{21} + a_{11} a_{22} - a_{12} b_{21} + a_{11} b_{22}$$

Thus for this special case the following relation holds

$$|A| + |B| == \left| \begin{pmatrix} a_{11} & a_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \end{pmatrix} \right|$$

True

This is a special case of the following general result

Theorem 4.44. *Determinant of a Sum*

Let A, B, and C be n×n matrices that differ only in a single row, say the rth, and assume that the rth row of C can be obtained by adding corresponding entries in the rth rows of A and B. Then

$$\det(C) = \det(A) + \det(B).$$

The same result holds for columns.■

4.5.2.2 Determinant of a Matrix Product

When one considers the complexity of the definitions of matrix multiplication and determinants, it would seem unlikely that any simple relationship should exist between them. This is what makes the elegant simplicity of the following result so surprising: we will show that if A and B are square matrices of the same size, then

$$\det(A.B) = \det(A)\det(B). \tag{4.119}$$

The proof of this relation is not straight forward, so we have to give some preliminary results first. We begin with the special case of (4.119) in which A is an elementary matrix. Because this special case is only a prelude to (4.119), we call it a corollary.

Corollary 4.1. *Determinant of a Product E.B*

If B is a $n \times n$ matrix and E is an $n \times n$ elementary matrix, then

$$\det(E.B) = \det(E)\det(B).\blacksquare$$

Proof 4.1. We shall consider three cases, each depending on the row operation that produces matrix E.

Case 1. If E results from multiplying a row of I_n by k, then by Theorem 4.28, $E.B$ results from B by multiplying a row by k; so from Theorem 4.41a we have

$$\det(E.B) = k\det(B).$$

But from Theorem 4.42a we have $\det(E) = k$, so

$$\det(E.B) = \det(E)\det(B).$$

Case 2 and 3. The proofs of these cases where E results from interchanging two rows of I_n or from adding a multiple of one row to another follows the same pattern as Case 1 and therefore are left as exercises.

QED

Remark 4.23. It follows by repeated application of Corollary 4.1 that if B is an $n \times n$ matrix and $E_1, E_2, \ldots, E_r$ are $n \times n$ elementary matrices, then

$$\det(E_1.E_2.E_3 \ldots E_r.B) = \det(E_1)\det(E_2) \ldots \det(E_r)\det(B).$$

4.5.2.3 Determinant Test for Invertibility

The next theorem provides an important criterion for invertibility in terms of determinants, and it will be used in proving (4.119).

Theorem 4.45. *Invertibility*

A square matrix A is invertible if and only if $\det(A) \neq 0.\blacksquare$

Proof 4.45. Let R be the reduced row-echelon form of A. As a preliminary step, we will show that $\det(A)$ and $\det(R)$ are both nonzero: Let $E_1, E_2, \ldots, E_r$ be the elementary matrices that correspond to the elementary row operations that produce R from A. Thus

$$R = E_r \ldots .E_2.E_1.A \tag{4.120}$$

and from the above remark we have

$$\det(R) = \det(E_r) \ldots \det(E_2)\det(E_1)\det(A). \tag{4.121}$$

But from Theorem 4.45 the determinants of the elementary matrices are all nonzero. Keep in mind that multiplying a row by zero is not an allowed elementary row operation, so $k \neq 0$ in this application of Theorem 4.45. Thus, it follows from (4.120) that $\det(A)$ and $\det(R)$ are both zero or both nonzero. Now to the main body of the proof.

If A is invertible, then by Theorem 4.34 we have $R = I$, so $\det(R) = 1 \neq 0$ and consequently $\det(A) \neq 0$. Conversely, if $\det(A) \neq 0$, then $\det(R) \neq 0$, so R cannot have a row of zeros. It follows from Theorem 4.20 that $R = I$, so A is invertible.

QED

It follows from Theorem 4.45 and 4.43 that a square matrix with two proportional rows or columns is not invertible.

Example 4.83. Determinant Test for A^{-1}

Since two rows are proportional to each other the following matrix is not invertible

$$\mathbf{A} = \begin{pmatrix} 1 & 6 & 7 \\ 3 & 0 & 6 \\ -2 & -12 & -14 \end{pmatrix};$$

This can be checked by determining the determinant of A

$$|A|$$

$$0$$

This means that A^{-1} does not exist.▲

We are now ready for the result concerning products of matrices.

Theorem 4.46. *Product Rule*

If A and B are square matrices of the same size, then

$$\det(A.B) = \det(A)\det(B).\blacksquare$$

We state this theorem without proof but motivate the result by an example.

Example 4.84. $\det(A.B) = \det(A)\det(B)$

Consider the matrices

$$\mathbf{A} = \begin{pmatrix} 2 & 5 \\ 6 & 8 \end{pmatrix}; \mathbf{B} = \begin{pmatrix} -1 & 5 \\ 2 & 3 \end{pmatrix};$$

The product is given by

$$\mathbf{AB} = \mathbf{A.B}$$

$$\begin{pmatrix} 8 & 25 \\ 10 & 54 \end{pmatrix}$$

The related determinants are

detA = |A|; detB = |B|; detAB = |AB|;

The values are

detA detB

182

and

detAB

182

which demonstrates that the relation $\det(A.B) = \det(A)\det(B)$ holds.▲

The following relation gives a useful relationship between the determinant of an invertible matrix and the determinant of its inverse.

Theorem 4.47. *$det(A^{-1})$*

If A is invertible, then

$$\det\left(A^{-1}\right) = \frac{1}{\det(A)}.■$$

Proof 4.46. Since $A^{-1}.A = I$, it follows that $\det\left(A^{-1}.A\right) = \det(I)$. Therefore, we must have $\det\left(A^{-1}\right)\det(A) = 1$. Since $\det(A) \neq 0$, the proof can be completed by dividing through by $\det(A)$.

QED

4.5.2.4 Eigenvalues and Eigenvectors

If A is a $n \times n$ matrix and $\vec{x}$ is a vector in $\mathbb{R}^n$, then $A.\vec{x}$ is also a vector in $\mathbb{R}^n$, but usually there is no simple geometric relationship between $\vec{x}$ and $A.\vec{x}$. However, in the special case where $\vec{x}$ is a nonzero vector and $A.\vec{x}$ is a scalar multiple of $\vec{x}$, a simple geometric relationship occurs. For example, if A is a 2×2 matrix, and if $\vec{x}$ is a nonzero vector such that $A.\vec{x}$ is a scalar multiple of $\vec{x}$, say $A.\vec{x} = \lambda\,\vec{x}$, then each vector on the line through the origin determined by $\vec{x}$ gets mapped back onto the same line under multiplication by A.

Nonzero vectors that get mapped into scalar multiples of themselves under a linear operator arise naturally in the study of genetics, quantum mechanics, vibrations, population dynamics, and economics, as well as in geometry.

Many applications of linear algebra are concerned with systems of n linear equations in n unknown that are expressed in the form

$$A.\vec{x} = \lambda\,\vec{x} \tag{4.122}$$

where λ is a scalar. Such systems are some sort of homogeneous linear systems, since equation (4.122) can be written as $\lambda \vec{x} - A.\vec{x} = 0$ or, by inserting an identity matrix and factoring, as

$$(\lambda I - A).\vec{x} = 0. \tag{4.123}$$

Here is an example:

Example 4.85. Equations of the form $(\lambda I - A).\vec{x} = 0$

The linear system

$$\begin{array}{rcrcl} x_1 & + & 3x_2 & = & \lambda x_1 \\ 4x_1 & + & 2x_2 & = & \lambda x_2 \end{array}$$

can be written in matrix form as

$$\textbf{eqs} = \begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix}.\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \lambda \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\begin{pmatrix} x_1 + 3x_2 \\ 4x_1 + 2x_2 \end{pmatrix} = \begin{pmatrix} \lambda x_1 \\ \lambda x_2 \end{pmatrix}$$

Which is equivalent to

$$\textbf{eqse} = \begin{pmatrix} \lambda - 1 & 3 \\ 4 & \lambda - 2 \end{pmatrix}.\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix};$$

which is of the form $(A - \lambda I).\vec{x} = 0$ defining a so called Eigenvalue problem.▲

The primary problem of interest for linear systems of the form (4.123) is to determine those values of λ for which the system has a nontrivial solution; such a value λ is called a characteristic value or an eigenvalue of A. If λ is an eigenvalue of A, then the nontrivial solution of (4.123) are called eigenvectors of A corresponding to λ.

It follows from Theorem 4.45 that the system $(\lambda I - A)\vec{x} = 0$ has a nontrivial solution if and only if

$$\det(\lambda I - A) = 0. \tag{4.124}$$

This is called the characteristic equation of A; the eigenvalues of A can be found by solving this equation for λ.

Example 4.86. Eigenvalues and Eigenvectors

Find the eigenvalues and corresponding eigenvectors of the matrix

$$\boldsymbol{A} = \begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix};$$

Solution 4.86. The characteristic equation of A is generated by subtracting the part λI from A and calculating the determinant.

$$\textbf{charEq} = \left| A - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right| = 0$$

$$\lambda^2 - 3\lambda - 10 = 0$$

which is a second order polynomial in λ. The solutions of this second order equation are

sol = Solve[charEq, λ]

$\{\{\lambda \to -2\}, \{\lambda \to 5\}\}$

So $\lambda = -2$ and $\lambda = 5$ are the eigenvalues of this matrix.

By definition $\vec{x} = (x_1, x_2)$ is an eigenvector of A if and only if $\vec{x}$ is a nontrivial solution of $(\lambda I - A).\vec{x} = 0$; that is

$$\mathbf{evEq} = \left(A - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right).\begin{pmatrix} \mathrm{x1} \\ \mathrm{x2} \end{pmatrix} == \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 3\,\mathrm{x2} + \mathrm{x1}\,(1-\lambda) \\ 4\,\mathrm{x1} + \mathrm{x2}\,(2-\lambda) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

So if $\lambda = -2$ is inserted into this equation we find

evEq /. sol⟦1⟧

$$\begin{pmatrix} 3\,\mathrm{x1} + 3\,\mathrm{x2} \\ 4\,\mathrm{x1} + 4\,\mathrm{x2} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Solving this system yields $x_1 = -t$ and $x_2 = t$, so the eigenvector corresponding to $\lambda = -2$ is $\vec{x}_{-2} = (-t, t)$ where $t \in \mathbb{R}$.

For the second eigenvalue we can derive the system

evEq /. sol⟦2⟧

$$\begin{pmatrix} 3\,\mathrm{x2} - 4\,\mathrm{x1} \\ 4\,\mathrm{x1} - 3\,\mathrm{x2} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

which results to the eigenvector $\vec{x}_5 = \left(\frac{3}{4}\,t,\, t\right)$ with $t \in \mathbb{R}$.▲

4.5.2.5 The Geometry of Eigenvectors

The definition of an eigenvector has a simple geometric interpretation. If $A.\vec{x} = \lambda\,\vec{x}$ then the vector $A.\vec{x}$ will point in the same or opposite direction as $\vec{x}$ (depending on whether $\lambda > 0$ or $\lambda < 0$), and the magnitude of the eigenvalue will correspond to the stretch/shrink factor. For example, the matrix $A = \begin{pmatrix} 5 & -3 \\ -4 & 9 \end{pmatrix}$ has the eigenvector $\vec{u} = (3, 2)$ with associated eigenvalue $\lambda = 3$. Here is a picture of the vectors $\vec{u}$ and $A.\vec{u}$ drawn in red and blue, respectively. Note that $A.\vec{u}$ is in the same direction as $\vec{u}$ but is 3 times as long.

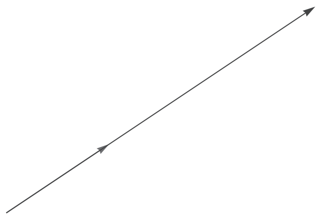


Figure 4.34. Direction of the eigenvector $\vec{u}$ and the vector $A.\vec{u}$.

Example 4.87. Eigenvalues and Eigenvectors

Here is an example where we can find the eigenvalues and eigenvectors of a matrix simply by picturing how the matrix behaves as a geometric transformation. Let R be the matrix that reflects vectors across the line $y = x$:

$$R = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

The figure below shows what R does to several vectors. Use this picture to determine all eigenvectors of R and their associated eigenvalues.

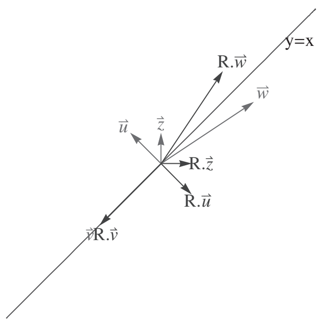


Figure 4.35. Reflection of different vectors (red) about the diagonal axis of the coordinate system. The transformed vectors are shown in blue.▲

Solution 4.87. R reflects vectors that lie along the line $y = -x$ to their negatives, and it keeps vectors that lie along the line $y = x$ fixed in place. Thus, every nonzero vector along the line $y = -x$ (such as $\vec{u}$ above) is an eigenvector with eigenvalue $\lambda = -1$, and every nonzero vector along the line $y = x$ (such as $\vec{v}$ above) is an eigenvector with eigenvalue $\lambda = 1$.▲

Even if we cannot figure out the eigenvalues of a matrix from our knowledge of its behavior as a geometric transformation, we will find pictures such as the one above quite helpful. Let's consider the matrix A given by

$$A = \begin{pmatrix} 1 & 1 \\ \frac{1}{2} & \frac{3}{2} \end{pmatrix} \tag{4.125}$$

as a matrix transformation. The vectors $\vec{u} = (-2, 1)$ and $\vec{v} = (1, 1)$ are eigenvectors of A with different eigenvalues. We will draw the input vectors u and v in red and their corresponding output vectors in blue. See if you can guess the eigenvalue for each eigenvector just by looking at the picture.

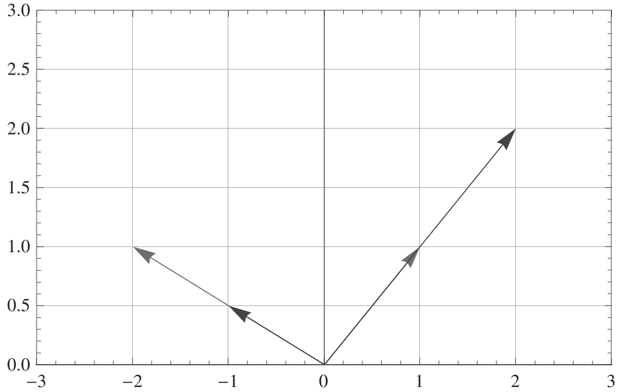


Figure 4.36. Structure of eigenvectors for matrix A given in (4.125).▲

Sometimes, it is easier to visualize eigenvectors by putting the tail of the output vector $A.\vec{x}$ at the head of the input vector $\vec{x}$ as shown in Figure 4.37.

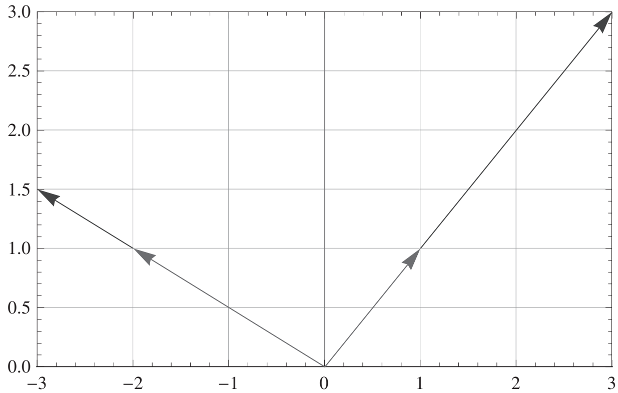


Figure 4.37. Vector space structure of functions as subspaces.▲

Notice that the blue output vector in the second quadrant is half the length of the corresponding red input vector, and the other output vector is twice the length of its input. Hence, the eigenvalues associated with $\vec{u}$ and $\vec{v}$ are $\frac{1}{2}$ and 2, respectively.

4.5.3 Tests and Exercises

The following two subsections serve to test your understanding of the last section. Work first on the test examples and then try to solve the exercises.

4.5.3.1 Test Problems

T1. What is a determinant?

T2. Is a determinant a scalar, a vector, or a matrix?

T3. What is Cramer's rule and what is it useful for?

T4. How are minors and cofactors defined?

T5. Explain how to calculate the adjoint matrix of a matrix.

T6. What are Eigenvalues and Eigenvectors?

4.5.3.2 Exercises

E1. Evaluate the determinants of the following 2×2 matrices.

a. $\begin{pmatrix} 3 & 6 \\ 7 & -1 \end{pmatrix}$,

b. $\begin{pmatrix} 5 & -3 \\ -2 & 1 \end{pmatrix}$,

c. $\begin{pmatrix} 1 & 6 \\ 2 & -1 \end{pmatrix}$,

d. $\begin{pmatrix} 1 & 6 \\ 2 & -1 \end{pmatrix}$,

e. $\begin{pmatrix} -2 & -4 \\ 5 & 2 \end{pmatrix}$,

E2. Evaluate the determinants of the following 2×2 matrices.

a. $\begin{pmatrix} 1 & 5 \\ 0 & 3 \end{pmatrix}$,

b. $\begin{pmatrix} 3 & -2 \\ -1 & 0 \end{pmatrix}$,

c. $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$,

d. $\begin{pmatrix} 1 & 5 \\ 1 & -3 \end{pmatrix}$,

E3. Given the following matrix A

$$A = \begin{pmatrix} 1 & 2 & -3 \\ 5 & 4 & 8 \\ -2 & 3 & -1 \end{pmatrix} \quad (1)$$

Find the following minors and cofactors of A

a. M_{11} and C_{11},

b. M_{23} and C_{23},

c. M_{21} and C_{21},

d. M_{33} and C_{33}.

Given the following matrix A

$$A = \begin{pmatrix} -1 & 0 & 7 \\ 2 & 8 & 9 \\ -3 & 0 & -1 \end{pmatrix} \tag{2}$$

Find the following minors and cofactors of A

a. M_{11} and C_{11},
b. M_{23} and C_{23},
c. M_{21} and C_{21},
d. M_{33} and C_{33}.

E5. Given the following matrix A

$$A = \begin{pmatrix} 1 & 2 & -3 & 0 \\ 5 & 4 & 8 & 2 \\ -2 & 3 & -1 & -3 \\ 3 & 6 & -2 & 7 \end{pmatrix} \tag{3}$$

Find the following minors and cofactors of A

a. M_{12} and C_{12},
b. M_{24} and C_{24},
c. M_{42} and C_{42},
d. M_{33} and C_{33}.

E6. Evaluate the determinants of the following matrices

a. $\begin{pmatrix} -2 & 3 & 4 \\ 8 & 2 & -1 \\ 5 & 7 & 1 \end{pmatrix}$,

b. $\begin{pmatrix} -2 & 3 & 4 & -1 \\ 8 & 2 & -1 & 6 \\ 5 & 7 & 1 & 3 \\ 3 & 9 & 12 & 0 \end{pmatrix}$,

c. $\begin{pmatrix} -2 & 3 & 4 & -1 & 1 \\ 8 & 2 & -1 & 6 & 0 \\ 5 & 7 & 1 & 3 & 2 \\ 3 & 9 & 12 & 0 & 0 \\ 3 & 6 & 5 & -6 & -3 \end{pmatrix}$,

d. $\begin{pmatrix} -1 & 2 & 4 & -1 \\ 8 & 0 & -9 & 4 \\ 5 & 7 & 0 & 3 \\ 5 & 7 & 12 & 5 \end{pmatrix}$,

e. $\begin{pmatrix} -1 & 3 & 7 \\ 8 & 1 & -6 \\ -5 & 0 & -1 \end{pmatrix}$,

E7. Evaluate the determinants of the following matrices using as little computations as possible.

a. $\begin{pmatrix} 1 & 6 & 7 & 0 \\ -4 & 3 & -2 & 0 \\ 1 & 2 & 1 & -1 \\ 0 & 0 & 2 & 0 \end{pmatrix}$

$$\begin{pmatrix} 1 & 6 & 7 & 6 \\ -4 & 3 & -2 & 2 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 2 & 0 \end{pmatrix}$$

c. $\begin{pmatrix} -1 & -3 & 0 & 1 \\ -4 & 3 & -2 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 2 & 2 & -1 \end{pmatrix}$

E8. Solve the following equations for x

a. $\det\begin{pmatrix} x+2 & 4 \\ -3 & x-2 \end{pmatrix} = 5.$

b. $\det\begin{pmatrix} x-3 & 4+x \\ 3 & x+5 \end{pmatrix} = 0.$

c. $\det\begin{pmatrix} x^2+3 & 4 \\ -2 & x^2-3 \end{pmatrix} = 0.$

E9. Solve the following equations for λ

a. $\det\begin{pmatrix} 5-\lambda & 4 & -3 \\ -3 & -2-\lambda & 1 \\ 6 & -2 & 1-\lambda \end{pmatrix} = 0.$

b. $\det\begin{pmatrix} 1-\lambda & 2 & -1 \\ -3 & 2-\lambda & -1 \\ 8 & -2 & 2-\lambda \end{pmatrix} = 0.$

c. $\det\begin{pmatrix} 6-\lambda & 1 & -13 \\ -3 & -6-\lambda & 15 \\ -6 & -2 & -7-\lambda \end{pmatrix} = 0.$

E10 Why would you expect the following determinant to have the same value whatever values are given t a and b?

$$\det\begin{pmatrix} -3 & 5 & a \\ 12 & 1 & b \\ 0 & 0 & 2 \end{pmatrix} \tag{4}$$

4.6 Row Space, Column Space, and Nullspace

In this section we will study three important vector spaces that are associated with matrices. Our work here will provide us with a deeper understanding of the relationships between the solutions of a linear system of equations and properties of its coefficient matrix.

We start wit some definitions.

Definition 4.25. *Row and Column Vectors*

For an $m \times n$ *matrix*

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$

the vectors

$\vec{r}_1 = (a_{11}, \ a_{12}, \ \dots, \ a_{1n}), \quad \vec{r}_2 = (a_{21}, \ a_{22}, \ \dots, \ a_{2n}), \quad \dots \vec{r}_m = (a_{m1}, \ a_{m2}, \ \dots, \ a_{mn})$ *in* $\mathbb{R}^n$ *formed from the rows of A are called the row vectors of A, and the vectors*

$$\vec{c}_1 = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix}, \quad \vec{c}_2 = \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix}, \dots \vec{c}_n = \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix}$$

in $\mathbb{R}^m$ *formed from the columns of A are called the column vectors of A.* ■

The following definition defines three important vector spaces associated with a matrix.

Definition 4.26. *Vector Spaces of A*

If A is an $m \times n$ *matrix, then the subspaces of* $\mathbb{R}^n$ *spanned by the row vectors of A is called the row space of A, and the subspace of* $\mathbb{R}^m$ *spanned by the column vectors of A is called the column space of A. The solution space of the homogeneous system of equations* $A.\vec{x} = 0$*, which is a subspace of* $\mathbb{R}^n$*, is called the nullspace of A.* ■

In this section and the next we will be concerned with the following two general questions:

- What relationship exist between the solutions of a linear system $A.\vec{x} = \vec{b}$ and the row space, column space, and nullspace of the coefficient matrix A?
- What relationship exists among the row space, column space, and nullspace of a matrix?

To investigate the first of these questions, suppose that

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \quad \text{and} \quad \vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}. \tag{4.126}$$

It follows that if $\vec{c}_1, \vec{c}_2, \dots, \vec{c}_n$ denote the column vectors of A, then the product $A.\vec{x}$ can be expressed as a linear combination of these column vectors with coefficients from $\vec{x}$; that is,

$$A.\vec{x} = x_1\,\vec{c}_1 + x_2\,\vec{c}_2 + \dots + x_n\,\vec{c}_n. \tag{4.127}$$

Thus a linear system, $A.\vec{x} = \vec{b}$, of m equations in n unknowns can be written as

$$x_1\,\vec{c}_1 + x_2\,\vec{c}_2 + \dots + x_n\,\vec{c}_n = \vec{b} \tag{4.128}$$

from which we conclude that $A.\vec{x} = \vec{b}$ is consistent if and only if $\vec{b}$ is expressible as a linear combination of the column vectors of A or, equivalently, if and only if $\vec{b}$ is in the column space of A. This yields the following theorem.

Theorem 4.48. *Consistent Equations*

A system of linear equations $A.\vec{x} = \vec{b}$ is consistent if and only if $\vec{b}$ is in the column space of A.■

Example 4.88. A Vector $\vec{b}$ in the Column Space of A

Let $A.\vec{x} = \vec{b}$ be the linear system with

$$\mathbf{A} = \begin{pmatrix} -1 & 3 & 2 \\ 1 & 2 & -3 \\ 2 & 1 & -2 \end{pmatrix}; \mathbf{b} = \{1, -9, -3\};$$

Show that $\vec{b}$ is in the column space of A, and express $\vec{b}$ as a linear combination of the column vectors in A.

Solution 4.88. Solving the system by Gaussian elimination yields

```
sol = Solve[Thread[A.{x, y, z} == b], {x, y, z}]
```

```
{{x → 2, y → −1, z → 3}}
```

Since the system is consistent, $\vec{b}$ is in the column space of A. Moreover, we find

```
x {−1, 1, 2} + y {3, 2, 1} + z {2, −3, −2} == b /. sol
```

```
{True}
```

which in detail is the left hand side

```
x {−1, 1, 2} + y {3, 2, 1} + z {2, −3, −2} /. sol
```

$$\begin{pmatrix} 1 & -9 & -3 \end{pmatrix}$$

which is equal to $\vec{b}$.▲

The next theorem establishes a fundamental relationship between the solutions of a non homogeneous linear system $A.\vec{x} = \vec{b}$ and those of the corresponding homogeneous linear system $A.\vec{x} = 0$ with the same coefficient matrix.

Theorem 4.49. *Consistent Equations*

If $\vec{x}_0$ denotes any single solution of a consistent linear system $A.\vec{x} = \vec{b}$, and if $\vec{v}_1, \vec{v}_2, \vec{v}_3, \ldots, \vec{v}_k$ form a basis for the nullspace of A—that is, the solution space of the homogeneous system $A.\vec{x} = 0$—then every solution of $A.\vec{x} = \vec{b}$ can be expressed in the form

$$\vec{x} = \vec{x}_0 + c_1\,\vec{v}_1 + c_2\,\vec{v}_2 + \ldots + c_k\,\vec{v}_k \tag{4.129}$$

and, conversely, for all choices of scalars $c_1, c_2, \ldots, c_k$, the vector $\vec{x}$ in this formula is a solution of $A.\vec{x} = \vec{b}$. ■

Proof 4.49. Assume that $\vec{x}_0$ is any fixed solution of $A.\vec{x} = \vec{b}$ and that $\vec{x}$ is an arbitrary solution. Then

$$A.\vec{x}_0 = \vec{b} \qquad \text{and} \quad A.\vec{x} = \vec{b}.$$

Subtracting these equations yields

$$A.\vec{x} - A.\vec{x}_0 = 0 \qquad \text{or} \qquad A.(\vec{x} - \vec{x}_0) = 0$$

which shows that $\vec{x} - \vec{x}_0$ is a solution of the homogeneous system $A.\vec{x} = 0$. Since $\vec{v}_1, \vec{v}_2, \vec{v}_3, \ldots, \vec{v}_k$ is a basis for the solution space of this system, we can express $\vec{x} - \vec{x}_0$ as a linear combination of these vectors, say

$$\vec{x} - \vec{x}_0 = c_1\,\vec{v}_1 + c_2\,\vec{v}_2 + \ldots + c_k\,\vec{v}_k.$$

Thus,

$$\vec{x} = \vec{x}_0 + c_1\,\vec{v}_1 + c_2\,\vec{v}_2 + \ldots + c_k\,\vec{v}_k$$

which proves the first part of the theorem. Conversely, for all choices of the scalars $c_1, c_2, \ldots, c_k$ in (4.127), we have

$$A.\vec{x} = A.(\vec{x}_0 + c_1\,\vec{v}_1 + c_2\,\vec{v}_2 + \ldots + c_k\,\vec{v}_k)$$

or

$$A.\vec{x} = A.\vec{x}_0 + c_1\,A.\vec{v}_1 + c_2\,A.\vec{v}_2 + \ldots + c_k\,A.\vec{v}_k.$$

But $\vec{x}_0$ is a solution of the non homogeneous system, and $\vec{v}_1, \vec{v}_2, \vec{v}_3, \ldots, \vec{v}_k$ are solutions of the homogeneous system, so that the last equation implies

$$A.\vec{x} = \vec{b} + 0 + 0 + \ldots + 0$$

which shows that $\vec{x}$ is a solution of $A.\vec{x} = \vec{b}$.

QED

There is some terminology associated with formula (4.127). The vector $\vec{x}_0$ is called a particular solution of $A.\vec{x} = \vec{b}$. The expression $\vec{x}_0 + c_1\,\vec{v}_1 + c_2\,\vec{v}_2 + \ldots + c_k\,\vec{v}_k$ is called the general solution of $A.\vec{x} = \vec{b}$, and the expression $c_1\,\vec{v}_1 + c_2\,\vec{v}_2 + \ldots + c_k\,\vec{v}_k$ is called the general solution of $A.\vec{x} = 0$. With this terminology relation (4.127) states that the general solution of $A.\vec{x} = \vec{b}$ is the sum of any particular solution of $A.\vec{x} = \vec{b}$ and the general solution of $A.\vec{x} = 0$.

For linear systems with two or three unknowns, Theorem 4.49 has a nice geometric interpretation in $\mathbb{R}^2$ and $\mathbb{R}^3$. For example, consider the case where $A.\vec{x} = 0$ and $A.\vec{x} = \vec{b}$ are linear systems with two unknowns. The solution of $A.\vec{x} = 0$ form a subspace of $\mathbb{R}^2$ and hence constitute a line through the origin, the origin only, or all of $\mathbb{R}^2$. From Theorem 4.49, the solutions of $A.\vec{x} = \vec{b}$ can be obtained by adding any particular solution of $A.\vec{x} = \vec{b}$, say $\vec{x}_0$, to the solution of $A.\vec{x} = 0$. Assuming that $\vec{x}_0$ is positioned with its initial point at the origin, this has the geometric effect of translating the solution space of $A.\vec{x} = 0$ so that the point at the origin is moved to the tip of $\vec{x}_0$. This means that the solution vectors of $A.\vec{x} = \vec{b}$ form a line through the tip of $\vec{x}_0$, the point at the tip of $\vec{x}_0$, or all of $\mathbb{R}^2$. Similarly, for linear systems with three unknowns, the solutions of $A.\vec{x} = \vec{b}$ constitute a plane through the tip of any particular solution $\vec{x}_0$, a line through the tip of $\vec{x}_0$, the point at the tip of $\vec{x}_0$, or all of $\mathbb{R}^3$.

Example 4.89. General Solution of a Linear System

The following system of linear equations are given

$$\begin{array}{cccccccc}
x_1 & +3x_2 & -2x_3 & & +2x_5 & & = & 0 \\
2x_1 & +6x_2 & -5x_3 & -2x_4 & +4x_5 & -3x_6 & = & -1 \\
 & & 5x_3 & +10x_4 & & +15x_6 & = & 5 \\
2x_1 & +6x_2 & & +8x_4 & +4x_5 & +18x_6 & = & 6
\end{array}$$

find the solution of this system.

Solution 4.89. If we apply Gauss-Jordan elimination on this set of equations we find for the augmented matrix

$$\mathbf{A} = \begin{pmatrix} \mathbf{1} & \mathbf{3} & \mathbf{-2} & \mathbf{0} & \mathbf{2} & \mathbf{0} & \mathbf{0} \\ \mathbf{2} & \mathbf{6} & \mathbf{-5} & \mathbf{-2} & \mathbf{4} & \mathbf{-3} & \mathbf{-1} \\ \mathbf{0} & \mathbf{0} & \mathbf{5} & \mathbf{10} & \mathbf{0} & \mathbf{15} & \mathbf{5} \\ \mathbf{2} & \mathbf{6} & \mathbf{0} & \mathbf{8} & \mathbf{4} & \mathbf{18} & \mathbf{6} \end{pmatrix};$$

the row reduced echelon form by

rowEchelon = RowReduce[*A*]

$$\begin{pmatrix} 1 & 3 & 0 & 4 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

This results means that the solution of this system is given by

$x_6 = 1/3$, $x_5 = t$, $x_4 = s$, $x_2 = r$, and $x_3 = -2s$, $x_1 = -3r - 4s - 2t$. In vector notation this result reads

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \frac{1}{3} \end{pmatrix} + r \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} -4 \\ 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} -2 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

which is the general solution of the system.▲

4.6.1 Bases for Row Space, Column Space, and Nullspace

We first developed elementary row operations for the purpose of solving linear systems, and we know from that performing an elementary row operation on an augmented matrix does not change the solution set of the corresponding linear system. It follows that applying an elementary row operation to a matrix A does not change the solution set of the corresponding linear system $A.\vec{x} = 0$, or, stated another way, it does not change the nullspace of A. Thus we have the following theorem.

Theorem 4.50. *Invariance of Nullspace*

Elementary row operations do not change the nullspace of a matrix.■

Example 4.90. Basis for Nullspaces

Find the basis for the nullspace of

$$\mathbf{A} = \begin{pmatrix} \mathbf{2} & \mathbf{2} & \mathbf{-1} & \mathbf{0} & \mathbf{1} \\ \mathbf{-1} & \mathbf{-1} & \mathbf{2} & \mathbf{3} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} & \mathbf{-2} & \mathbf{0} & \mathbf{-1} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{1} \end{pmatrix};$$

This means find the solutions of $A.\vec{x} = 0$.

Solution 4.90. The nullspace is the solution of the homogeneous system $A.\vec{x} = 0$ with $\vec{x} = (x_1, x_2, x_3, x_4, x_5)$. These solution follow with

$$\mathbf{RowReduce}\left[\begin{pmatrix} \mathbf{2} & \mathbf{2} & \mathbf{-1} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{-1} & \mathbf{-1} & \mathbf{2} & \mathbf{3} & \mathbf{1} & \mathbf{0} \\ \mathbf{1} & \mathbf{1} & \mathbf{-2} & \mathbf{0} & \mathbf{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{0} \end{pmatrix}\right]$$

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

which tells us that the solution is given by

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -t-s \\ t \\ -s \\ 0 \\ s \end{pmatrix} = t \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} = t\,\vec{v}_1 + s\,\vec{v}_2.$$

So the two vectors $\vec{v}_1 = (-1, 1, 0, 0, 0)$ and $\vec{v}_2 = (-1, 0, -1, 0, 1)$ form a basis of the nullspace.▲

The following theorem is a companion to Theorem 4.50.

Theorem 4.51. *Invariance of Nullspace*

Elementary row operations do not change the row space of a matrix.■

Proof 4.51. Suppose that the row vectors of a matrix A are $\vec{r}_1$, $\vec{r}_2$, ..., $\vec{r}_m$, and let B be obtained from A by performing an elementary row operation. We shall show that every vector in the row space of B is also in the row space of A and that, conversely, every vector in the row space of A is in the row space of B. We can then conclude that A and B have the same row space.

Consider the possibilities: If the row operation is a row interchange, then B and A have the same row vectors and consequently have the same row space. If the row operation is multiplication of a row by a nonzero scalar or the addition of a multiple of one row to another, then the row vectors $\vec{r}_1^{\,'}$, $\vec{r}_2^{\,'}$, ..., $\vec{r}_m^{\,'}$ of B are linear combinations of $\vec{r}_1$, $\vec{r}_2$, ..., $\vec{r}_m$; thus they lie in the row space of A. Since a vector space is closed under addition and scalar multiplication, all linear combinations of $\vec{r}_1^{\,'}$, $\vec{r}_2^{\,'}$, ..., $\vec{r}_m^{\,'}$ will also lie in the row space of A. Therefore, each vector in the row space of B is in the row space of A.

Since B is obtained from A by performing a row operation, A can be obtained from B by performing the inverse operations. Thus the argument above shows that the row space of A is contained in the row space of B.

QED

In light of Theorem 4.50 and 4.51, one might anticipate that elementary row operations should not change the column space of a matrix. However, this is not so—elementary row operations can change the column space. For example, consider the matrix

$$A = \begin{pmatrix} 2 & 4 \\ 4 & 8 \end{pmatrix};$$

The second column is a scalar multiple of the first, so the column space of A consist of all scalar multiples of the first column vector. However, if we add -2 times the first row of A to the second row, we obtain

$$B = \begin{pmatrix} 2 & 4 \\ 0 & 0 \end{pmatrix};$$

Here again the second column is a scalar multiple of the first, so the column space of B consists of all scalar multiples of the first column vector. This is not the same as the column space of A.

Although elementary row operations can change the column space of a matrix, we shall show that whatever relationship of linear independence or linear dependence exist among the column vectors prior to the row operations will also hold for the corresponding columns of the matrix that results from that operation. To make this more precise, suppose a matrix B results from performing an elementary row operation on an $m \times n$ matrix A. By Theorem 4.49, the two homogeneous linear systems

$$A.\vec{x} = 0 \qquad \text{and} \qquad B.\vec{x} = 0 \tag{4.130}$$

have the same solution set. Thus the first system has a nontrivial solution if and only if the same is true of the second. But if the column vectors of A and B, respectively, are

$$\vec{c}_1, \vec{c}_2, \ldots, \vec{c}_m \qquad \text{and} \qquad \vec{c}_1^{\,'}, \vec{c}_2^{\,'}, \ldots, \vec{c}_m^{\,'} \tag{4.131}$$

Then from (4.128) the two systems can be written as

$$x_1 \vec{c}_1 + x_2 \vec{c}_2 + \ldots + x_n \vec{c}_n = 0 \tag{4.132}$$

and

$$x_1 \vec{c}_1^{\,'} + x_2 \vec{c}_2^{\,'} + \ldots + x_n \vec{c}_n^{\,'} = 0. \tag{4.133}$$

Thus (4.132) has a nontrivial solution for $x_1, x_2, \ldots, x_n$ if and only if the same is true of (4.133). This implies that the column vectors of A are linearly independent if and only if the same is true of B. Although we shall omit the proof, this conclusion also applies to any subset of column vectors. Thus we have the following result.

Theorem 4.52. *Column Independence*

If A and B are row equivalent matrices, then

a) A given set of column vectors of A is linearly independent if and only if the corresponding column vectors of B are linearly independent.

b) A given set of column vectors of A forms a basis for the column space of A if and only if the corresponding column vectors of B form a basis for the column space of B. ■

The following theorem makes it possible to find bases for the row and column spaces of a matrix in row-echelon form by inspection.

Theorem 4.53. *Bases*

If a matrix R is in row-echelon form, then the row vectors with the leading 1's (the nonzero row vectors) form a basis for the row space of R, and the column vectors with the leading 1's of the row vectors form a basis for the column space of R. ■

Since this result is virtually self-evident when one looks at numerical examples, we shall omit the proof; the proof involves little more than an analysis of the positions of the 0's and 1's of R.

Example 4.91. Bases for Row and Column Spaces I

The matrix

$$R = \begin{pmatrix} 1 & -2 & 5 & 0 & 3 \\ 0 & 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

is given in row-echelon form.

Solution 4.91. From Theorem 4.52, the vectors

$$\vec{r}_1 = (1, -2, 5, 0, 3),\ \vec{r}_2 = (0, 1, 3, 0, 0),\ \text{and}\ \vec{r}_3 = (0, 0, 0, 1, 0)$$

form a basis for the row space of R, and the vectors

$$\vec{c}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix},\ \vec{c}_2 = \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix},\ \text{and}\ \vec{c}_4 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

form a basis for the column space of R.▲

Example 4.92. Bases for Row and Column Spaces II

Find bases for the row and column space of

$$A = \begin{pmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 2 & -6 & 9 & -1 & 8 & 2 \\ 2 & -6 & 9 & -1 & 9 & 7 \\ -1 & 3 & -4 & 2 & -5 & -4 \end{pmatrix}$$

Solution 4.92. Since elementary row operations do not change the row space of a matrix, we can find a basis for the row space of A by finding a basis for the row space of any row-echelon form of A. Reducing A to row-echelon form, we obtain

$$R = \begin{pmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 0 & 0 & 1 & 3 & -2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

By Theorem 4.53, the nonzero row vectors of R form a basis for the row space of R and hence form a basis for the row space of A. These basis vectors are

$$\vec{r}_1 = (1, -3, 4, -2, 5, 4),\ \vec{r}_2 = (0, 0, 1, 3, -2, -6),\ \text{and}\ \vec{r}_3 = (0, 0, 0, 0, 1, 5)$$

Keeping in mind that A and R may have different column spaces, we cannot find a basis for the column space of A directly from the column vectors of R. However, it follows from Theorem 4.52b that if we can find a set of column vectors of R that forms a basis for the column space of R, then the corresponding column vectors of A will form a basis for the column space of A.

The first, third, and fifth columns of R contain the leading 1's of the row vector, so

$$\vec{c}_1' = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \ \vec{c}_3' = \begin{pmatrix} 4 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \ \text{and } \vec{c}_5' = \begin{pmatrix} 5 \\ -2 \\ 1 \\ 0 \end{pmatrix}$$

form a basis for the column space of R; thus the corresponding column vectors of A—namely

$$\vec{c}_1 = \begin{pmatrix} 1 \\ 2 \\ 2 \\ -1 \end{pmatrix}, \ \vec{c}_3 = \begin{pmatrix} 4 \\ 9 \\ 9 \\ -4 \end{pmatrix}, \ \text{and } \vec{c}_5 = \begin{pmatrix} 5 \\ 8 \\ 9 \\ -5 \end{pmatrix}$$

form a basis for the column space of A.▲

Example 4.93. Bases for a Vector Space Using Row Operations

Find a basis for the space spanned by the vectors

$$\vec{v}_1 = (1, -2, 0, 0, 3), \ \vec{v}_2 = (2, -5, -3, -2, 6), \ \vec{v}_3 = (0, 5, 15, 10, 0) \text{ and } \vec{v}_4 = (2, 6, 18, 8, 6).$$

Solution 4.93. Except for a variation in notation, the space spanned by these vectors is the row space of the matrix

$$\boldsymbol{A} = \begin{pmatrix} \mathbf{1} & \mathbf{-2} & \mathbf{0} & \mathbf{0} & \mathbf{3} \\ \mathbf{2} & \mathbf{-5} & \mathbf{-3} & \mathbf{-2} & \mathbf{6} \\ \mathbf{0} & \mathbf{5} & \mathbf{15} & \mathbf{10} & \mathbf{0} \\ \mathbf{2} & \mathbf{6} & \mathbf{18} & \mathbf{8} & \mathbf{6} \end{pmatrix};$$

Reducing this matrix to row-echelon form, we obtain

$$\boldsymbol{R} = \begin{pmatrix} \mathbf{1} & \mathbf{-2} & \mathbf{0} & \mathbf{0} & \mathbf{3} \\ \mathbf{0} & \mathbf{1} & \mathbf{3} & \mathbf{2} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix};$$

The nonzero row vectors in this matrix are

$$\vec{r}_1 = (1, -2, 0, 0, 3), \ \vec{r}_2 = (0, 1, 3, 2, 0), \text{and } \vec{r}_3 = (0, 0, 1, 1, 0).$$

These vectors form a basis for the row space and consequently form a basis for the subspace of $\mathbb{R}^5$ spanned by $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$, and $\vec{v}_4$.▲

Observe that in Example 4.93 the basis vectors obtained for the column space of A consisted of column vectors of A. The following example illustrates a procedure for finding a basis for the row space of a matrix A that consists entirely of row vectors of A.

Example 4.94. Basis for the Row Space of a Matrix

Find a basis for the row space of

$$\mathbf{A} = \begin{pmatrix} \mathbf{1} & \mathbf{-2} & \mathbf{0} & \mathbf{0} & \mathbf{3} \\ \mathbf{2} & \mathbf{-5} & \mathbf{-3} & \mathbf{-2} & \mathbf{6} \\ \mathbf{0} & \mathbf{5} & \mathbf{15} & \mathbf{10} & \mathbf{0} \\ \mathbf{2} & \mathbf{6} & \mathbf{18} & \mathbf{8} & \mathbf{6} \end{pmatrix};$$

consisting entirely of row vectors from A.

Solution 4.94. We will transpose A, thereby converting the row space of A into the column space of A^T; then we will use the method of Example 4.93 to find a basis for the column space of A^T; and then we will transpose again to convert column vectors back to row vectors.

Transposing A yields

$$\mathbf{At} = \boldsymbol{A}^{\mathrm{T}}$$

$$\begin{pmatrix} 1 & 2 & 0 & 2 \\ -2 & -5 & 5 & 6 \\ 0 & -3 & 15 & 18 \\ 0 & -2 & 10 & 8 \\ 3 & 6 & 0 & 6 \end{pmatrix}$$

Reducing this matrix to row-echelon form yields

$$R = \begin{pmatrix} 1 & 2 & 0 & 2 \\ 0 & 1 & -5 & -10 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The first second, and fourth columns contain the leading 1's, so the corresponding column vectors in A^T form a basis for the column space of A^T; these are

$$\vec{c}_1 = \begin{pmatrix} 1 \\ -2 \\ 0 \\ 0 \\ 3 \end{pmatrix}, \ \vec{c}_2 = \begin{pmatrix} 2 \\ -5 \\ -3 \\ -2 \\ 6 \end{pmatrix}, \text{ and } \vec{c}_4 = \begin{pmatrix} 2 \\ 6 \\ 18 \\ 8 \\ 6 \end{pmatrix}$$

Transposing again and adjusting the notation appropriately yields the basis vectors

$$\vec{r}_1 = (1, -2, 0, 0, 3), \ \vec{r}_2 = (2, -5, -3, -2, 6), \text{ and } \vec{r}_4 = (2, 6, 18, 8, 6)$$

for the row space of A.▲

We know from Theorem 4.41 that elementary row operations do not alter relationships of linear independence and linear dependence among the column vectors; however, formulas (4.132) and (4.133) imply an even deeper result. Because these formulas actually have the same scalar coefficient $x_1, x_2, \ldots, x_n$, it follows that elementary row operations do not alter the formulas (linear combinations) that relate linearly dependent column vectors.

Example 4.95. Basis and Linear Combinations

a) Find a subset of the vectors

$$\vec{v}_1 = (1, -2, 0, 3),\ \vec{v}_2 = (2, -5, -3, 6),\ \vec{v}_3 = (0, 1, 3, 0),$$
$$\vec{v}_4 = (2, -1, 4, -7),\ \text{and}\ \vec{v}_5 = (5, -8, 1, 2)$$

that forms a basis for the space spanned by these vectors.

b) Express each vector not in the basis as a linear combination of the basis vectors.

Solution 4.95. a) We begin by constructing a matrix that has $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_5$ as its column vectors:

$$\mathbf{A} = \begin{pmatrix} \mathbf{1} & \mathbf{2} & \mathbf{0} & \mathbf{2} & \mathbf{5} \\ \mathbf{-2} & \mathbf{-5} & \mathbf{1} & \mathbf{-1} & \mathbf{-8} \\ \mathbf{0} & \mathbf{-3} & \mathbf{3} & \mathbf{4} & \mathbf{1} \\ \mathbf{3} & \mathbf{6} & \mathbf{0} & \mathbf{-7} & \mathbf{2} \end{pmatrix};$$

The first part of our problem can be solved by finding a basis for the column space of this matrix. Reducing the matrix to reduced row-echelon form and denoting the column vectors of the resulting matrix by $\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_5$ yields

RowReduce[*A*]

$$\begin{pmatrix} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The leading 1's occur in column 1, 2, and 4, so by Theorem 4.53 the basis is $\vec{w}_1$, $\vec{w}_2$, and $\vec{w}_4$, and consequently $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_4$ is a basis for the column space of A.

b) We shall start by expressing $\vec{w}_3$ and $\vec{w}_5$ as linear combinations of the basis vectors $\vec{w}_1$, $\vec{w}_2$, and $\vec{w}_4$. The simplest way of doing this is to express $\vec{w}_3$ and $\vec{w}_5$ in terms of basis vectors. By inspection these linear combinations are

$$\vec{w}_3 = 2\,\vec{w}_1 - \vec{w}_2 \qquad \text{and} \quad \vec{w}_5 = \vec{w}_1 + \vec{w}_2 + \vec{w}_4.$$

We call these the dependency equations. The corresponding relationships for A are

$$\vec{v}_3 = 2\,\vec{v}_1 - \vec{v}_2 \qquad \text{and} \quad \vec{v}_5 = \vec{v}_1 + \vec{v}_2 + \vec{v}_4.\blacktriangle$$

The procedure illustrated in the preceding example is sufficiently important that we shall summarize the steps:

Given a set of vectors $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ in $\mathbb{R}^n$, the following procedure produces a subset of these vectors that forms a basis for span(S) and expresses those vectors of S that are not in the basis as linear combinations of the basis vectors.

1. **Step**: Form the matrix A having $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ as its column vectors.
2. **Step**: Reduce the matrix A to its reduced row-echelon form R, and let $\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k$ be the column vectors of R.
3. **Step**: Identify the columns that contain the leading 1's in R. The corresponding vectors of A are the basis vectors for span(S).
4. **Step**: Express each column vector of R that does not contain a leading 1 as a linear combination of preceding column vectors that do contain leading 1's. This yields a set of dependency equations involving the column vectors of R. The corresponding equations for the column vectors of A express the vectors that are not in the basis as linear combinations of the basis vectors.

4.6.2 Rank and Nullity

In the preceding sections we examined the relationship between systems of linear equations and the row space, column space, and nullspace of the coefficient matrix. In this section we shall be concerned with relationships between the dimensions of the row space, column space, and null space of a matrix and its transpose. The result we will obtain are fundamental and will provide deeper insights into linear systems and linear transformations.

If we consider a matrix A and its transpose A^T together, then there are six vector spaces of interest:

row space of A	row space of A^T
column space of A	column space of A^T
nullspace of A	nullspace of A^T

However, transposing a matrix converts row vectors into column vectors and column vectors into row vectors, so except for a difference in notation, the row space of A^T is the same as the column space of A, and the column space of A^T is the same as the row space of A. This leaves the four vector spaces of interest:

row space of A	column space of A
nullspace of A	nullspace of A^T

These four spaces are known as the fundamental matrix spaces associated with A. If A is an $m \times n$ matrix, then the row space of A and the nullspace of A are subspaces of $\mathbb{R}^n$, and the column space of A and the nullspace of A^T are subspaces of $\mathbb{R}^m$. Our primary goal in this section is to establish relationships between the dimensions of these four vector spaces.

In Example 4.93 of Section 4.6.1, we found that the row and column spaces of the matrix

$$A = \begin{pmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 2 & -6 & 9 & -1 & 8 & 2 \\ 2 & -6 & 9 & -1 & 9 & 7 \\ -1 & 3 & -4 & 2 & -5 & -4 \end{pmatrix}$$

each have three basis vectors; that is, both are three-dimensional. It is not accidental that these dimensions are the same; it is a consequence of the following general result.

Theorem 4.54. *Dimension of Row and Column Space*

If A is any matrix, then the row space and column space of A have the same dimension. ■

Proof 4.53. Let R be any row-echelon form of A. It follows from Theorem 4.51 that

$$\dim(\text{row space of } A) = \dim(\text{row space of } R)$$

and it follows from Theorem 4.52b that

$$\dim(\text{column space of } A) = \dim(\text{column space of } R).$$

Thus the proof will be complete if we can show that the row space and column space of R have the same dimension. But the dimension of the row space of R is the number of columns that contain leading 1's. However, the nonzero rows are precisely the rows in which the leading 1's occur, so the number of leading 1's and the number of nonzero rows are the same. This shows that the row space and column space of R have the same dimension.

QED

The dimensions of the row space, column space, and nullspace of a matrix are such important numbers that there is some notation and terminology associated with them.

Definition 4.27. *Rank and Nullity*

The common dimension of the row space and column space of a matrix A is called the rank of A and is denoted by rank(A); the dimension of the nullspace of A is called the nullity of A and is denoted by nullity(A). ■

Example 4.96. Rank and Nullity of a 4×6 Matrix

Find the rank and nullity of the matrix

$$\mathbf{A} = \begin{pmatrix} \mathbf{-1} & \mathbf{2} & \mathbf{0} & \mathbf{4} & \mathbf{5} & \mathbf{-3} \\ \mathbf{3} & \mathbf{-7} & \mathbf{2} & \mathbf{0} & \mathbf{1} & \mathbf{4} \\ \mathbf{2} & \mathbf{-5} & \mathbf{2} & \mathbf{4} & \mathbf{6} & \mathbf{1} \\ \mathbf{4} & \mathbf{-9} & \mathbf{2} & \mathbf{-4} & \mathbf{-4} & \mathbf{7} \end{pmatrix};$$

Solution 4.96. The reduced row-echelon form of A is

GaussEchelon(A)

$$\begin{pmatrix} -1 & 2 & 0 & 4 & 5 & -3 \\ 3 & -7 & 2 & 0 & 1 & 4 \\ 2 & -5 & 2 & 4 & 6 & 1 \\ 4 & -9 & 2 & -4 & -4 & 7 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -2 & 0 & -4 & -5 & 3 \\ 0 & -1 & 2 & 12 & 16 & -5 \\ 0 & -1 & 2 & 12 & 16 & -5 \\ 0 & -1 & 2 & 12 & 16 & -5 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & -4 & -28 & -37 & 13 \\ 0 & 1 & -2 & -12 & -16 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & -4 & -28 & -37 & 13 \\ 0 & 1 & -2 & -12 & -16 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Since there are two nonzero rows; i.e. two leading 1's, the row space and column space are both two-dimensional, so rank(A) = 2. To find the nullity of A, we must find the dimension of the solution space of the linear system $A.\vec{x} = 0$. This system can be solved by reducing the augmented matrix to reduced row-echelon form. The resulting matrix will be identically to the result already derived. So the augmented matrix for the homogeneous system is

$$\mathbf{Ag} = \begin{pmatrix} -1 & 2 & 0 & 4 & 5 & -3 & 0 \\ 3 & -7 & 2 & 0 & 1 & 4 & 0 \\ 2 & -5 & 2 & 4 & 6 & 1 & 0 \\ 4 & -9 & 2 & -4 & -4 & 7 & 0 \end{pmatrix};$$

and the related reduced row-echelon form is

RowReduce[Ag]

$$\begin{pmatrix} 1 & 0 & -4 & -28 & -37 & 13 & 0 \\ 0 & 1 & -2 & -12 & -16 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

So the general solution for the system is

$$x_1 = 4\,r + 28\,s + 37\,t - 13\,u$$

$$x_2 = 2\,r + 12\,s + 16\,t - 5\,u$$

$$x_3 = r,\;\; x_4 = s,\;\; x_5 = t,\;\; x_6 = u.$$

or equivalently

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = r\begin{pmatrix} 4 \\ 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + s\begin{pmatrix} 28 \\ 12 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} 37 \\ 16 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + u\begin{pmatrix} -13 \\ -5 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Because the four vectors on the right form a basis for the solution space, nullity is nullity$(A) = 4$.▲

The following theorem states that a matrix and its transpose have the same rank.

Theorem 4.55. *Rank and Dimension*

If A is any matrix, the rank$(A) = $ *rank*$\left(A^T\right)$.■

Proof 4.54. We examine

$$\text{rank}(A) = \dim(\text{row space of } A) = \dim\left(\text{column space of } A^T\right) = \text{rank}\left(A^T\right)$$

QED

The following theorem establishes an important relationship between the rank and the nullity of a matrix.

Theorem 4.56. *Dimension Theorem for Matrices*

If A is a matrix with n columns, then

$$\text{rank}(A) + \text{nullity}(A) = n.■$$

Proof 4.55. Since A has n columns, the homogeneous linear system $A.\vec{x} = 0$ has n unknowns. These fall into two categories: the leading variables and the free variables. Thus

$$\text{number of leading variables} + \text{number of free variables} = n.$$

But the number of leading variables is the same as the number of leading 1's in the reduced row-echelon form of A, and this is the rank(A). Thus

$$\text{rank}(A) + \text{number of free variables} = n.$$

The number of free variables is equal to the nullity of A. This is so because the nullity of A is the dimension of the solution space of $A.\vec{x} = 0$, which is the same as the number of parameters in the general solution, which is the same as the number of free variables. Thus

$$\text{rank}(A) + \text{nullity}(A) = n.$$

QED

The proof of the preceding theorem contains two results that are of importance in their own right.

Theorem 4.57. *Rank and Leading Variables*

If A is an $m \times n$ matrix, then

a) rank(A) = the number of leading variables in the solution of $A.\vec{x} = 0$.

b) nullity(A) = the number of parameters in the general solution of $A.\vec{x} = 0$. ■

Example 4.97. The Sum of Rank and Nullity

The following matrix is given

$$A = \begin{pmatrix} -1 & 2 & 0 & 4 & 5 & -3 \\ 3 & -7 & 2 & 0 & 1 & 4 \\ 2 & -5 & 2 & 4 & 6 & 1 \\ 4 & -9 & 2 & -4 & -4 & 7 \end{pmatrix};$$

Determine the rank and the nullity of the matrix.

Solution 4.97. The rank and nullity are defined as the number of leading variables and the number of parameters in the general solution of the homogeneous system. The rank can be calculated by

MatrixRank[*A*]

2

the nullity follows by

Nullity(*A*)

4

The size of the matrix is

Size(*A*)

{6, 4}

so the condition

MatrixRank[*A*] + Nullity(*A*) == Size(*A*)⟦1⟧

True

is satisfied.▲

Example 4.98. Number of Parameters in a General Solution

Find the number of parameters in the general solution $A.\vec{x} = 0$ if A is a 5×7 matrix of rank 3.

Solution 4.97. From the dimensional relation we have

$$\text{rank}(A) + \text{nullity}(A) = n.$$

Here we need the nullity, so

$$\text{nullity}(A) = n - \text{rank}(A) = 7 - 3 = 4. \blacktriangle$$

Suppose now that A is an $m \times n$ matrix of rank r, it follows from Theorem 4.55 that A^T is an $n \times m$ matrix of rank r. Applying Theorem 4.56 to A and A^T yields

$$\text{nullity}(A) = n - r \qquad \text{and} \qquad \text{nullity}\left(A^T\right) = m - r \tag{4.134}$$

from which we deduce the following table relating the dimensions of the four fundamental spaces of an $m \times n$ matrix A of rank r.

Table 4.2. Fundamental relations between dimensions of matrix A

Fundamental space	Dimension
Row space of A	r
Column space of A	r
Nullspace of A	$n - r$
Nullspace of A^T	$m - r$

If A is an $m \times n$ matrix, then the row vectors lie in $\mathbb{R}^n$ and the column vectors lie in $\mathbb{R}^m$. This implies that the row space of A is at most n-dimensional. Since the row and column spaces have the same dimension (the rank of A), we must conclude that if $m \neq n$, then the rank of A is at most the smaller of the values of m and n. We denote this by writing

$$\text{rank}(A) \leq \min(m, n) \tag{4.135}$$

where $\min(m, n)$ denotes the smaller of the numbers m and n if $m \neq n$ or denotes their common value if $m = n$.

Example 4.99. Maximum Value of Rank for a 7×4 Matrix

If A is a 7×4 matrix, then the rank of A is at most 4, and consequently, the seven row vectors must be linearly dependent. If A is a 7×4 matrix, then again the rank of A is at most 4, and consequently, the seven column vectors must be linearly dependent.▲

In earlier sections we obtained a wide range of theorems concerning linear systems of n equations in n unknowns. We shall now turn our attention to linear systems of m equations in n unknowns in which m and n need not be the same.

The following theorem specifies conditions under which a linear system of m equations in n unknowns is guaranteed to be consistent.

Theorem 4.58. *Consistency Theorem*

If $A.\vec{x} = \vec{b}$ *is a linear system of m equations in n unknowns, then the following are equivalent.*

a) $A.\vec{x} = \vec{b}$ *is consistent.*

b) $\vec{b}$ is in the column space of A.

c) The coefficient matrix A and the augmented matrix $(A \mid b)$ have the same rank. ∎

It is not hard to visualize why this theorem is true if one views the rank of a matrix as the number of nonzero rows in its reduced row-echelon form. For example, the augmented matrix for the system

$$\begin{array}{rrrrcr} x_1 & -2\,x_2 & -3\,x_3 & +2\,x_4 & = & -4 \\ -3\,x_1 & +7\,x_2 & -x_3 & +x_4 & = & -3 \\ 2\,x_1 & -5\,x_2 & +4\,x_3 & -3\,x_4 & = & 7 \\ -3\,x_1 & +6\,x_2 & +9\,x_3 & -6\,x_4 & = & -1 \end{array}$$

is

$$\mathbf{Ag} = \begin{pmatrix} 1 & -2 & -3 & 2 & -4 \\ -3 & 7 & -1 & 1 & -3 \\ 2 & -5 & 4 & -3 & 7 \\ -3 & 6 & 9 & -6 & -1 \end{pmatrix};$$

which has the following reduced row-echelon form

RowReduce[Ag]

$$\begin{pmatrix} 1 & 0 & -23 & 16 & 0 \\ 0 & 1 & -10 & 7 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

We see from the third row in this matrix that the system is inconsistent. However, it is also because of this row that the reduced row-echelon form of the augmented matrix has fewer zero rows than the reduced row-echelon form of the coefficient matrix, This forces the coefficient matrix and the augmented matrix for the system to have different ranks.

The consistency theorem is concerned with conditions under which a linear system $A.\vec{x} = \vec{b}$ is consistent for a specific vector $\vec{b}$. The following theorem is concerned with conditions under which a linear system is consistent for all possible choices of $\vec{b}$.

Theorem 4.59. *Consistency Theorem*

If $A.\vec{x} = \vec{b}$ is a linear system of m equations in n unknowns, then the following are equivalent.

a) $A.\vec{x} = \vec{b}$ is consistent for every $m \times 1$ matrix $\vec{b}$.

b) The column vectors of A span $\mathbb{R}^m$.

c) $rank(A) = m$. ∎

A linear system with more equations than unknowns is called an overdetermined linear system. If $A.\vec{x} = \vec{b}$ is an overdetermined linear system of m equations in n unknowns, so that $m > n$, then the column vectors of A cannot span $\mathbb{R}^m$; it follows from the last theorem that for a fixed $m \times n$ matrix A with $m > n$, the overdetermined linear system $A.\vec{x} = \vec{b}$ cannot be consistent for every possible $\vec{b}$.

Example 4.100. An Overdetermined System

The linear system

$$\begin{aligned} x_1 \; -2\,x_2 &= b_1 \\ x_1 \; -x_2 &= b_2 \\ x_1 \; +x_2 &= b_3 \end{aligned}$$

is overdetermined, so it cannot be consistent for all possible values of b_1, b_2, and b_3.

Solution 4.100. Exact conditions under which the system is consistent can be obtained by solving the linear system by Gauss-Jordan elimination. The reduced row-echelon for of the augmented matrix

$$\mathbf{Ag} = \begin{pmatrix} \mathbf{1} & \mathbf{-2} & \mathbf{b1} \\ \mathbf{1} & \mathbf{-1} & \mathbf{b2} \\ \mathbf{1} & \mathbf{1} & \mathbf{b3} \end{pmatrix};$$

The reduced row-echelon form is given by

$$\begin{pmatrix} 1 & 0 & 3\,b_1 - 2\,b_3 \\ 0 & 1 & b_1 - b_3 \\ 0 & 0 & b_2 + 2\,b_1 - 3\,b_3 \end{pmatrix}$$

or solving the related homogeneous system of equations for the b_i's by

$$\boldsymbol{B} = \begin{pmatrix} \mathbf{3} & \mathbf{0} & \mathbf{-1} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} & \mathbf{-1} & \mathbf{0} \\ \mathbf{1} & \mathbf{1} & \mathbf{-3} & \mathbf{0} \end{pmatrix};$$

RowReduce[*B*]

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

This shows us that a consistent solution is only possible if $b_i = 0$ for $i = 1, 2, 3$.▲

The following theorem tells us about the relation between the number of parameters in a solution and the size respectively the rank of the related coefficient matrix.

Theorem 4.60. *Number of Parameters in a Solution*

If $A.\vec{x} = \vec{b}$ is a consistent linear system of m equations in n unknowns, and if A has rank r, then the general solution of the system contains $n - r$ parameters.■

Example 4.101. Number of Parameters

If A is a 5×7 matrix with rank 4, and if $A.\vec{x} = \vec{b}$ is a consistent linear system, then the general solution of the system contains $7 - 4 = 3$ parameters.▲

In earlier sections we obtained a wide range of conditions under which a homogeneous liner system $A.\vec{x} = 0$ of n equations in n unknowns is guaranteed to have only the trivial solution. The following theorem obtains some corresponding results for systems of m equations in n unknowns, where m and n may differ.

Theorem 4.61. *Equivalent Statements*

If A is an $m \times n$ matrix, then the following are equivalent.

a) $A.\vec{x} = 0$ has only the trivial solution.

b) The column vectors of A are linearly independent.

c) $A.\vec{x} = \vec{b}$ has at most one solution (none or one) for every $m \times 1$ matrix $\vec{b}$.■

A linear system with more unknown than equations is called an under determined linear system. If $A.\vec{x} = \vec{b}$ is a consistent under determined linear system of m equations in n unknowns (so that $m < n$), then it follows that the general solution has at least one parameter; hence a consistent under determined linear system must have infinitely many solutions. In particular, an under determined homogeneous linear system has infinitely many solutions, though this was already proven.

Example 4.102. An Underdetermined System

If A is a 5×7 matrix, then for every 7×1 matrix $\vec{b}$, the linear system $A.\vec{x} = \vec{b}$ is under determined. Thus $A.\vec{x} = \vec{b}$ must be consistent for some $\vec{b}$, and for each such $\vec{b}$ the general solution must have $7 - r$ parameters, where r is the rank of A.▲

4.6.3 Tests and Exercises

The following two subsections serve to test your understanding of the last section. Work first on the test examples and then try to solve the exercises.

4.6.3.1 Test Problems

T1. How is a vector space defined?

T2. What is the span? Give examples.

T3. What is a column space?

T4. Which meaning is assigned to the row space?

T5. How is the Nullity defined?

T6. What is a rank of a matrix?

T7. How is the rank related to the dimension of a solution space?

T8. What is a null space?

4.6.3.2 Exercises

E1. Prove the following properties of vector addition and scalar multiplication that were introduced in the previous sections.

a. $\vec{u} + \left(\vec{v} + \vec{w}\right) = \left(\vec{u} + \vec{v}\right) + \vec{w}$,

b. $\vec{u} + \left(-\vec{u}\right) = 0$,

c. $(c + d)\,\vec{u} = c\,\vec{u} + d\,\vec{u}$,

d. $1\,\vec{u} = \vec{u}$.

E2. Which of the following physical quantities are scalars, and which are vectors?

a. acceleration,

b. temperature,

c. pressure,

d. position,

e. time,

f. gravity,

g. sound,

h. light intensity,

i. electrical field,

j. velocity.

E3. Consider the set of all continuous functions with operations of point wise addition and scalar multiplication, having domain [0, 1]. Is this set a vector space?

E4. Prove that the following sets of vectors are linearly dependent in $\mathbb{R}^3$. Express one vector in each set as a linear combination of the other vectors.

a. $\{(1, -2, 3), (-2, 4, 1), (-4, 8, 9)\}$,

b. $\{(1, 2, -3), (2, 1, -1), (0, 0, 0)\}$,

c. $\{(3, 4, 1), (2, 1, 0), (9, 7, 1)\}$,

d. $\{(1, 0, 2), (2, 6, 4), (1, 12, 2)\}$.

E5. Prove that the following sets of vectors are linearly independent in $\mathbb{R}^3$.

a. $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$,

b. $\{(1, 2, 5), (2, 1, 4), (1, -2, 1)\}$,

c. $\{(5, 2, 1), (10, 6, 9), (-2, 0, 3)\}$,

d. $\{(1, 1, 1), (4, 1, 2), (-4, 3, 2)\}$.

E6. Consider the following matrix, which is in reduced echelon form.

$$A = \begin{pmatrix} 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -1 \end{pmatrix} \tag{1}$$

Show that the row vectors form a linearly independent set. Is the set of nonzero row vectors of any matrix in reduced echelon form linearly independent?

E7. Let the set $\left\{\vec{v}_1, \vec{v}_2, \vec{v}_3\right\}$ be linearly independent in a vector space V. Let c be a nonzero scalar. Prove that the following sets are linearly independent.

a. $\left\{\vec{v}_1, \vec{v}_1 + \vec{v}_2, \vec{v}_3\right\}$,

b. $\left\{\vec{v}_1, \vec{v}_1 + c\,\vec{v}_2, \vec{v}_3\right\}$,

c. $\left\{\vec{v}_1, c\,\vec{v}_2, \vec{v}_3\right\}$.

E8. Determine the ranks of the following matrices using the definition of the rank.

$\begin{pmatrix} 2 & 4 \\ 3 & 6 \end{pmatrix}$,

b. $\begin{pmatrix} 4 & 0 \\ 1 & 5 \end{pmatrix}$,

c. $\begin{pmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{pmatrix}$,

d. $\begin{pmatrix} 2 & 1 & 3 \\ 4 & 2 & 6 \\ 2 & 1 & 3 \end{pmatrix}$.

E9. Find the reduced echelon form for each of the following matrices. Use the echelon form to determine a basis for row space and the rank of each matrix.

a. $\begin{pmatrix} 1 & 2 & -1 \\ 2 & 5 & 2 \\ 0 & 2 & 9 \end{pmatrix}$,

b. $\begin{pmatrix} 1 & -3 & 2 \\ -2 & 6 & -4 \\ -1 & 3 & -2 \end{pmatrix}$,

c. $\begin{pmatrix} 1 & 1 & 8 \\ 0 & 1 & 3 \\ -1 & 1 & 2 \end{pmatrix}$,

d. $\begin{pmatrix} 1 & 4 & 0 \\ -1 & -3 & 3 \\ 2 & 9 & 5 \end{pmatrix}$.

E10 Let A be an $n \times n$ matrix. Prove that columns of A are linearly independent if and only if $\text{rank}(A) = n$.

E11 Let A be an $n \times n$ matrix. Prove that the columns of A span $\mathbb{R}^n$ if and only if the rows of A are linearly independent.

E12 Which of the following vectors are orthogonal sets of vectors?

a. $\{(1, 2), (2, -1)\}$,

b. $\{(4, 1), (2, -3)\}$,

c. $\{(-3, 2), (2, 3)\}$,

d. $\{(3, -1), (0, 5)\}$.

E13 The following vectors form a basis for $\mathbb{R}^3$. Use these vectors in the Gram-Schmidt process to construct an orthonormal basis for $\mathbb{R}^3$.

a. $\{(1, 1, 1), (2, 0, 1), (2, 4, 5)\}$,

b. $\{(3, 2, 0), (5, -1, 2), (1, 5, -1)\}$.

4.7 Linear Transformations

In this section we will define and study linear transformations from an arbitrary vector space V to another arbitrary vector space W. The results we obtain here have important applications in physics, engineering, and various branches of mathematics.

4.7.1 Basic Definitions on Linear Transformations

Recall that a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ is defined as a function

$$T(x_1, x_2, \ldots x_n) = (w_1, w_2, \ldots, w_m) \tag{4.136}$$

for which the equations relating $w_1, w_2, \ldots, w_m$ and $x_1, x_2, \ldots x_n$ are linear. We also showed in the preceding sections that a transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is linear if and only if the two relationships

$$T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v}) \qquad \text{and} \qquad T(c\,\vec{u}) = \mathrm{c}\mathrm{T}(\vec{u}) \tag{4.137}$$

hold for all vectors $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^n$ and every scalar c. We shall use these properties as the starting point for general linear transformations.

Definition 4.28. *Linear Transformation*

If $T : V \to W$ is a function from a vector space V into a vector space W, then T is called a linear transformation from V to W if, for all vectors $\vec{u}$ ad $\vec{v}$ in V and all scalars c

a) $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$

b) $T(c\,\vec{u}) = cT(\vec{u})$.

In the special case where $V = W$, the linear transformation $T : V \to V$ is called a linear operator on V. ■

Because the definition of a linear transformation based on linear Transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$ are linear transformations under this more general definition as well. We will call linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$ matrix transformations, since they can be carried out by matrix multiplications.

Let V and W be any two vector spaces. The mapping $T : V \to W$ such that $T(\vec{v}) = 0$ for every $\vec{v}$ in V is a linear transformation called the zero transformation. To see that T is linear, observe that

$$T(\vec{u} + \vec{v}) = 0, \; T(\vec{u}) = 0, \; T(\vec{v}) = 0, \qquad \text{and} \qquad T(k\,\vec{u}) = 0. \tag{4.138}$$

Therefore,

$$T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v}) \qquad \text{and} \qquad T(k\,\vec{u}) = k\,T(\vec{u}). \tag{4.139}$$

Let V be any vector space. The mapping $I : V \to V$ defined by $I(\vec{v}) = \vec{v}$ is called the identity operator on V. The verification that I is linear is left for the reader.

Example 4.103. Dilation and Contraction Operators

Let V be any vector space and k any fixed scalar. We leave it as an exercise to check that the mapping $T : V \to V$ defined by

$$T(\vec{v}) = k\,\vec{v}$$

is a linear operator on V. This linear operator is called a dilation of V with factor k if $k > 1$ and is called a contraction of V with factor k of $0 < k < 1$. Geometrically, the dilation stretches each vector in V by a factor of k, and the contraction of V compresses each vector by a factor of k.

Solution 4.103. Recall that a 2×2 dilation matrix has the form

$$D = \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix}.$$

We construct an example as follows:

$$\boldsymbol{A = \text{DiagonalMatrix}\left[\left\{\frac{3}{4}, \frac{3}{4}\right\}\right]}$$

$$\begin{pmatrix} \frac{3}{4} & 0 \\ 0 & \frac{3}{4} \end{pmatrix}$$

Here is the above dilation transformation applied to the vector $\vec{u} = (1, 2)$:

u = {1, 2}
Au = $A.u$

{1, 2}

$$\left\{\frac{3}{4}, \frac{3}{2}\right\}$$

Alternatively, we can use function notation. Let T be the transformation that takes as input the vector $\vec{u}$ and returns as output the vector $A.\vec{u}$. Symbolically, we write $T(\vec{u}) = A.\vec{u}$. We can define this function in *Mathematica* as shown on the next line.

T(u_) := $A.u$

Now we find $A.\vec{u}$ again, this time by applying the transformation T to $\vec{u}$:

$T(u)$

$$\left\{\frac{3}{4}, \frac{3}{2}\right\}$$

Thus, T maps {1, 2} to $\left\{\frac{3}{4}, \frac{2}{2}\right\}$.

We can draw the original and transformed vectors to picture the input vectors $u = \{1, 2\}$, $v = \{3, 1\}$, and their corresponding outputs.

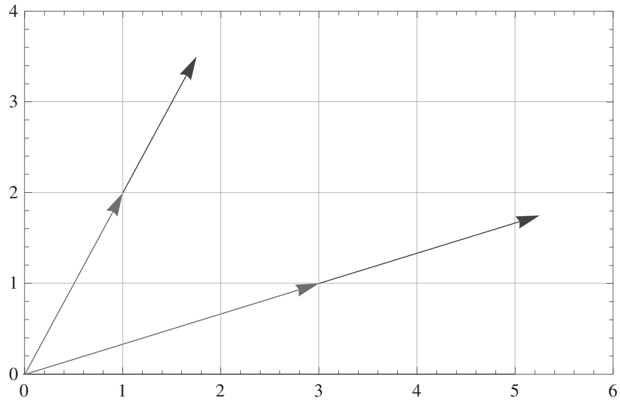


Figure 4.38. The input vectors $\vec{u}$ and $\vec{v}$ are colored red and their corresponding output vectors $A.\vec{u}$ and $A.\vec{v}$ are blue.▲

Example 4.104. Linear Transformation from P_n to P_{n+1}

Let $\vec{p} = p(x) = c_0 + c_1\, x + c_2\, x^2 + \ldots + c_n\, x^n$ be a polynomial in P_n and define the mapping $T : P_n \to P_{n+1}$ by

$$T(\vec{p}) = T(p(x)) = x\, p(x) \;= x\, c_0 + c_1\, x^2 + \ldots + c_n\, x^{n+1}.$$

The mapping T is a linear transformation, since for any scalar k and any polynomial $\vec{p}_1$ and $\vec{p}_2$ in P_n we have

$$T(\vec{p}_1 + \vec{p}_2) = T(p_1(x) + p_2(x)) \;=\; x(p_1(x) + p_2(x)) = \; x\, p_1(x) + x\, p_2(x) = T(\vec{p}_1) + T(\vec{p}_2)$$

and

$$T(k\,\vec{p}) = T(k\; p(x)) = x(k\; p(x)) = k\,(x\; p(x)) = k\; T(\vec{p}).\blacktriangle$$

Example 4.105. A Linear Transformation from $C^1(-\infty,\,\infty)$ to $F(-\infty,\,\infty)$

Let $V = C^1(-\infty,\,\infty)$ be the vector space of functions with continuous first derivatives on $(-\infty,\,\infty)$, and let $W = F(-\infty,\,\infty)$ be the vector space of all real-valued functions defined on $(-\infty,\,\infty)$. Let $D : V \to W$ be the transformation that maps a function $\vec{f} = f(x)$ into its derivative—that is,

$$D(\vec{f}) = f\,'(x).$$

Solution 4.105. From the properties of differentiation, we have

$$D\left(\vec{f} + \vec{g}\right) = D(\vec{f}) + D(\vec{g}) \qquad \text{and} \qquad D\left(k\,\vec{f}\right) = k\,D(\vec{f}).$$

Thus, D is a linear transformation.▲

Example 4.106. A Linear Transformation from $C(-\infty,\,\infty)$ to $C^1(-\infty,\,\infty)$

Let $V = C(-\infty, \infty)$ be the vector space of continuous functions on $(-\infty, \infty)$, and let $W = C^1(-\infty, \infty)$ be the vector space of functions with continuous first derivatives on $(-\infty, \infty)$. Let $J : V \to W$ be the transformation that maps $\vec{f} = f(x)$ into the integral $\int_0^x f(t)\,dt$. For example, if $\vec{f} = x^2$, then

$$J(\vec{f}) = \int_0^x t^2\,dt = \frac{t^3}{3}\Big|_0^x = \frac{x^3}{3}.$$

Solution 4.106. From the properties of integration, we have

$$J\left(\vec{f} + \vec{g}\right) = \int_0^x f(t) + g(t)\,dt = \int_0^x f(t)\,dt + \int_0^x g(t)\,dt = J(\vec{f}) + J(\vec{g})$$

$$J\left(k\,\vec{f}\right) = \int_0^x k\,f(t)\,dt = k\int_0^x f(t)\,dt = k\,J(\vec{f})$$

so J is a linear transformation.▲

Example 4.107. A Transformation That is Not Linear

Let $T : M_{nn} \to R$ be the transformation that maps an $n \times n$ matrix into its determinant:

$$T(A) = \det(A).$$

Solution 4.107. If $n > 1$, then this transformation does not satisfy either of the properties required of a linear transformation. For example, we know that

$$\det(A + B) \neq \det(A) + \det(B)$$

in general. Moreover, $\det(c\,A) = c^n \det(A)$, so

$$\det(c\,A) \neq c\det(A)$$

in general. Thus T is not a linear Transformation.▲

If $T : V \to W$ is a linear transformation, then for any vector $\vec{v}_1$ and $\vec{v}_2$ in V and any scalar c_1 and c_2, we have

$$T(c_1\,\vec{v}_1 + c_2\,\vec{v}_2) = T(c_1\,\vec{v}_1) + T(c_2\,\vec{v}_2) = c_1\,T(\vec{v}_1) + c_2\,T(\vec{v}_2) \tag{4.140}$$

and more generally, if $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ are vectors in V and $c_1, c_2, \ldots, c_n$ are scalars, then

$$T(c_1\,\vec{v}_1 + c_2\,\vec{v}_2 + \ldots + c_n\,\vec{v}_n) = c_1\,T(\vec{v}_1) + c_2\,T(\vec{v}_2) + \ldots + c_n\,T(\vec{v}_n) \tag{4.141}$$

Formula (4.141) is sometimes described by saying that linear transformations preserve linear combinations.

The following theorem lists three basic properties that are common to all linear transformations.

Theorem 4.62. *Properties of Linear Transformations*

If $T : V \to W$ is a linear transformation, then *a)* $T(0)=0$

b) $T(-\vec{v}) = -T(\vec{v})$ *for all* $\vec{v}$ *in V*

c) $T(\vec{v}-\vec{w}) = T(\vec{v}) - T(\vec{w})$ *for all* $\vec{v}$ *and* $\vec{w}$ *in* V.■

The proofs of these properties follow directly from the definition of a linear transformation, try it. In words, part a) of the preceding theorem states that a linear transformation maps 0 to 0. This property is useful for identifying transformations that are not linear. For example, if $\vec{x}_0$ is a fixed nonzero vector in $\mathbb{R}^2$, then the transformation

$$T(\vec{x}) = \vec{x} + \vec{x}_0 \tag{4.142}$$

has the geometric effect of translating each point $\vec{x}$ in a direction parallel to $\vec{x}_0$ through a distance of $\|\vec{x}_0\|$. This cannot be a linear transformation, since $T(0) = \vec{x}_0$, so T does not map 0 to 0.

If T is a matrix transformation, then the standard matrix for T can be obtained from the images of the standard basis vectors. Stated another way, a matrix transformation is completely determined by its images of the standard basis vectors. This is a special case of a more general result: If $T : V \to W$ is a linear transformation, and if $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is any basis for V, then the image $T(\vec{v})$ of any vector $\vec{v}$ in V can be calculated from the images

$$T(\vec{v}_1),\ T(\vec{v}_2),\ \ldots,\ T(\vec{v}_n) \tag{4.143}$$

of the basis vectors. This can be done by first expressing $\vec{v}$ as a linear combination of the basis vectors, say

$$\vec{v} = c_1\,\vec{v}_1 + c_2\,\vec{v}_2 + \ldots + c_n\,\vec{v}_n \tag{4.144}$$

and then using Formula (4.141) to write

$$T(\vec{v}) = c_1\,T(\vec{v}_1) + c_2\,T(\vec{v}_2) + \ldots + c_n\,T(\vec{v}_n). \tag{4.145}$$

In words, a linear transformation is completely determined by the images of any set of basis vectors.

Example 4.108. Computing with Images of Basis Vectors

Consider the basis $S = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ for $\mathbb{R}^3$, where $\vec{v}_1 = (1, 1, 1)$, $\vec{v}_2 = (1, 1, 0)$, and $\vec{v}_3 = (1, 0, 0)$. Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation such that

$$T(\vec{v}_1) = (1, 0), \qquad T(\vec{v}_2) = (2, -1), \qquad T(\vec{v}_3) = (4, 3).$$

Find the formula for $T(x_1, x_2, x_3)$; then use this formula to compute $T(2, -3, 5)$.

Solution 4.108. We first express $\vec{x} = (x_1, x_2, x_3)$ as a linear combination of $\vec{v}_1 = (1, 1, 1)$, $\vec{v}_2 = (1, 1, 0)$, and $\vec{v}_3 = (1, 0, 0)$. If we write

$$(x_1, x_2, x_3) = c_1(1, 1, 1) + c_2(1, 1, 0) + c_3(1, 0, 0)$$

then on equating corresponding components, we obtain

$$\begin{array}{llll} c_1 & +c_2 & +c_3 & = x_1 \\ c_1 & +c_2 & & = x_2 \\ c_1 & & & = x_3 \end{array}$$

which yields $c_1 = x_3$, $c_2 = x_2 - x_3$, $c_3 = x_1 - x_2$, so

$$(x_1, x_2, x_3) = x_3(1, 1, 1) + (x_2 - x_3)(1, 1, 0) + (x_1 - x_2)(1, 0, 0) = x_3 \vec{v}_1 + (x_2 - x_3) \vec{v}_2 + (x_1 - x_2) \vec{v}_3$$

Thus we get

$$T(x_1, x_2, x_3) =$$
$$x_3 T(\vec{v}_1) + (x_2 - x_3) T(\vec{v}_2) + (x_1 - x_2) T(\vec{v}_3) = x_3(1, 0) + (x_2 - x_3)(2, -1) + (x_1 - x_2)(4, 3) =$$
$$(x_3 + 2(x_2 - x_3) + 4(x_1 - x_2), -(x_2 - x_3) + 3(x_1 - x_2)) = (4x_1 - 2x_2 - x_3, 3x_1 - 4x_2 + x_3).$$

From this formula, we find

$$T(2, -3, 5) = (9, 23).\blacktriangle$$

The following definition extends the concept of matrix transformations to general linear transformations.

Definition 4.29. *Composition of Linear Transformations*

If $T_1 : U \to V$ and $T_2 : V \to W$ are linear transformations, then the composition of T_2 with T_1 denoted by $T_2 \circ T_1$ (which is read T_2circle T_1), is the function defined by the formula

$$(T_2 \circ T_1)(\vec{u}) = T_2(T_1(\vec{u}))$$

where $\vec{u}$ is a vector in U.■

The next result shows that the composition of two transformations is itself a linear transformation.

Theorem 4.63. *Linearity of Compositions*

If $T_1 : U \to V$ and $T_2 : V \to W$ are linear transformations, then $(T_2 \circ T_1) : U \to W$ is also a linear transformation.■

Example 4.109. Composition of Linear Transformations

Let $T_1 : P_1 \to P_2$ and $T_2 : P_2 \to P_2$ be the linear transformations given by the formulas

$$T_1(p(x)) = x\, p(x) \qquad \text{and} \qquad T_2(p(x)) = p(2x + 4).$$

Then the composition $(T_2 \circ T_1) : P_1 \to P_2$ is given by the formula

$$(T_2 \circ T_1)(p(x)) = T_2(T_1(p(x))) = T_2(x\, p(x)) = (2x + 4)\, p(2x + 4).$$

In particular, if $p(x) = c_0 + c_1 x$, then

$$(T_2 \circ T_1)(p(x)) = T_2(T_1(c_0 + c_1 x)) = (2x + 4)(c_0 + c_1(2x + 4)) = c_0(2x + 4) + c_1(2x + 4)^2.\blacktriangle$$

Example 4.110. Composition with the Identity Operator

If $T : V \to V$ is any linear operator, and if $I : V \to V$ is the identity operator, then for all vectors $\vec{v}$ in V, we have

$$(T \circ I)(\vec{v}) = T(I(\vec{v})) = T(\vec{v})$$

$$(I \circ T)(\vec{v}) = I(T(\vec{v})) = T(\vec{v}).$$

It follows that $T \circ I$ and $I \circ T$ are the same as T; that is

$$T \circ I = T \qquad \text{and} \qquad I \circ T = T. \blacktriangle$$

We conclude this section by noting that comparisons can be defined for more than two linear transformations. For example, if

$$T_1 : U \to V, \qquad T_2 : V \to W, \qquad \text{and } T_3 : W \to Y \tag{4.146}$$

are linear transformations, then the composition $T_3 \circ T_2 \circ T_1$ is defined by

$$(T_3 \circ T_2 \circ T_1)(\vec{u}) = T_3(T_2(T_1(\vec{u}))). \tag{4.147}$$

This relation can be generalized to any number of mappings.

4.7.2 Kernel and Range

In this section we shall develop some basic properties of linear transformations that generalize properties of matrix transformations.

Recall that if A is an $m \times n$ matrix, then the nullspace of A consists of all vectors $\vec{x}$ in $\mathbb{R}^n$ such that $A.\vec{x} = 0$, and the column space of A consists of all vectors $\vec{b}$ in $\mathbb{R}^m$ for which there is at least one vector $\vec{x}$ in $\mathbb{R}^n$ such that $A.\vec{x} = \vec{b}$. From the viewpoint of matrix transformations, the nullspace of A consists of all vectors in $\mathbb{R}^n$ that multiplication by A maps into 0, and the column space of A consists of all vectors in $\mathbb{R}^m$ that are images of at least one vector in $\mathbb{R}^n$ under multiplication by A. The following definition extends these ideas to general linear transformations.

Definition 4.30. *Kernel and Range of a Linear Transformation*

If $T : V \to W$ is a linear transformation, then the set of vectors in V that T maps into 0 is called the kernel of T; it is denoted by $ker(T)$. The set of all vectors in W that are images under T of at least one vector in V is called the range of T; it is denoted as $\mathcal{R}(T)$. ■

Example 4.111. Kernel and Range of a Matrix Transformation

If $T_A : \mathbb{R}^n \to \mathbb{R}^m$ is multiplication by the $m \times n$ matrix A, then from the discussion preceding the definition above, the kernel of T_A is the nullspace of A, and the range of T_A is the column space of A.▲

Example 4.112. Kernel and Range of the Zero Transformation

Let $T : V \to W$ be the zero transformation. Since T maps every vector in V into 0, it follows that $\ker(T) = V$. Moreover, since 0 is the only image under T of vectors in V, we have $\mathcal{R}(T) = \{0\}$.▲

Example 4.113. Kernel and Range of the Identity Operator

Let $I : V \to V$ be the identity operator. Since $I(\vec{v}) = \vec{v}$ for all vectors in V, every vector in V is the image of some vector (namely, itself); thus $\mathcal{R}(I) = V$. Since the only vector that I maps into 0 is 0, it follows that $\ker(I) = \{0\}$.▲

Example 4.114. Kernel and Range of a Rotation

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear operator that rotates each vector in the (x, y)-plane through the angle θ. Since every vector is the (x, y)-plane can be obtained by rotating some vector through the angle θ, we have $\mathcal{R}(T) = \mathbb{R}^2$, Moreover, the only vector that rotates into 0 is 0, so $\ker(T) = \{0\}$.▲

Example 4.115. Kernel of a Differentiation Transformation

Let $V = C^1(-\infty, \infty)$ be the vector space of functions with continuous first derivatives on (-∞,∞), let $W = F(-\infty, \infty)$ be the vector space of all real-valued functions defined on (-∞,∞), and let $D : V \to W$ be the differentiation transformation $D(\vec{f}) = f'(x)$. The kernel of D is the set of functions in V with derivative zero. From calculus, this is the set of constant functions on (-∞,∞).▲

In all of the preceding examples $\ker(T)$ and $\mathcal{R}(T)$ turned out to be subspaces. This is not accidental; it is a consequence of the following general result.

Theorem 4.64. *Subspaces*

If $T : V \to W$ is a linear transformation, then

a) The kernel of T is a subspace of V.

b) The range of T is a subspace of W. ■

In Section 4.6.2 we defined the rank of a matrix to be the dimension of its column (or row) space and the nullity to be the dimension of its nullspace. We now extend these definitions to general linear transformations.

Definition 4.31. *Rank and Nullity of a Linear Transformation*

If $T : V \to W$ is a linear transformation, then the dimension of the range of T is called the rank of T and is denoted by rank(T); the dimension of the kernel is called the nullity of T and is denoted by nullity(T). ■

If A is a $m \times n$ matrix and $T_A : \mathbb{R}^n \to \mathbb{R}^m$ is multiplication by A, then we know from Example 4.111 that the kernel of T_A is the nullspace of A and the range of T_A is the column space of A. Thus we have the following relationship between the rank and nullity of a matrix and the rank and nullity of the corresponding matrix transformation.

Theorem 4.65. *Matrix Transformation and Linear Transformation (Rank)*

If A is an $m \times n$ matrix and $T_A : \mathbb{R}^n \to \mathbb{R}^m$ is multiplication by A, then

a) nullity(T_A) *= nullity*(A)

b) rank(T_A) *= rank*(A).■

Example 4.116. Rank and Nullity

Let $T_A : \mathbb{R}^6 \to \mathbb{R}^4$ be multiplication by

$$A = \begin{pmatrix} -1 & 0 & 0 & 4 & 5 & -3 \\ 6 & -7 & -2 & 0 & 1 & 4 \\ 2 & -1 & 8 & 4 & 6 & 0 \\ 2 & -9 & 2 & -4 & -4 & 7 \end{pmatrix};$$

Find the rank and nullity of T_A.

Solution 4.116. The rank and nullity of T_A are equal to the rank ad nullity of A so we can use the procedure of Gauß to find the row echelon form and identify the leading 1's in the matrix. However, in *Mathematica* there is a function MatrixRank available which allows us to derive the rank

MatrixRank[*A*]

4

The nullity is given by the length of the null space

Nullity(*A*)

2

Thus the rank(T_A) = 4 and the nullity(T_A) = 2.▲

Recall from the dimension theorem for matrices that if A is a matrix with n columns, then

$$\text{rank}(A) + \text{nullity}(A) = n \tag{4.148}$$

The following theorem extends this result to general linear transformations.

Theorem 4.66. *Dimension of Linear Transformations*

If $T : V \to W$ is a linear transformation from an n-dimensional vector space V to a vector space W, then

$$\text{rank}(T) + \text{nullity}(T) = n.■$$

In words, this theorem states that for linear transformations the rank plus the nullity is equal to the dimension of the domain. This theorem is also known as Rank Theorem.

Remark 4.24. If A is an $m \times n$ matrix and $T_A : \mathbb{R}^n \to \mathbb{R}^m$ is multiplication by A, then the domain of T_A has dimension n, so the theorems on matrices and linear transformations agree on each other.

Example 4.117. Using the Rank Theorem

Let $T:\mathbb{R}^2 \to \mathbb{R}^2$ be the linear operator that rotates each vector in the (x, y)-plane through the angle θ. We discussed that for this mapping $\ker(T) = \{0\}$ and $\mathcal{R}(T) = \mathbb{R}^2$. Thus

$$\operatorname{rank}(T) + \operatorname{nullity}(T) = 2 + 0 = 2$$

which is consistent with the fact that the domain of T is two-dimensional.▲

4.7.3 Tests and Exercises

The following two subsections serve to test your understanding of the last section. Work first on the test examples and then try to solve the exercises.

4.7.3.1 Test Problems

T1. What are linear transformations?

T2. State the properties of linear transformations.

T3. How is the rank and the nullity of a linear transformation determined?

T4. Is there a relation between the rank and the nullity of a linear transformation?

T5. Is a composition of linear transformations again a linear transformation?

4.7.3.2 Exercises

E1. Prove that the transformation $T:\mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x, y) = (3x, y - x)$ is linear. Find the images of the vectors $(1, 4)$ and $(-2, 5)$ under this transformation.

E2. Determine whether the given transformations are linear or not.

a. $T(x, y) = (x, z, y)$ of $\mathbb{R}^2 \to \mathbb{R}^3$ when $z = 0$ or $z = 2$,

b. $T(x) = (x, 3x, -2x)$ of $\mathbb{R} \to \mathbb{R}^3$,

c. $T(x, y) = (x^2, y)$ of $\mathbb{R}^2 \to \mathbb{R}^2$,

d. $T(x, y, z) = (x + 3y, x - y, 4z)$ of $\mathbb{R}^3 \to \mathbb{R}^3$.

E3. Prove that $T:\mathbb{R}^2 \to \mathbb{R}$ defined by $T(x, y) = x + a$ where a is a nonzero scalar, is not linear.

E4. Consider the linear operator T on $\mathbb{R}^2$ defined by the following matrix A.

$$A = \begin{pmatrix} 0 & 5 \\ -2 & 7 \end{pmatrix} \tag{1}$$

Determine the images of the given vectors $\vec{x}$, $\vec{y}$, and $\vec{z}$.

$$\vec{x} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \ \vec{y} = \begin{pmatrix} -3 \\ 7 \end{pmatrix}, \text{ and } \vec{z} = \begin{pmatrix} -2 \\ -1 \end{pmatrix}. \tag{2}$$

E5. Let $T_1(\vec{x}) = A_1.\vec{x}$ and $T_2(\vec{x}) = A_2.\vec{x}$ be defined by the following matrices A_1 and A_2. Let $T = T_2 \circ T_1$. Find the matrix that defines T and use it to determine the image of the vector $\vec{x}$ under T.

a. $A_1 = \begin{pmatrix} 1 & -2 \\ 2 & 2 \end{pmatrix}$, $A_2 = \begin{pmatrix} -1 & 1 \\ 0 & 2 \end{pmatrix}$, and $\vec{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$,

b. $A_1 = \begin{pmatrix} 1 & -6 & 1 \\ -1 & 1 & 2 \\ 0 & -1 & 1 \end{pmatrix}$, $A_2 = \begin{pmatrix} -1 & 1 & 3 \\ 0 & 2 & -2 \\ -2 & 1 & -2 \end{pmatrix}$, and $\vec{x} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$.

E6. Let $\vec{x}$ be an arbitrary vector in $\mathbb{R}^n$ and $\vec{y}$ is a fixed vector in that space. Prove that the dot product $T(\vec{x}) = \vec{x}.\vec{y}$ is a linear transformation of $\mathbb{R}^n \to \mathbb{R}$.

E7. Find a single matrix that defines a rotation in a plane through an angle of $\pi/2$ about the origin, while at the same time moves points to twice their original distance from the origin. Hint: use a composite transformation.

E8. Determine the matrix that defines a reflection on the line $y = x$. Find the image of the vector (2, 1) under this transformation.

E9. Find the equation of the image of the circle $x^2 + y^2 = 4$, under a scaling of factor 5 in the x direction and factor 2 in the y direction.

E10 Show that the following matrices A and B, that $A^2\,B\,A^2 = I$.

$$A = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}, \text{ and } B = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3}$$

Give a geometrical reason for expecting this result.

A. Functions Used

Differential Equations Chapter 3

The following function generates the direction fields used in Chapter 3 to represent differential equations in the plane

```
DirectionField[equation_, dependentVariable_, independentVariable_,
  {x0_, y0_}] := Block[{sol1}, sol1 = Flatten[Solve[equation,
     ∂independentVariable dependentVariable[independentVariable]]];
  y0 + (∂independentVariable dependentVariable[independentVariable] /. sol1 /.
       {independentVariable -> x0,
        dependentVariable[independentVariable] -> y0}) (x - x0)
 ]
```

```
PlotDirectionField[field_, {x_, xu_, xm_}, {y_, yu_, ym_}, options___] :=
 Block[{nxPoints = 20, nyPoints = 20, ϵx, ϵy, x0ts, y0ts, yfields},
  ϵx = (xm - xu)/nxPoints;
  ϵy = (ym - yu)/nyPoints;
  x0ts = Table[xu + ϵx i, {i, 1, nxPoints}];
  yfields = Map[(field /. x0 -> #) &, x0ts];
  x0ts = Table[{x0ts[[i]] - ϵx/5, x0ts[[i]] + ϵx/5}, {i, 1, nxPoints}];
  y0ts = Table[yu + ϵy i, {i, 1, nyPoints}];
  yfields =
   Table[Map[(yfields[[i]] /. y0 -> #) &, y0ts], {i, 1, Length[yfields]}];
  plots = MapThread[(Plot[Evaluate[#2], {x, #1[[1]], #1[[2]]},
       DisplayFunction -> Identity, PlotStyle -> Hue[(#1[[1]] + #1[[2]]) 2/nxPoints],
       options]) &, {x0ts, yfields}];
  Show[Flatten[plots], DisplayFunction -> $DisplayFunction,
   PlotRange -> All]
 ]
```

Linear Algebra Chapter 4

The following functions are useful in handling some of the linear algebra tasks of Chapter 4.

The function Size determines the dimension of a matrix

```
Size[A_] := {Length[First[A]], Length[A]}
```

The function MatrixMinors determines the minors of a square matrix A

```
MatrixMinors[A_] := Map[Reverse, Minors[A], {0, 1}]
```

Cofactors determines the cofactors of a matrix A. This function is based on the function MatrixMinors.

```
Cofactors[A_] := Block[{minors}, minors = MatrixMinors[A];
  Table[(-1)^(i+j) minors[[i, j]], {i, 1, Length[A]}, {j, 1, Length[A]}]]
```

The function adj determines the adjoint matrix of a square matrix A

```
adj[A_] := Transpose[Cofactors[A]]
```

This function determines the nullity of a square matrix A

```
Nullity[A_] := Length[NullSpace[A]]
```

The following function determines the row-reduced echelon form of a matrix

```
GaussEchelon[A0_] :=
 Block[ {A = A0, i, p},
  Print[ MatrixForm[A] ];
  n = Length[A[[1]]];
  m = Length[A];
  For[ p = 1, p ≤ n, p++,
   If[A[[p,p]] == 0, Return[A]];
   A[[p]] = A[[p]]/A[[p,p]];
   For[ i = 1, i ≤ m, i++,
    If[ i ≠ p,
      A[[i]] = A[[i]] - A[[i,p]] A[[p]] ]; ];
   Print[MatrixForm[A]]; ]; A ]
```

B. Notations

This section collects some notations used in the book to make the *Mathematica* expressions compatible with the mathematical notation.

The following notation introduces a general sequence

$$\{a_n\}_{n=1}^{+\infty}. \tag{B.1}$$

To have the same notation in *Mathematica* available the following steps are needed

```
<< "Notation`"
```

Here the notation is defined as an equivalent with a table

```
Notation[{f_}^a_ _(n_=m_) ⟺ Table[f_, {n_, m_, a_}]]
```

The corresponding alias is set her as Seq

```
AddInputAlias[{□}^□ _(□=□), "Seq"]
```

C. Options

This section defines options used in the notebooks to change the formating and style of plots.

This line switches off the messages generated by *Mathematica*

```
Off[General::"spell1"]; Off[General::"spell"]; Off[Solve::"ifun"];
```

The following line changes the global text style settings

```
$TextStyle = {FontFamily → "Arial", FontSize → 12};
```

The next line sets options for the Plot function

```
SetOptions[Plot, GridLines → Automatic,
  Frame → True, PlotStyle -> RGBColor[0.501961, 0, 0.25098]];
```

Here are options for ListPlot set

```
SetOptions[ListPlot, GridLines → Automatic,
  Frame → True, PlotStyle -> RGBColor[0.501961, 0, 0.25098]];
```

The options for ParametricPlot are changed

```
SetOptions[ParametricPlot, GridLines → Automatic,
  Frame → True, PlotStyle -> RGBColor[0.501961, 0, 0.25098]];
```

The options for FilledPlot are changed

```
SetOptions[FilledPlot, GridLines → Automatic,
  Frame → True, PlotStyle -> RGBColor[0.501961, 0, 0.25098]];
```

References

[1] Abell, M. L. & Braselton, J. P. Mathematica by example. 4 th ed. (Elsevier, Amsterdam, 2009).

[2] Axler, S. J. Linear algebra done right. 2 nd ed. (New YorkSpringer, , 2002).

[3] Ayres, F. & Mendelson, E. Schaum's outlines calculus. 5 th ed. (McGraw - Hill, New York, 2009).

[4] Banner, A. D. The calculus lifesaver. All the tools you need to excel at calculus (Princeton University Press, Princeton, 2007).

[5] Baumann, G. Classical mechanics and nonlinear dynamics. 2 nd ed. (Springer, New York, NY, 2005).

[6] Baumann, G. Electrodynamics, quantum mechanics, general relativity, and fractals. 2 nd ed. (Springer, New York, NY, 2005).

[7] Berresford, G. Brief applied calculus. 4 th ed. (Houghton Mifflin Co., Boston MA, 2006).

[8] Bittinger, M. L. & Ellenbogen, D. Calculus and its applications. 9 th ed. (Pearson Addison Wesley, Boston, 2008).

[9] Bleau, B. L. Forgotten calculus. A refresher coursewith applications to economics and business and the optional use of the graphing calculator. 3 rd ed. (Barron' s Educational Series, Hauppauge N.Y., 2002).

[10] Bronson, R. & Costa, G. B. Linear algebra. An introduction. 2 nd ed. (Elsevier/AP, Amsterdam, 2007).

[11] Friedberg, S. H., Insel, A. J. & Spence, L. E. Linear algebra. 4 th ed. (Prentice Hall, Upper Saddle River, N.J., 2003).

[12] Goldstein, L. J., Lay, D. C. & Schneider, D. I. Calculus & its applications (Pearson Education, Upper Saddle River NJ, 2006).

[13] Hass, J. & Weir, M. D. Thomas' calculus. Early transcendentals (Pearson Addison - Wesley, Boston, 2008).

[14] Huettenmueller, R. Precalculus demystified (McGraw - Hill, New York, 2005).

[15] Hughes - Hallett, D. Calculus. 4 th ed. (J. Wiley, Hoboken N.J., 2005).

[16] Hughes - Hallett, D., Gleason, A. M. & Marks, E. J. Applied calculus. 3 rd ed. (Wiley, Hoboken, NJ, 2006).

[17] Kelley, W. M. The complete idiot' s guide to calculus. 2 nd ed. (Alpha, Indianapolis IN, 2006).

[18] Kelley, W. M. The humongous book of calculus problems. Translated for people who don' t speak math!! (Alpha Books, New Yoprk, NY, 2006).

[19] Kolman, B. & Hill, D. R. Elementary linear algebra (Pearson Education, Upper Saddle River N.J., 2004).

[20] Kolman, B. & Hill, D. R. Introductory linear algebra. An applied first course. 8 th ed (Pearson/Prentice Hall, Upper Saddle River N.J., 2005).

[21] Kolman, B. & Hill, D. R. Elementary linear algebra with applications. 9 th ed. (Pearson Prentic Hall, Upper Saddle River N.J., 2008).

[22] Lang, S. Introduction to linear algebra. 2 nd ed. (Springer, New York, 1997).

[23] Larson, R. Brief calculus. An applied approach. 8 th ed. (Houghton Mifflin, Boston MA, 2007).

[24] Larson, R. Calculus. An applied approach. 8 th ed. (Houghton Mifflin Co., Boston MA, 2007).

[25] Larson, R. Elementary linear algebra. 6 th ed. (Houghton Mifflin, Boston MA, 2008).

[26] Larson, R., Hostetler, R. P. & Edwards, B. H. Calculus with analytic geometry. 8 th ed. (Houghton Mifflin, Boston, 2007).

[27] Lay, D. C. Linear algebra and its applications. 3 rd ed. (Pearson/Addison - Wesley, Boston, Mass 2006).

[28] Lial, M. L., Greenwell, R. N. & Ritchey, N. P. Calculus with applications. 9 th ed (Pearson/Addison Wesley, Boston MA, 2008).

[29] Lipschutz, S. 3000 solved problems in linear algebra (McGraw - Hill, New York, 1989).

[30] Lipschutz, S. Schaum' s outline of theory and problems of beginning linear algebra (McGraw - Hill New York, 1997).

[31] Lipschutz, S. & Lipson, M. L. Linear algebra. [612 fully solved problems ; concise explanations of all course concepts ; information on algebraic systems, polynomials, and matrix applications]. 4 th ed. (McGraw - Hill, New York, 2009).

[32] Poole, D. Linear algebra. A modern introduction. 2 nd ed. (Thomson/Brooks/Cole, Belmont, Calif 2006).

[33] Rumsey, D. Pre - calculus for dummies. 1 st ed. (Wiley Pub. Inc., Indianapolis IN, 2008).

[34] Ruskeepää, H. Mathematica navigator. Mathematics, statistics, and graphics. 3 rd ed. (Elsevier/Academic Press, Amsterdam, Boston, 2009).

[35] Ryan, M. Calculus for dummies (Wiley, Hoboken, 2003).

[36] Silov, G. E. Linear algebra (Dover Publ., New York, 1977).

[37] Simmons, G. F. Calculus with analytic geometry. 2 nd ed. (McGraw - Hill, New York, NY, 1996).

[38] Spence, L. E., Insel, A. J. & Friedberg, S. H. Elementary linear algebra. A matrix approach. 2 nd ed. (Pearson/Prentice [38] Hall, Upper Saddle River N.J., 2008).

[39] Stewart, J. Single variable Calculus. Concepts and contexts. 4 th ed. (Brooks/Cole Cengage Learn ing, Belmont CA, 2009).

[40] Strang, G. Introduction to linear algebra. 4 th ed. (Wellesley - Cambridge Press, Wellesley, Mass 2009).

[41] Varberg, D. E., Purcell, E. J. & Rigdon, S. E. Calculus (Pearson Prentice Hall, Upper Saddle River N.J., 2007).

[42] Washington, A. J. Basic technical mathematics with calculus. 9 th ed. (Pearson/Prentice Hall Upper Saddle River N.J., 2009).

[43] Wellin, P. R., Gaylord, R. J. & Kamin, S. N. An introduction to programming with Mathematica. 3rd ed. (Cambridge Univ. Press, Cambridge, 2005).

Index