**11.8- misol.** $\displaystyle \lim_{x\to 0} x \int_x^1 \frac{\cos^2 t}{t^2}\, dt$ integralni hisoblang.

**Yechilishi.** Berilgan integralda $f(t)=\cos^2 t,\ g(t)=\dfrac{1}{t^2}$ deb, o'rta qiymat haqidagi teoremaga asosan,

$$\int_x^1 \frac{\cos^2 t}{t^2}\, dt = \left(\frac{1}{x}-1\right)\cdot \cos^2 \xi,\quad x<\xi<1$$

ni hosil qilamiz. $\forall \varepsilon >0$ son berilgan bo'lsin, u holda $\forall x \in \left(0;\ \dfrac{\varepsilon}{1+\varepsilon}\right)$ lar uchun

$$\cos^2 \xi \left(\frac{1}{x}-1\right) > \frac{\cos^2 \xi}{\varepsilon}$$

bo'lgani sababli $x\to +0$ da $\displaystyle\int_x^1 \frac{\cos^2 t}{t^2}\, dt \to +\infty$. Berilgan limitni hisoblash uchun Lopital qoidasini qo'llaymiz:

$$\lim_{x\to 0} x\cdot \int_x^1 \frac{\cos^2 t}{t^2}\, dt = \lim_{x\to 0} \frac{\dfrac{\cos^2 x}{x^2}}{-\dfrac{1}{x^2}} = \lim_{x\to 0}\cos^2 x = 1.$$

## 11.3. Chegaralanmagan funksiya xosmas integrallarining ba'zi bir tatbiqlari.

**11.3.1. Yuzni xosmas integral yordamida hisoblash.** $f(x)$ funksiya $[a;b)$ da aniqlangan, uzluksiz va $\forall x \in [a;b)$ uchun $f(x)\ge 0$ bo'lsin. Unda $D=\{(x;y): a\le x<b,\ 0\le y \le f(x)\}$ sohaning yuzi ushbu

$$S=\int_a^b f(x)dx$$

xosmas integral orqali ifoda qilinadi.

**11.9- misol.** $y=x^{-\frac{2}{3}}$, $y=0$, $x=-1$, $x=1$ chiziqlar bilan chegaralangan shaklning yuzini hisoblang.

**Yechilishi.** Talab qilingan yuzni quyidagi xosmas integral orqali hisoblaymiz:

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$$S = \int_{-1}^{1} x^{-\frac{2}{3}} dx = \lim_{\substack{\varepsilon \to 0+0 \\ \eta \to 0-0}} \left( \int_{-1}^{\varepsilon} x^{-\frac{2}{3}} dx + \int_{\eta}^{1} x^{-\frac{2}{3}} dx \right) =$$

$$= \lim_{\substack{\varepsilon \to 0+0 \\ \eta \to 0-0}} \left( 3\left(\varepsilon^{\frac{1}{3}} + 1\right) + 3\left(1 - \eta^{1/3}\right) \right) = 6 \text{ (kv.birl.)}.$$

## 11.3.2. Aylanma jismning hajmini xosmas integral yordamida hisoblash.

Ushbu $D = \{(x; y) : a \le x < b, \quad 0 \le y \le f(x)\}$ egri chiziqli trapetsiyani $Ox$ va $Oy$ o'qlar atrofida aylantirish natijasida hosil bo'lgan aylanma jismlarning hajmi, mos ravishda,

$$V_x = \pi \int_a^b f^2(x) dx, \quad V_y = 2\pi \int_a^b |xy| dx \qquad (11.3)$$

xosmas integrallar orqali hisoblanadi.

**11.10- misol.** $y = \dfrac{1}{\sqrt{x-1}}$, $x \in (1; 2]$ chiziq bilan chegaralangan shaklni $Ox$ o'q atrofida aylantirish natijasida hosil bo'lgan aylanma jism hajmini toping.

**Yechilishi.** Talab qilingan aylanma jismning hajmini (11.3) formula orqali topamiz:

$$V_x = \pi \int_1^2 \frac{dx}{\sqrt{x-1}} = \lim_{\varepsilon \to 0} \pi \int_{1+\varepsilon}^2 \frac{dx}{\sqrt{x-1}} = \pi \lim_{\varepsilon \to 0} 2\sqrt{x-1} \Big|_{1+\varepsilon}^2 = 2\pi \text{ (kv.birl.)}.$$

## 11.3.3. Aylanma sirtning yuzini xosmas integrallar yordamida hisoblash.

$f(x)$ funksiya $[a; b)$ da aniqlangan, uzluksiz va uzluksiz $f'(x)$ hosilaga ega bo'lib, u $f(x) \ge 0$ bo'lsin. $f(x)$ funksiya grafigini $Ox$ o'q atrofida aylantirish natijasida hosil bo'lgan aylanma sirtning yuzi ushbu formula orqali hisoblanadi:

$$S = 2\pi \int_a^b f(x)\sqrt{1+(f'(x))^2}\, dx . \qquad (11.4)$$

Shunga o'xshash, $Oy$ o'q atrofida aylantirish natijasida hosil bo'lgan aylanma sirtning yuzi ushbu formula orqali hisoblanadi:

$$S = 2\pi \int_c^d x(y)\sqrt{1+(x'(y))^2}\,dy \,. \tag{11.5}$$

**Mustaqil yechish uchun misollar**

Xosmas integrallarning yaqinlashuvchiligini ko'rsating va qiymatini toping:

**11.1.** $\displaystyle\int_0^1 \frac{dx}{\sqrt[3]{x}}$. **11.2.** $\displaystyle\int_0^1 \frac{dx}{\sqrt{1-x^2}}$. **11.3.** $\displaystyle\int_1^e \frac{dx}{x\sqrt{\ln x}}$. **11.4.** $\displaystyle\int_0^4 \frac{dx}{x+\sqrt{x}}$.

**11.5.** $\displaystyle\int_1^2 \frac{x\,dx}{\sqrt{x-1}}$. **11.6.** $\displaystyle\int_{-1}^1 \frac{x+1}{\sqrt[5]{x^3}}\,dx$. **11.7.** $\displaystyle\int_0^1 \frac{\arcsin x}{\sqrt{1-x^2}}\,dx$.

**11.8.** $\displaystyle\int_0^{\frac{1}{2}} \frac{dx}{x\ln^2 x}$. **11.9.** $\displaystyle\int_0^3 \frac{x^2\,dx}{\sqrt{9-x^2}}$. **11.10.** $\displaystyle\int_{-1}^1 \frac{\arccos x}{\sqrt{1-x^2}}\,dx$.

**11.11.** $\displaystyle\int_{-1}^0 \frac{e^{\frac{1}{x}}}{x^3}\,dx$. **11.12.** $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\cos x}{\sqrt{\sin x}}\,dx$.

Xosmas integrallarning uzoqlashuvchi ekanligini isbotlang:

**11.13.** $\displaystyle\int_{-1}^3 \frac{dx}{x}$. **11.14.** $\displaystyle\int_0^e \frac{dx}{e^r-1}$. **11.15.** $\displaystyle\int_{-3}^3 \frac{x\,dx}{x^2-1}$. **11.16.** $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\cos x}{\sqrt{\sin^3 x}}\,dx$.

**11.17.** $\displaystyle\int_0^{\frac{1}{2}} \frac{dx}{x\ln x}$. **11.18.** $\displaystyle\int_{-1}^1 \frac{e^{\frac{1}{x}}}{x^3}\,dx$. **11.19.** $\displaystyle\int_0^1 \frac{e^{\frac{1}{x}}}{x^3}\,dx$. **11.20.** $\displaystyle\int_0^{\frac{\pi}{2}} \mathrm{tg}\,x\,dx$.

Xosmas integrallarni hisoblang:

**11.21.** $\displaystyle\int_0^1 \frac{2-\sqrt[3]{x}-x^3}{\sqrt[5]{x^3}}\,dx$. **11.22.** $\displaystyle\int_{\sqrt{2}}^2 \frac{dx}{(x-1)\sqrt{x^2-2}}$.

**11.23** $\displaystyle\int_0^1 \frac{dx}{\sqrt{x\sqrt{x}}}$. **11.24.** $\displaystyle\int_0^{\frac{\pi}{2}} \sqrt{\mathrm{tg}x}\,dx$. **11.25.** $\displaystyle\int_0^{\pi} \ln\cos x\,dx$.

**11.26.** $\displaystyle\int_0^{\pi} x\ln\sin x\,dx$. **11.27.** $\displaystyle\int_0^{\frac{\pi}{4}} \sqrt{\mathrm{ctg}x}\,dx$.

**11.28.** $\displaystyle\int_{-1}^{1}\frac{dx}{(16-x^2)\sqrt{1-x^2}}$. **11.29.** $\displaystyle\int_{a}^{b}\frac{xdx}{\sqrt{(x-a)(b-x)}}$.

**11.30.** $\displaystyle\int_{0}^{1}\frac{x^3\arcsin x}{\sqrt{1-x^2}}\,dx$.

Limitlarni hisoblang:

**11.31.** $\displaystyle\lim_{x\to+\infty}\frac{\displaystyle\int_{0}^{x}\sqrt{1+t^{10}}\,dt}{x^6}$. **11.32.** $\displaystyle\lim_{x\to+\infty}\frac{\displaystyle\int_{0}^{x}t^{-1}e^{-t}\,dt}{\ln\dfrac{1}{x}}$.

**11.33.** $\displaystyle\lim_{x\to0}x^2\int_{x}^{1}\frac{\sin 2t}{t^3}\,dt$. **11.34.** $\displaystyle\lim_{x\to0}\sqrt{x}\int_{x}^{1}\frac{e^{-t}}{t^{3/2}}\,dt$.

Berilgan funksiyaning grafigi va abssissalar o'qi bilan chegaralangan shaklning yuzini toping:

**11.35.** $y=\dfrac{-x}{\sqrt{x+1}}$, $x\in(-1;0]$.

**11.36.** $y=\dfrac{1}{\sqrt{2-5x}}$, $x\in[0;0,4)$.

**11.37.** $y=\dfrac{x}{\sqrt{(x-2)(5-x)}}$, $x\in(2;5)$.

**11.38.** $y=\dfrac{1}{\sqrt{1-x}}$, $x\in[0;1)$.

**11.39.** $y=\dfrac{\arcsin\sqrt{x}}{\sqrt{1-x}}$, $x\in[0;1)$.

**11.40.** $y=\sqrt{\dfrac{x^3}{2a-x}}$, $x\in[0;2a)$.

**11.41.** $y=x\ln\dfrac{1+x}{1-x}$, $x\in[0;1)$.

Berilgan chiziq va uning asimptotalari bilan chegaralangan shaklning yuzini toping.

**Yechilishi.** $x = \dfrac{\pi}{2}$ nuqta atrofida integral ostidagi funksiya

chegaralanmagan. Berilgan integralda $x = \dfrac{\pi}{2} - t$ almashtirish

bajaramiz. Natijada

$$\int\limits_0^{\frac{\pi}{2}} \ln\cos x\,dx = \int\limits_0^{\frac{\pi}{2}} \ln\sin t\,dt \qquad (*)$$

hosil bo'ladi. Tenglikning o'ng tomonidagi integral ostidagi

funksiya uchun $t=0$ nuqta maxsus nuqta bo'ladi. $\int\limits_0^{\frac{\pi}{2}} \ln\sin t\,dt$

integralning mavjudligini ko'rsatish uchun bo'laklab integrallash

formulasidan foydalanamiz: $u = \ln\sin t$, $dv = dt$, $du = \text{ctg}\,t\,dt$,

$v = t$, u holda

$$\int\limits_0^{\frac{\pi}{2}} \ln\sin t\,dt = t\cdot\ln\sin t\,\Big|_0^{\frac{\pi}{2}} - \int\limits_0^{\frac{\pi}{2}} t\cdot\text{ctg}\,t\,dt = \lim_{t\to 0}(t\cdot\ln\sin t) -$$

$$- \int\limits_0^{\frac{\pi}{2}} t\cdot\text{ctg}\,t\,dt = -\int\limits_0^{\frac{\pi}{2}} t\cdot\text{ctg}\,t\,dt.$$

$t\cdot\text{ctg}\,t$ funksiya $\left[0; \dfrac{\pi}{2}\right]$ da chegaralanganligi uchun oxirgi integral

mavjud.

Demak, $\int\limits_0^{\frac{\pi}{2}} \ln\sin t\,dt$ integral ham mavjud bo'ladi, (*) ga asosan

berilgan integral ham mavjud. $J = \int\limits_0^{\frac{\pi}{2}} \ln\sin t\,dt$ integralda $t = 2u$

almashtirishni bajaramiz:

$$J = 2\int\limits_0^{\pi/4} \ln\sin 2u\, du = 2\int\limits_0^{\pi/4}(\ln 2 + \ln\sin u + \ln\cos u)du =$$

$$= \frac{\pi}{2}\ln 2 + \int\limits_0^{\pi/4}\ln\sin u\, du + 2\int\limits_0^{\pi/4}\ln\cos u\, du.$$

Keyingi integralda $u = \dfrac{\pi}{2} - z$ almashtirishni bajarib, uni

$\int\limits_0^{\frac{\pi}{2}}\ln(\sin z)dz$ ko'rinishga keltiramiz. Natijada

$$J = \frac{\pi}{2}\ln 2 + 2\int\limits_0^{\frac{\pi}{2}}\ln\sin u\, du + 2\int\limits_0^{\frac{\pi}{2}}\ln\sin z\, dz =$$

$$= \frac{\pi}{2}\ln 2 + 2\int\limits_0^{\frac{\pi}{2}}\ln\sin z\, dz = \frac{\pi}{2}\ln 2 + 2J.$$

Bu tenglamadan $J = -\dfrac{\pi}{2}\ln 2$ bo'ladi. Shunday qilib,

$$\int\limits_0^{\frac{\pi}{2}}\ln\cos x\, dx = -\frac{\pi}{2}\ln 2 .$$

**11.7- misol.** $\int\limits_0^2 \dfrac{dx}{(4-x)\sqrt{2-x}}$ integralni hisoblang.

**Yechilishi.** Berilgan xosmas integralni hisoblash uchun o'zgaruvchilarni almashtirish formulasidan foydalanamiz. $2 - x = t^2$ deb belgilaymiz, $t > 0$, bu yerdan $x = 2 - t^2$, $dx = 2tdt$, $\alpha = \sqrt{2}$, $\beta = 0$ bo'ladi. Demak, quyidagiga ega bo'lamiz:

$$\int\limits_0^2 \frac{dx}{(4-x)\sqrt{2-x}} = -2\int\limits_{\sqrt{2}}^0 \frac{tdt}{t(t^2+2)} = 2\int\limits_0^{\sqrt{2}} \frac{dt}{t^2+2} = \frac{\sqrt{2}}{4}\pi.$$

Bu yerda o'zgaruvchilarni almashtirgandan so'ng, xosmas integral xos integralga aylantirildi.

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**11.8- misol.** $\lim\limits_{x\to 0} x\int\limits_{x}^{1} \dfrac{\cos^2 t}{t^2}\, dt$ integralni hisoblang.

**Yechilishi.** Berilgan integralda $f(t)=\cos^2 t,\ g(t)=\dfrac{1}{t^2}$ deb, o'rta qiymat haqidagi teoremaga asosan,

$$\int\limits_{x}^{1}\frac{\cos^2 t}{t^2}\, dt=\left(\frac{1}{x}-1\right)\cdot\cos^2\xi,\quad x<\xi<1$$

ni hosil qilamiz. $\forall\varepsilon>0$ son berilgan bo'lsin, u holda $\forall x\in\left(0;\dfrac{\varepsilon}{1+\varepsilon}\right)$ lar uchun

$$\cos^2\xi\left(\frac{1}{x}-1\right)>\frac{\cos^2\xi}{\varepsilon}$$

bo'lgani sababli $x\to+0$ da $\int\limits_{x}^{1}\dfrac{\cos^2 t}{t^2}\, dt\to+\infty$. Berilgan limitni hisoblash uchun Lopital qoidasini qo'llaymiz:

$$\lim\limits_{x\to 0} x\cdot\int\limits_{x}^{1}\frac{\cos^2 t}{t^2}\, dt=\lim\limits_{x\to 0}\frac{\dfrac{\cos^2 x}{x^2}}{\dfrac{1}{x^2}}=\lim\limits_{x\to 0}\cos^2 x=1$$

### 11.3. Chegaralanmagan funksiya xosmas integrallarining ba'zi bir tatbiqlari.

**11.3.1. Yuzni xosmas integral yordamida hisoblash.** $f(x)$ funksiya $[a;\, b)$ da aniqlangan, uzluksiz va $\forall x\in[a;b)$ uchun $f(x)\geq 0$ bo'lsin. Unda $D=\{(x;y): a\leq x<b,\ 0\leq y\leq f(x)\}$ sohaning yuzi ushbu

$$S=\int\limits_{a}^{b} f(x)dx$$ xosmas integral orqali ifoda qilinadi.

**11.9- misol.** $y=x^{-\frac{2}{3}},\ y=0,\ x=-1,\ x=1$ chiziqlar bilan chegaralangan shaklning yuzini hisoblang.

**Yechilishi.** Talab qilingan yuzni quyidagi xosmas integral orqali hisoblaymiz:

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$$S = \int_{-1}^{1} x^{-\frac{2}{3}} dx = \lim_{\substack{\varepsilon \to 0+0 \\ \eta \to 0-0}} \left( \int_{-1}^{\varepsilon} x^{-\frac{2}{3}} dx + \int_{\eta}^{1} x^{-\frac{2}{3}} dx \right) =$$

$$= \lim_{\substack{\varepsilon \to 0+0 \\ \eta \to 0-0}} \left( 3\left(\varepsilon^{\frac{1}{3}}+1\right) + 3\left(1 - \eta^{1/3}\right) \right) = 6 \ (\text{kv.birl.}).$$

**11.3.2. Aylanma jismning hajmini xosmas integral yordamida hisoblash.** Ushbu $D = \{(x; y) : a \le x < b, \ 0 \le y \le f(x)\}$ egri chiziqli trapetsiyani $Ox$ va $Oy$ o'qlar atrofida aylantirish natijasida hosil bo'lgan aylanma jismlarning hajmi, mos ravishda,

$$V_x = \pi \int_a^b f^2(x)dx, \quad V_y = 2\pi \int_a^b |xy| dx \qquad (11.3)$$

xosmas integrallar orqali hisoblanadi.

**11.10- misol.** $y = \dfrac{1}{\sqrt{x-1}}$, $x \in (1; 2]$ chiziq bilan chegaralangan shaklni $Ox$ o'q atrofida aylantirish natijasida hosil bo'lgan aylanma jism hajmini toping.

**Yechilishi.** Talab qilingan aylanma jismning hajmini (11.3) formula orqali topamiz:

$$V_x = \pi \int_1^2 \frac{dx}{\sqrt{x-1}} = \lim_{\varepsilon \to 0} \pi \int_{1+\varepsilon}^2 \frac{dx}{\sqrt{x-1}} = \pi \lim_{\varepsilon \to 0} 2\sqrt{x-1} \bigg|_{1+\varepsilon}^2 = 2\pi \ (\text{kv.birl.}).$$

**11.3.3. Aylanma sirtning yuzini xosmas integrallar yordamida hisoblash.** $f(x)$ funksiya $[a;b)$ da aniqlangan, uzluksiz va uzluksiz $f'(x)$ hosilaga ega bo'lib, u $f(x) \ge 0$ bo'lsin. $f(x)$ funksiya grafigini $Ox$ o'q atrofida aylantirish natijasida hosil bo'lgan aylanma sirtning yuzi ushbu formula orqali hisoblanadi:

$$S = 2\pi \int_a^b f(x)\sqrt{1+(f'(x))^2}\, dx. \qquad (11.4)$$

Shunga o'xshash, $Oy$ o'q atrofida aylantirish natijasida hosil bo'lgan aylanma sirtning yuzi ushbu formula orqali hisoblanadi:

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