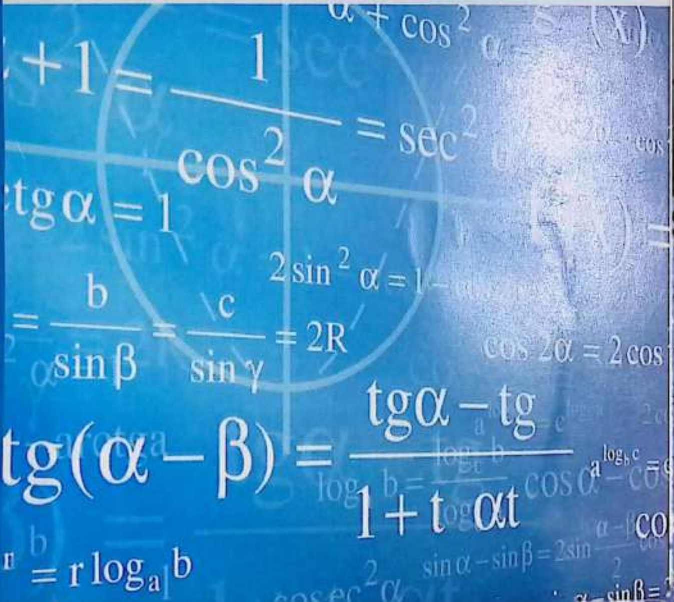
D. N. SHAMSIYEV

# OLIY MATEMATIKA

AMALIY MASHG'ULOTLAR

O'quv qo'llanma



O'ZBEKISTON RESPUBLIKASI
OLIY VA O'RTA MAXSUS TA'LIM VAZIRLIGI

TOSHKENT DAVLAT TEXNIKA UNIVERSITETI

D. N. Shamsiyev

# OLIY MATEMATIKA

# AMALIY MASHG'ULOTLAR

(o'quv qo'llanma)



Toshkent
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Mazkur o‘quv qo‘llanma oliy texnika o‘quv yurtlari uchun “Oliy matematika” fani bo‘yicha tuzilgan dastur doirasida tayyorlangan bo‘lib, oliy matematika kursining bir argumentli funksiyaning integral hisobi, ko‘p argumentli funksiyalar, oddiy differensial tenglamalar hamda qatorlar kabi mavzulari yoritilgan. Qo‘llanma texnika va iqtisodiyot yo‘nalishlari bo‘yicha bakalavriat bosqichidagi talabalar uchun mo‘ljallanib tayyorlangan.

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## SO‘Z BOSHI

O‘quv qo‘llanma O‘zbekiston Respubliksi Vazirlar mahkamasining 2018-yil 10-oktyabrdagi Oliy ta’lim muassasalarini o‘quv adabiyotlari bilan ta’minlash to‘g‘risidagi 816 sonli qarori bajarilishini ta’minlash uchun texnika yo‘nalishlari o‘quv rejalariga asosan yozilgan. Mavzularni yoritishda, O‘zbekiston Respublikasi Oliy va o‘rta maxsus ta’lim Vazirligi tomonidan texnika oliy o‘quv yurtlarining bakalavrlar tayyorlash borasidagi oliy matematika fani bo‘yicha tasdiqlangan dasturi asos qilib olindi.

Ushbu o‘quv qo‘llanma mazmunan oliy o‘quv yurtlarining texnika yo‘nalishlari I kursi talabalarining II semestri oliy matematika kursi uchun mo‘ljallab tayyorlangan. Unda bir argumentli funksiyaning integral hisobi, ikki argumentli funksiyaning differensial hisobi, oddiy differensial tenglamalar hamda qatorlar kabi muhim mavzular yoritilgan bo‘lib, ular amaliy mashg‘ulot darslari uchun mo‘ljallangan.

Har bir mavzuga doir misol va masalalar yechilishidan oldin kerakli nazariy tushunchalar hamda formulalar qisqacha bayon etiladi. Shuningdek, har bir mavzuga doir mustaqil yechish uchun misol va masalalar ham berilgan.

O‘quv qo‘llanmadan nafaqat texnikaviy yo‘nalishdagi talabalar, balki iqtisodiyot yo‘nalishi bo‘yicha tahsil oladigan talabalar hamda akademik litseylar va kasb- hunar kollejlarining o‘quvchilari ham foydalanishlari mumkin.

O‘quv qo‘llanmada ayrim xato va kamchiliklar uchrashi tabiiy. Ularni bartaraf etish borasidagi barcha takliflarni muallif mamnuniyat bilan qabul qiladi.

# 1. ANIQMAS INTEGRAL

## 1.1. Boshlang'ich funksiya va aniqmas integral tushunchasi, xossalari.

### Asosiy integrallar jadvali.

1. Ta'riflar. Agar $(a;\ b)$ oraliqda berilgan $f(x)$ funksiya uchun $F'(x)=f(x)$ yoki $dF(x)=f(x)dx$ kabi munosabat o'rinli bo'lsa, $F(x)$ funksiyani u oraliqda $f(x)$ funksiya uchun boshlang'ich funksiya deb yuritiladi. Berilgan $f(x)$ funksiyaning har qanday ikkita boshlang'ich funksiyalari bir-biridan ixtiyoriy o'zgarmas son bilan farqlanadi. Agar $F(x)$ funksiya $f(x)$ funksiyaning boshlang'ich funksiyasi bo'lsa $(x\in(a;b))$, u holda $F(x)+C$ (bu yerda C- ixtiyoriy o'zgarmas son) funksiyalar ham $f(x)$ funksiya uchun boshlang'ich funksiya bo'ladi va ular $f(x)$ funksiyaning $(a;b)$ oraliqdagi aniqmas integrali deb ataladi va u quyidagicha yoziladi.

$$\int f(x)dx = F(x)+C$$

Bu yerda $\int$ - belgisi, $f(x)$ – integrallanuvchi funksiya, $f(x)dx$ – integral belgisi ostidagi ifoda, $x$ – integrallash o'zgaruvchisidir.

Funksiyaning aniqmas integralini hisoblashni uni integrallash deb yuritiladi.

2. Aniqmas integralning asosiy xossalari quyidagicha yoziladi:

2.1. $\left[\int f(x)dx\right]' = f(x)$,

2.2. $d\left[\int f(x)dx\right] = f(x)dx$,

2.3. $\int d\,f(x) = f(x)+C$,

2.4. $\int af(x)dx = a\int f(x)dx \qquad (a$ - o'zgarmas son),

2.5. $\int[f_1(x)\pm f_2(x)\pm\ldots\pm f_n(x)]dx = \int f_1(x)dx \pm \int f_2(x)dx \pm\ldots\pm \int f_n(x)dx$.

Integrallash natijasining to'g'riligini topilgan boshlang'ich funksiyani diferentsiallash orqali tekshiriladi, ya'ni

$$(F(x)+C)' = f(x).$$

1. Asosiy integrallar jadvali

1. $\int dx = x+C$,

2. $\int x^a dx = \dfrac{x^{a+1}}{a+1}+C \qquad (a\neq -1)$,

3. $\int \dfrac{dx}{x} = \ln|x|+C$,

4. $\int e^x dx = e^x + C$,

5. $\int a^x dx = \dfrac{a^x}{\ln a}+C$,

6. $\int \sin x dx = -\cos x + C$,

7. $\int \cos x dx = \sin x + C,$

8. $\int \frac{dx}{\cos^2 x} = tgx + C,$

9. $\int \frac{dx}{\sin^2 x} = -ctgx + C,$

10. $\int \frac{dx}{1+x^2} = arctgx + C,$

11. $\int \frac{dx}{a^2+x^2} = \frac{1}{a} arctg \frac{x}{a} + C,$

12. $\int \frac{dx}{\sqrt{1-x^2}} = \arcsin x + C,$

13. $\int \frac{dx}{\sqrt{a^2-x^2}} = \arcsin \frac{x}{a} + C,$

14. $\int \frac{dx}{a^2-x^2} = \frac{1}{2a} \ln\left|\frac{a+x}{a-x}\right| + C,$

15. $\int \frac{dx}{\sqrt{x^2 \pm k}} = \ln\left|x + \sqrt{x^2 \pm k}\right| + C,$

16. $\int \frac{f'(x)}{f(x)} dx = \ln|f(x)| + C,$

17. $\int \frac{dx}{\sin x} = \ln\left|tg \frac{x}{2}\right| + C,$

18. $\int \frac{dx}{\cos x} = \ln\left|tg\left(\frac{x}{2} + \frac{\pi}{4}\right)\right| + C,$

19. $\int tgxdx = -\ln|\cos x| + C,$

20. $\int ctgxdx = \ln|\sin x| + C,$

Quyida keltiriladigan misollardagi aniqmas integrallarni hisoblashda aniqmas integrallarning jadvali hamda uning xossalaridan foydalanamiz.

**1-Misol.** $\int (x^2 + 2x + \frac{1}{x})dx = \int x^2 dx + 2\int xdx + \int \frac{dx}{x} = \frac{x^3}{3} + x^2 + \ln|x| + C$.

**2-Misol.** $\int \left(\sqrt{x} + \sqrt[3]{x}\right)dx = \int \sqrt{x}dx + \int \sqrt[3]{x}dx = \int x^{1/2}dx + \int x^{1/3}dx = \frac{x^{3/2}}{3/2} + \frac{x^{4/3}}{4/3} + C =$

$= \frac{2\sqrt{x^3}}{3} + \frac{3\sqrt[3]{x^4}}{4} + C.$.

**3-Misol.** $\int \frac{(\sqrt{x}-1)^3}{x} dx = \int \frac{\sqrt{x^3} - 3\sqrt{x^2} + 3\sqrt{x} - 1}{x} dx = \int \sqrt{x}dx - 3\int dx + 3\int x^{-\frac{1}{2}}dx - \int \frac{dx}{x} =$

$= \frac{2}{3}\sqrt{x^3} - 3x + 6\sqrt{x} - \ln|x| + C$.

**4-Misol.**

$$\int \frac{\cos 2x dx}{\cos^2 x \cdot \sin^2 x} = \int \frac{\cos^2 x - \sin^2 x}{\cos^2 x \cdot \sin^2 x} dx = \int \frac{dx}{\sin^2 x} - \int \frac{dx}{\cos^2 x} = -ctgx - tgx + C.$$

**5-Misol.** $\int \sin^2 \frac{x}{2} dx = \int \frac{1-\cos x}{2} dx = \frac{1}{2}\int dx - \frac{1}{2}\int \cos x dx = \frac{x}{2} - \frac{\sin x}{2} + C$.

**6-Misol.** $\int 3^x 4^{2x} 5^{3x} dx = \int \left(3 \cdot 4^2 \cdot 5^3\right)^x dx = \int 6000^x dx = \frac{6000^x}{\ln 6000} + C$.

**7-Misol.** $\int \frac{x^4}{1+x^2} dx = \int \frac{(x^4 - 1 + 1)}{1+x^2} dx = \int \frac{(x^2+1)(x^2-1)}{1+x^2} dx + \int \frac{dx}{1+x^2} =$

$= \int x^2 dx - \int dx + \int \frac{dx}{1+x^2} = \frac{x^3}{3} - x + arctgx + C.$

## MASHQLAR

Quyidagi integrallar hisoblansin.

1. $\int \frac{10x^5+5}{x^4}dx$
2. $\int \frac{(x^2+1)^2}{x^3}dx$
3. $\int \left(\frac{1}{\sqrt{x}}-\frac{1}{\sqrt[4]{x^3}}\right)dx$
4. $\int \frac{x-1}{\sqrt[3]{x^2}}dx$
5. $\int 7^x\left(1+\frac{7^{-x}}{\sqrt{x^3}}\right)dx$
6. $\int ctg^2 x dx$
7. $\int \frac{3-2ctg^2 x}{\cos^2 x}dx$
8. $\int \cos^2 \frac{x}{2}dx$
9. $\int \frac{1-\sin^3 x}{\sin^2 x}dx$
10. $\int \frac{1+2x^2}{x^2(1+x^2)}dx$
11. $\int \frac{\cos 2x dx}{\cos^2 x \sin^2 x}$
12. $\int (2^x 3^{2x}+3\sin x+\frac{1}{\sqrt{1-x^2}})dx$
13. $\int \frac{dx}{\sqrt{4-x^2}}$
14. $\int \frac{\cos 2x dx}{\cos x+\sin x}$
15. $\int \frac{dx}{25+x^2}$
16. $\int \frac{dx}{\sqrt{x^2-8}}$

### 1.2. Aniqmas integralni o‘rniga o‘yishusuli bilan integrallash

Agar $x=\varphi(t)$ funksiya uzluksiz differensiallanuvchi bo‘lsa, u holda $\int f(x)dx$ ni har doim o‘zgaruvchi $t$ ga nisbatan o‘zgartirish mumkin bo‘ladi:

$$\int f(x)dx=\int f[\varphi(x)]\cdot\varphi'(t)dt.$$

Bu yerda, o‘ng tomondagi integral hisoblangandan keyin hosil bo‘lgan natijada avvalgi $x$ o‘zgaruvchiga qaytiladi. Aniqmas integralni mazkur usul bilan hisoblash usulini o‘rniga qo‘yish yoki o‘zgaruvchini almashtirish usuli deb yuritiladi.

Ta’kidlash lozimki, $x=\varphi(t)$ almashtirish bajarilayotganda, $\varphi(t)$ va $f(x)$ funksiyalarning aniqlanish sohalari orasida o‘zaro bir qiymatli moslik shunday bajarilishi kerakki, $\varphi(t)$ funksiya, $x$ ning barcha qiymatlarini qabul qilishi lozim bo‘ladi.

Ayrim hollarda aniqmas integrallarni hisoblash jarayonida integral belgisi ostidagi ifodalarning ayrim qismlarini differensial belgisi ostiga kiritib, undan keyin integralning invariantlik xossasidan foydalanish lozimligini ham eslatib ketamiz.

**1-Misol.** $\int \frac{\sin\sqrt[3]{x}}{\sqrt[3]{x^2}}dx$ hisoblansin.

__Yechish.__ $x=t^3$ desak, $dx=3t^2dt$ bo'lib, integral esa $\int \frac{3t^2\sin t}{t^2}dt=3\int \sin tdt$ ko'rinishga keladi. Uni hisoblaymiz:

$$3\int \sin tdt=-3\cos t+C.$$

Javobda, $t$ ning o'rniga uning $\sqrt[3]{x}$ qiymatini qo'yamiz.

Shuning uchun $\int \frac{\sin\sqrt[3]{x}}{\sqrt[3]{x^2}}dx=-3\cos\sqrt[3]{x}+C.$

**2-Misol.** $\int (3x-1)^{15}dx=\frac{1}{3}\int (3x-1)^{15}d(3x-1)=\frac{(3x-1)^{16}}{48}+C.$

**3-Misol.** $\int x^2\sqrt{x^3-7}dx$ hisoblansin.

__Yechish.__ $x^3-7=t^2$ desak, $3x^2dx=2tdt,\quad x^2dx=\frac{2}{3}tdt.$ demak,

$$\int x^2\sqrt{x^3-7}dx=\int t\frac{2}{3}tdt=\frac{2}{3}\int t^2dt=\frac{2}{9}t^3+C=\frac{2}{9}\sqrt{(x^3-7)^2}+C=\frac{2}{9}(x^3-7)\sqrt{x^3-7}+C$$

. Bu integralni hisoblashda ifodani differensial belgisi ostiga kiritish usulidan foydalanildi.

**4-Misol.** $\int \frac{(3\ln x-5)^2}{x}dx$ hisoblansin.

__Yechish.__ $3\ln x-5=t,\ 3\frac{dx}{x}=dt,\ \frac{dx}{x}=\frac{dt}{3}$, bulardan

$$\int \frac{(3\ln x-5)^2}{x}dx=\int t^2\frac{dt}{3}=\frac{1}{3}\int t^2dt=\frac{1}{9}t^3+C=\frac{(3\ln x-5)^3}{9}+C.$$

**5-Misol.** $\int \frac{dx}{x\sqrt{3x+5}}$ hisoblansin.

__Yechish.__ $3x+5=t^2$ desak, $3dx=2tdt,\ dx=\frac{2}{3}tdt$, hamda $x=\frac{t^2-5}{3}$ bo'lgani uchun

$$\int \frac{dx}{x\sqrt{3x+5}}=\frac{2}{3}\int \frac{3tdt}{(t^2-5)t}=2\int \frac{dt}{t^2-5}=\frac{1}{2\sqrt{5}}\ln\left|\frac{t-\sqrt{5}}{t+\sqrt{5}}\right|+C=\frac{1}{\sqrt{5}}\ln\left|\frac{\sqrt{3x+3}-\sqrt{5}}{\sqrt{3x+5}+\sqrt{5}}\right|+C.$$

Bu integral (14) formulaga ko'ra hisoblandi.

**6-Misol.** $\int \frac{xdx}{x^4+2x^2+10}$ hisoblansin.

__Yechish.__ $x^4+2x^2+10=(x^2+1)^2+9,\quad x^2+1=t,\ 2xdx=dt$ larga ko'ra,

$$\int \frac{xdx}{x^4+2x^2+10}=\int \frac{dt}{2(t^2+9)}=\frac{1}{2}\int \frac{dt}{t^2+3^2}=\frac{1}{6}arctg\frac{t}{3}+C=\frac{1}{6}arctg\frac{x^2+1}{3}+C.$$

**7-Misol.** $\int x^3(1-2x^4)^3dx$ hisoblansin.

**Yechish.** $1-2x^4=t,\quad -8x^3dx=dt,\ x^3dx=-\frac{dt}{8}$ larga asosan,

$$\int x^3(1-2x^4)^3dx=-\frac{1}{8}\int t^3dt=-\frac{1}{32}t^4+C=-\frac{1}{32}(1-2x^4)^4+C.$$

## MASHQLAR

Quyidagi integrallar hisoblansin

1. $\int\sqrt[3]{5-6x}dx$
2. $\int\frac{dx}{\sqrt{3x+5}}$
3. $\int\frac{e^{2x}dx}{1-3e^{2x}}$
4. $\int\frac{\cos x}{12\sin x}dx$
5. $\int\frac{dx}{x(1+\ln x)}$
6. $\int\frac{e^{\sqrt{x}}}{\sqrt{x}}dx$
7. $\int\frac{x^2dx}{\sqrt[3]{1+x^3}}$
8. $\int\sqrt{1+\sin x}\cos xdx$
9. $\int e^{\sin x}\cos xdx$
10. $\int\frac{dx}{x^2+4x+5}$
11. $\int\frac{x^4dx}{x^2-3}$
12. $\int\frac{3x-4}{x^2-4}dx$
13. $\int\frac{xdx}{\sqrt{4-x^4}}$
14. $\int\frac{\sin\sqrt{x}dx}{\sqrt{x}}$
15. $\int\frac{x^{n-1}dx}{a^2+x^{2n}}dx$
16. $\int x^2\sqrt[3]{1-x}dx$
17. $\int\frac{\arcsin xdx}{\sqrt{1-x^2}}$
18. $\int\frac{\sin 2xdx}{1+\sin^2 x}$
19. $\int\left(\ln x+\frac{1}{\ln x}\right)\frac{dx}{x}$
20. $\int\frac{(x^2+1)dx}{\sqrt[4]{x^3+3x-6}}$

### 1.3. Bo'laklab integrallash usuli

Bo'laklab integrallash usuli, bo'laklab integrallash formulasi deb ataluvchi ushbu $\int udv=uv-\int vdu$ formulaga asoslangan bo'lib, u yerda $u=u(x)$ va $\upsilon=\upsilon(x)$ lar uzluksiz differensiallanuvchi funksiyalardir. Mazkur usulni, $x^k\sin ax, x^k\cos ax, x^ke^{ax}, x^n\ln^k x, x^k chax,$

$x^k shax, a^{\beta x}\sin ax, a^{\beta x}\cos ax, \arcsin x, arctgx$ (bu yerda: $n$, $k$ – butun musbat sonlar, $a,\beta\in R$) kabi va boshqa funksiyalarning aniqmas integrallarini hisoblash uchun qo'llash tavsiya etiladi.

Ayrim hollarda bo'laklab integrallash formulasini bir necha marta qo'llash hollari ham uchrashini ta'kidlab o'tish joizdir.

**1-Misol.** $\int x\sin xdx$ hisoblansin.

**Yechish.** $u=x, dv=\sin xdx$ lardan $du=dx, v=-\cos x$

$$\int x\sin xdx=-x\cos x+\int\cos xdx=-x\cos x+\sin x+C.$$

**2-Misol.** $\int\ln xdx$ hisoblansin.

**Yechish.** $u=\ln x,\ dv=dx$ lardan $du=\frac{dx}{x},\ v=x$

$$\int \ln xdx = x\ln x - \int x\frac{dx}{x} = x\ln x - \int dx = x\ln x - x + C = x(\ln x - 1) + C.$$

**3-Misol.** $\int x^2 e^x dx$ hisoblansin

**Yechish.** $u=x^2,\ dv=e^x dx$ desak, $du=2xdx,\ v=e^x$

$$\int x^2 e^x dx = x^2 e^x - 2\int xe^x dx.$$

Oxirgi integralni hisoblash uchun yana bo'laklab integrallash usulidan foydalanamiz

$u=x^2,\ dv=e^x dx$ lardan $du=2xdx,\ v=e^x$

$$\int x^2 e^x dx = x^2 e^x - 2(xe^x - \int e^x dx) = x^2 e^x - 2xe^x + 2e^x + C = e^x(x^2-2x+2)+C \ .$$

**4-Misol.** $\int \arcsin xdx$ hisoblansin.

**Yechish.** $u=\arcsin x,\ dv=dx$ lardan $du=\frac{dx}{\sqrt{1-x^2}},\ v=x$

$$\int \arcsin xdx = x\arcsin x - \int\frac{xdx}{\sqrt{1-x^2}} = x\arcsin x + \sqrt{1-x^2} + C.$$

**5-Misol.** $\int e^x \sin xdx$ hisoblansin.

**Yechish.** $u=e^x, dv=\sin xdx$ desak $du=e^x dx,\ v=-\cos x$

$$\int e^x \sin xdx = -e^x\cos x + e^x \sin x - \int e^x \sin dx.$$

Oxirgi integralni yana bo'laklab integrallaymiz. $u=e^x,\ dv=\cos xdx$ lardan $du=e^x dx,\ v=\sin x$

$$\int e^x \sin xdx = -e^x\cos x + e^x\sin x - \int e^x \sin x,$$

bundan,

$$2\int e^x \sin xdx = e^x \sin x - e^x\cos x$$

ya'ni

$$\int e^x \sin xdx = \frac{e^x}{2}(\sin x - \cos x) + C.$$

Umuman, $\int e^{ax}\sin bxdx,\ \int e^{ax}\cos bxdx$ kabi integrallarni yuqoridagi misoldagiga o'xshash yo'l bilan integrallab, quyidagilarni yozish mumkin:

$$\int e^{ax}\sin bxdx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx) + C,$$

$$\int e^{ax}\cos bxdx = \frac{e^{ax}}{a^2+b^2}(b\sin bx + a\cos bx) + C.$$

## MASHQLAR

Quyidagi integrallar hisoblansin

1. $\int x\ln(x-1)dx$
2. $\int x\,arctgx\,dx$
3. $\int\frac{xdx}{\sin^2 x}$
4. $\int\frac{\arcsin xdx}{\sqrt{1-x}}$
5. $\int\ln^2 xdx$
6. $\int x^3e^{-x}dx$
7. $\int\ln(x^2+1)dx$
8. $\int\cos(\ln x)dx$
9. $\int e^x\cos xdx$
10. $\int\frac{x\cos xdx}{\sin^2 x}$
11. $\int arctg\sqrt{2x-1}dx$
12. $\int\sin\sqrt{x}dx$
13. $\int x2^{3x}dx$
14. $\int e^{\sqrt{x}}dx$
15. $\int x^2\ln^2 xdx$
16. $\int\frac{\ln xdx}{\sqrt{x}}$
17. $\int e^{2x}\cos xdx$
18. $\int\sin x(\ln x)dx$
19. $\int\frac{\ln(\sin x)dx}{\cos^2 x}$
20. $\int\sin x\ln(\cos x)dx$
21. $\int e^x\ln(e^x+1)dx$
22. $\int x^4\ln xdx$
23. $\int\arccos 2xdx$
24. $\int x^3\cos xdx$
25. $\int\frac{\ln(\cos x)dx}{\sin^2 x}$

## 1.4. Ratsional funksiyalarni integrallash

### 1.4.1. Eng sodda ratsional kasrlarni integrallash

Eng sodda ratsional kasrlar deb quyidagilarga aytiladi:

1. $\frac{A}{x-a}$
2. $\frac{A}{(x-a)^k}$, ($k\geq 2$ ва бутун сон)
3. $\frac{Ax+B}{x^2+px+q}$
4. $\frac{Ax+B}{(x^2+px+q)^n}$, ($n\geq 2$ ва бутун сон)

Bu yerda A, B, $a$, $p$ va $q$ lar haqiqiy sonlar bo'lib, $D=\frac{p^2}{4}-q<0$.

Birinchi va ikkinchi turdagi kasrlarni integrallash osongina jadval integrallariga keltiriladi:

$$\int\frac{Adx}{x-a}=A\int\frac{d(x-a)}{x-a}=A\ln|x-a|+C,$$

$$\int\frac{Adx}{(x-a)^n}=A\int(x-a)^{-n}d(x-a)=A\frac{(x-a)^{-n+1}}{-n+1}+C=\frac{A}{(1-n)(x-a)^{n-1}}+C.$$

Quyidagi misollarni ko'rib chiqamiz.

**1-Misol.** $\int\frac{dx}{x-3}=\ln|x-3|+C$.

**2-Misol.** $\int\frac{dx}{5-x}=-\int\frac{dx}{x-5}=-\ln|x-5|+C=\ln\left|\frac{1}{x-5}\right|+C$.

**3-Misol.** $\int\frac{4dx}{3x-1}=\frac{4}{3}\int\frac{d(3x-1)}{3x-1}=\frac{4}{3}\ln|3x-1|+C$.

**4-Misol.** $\int\frac{6dx}{4-7x}=-\frac{6}{7}\int\frac{d(4-7x)}{4-7x}=-\frac{6}{7}\ln|4-7x|+C$.

**5-Misol.** $\int\frac{3dx}{(x-2)^3}=3\int(x-2)^{-3}d(x-2)=3\frac{(x-2)^{-2}}{-2}+C=-\frac{3}{2(x-2)^2}+C.$

**6-Misol.** $\int\frac{dx}{(2x-1)^4}=\frac{1}{2}\int(2x-1)^{-4}d(2x-1)=\frac{1}{2}\frac{(2x-1)^{-3}}{-3}+C=-\frac{1}{6(2x-1)^3}+C.$

**7-Misol.** $\int\frac{5dx}{(4-3x)^6}=5\int(4-3x)^{-6}dx=-\frac{5}{3}\int(4-3x)^{-6}d(4-3x)=$

$$=-\frac{5}{3}\frac{(4-3x)^{-5}}{-5}+C=\frac{1}{3(4-3x)^5}+C.$$

Uchinchi turdagi kasrni integrallash uchun kasrning suratiga maxrajning hosilasi $2x+p$ yoziladi va ayniy almashtirishlar orqali $2x+p$ dan $Ax+B$ ni hosil qilinadi, ya'ni $Ax+B=(2x+p)\frac{A}{2}+B-\frac{Ap}{2}$

U holda

$$\int\frac{(Ax+B)dx}{x^2+px+q}=\int\frac{\frac{A}{2}(2x+p)+\left(B-\frac{Ap}{2}\right)}{x^2+px+q}dx=\frac{A}{2}\int\frac{(2x+p)dx}{x^2+px+q}+\left(B-\frac{Ap}{2}\right)\int\frac{dx}{x^2+px+q}.$$

Integrallardan birinchisi $\ln\left|x^2+px+q\right|$ ga teng. Ikkinchi integralning maxrajida esa, to'la kvadrat ajratamiz: $x^2+px+q=\left(x+\frac{p}{2}\right)^2+q-\frac{p^2}{4}$, bu yerda $q-\frac{p^2}{4}>0$, chunki shartga ko'ra $D=\frac{p^2}{4}-q<0$ edi. Demak, ikkinchi integral ham jadval integraliga keladi. Yuqoridagi mulohazalarga asosan

$$\int\frac{(Ax+B)dx}{x^2+px+q}=\frac{A}{2}\int\frac{d\left(x^2+px+q\right)}{x^2+px+q}+\left(B-\frac{Ap}{2}\right)\int\frac{d\left(x+\frac{p}{2}\right)}{\left(x+\frac{p}{2}\right)^2+q-\frac{p^2}{4}}=$$

$$=\frac{A}{2}\ln\left|x^2+px+q\right|+\left(B-\frac{Ap}{2}\right)\frac{1}{\sqrt{q-\frac{p^2}{4}}}arctg\frac{x+\frac{p}{2}}{\sqrt{q-\frac{p^2}{4}}}+C.$$

Ta'kidlash joizki, agar yuqorida $A=0$ bo'lsa, suratda maxrajning hosilasini ajratish shart emas, maxrajda darhol to'la kvadrat ajratish lozim.

<u>**Eslatma:**</u> Agar ikkinchi tur kasrning maxrajidagi $x^2+px+q$ kvadrat uch had o'rnida $ax^2+bx+c\ (a\neq0)$ kabi kvadrat uch had qatnashganda ham yuqorida bayon etilganlar asosida ish yuritiladi, farqi shuki, bu yerda koeffitsient $a$ ni qavsdan tashqariga chiqariladi.

**8-Misol.** $\int\frac{(3x+4)dx}{x^2+7x+14}$ hisoblansin.

Yechish.

$$\int\frac{(3x+4)dx}{x^2+7x+14}=\int\frac{(2x+7)\frac{3}{2}+4-\frac{21}{2}}{x^2+7x+14}dx=\frac{3}{2}\int\frac{(2x+7)dx}{x^2+7x+14}-\frac{13}{2}\int\frac{dx}{x^2+7x+14}=$$

$$=\frac{3}{2}\int\frac{d(x^2+7x+14)}{x^2+7x+14}-\frac{13}{2}\int\frac{d\left(x+\frac{7}{2}\right)}{\left(x+\frac{7}{2}\right)^2+\frac{7}{4}}=\frac{3}{2}\ln(x^2+7x+14)-\frac{13}{2}\frac{1}{\frac{\sqrt{7}}{2}}arctg\frac{x+\frac{7}{2}}{\frac{\sqrt{7}}{2}}+C=$$

$$=\frac{3}{2}\ln(x^2+7x+14)-\frac{13}{\sqrt{7}}arctg\frac{2x+7}{\sqrt{7}}+C.$$

**9-Misol.** $\int\frac{(5x-7)dx}{x^2+3x+8}$ hisoblansin.

Yechish.

$$\int\frac{(5x-7)dx}{x^2+3x+8}=\int\frac{(2x+3)\frac{5}{2}-7-\frac{15}{2}}{x^2+3x+8}dx=\frac{5}{2}\int\frac{d(x^2+3x+8)}{x^2+3x+8}-\frac{29}{2}\int\frac{dx}{\left(x+\frac{3}{2}\right)^2+8-\frac{9}{4}}=$$

$$=\frac{5}{2}\ln(x^2+3x+8)-\frac{29}{2}\int\frac{d\left(x+\frac{3}{2}\right)}{\left(x+\frac{3}{2}\right)^2+\frac{23}{4}}=\frac{5}{2}\ln(x^2+3x+8)-\frac{29}{2}\frac{1}{\frac{\sqrt{23}}{2}}arctg\frac{x+\frac{3}{2}}{\frac{\sqrt{23}}{2}}+C=$$

$$=\frac{5}{2}\ln(x^2+3x+8)-\frac{29}{\sqrt{23}}arctg\frac{2x+3}{\sqrt{23}}+C.$$

**10-Misol.** Integralni hisoblang.

$$\int\frac{dx}{x^2+6x+19}=\int\frac{d(x+3)}{(x+3)^2+10}=\frac{1}{\sqrt{10}}arctg\frac{x+3}{\sqrt{10}}+C.$$

**11-Misol.** Integralni hisoblang.

$$\int\frac{dx}{3x^2-11x+17}=\frac{1}{3}\int\frac{dx}{x^2-\frac{11}{3}x+\frac{17}{3}}=\frac{1}{3}\int\frac{dx}{\left(x-\frac{11}{6}\right)^2+\frac{17}{3}-\frac{121}{36}}=$$

$$=\frac{1}{3}\int\frac{d\left(x-\frac{11}{6}\right)}{\left(x-\frac{11}{6}\right)^2+\frac{83}{36}}=\frac{1}{3}\frac{1}{\frac{\sqrt{83}}{6}}arctg\frac{x-\frac{11}{6}}{\frac{\sqrt{83}}{6}}+C=\frac{2}{\sqrt{83}}arctg\frac{6x-11}{\sqrt{83}}+C.$$

To‘rtinchi turdagi kasrni integrallash ham uchinchi turdagi kasrni integrallashga o‘xshash olib boriladi:

$$\int\frac{(Ax+B)dx}{(x^2+px+q)^n}=\frac{A}{2}\int\frac{(2x+p)dx}{(x^2+px+q)^n}+\left(B-\frac{Ap}{2}\right)\int\frac{dx}{(x^2+px+q)^n}=$$

$$=\frac{A}{2}\int(x^2+px+q)^{-n}d(x^2+px+q)+\left(B-\frac{Ap}{2}\right)\int\frac{dx}{\left[\left(x+\frac{p}{2}\right)^2+\left(q-\frac{p^2}{4}\right)\right]^n}=$$

$$=\frac{A}{2(1-n)(x^2+px+q)^{n-1}}+\left(B-\frac{Ap}{2}\right)\int\frac{d\left(x+\frac{p}{2}\right)}{\left[\left(x+\frac{p}{2}\right)^2+\left(q-\frac{p^2}{4}\right)\right]^n}.$$

Agar ikkinchi integralda $x+\frac{p}{2}=z$ va $q-\frac{p^2}{4}=m^2$ deb belgilash kiritsak (chunki $q-\frac{p^2}{4}>0$ edi), u integral $I_n=\int\frac{dz}{(m^2+z^2)^n}$ ni integrallashga keltiriladi va u $I_n=\frac{z}{2(n-1)m^2(m^2+z^2)^{n-1}}+\frac{2n-3}{2(n-1)m^2}I_{n-1}$ kabi rekurrent formula bilan hisoblanadi, bu yerda: $I_{n-1}=\int\frac{dz}{(m^2+z^2)^{n-1}}$. Bu formula bo'yicha $I_{n-1}$ni $I_{n-2}$ orqali, sungra $I_{n-2}$ni $I_{n-3}$ orqali ifodalaymiz va hokazo. Bu jarayon $I_1=\int\frac{dz}{m^2+z^2}=\frac{1}{m}arctg\frac{z}{m}+C$ ni hosil qilingunicha davom ettiriladi.

**12-Misol.** $I_3=\int\frac{dx}{(1+x^2)^3}$ hisoblansin.

<u>Yechish.</u> Yuqoridagi rekurrent formulaga binoan,

$$I_3=\frac{x}{2(3-1)(1+x^2)^2}+\frac{2\cdot3-3}{2(3-1)}=I_2=\frac{x}{4(1+x^2)^2}+\frac{3}{4}I_2.$$

Endi rekurrent formulani $I_2$ ga qo'llaymiz.

$$I_3=\frac{x}{4(1+x^2)^2}+\frac{3}{4}\left[\frac{x}{2(1+x^2)}+\frac{1}{2}I_1\right]=\frac{x}{4(1+x^2)^2}+\frac{3x}{8(1+x^2)}+\frac{3}{8}I_1.$$

Agar $I_1=\int\frac{dx}{1+x^2}=arctgx+C$ ekanligini inobatga olsak,

$$I_3=\int\frac{dx}{(1+x^2)^3}=\frac{1}{4}\frac{x}{(1+x^2)^2}+\frac{3}{8}\frac{x}{1+x^2}+\frac{3}{8}arctgx+C.$$

**13-Misol.** Integralni hisoblang.

$$\int\frac{dx}{(3x^2+x+7)^2}=\int\frac{dx}{3^2(x^2+\frac{x}{3}+\frac{7}{3})^2}=\frac{1}{9}\int\frac{d(x+\frac{1}{6})}{\left[(x+\frac{1}{6})^2+\frac{83}{36}\right]^2}=\frac{1}{9}\frac{x+\frac{1}{6}}{2\cdot\frac{83}{36}\left[(x+\frac{1}{6})^2+\frac{83}{36}\right]}+$$

$$+\frac{1}{2\cdot\frac{83}{36}}\int\frac{d(x+\frac{1}{6})}{(x+\frac{1}{6})^2+\frac{83}{36}}=\frac{6x+1}{83(3x^2+x+7)}+\frac{12}{83\sqrt{83}}arctg\frac{6x+1}{\sqrt{83}}+C.$$

**14-Misol.** $\int\frac{(3x+5)dx}{(x^2+2x+5)^4}$ ni hisoblang.

**Yechish.**

$$\int\frac{(3x+5)dx}{(x^2+2x+5)^4}=\int\frac{(2x+2)\frac{3}{2}+2}{(x^2+2x+4)^4}dx=\frac{3}{2}\int\frac{d(x^2+2x+4)}{(x^2+2x+4)^4}+2\int\frac{d(x+1)}{\left[(x+1)^2+2^2\right]^4}=$$

$$=\frac{1}{2(x^2+2x+4)^3}+2I_4.$$

Endi $I_4$ ni rekurrent formula yordamida hisoblaymiz:

$$I_4=\frac{(x+1)}{2\cdot3\cdot4(x^2+2x+5)^3}+\frac{5}{2\cdot3\cdot4}I_3=\frac{x+1}{24(x^2+2x+5)^3}+$$

$$+\frac{5}{24}\left[\frac{x+1}{2\cdot2\cdot4(x^2+2x+5)^2}+\frac{3}{2\cdot2\cdot4}I_2\right]=\frac{x+1}{24(x^2+2x+5)^3}+\frac{5(x+1)}{384(x^2+2x+5)^2}+$$

$$+\frac{15}{384}\frac{x+1}{2\cdot4\cdot(x^2+2x+5)}+\frac{15}{384}\frac{1}{2\cdot4}\int\frac{d(x+1)}{(x+1)^2+4}=\frac{x+1}{24(x^2+2x+5)^3}+\frac{5(x+1)}{384(x^2+2x+5)^2}+$$

$$+\frac{5(x+1)}{1024(x^2+2x+5)}+\frac{5}{2048}arctg\frac{x+1}{2}+C$$

## MASHQLAR

1. $\int\frac{dx}{2x-1}$
2. $\int\frac{2dx}{x-13}$
3. $\int\frac{4dx}{15-3x}$
4. $\int\frac{3dx}{3-8x}$
5. $\int\frac{dx}{4x-9}$
6. $\int\frac{4dx}{(x+3)^5}$
7. $\int\frac{2dx}{(4x-5)^5}$
8. $\int\frac{6dx}{(5-6x)^4}$
9. $\int\frac{7dx}{(1-8x)^2}$
10. $\int\frac{dx}{x^2-2x+5}$
11. $\int\frac{dx}{7+x+x^2}$
12. $\int\frac{dx}{3x^2-5x+3}$
13. $\int\frac{(2x-3)dx}{x^2+4x+5}$
14. $\int\frac{(4x+5)dx}{3x^2-5x+4}$
15. $\int\frac{(5-3x)dx}{4x^2+x+5}$
16. $\int\frac{(6x-5)dx}{(3x^2+4x+5)^2}$
17. $\int\frac{(4x+13)dx}{(x^2-6x+10)^3}$
18. $\int\frac{(3x-11)dx}{(x^2+8x+18)^2}$

### 1.4.2. Kasr-ratsional funksiyalarni integrallash

1. Ma'lumki, $P_n(x)=a_nx^n+a_{n-1}x^{n-1}+...+a_1x+a_0$ kabi funksiya, $n$-darajali ko'phad deyiladi, bu yerda $a_0, a_1, ..., a_n$ lar haqiqiy sonlar bo'lib, ular ko'phadning koeffitsientlaridir, $n$-daraja ko'rsatkichi.

Ikki ko'phadning nisbati kasr-ratsional funksiya yoki ratsional kasr deb yuritiladi, ya'ni:

$R(x)=\dfrac{Q_m(x)}{P_n(x)}$. Agar m<n bo'lsa, uni to'g'ri ratsional kasr, agar m≥n bo'lsa – noto'g'ri ratsional kasr deb aytiladi.

Noto'g'ri ratsional kasr funksiyaning suratidagi ko'phadni maxrajidagi ko'phadga bo'lish yo'li bilan uni ko'phad bilan to'g'ri ratsional kasrning yig'indisi shaklida ifodalash mumkin bo'ladi. Demak, har qanday kasr-ratsional funksiyani integrallash har doim ko'phad bilan to'g'ri ratsional kasrni integrallashga keltiriladi.

2. Agar $x=x_1$ da $P_n(x_1)=0$ bo'lsa, $x_1$ ni $P_n(x)$ ko'phadning ildizi deb ataladi.

a) Agar $x_1, x_2, ..., x_n$ sonlari $P_n(x)$ ko'phadning ildizlari bo'lsa, u holda uni quyidagicha ifodalash mumkin:

$$P_n(x)=a_n(x-x_1)(x-x_2)(x-x_3)...(x-x_n)$$

Bu yoyilmada x-$x_1$ ko'paytuvchi bir marta qatnashsa, $x_1$ ni oddiy ildiz, agar u $\alpha_1$ marta qatnashsa, $x_1$ ni $\alpha_1$ karrali ildiz deb ataladi.

Agar $x_1$, $P_n(x)$ ning $\alpha_1$ karrali, $x_2$, $a_2$ karrali va hokazo $x_k$ esa $a_k$ karrali ildizi bo'lsa, $P_n(x)=a_n(x-x_1)^{\alpha_1}(x-x_2)^{\alpha_2}...(x-x_k)^{\alpha_k}$ deb yozish mumkin. Bu yerda: $\alpha_1+\alpha_2+...\alpha_k=n.$

b) Agar $P_n(x)$ ko'phad nafaqat haqiqiy ildizlarga, balki o'zaro qo'shma bo'lgan kompleks son ildizlarga ham ega bo'lsa, uni

$$P_n(x)=a_n(x-x_1)^{\alpha_1}(x-x_2)^{\alpha_2}...(x-x_k)^{\alpha_k}(x^2+p_1x+q_1)^{\beta_1}$$
$$(x^2+p_2x+q_2)^{\beta_2}...(x^2+p_kx+q_k)^{\beta_k} \qquad (*)$$

kabi ifodalash mumkin, bu yerda $a_1+a_2+...+a_k+2\beta_1+2\beta_2+...+2\beta_k=n.$

<u>**Teorema.**</u> Har qanday $R(x)=\dfrac{Q_m(x)}{P_n(x)}$ to'g'ri ratsional kasrning maxraji (*) kabi yoyilmaga ega bo'lsa, uni I, II, III va IV turdagi eng sodda kasrlarning yig'indisi ko'rinishida ifoda etish mumkin. Bunda:

1. (*) yoyilmaning (x-$x_1$) ko'rinishdagi ko'paytuvchisiga I-turdagi bitta $\dfrac{A}{x-x_1}$ kasr mos keladi;

2. (*) yoyilmaning $(x-x_1)^{\alpha}$ kabi ko'paytuvchisiga I va II turdagi $\alpha$ dona kasr yig'indisi mos keladi:

$$\frac{A_1}{(x-x_1)^{\alpha}}+\frac{A_2}{(x-x_1)^{\alpha-1}}+\frac{A_3}{(x-x_1)^{\alpha-2}}+...+\frac{A_{\alpha}}{x-x_1};$$

3. (*) yoyilmaning $x^2+px+q$ kabi ko'paytuvchisiga III turdagi $\frac{Ax+B}{x^2+px+q}$ kasr mos keladi;

4. (*) yoyilmaning $(x^2+px+q)^{\beta}$ kabi ko'paytuvchisiga III va IV turdagi $\beta$ dona kasr yig'indisi mos keladi:

$$\frac{A_1x+B_1}{(x^2+px+q)^{\beta}}+\frac{A_2x+B_2}{(x^2+px+q)^{\beta-1}}+...+\frac{A_{\beta}x+B_{\beta}}{x^2+px+q}.$$

To'g'ri ratsional kasrning eng sodda ratsional kasrlar yig'indisiga bo'lgan yoyilmasining suratida qatnashuvchi koeffitsientlarni aniqlashning son qiymatlarni o'rniga qo'yish va noma'lum koeffitsientlar usuli deb ataluvchi usullari mavjud.

3. Shunday qilib, to'g'ri ratsional kasr funksiyani integrallash, 4 xildagi eng sodda kasrlarni integrallashga keltirilar ekan. To'g'ri ratsional kasr funksiyalarni integralash masalasini quyida keltirilgan misollarni yechish jarayonida o'rganamiz. Shu maqsadda quyidagi 4 holni ko'rib o'tamiz:

__Birinchi hol.__ To'g'ri ratsional kasr funksiya maxrajidagi ko'phad faqat haqiqiy har xil ildizlarga ega bo'lsin.

**1-Misol.** $\int\frac{(2x+3)dx}{x^2-7x+12}$

__Yechish.__ $x^2-7x+12=(x-3)(x-4)$ bo'lganligi uchun

$$\frac{2x+3}{x^2-7x+12}=\frac{2x+3}{(x-3)(x-4)}=\frac{A}{x-3}+\frac{B}{x-4}$$

va bu tenglikning har ikkala tomonini $(x-3)(x-4)$ ga ko'paytirib, $2x+3=A(x-4)+B(x-3)$ ni hosil qilamiz. Noma'lum koeffitsietlarni aniqlash uchun xususiy qiymatllarni o'rniga qo'yish usulidan foydalanamiz.

Agar $x=3$ bo'lsa, $A=-9$ va $x=4$ bo'lsa, $B=11$

Demak,

$$\int\frac{(2x+3)dx}{x^2-7x+12}=\int\frac{(2x+3)dx}{(x-3)(x-4)}=-9\int\frac{dx}{x-3}+11\int\frac{dx}{x-4}=-9\ln|x-3|+11\ln|x-4|+C=$$

$$=\ln\frac{(x-4)^{11}}{(x-3)^9}+C.$$

**2-Misol.** $\int \frac{x^4-3x^3-11x^2+4x+15}{x^3-5x^2-x+5}dx$

<u>**Yechish.**</u> Integral ishorasi ostida noto'g'ri ratsional kasr funksiya qatnashganligi uchun suratdagi ko'phadni maxrajdagi ko'phadga bo'lib, uning butun qismini ajratamiz. Maxraj esa

$$x^3-5x^2-x+5=(x+1)(x-1)(x-5).$$

$$\frac{x^4-3x^3-11x^2+4x+15}{x^3-5x^2-x+5}=x+2+\frac{x+5}{(x+1)(x-1)(x-5)},$$

$$\frac{x+5}{(x+1)(x-1)(x-5)}=\frac{A}{x+1}+\frac{B}{x-1}+\frac{C}{x-5}$$

dan

$$x+5=A(x-1)(x-5)+B(x+1)(x-5)+C(x+1)(x-1).$$

Agar x=-1 bo'lsa, $A=\frac{1}{3}$; agar x=1 bo'lsa, $B=-\frac{3}{4}$ va agar x=5 bo'lsa, $C=\frac{5}{12}$.

Natijada:

$$\int\frac{(x^4-3x^3-11x^2+4x+15)dx}{x^3-5x^2-x-5}=\int(x+2)dx+\frac{1}{3}\int\frac{dx}{x+1}-\frac{3}{4}\int\frac{dx}{x-1}+\frac{5}{12}\int\frac{dx}{x-5}=$$

$$=\frac{(x+2)^2}{2}+\frac{1}{3}\ln|x+1|-\frac{3}{4}\ln|x-1|+\frac{5}{12}\ln|x-5|+C.$$

**3-Misol.** $\int\frac{(2x^2-7x+8)dx}{x^4-10x^2+9}=\int\frac{(2x^2-7x+8)dx}{(x-1)(x+1)(x-3)(x+3)}$ hisoblansin.

<u>**Yechish.**</u> $\frac{2x^2-7x+8}{(x-1)(x+1)(x-3)(x+3)}=\frac{A}{x-1}+\frac{B}{x+1}+\frac{C}{x-3}+\frac{D}{x+3}$ dan

$$2x^2-7x+8=A(x+1)(x^2-9)+B(x-1)(x^2-9)+C(x+3)(x^2-1)+D(x-3)(x^2-1)$$

Bu yerda ham A,B,C va D larni aniqlash uchun xususiy qiymatlar berish usulidan foydalanamiz (umuman, agar maxraj faqat haqiqiy va har xil ildizlarga ega bo'lsa, mazkur usulni qo'llash eng maqsadga muvofiq usuldir).

Agar x=1 bo'lsa, $A=-\frac{3}{16}$; x=-1 da $B=\frac{17}{16}$; x=3 da $C=\frac{5}{48}$ va $x=-3$ ва $D=-\frac{47}{48}$

Demak,

$$\int\frac{(2x^2-7x+8)dx}{x^4-10x^2+9}=-\frac{3}{16}\ln|x-1|+\frac{17}{16}\ln|x+1|+\frac{5}{48}\ln|x-3|-\frac{47}{48}\ln|x+3|+C.$$

<u>**Ikkinchi hol.**</u> Maxrajdagi ko'phad faqat haqiqiy ildizlarga ega bo'lib, ulardan ayrimlari karrali ildizlar bo'lsin.

**4-Misol.** $\int\frac{(x^2+x-1)dx}{x^2(x-2)}$ hisoblansin.

<u>**Yechish.**</u> $\frac{x^2+x-1}{x^2(x-2)}=\frac{A}{x^2}+\frac{B}{x}+\frac{C}{x-2}$ dan

$x^2+x-1=A(x-2)+Bx(x-2)+Cx^2$ ni yozamiz. Agar x=0 bo'lsa, $A=\frac{1}{2}$ ni va x=2 bo'lsa, $C=\frac{5}{4}$ ni aniqlaymiz. B koeffitsientini aniqlash uchun $x^2$ oldidagi koeffitsientlarni tenglashtiramiz:

$$1=B+C,\qquad B=1-C=1-\frac{5}{4}=-\frac{1}{4};\qquad B=-\frac{1}{4}.$$

Natijada:

$$\int\frac{(x^2+x-1)dx}{x^2(x-2)}=\frac{1}{2}\int\frac{dx}{x^2}-\frac{1}{4}\int\frac{dx}{x}+\frac{5}{4}\int\frac{dx}{x-2}=-\frac{1}{2x}-\frac{1}{4}\ln|x|+\frac{5}{4}\ln|x-2|+C.$$

**5-Misol.** $\int\frac{dx}{(x-1)^2(x+4)^2}$ hisoblansin.

<u>**Yechish.**</u> $\frac{1}{(x-1)^2(x+4)^2}=\frac{A}{(x-1)^2}+\frac{B}{x-1}+\frac{C}{(x+4)^2}+\frac{D}{x+4}$ dan

$$1=A(x+4)^2+B(x-1)(x+4)^2+C(x-1)^2+D(x-1)^2(x+4)\text{ ni}$$

yoki

$$Ax^2+8Ax+16A+Bx^3+7Bx^2+8Bx-16B+Cx^2-$$
$$-2Cx+Dx^3+2Dx^2-7Dx+4D=1$$

Noma'lum koeffitsientlar usuliga binoan x ning darajalariga ko'ra, tenglikning har ikkala tomonida ular oldidagi koeffitsientlarni tenglashtiramiz:

$x^3$: $B+D=0,$
$x^2$: $A+7B+C+2D=0,$
$x$: $8A+8B-2C-7D=0,$
$x^0$: $16A-16B+C+4D=1$

Ushbu sistemani yechib, $A=C=\frac{1}{25}$, $B=-\frac{1}{125}$, $D=\frac{2}{25}$ larni topamiz. U holda

$$\int\frac{dx}{(x-1)^2(x+4)^2}=\frac{1}{25}\int\frac{dx}{(x-1)^2}-\frac{2}{125}\int\frac{dx}{x-1}+\frac{1}{25}\int\frac{dx}{(x+4)^2}+$$
$$+\frac{2}{125}\int\frac{dx}{x+4}=-\frac{1}{25(x-1)}-\frac{2}{125}\ln|x-1|-\frac{1}{25(x+4}+\frac{2}{125}\ln|x+4|+C=$$
$$=\frac{2}{125}\ln\left|\frac{x+4}{x-1}\right|-\frac{2x-3}{25(x^2+3x-4)}+C.$$

**6-Misol.** $\int\frac{(x^2-x+14)dx}{(x-4)^3(x-2)}$ hisoblansin.

**Yechish.** $\frac{x^2-x+14}{(x-4)^3(x-2)}=\frac{A}{(x-4)^3}+\frac{B}{(x-4)^2}+\frac{C}{x-4}+\frac{D}{x-2}$ dan

$$x^2-x+14=A(x-2)+B(x-2)(x-4)+C(x-2)(x-4)^2+D(x-4)^3$$

ni yoki

$$Ax-2A+Bx^2-6Bx+8B+Cx^3-10Cx^2+32Cx-32C+Dx^3-12Dx^2+48Dx-64D=x^2-x+14$$

ni yozib olamiz. Bundan :

$x^3$: $C+D=0$,

$x^2$: $B-10C-12D=1$,

$x$: $A-6B+32C+48D=-1$,

$x^0$: $-2A+8B-32C-64D=14$

va $A=13$; $B=-3$; $C=2$; $D=-2$ larni hosil qilamiz.

Natijada:

$$\int\frac{(x^2-x+14)dx}{(x-4)^3(x-2)}=13\int\frac{dx}{(x-4)^3}-3\int\frac{dx}{(x-4)^2}+2\int\frac{dx}{x-4}-2\int\frac{dx}{(x-2)}=$$

$$-\frac{13}{2(x-4)^2}+\frac{3}{x-4}+2\ln|x-4|-2\ln|x-2|+C=-\frac{13}{2(x-4)^2}+\frac{3}{x-4}+2\ln\left|\frac{x-4}{x-2}\right|+C.$$

**Uchinchi hol**. To'g'ri ratsional kasr funksiya maxrajidagi ko'phad ildizlari orasida haqiqiy ildizlardan tashqari bir-biriga teng bo'lmagan kompleks ildizlar ham mavjud bo'lsin.

**7-Misol.** $\int\frac{dx}{x^4-1}$ hisoblansin.

**Yechish.** $x^4-1=(x^2-1)(x^2+1)=(x-1)(x+1)(x^2+1)$ bo'lgani uchun

$$\frac{1}{x^4-1}=\frac{A}{x-1}+\frac{B}{x+1}+\frac{Cx+D}{x^2+1} \text{ va}$$

$$1=A(x+1)(x^2+1)+B(x-1)(x^2+1)+(Cx+D)(x^2-1) \quad \text{yoki}$$

$$Ax^3+Ax^2+Ax+A+Bx^3-Bx^2+Bx-B+Cx^3+Dx^2-Cx-D=1$$

Bu yerda mos koeffitsientlarni tenglashtirsak

$x^3$: $A+B+C=0$,

$x^2$: $A-B+D=0$,

$x$: $A+B-C=0$,

$x^0$: $A-B-D=1$.

Bundan $A=\frac{1}{4}$; $B=-\frac{1}{4}$; $C=0$ ва $D=-\frac{1}{2}$.

Demak

$$\int\frac{dx}{x^4-1}=\frac{1}{4}\int\frac{dx}{x-1}-\frac{1}{4}\int\frac{dx}{x+1}-\frac{1}{2}\int\frac{dx}{x^2+1}=\frac{1}{4}\ln|x-1|-\frac{1}{4}\ln|x+1|-\frac{1}{2}arctg+C=$$

$$=\ln\sqrt[4]{\frac{x-1}{x+1}}-\frac{1}{2}arctg+C.$$

Bundan: $\frac{7x+1}{x(x-1)}=\frac{A}{x}+\frac{B}{x-1}$ ;$A=-1,\ B=8.$

Demak

$$\frac{7x+1}{x^2-x}=-\frac{1}{x}+\frac{8}{x-1}$$

$$\int\frac{(x+1)^3}{x^2-x}dx=\int xdx+4\int dx-\int\frac{dx}{x}+8\int\frac{dx}{x-1}=\frac{x^2}{2}+4x-\ln|x|+8\ln|x-1|+C.$$

**14-Misol.** $\int\frac{x^2+2x+6}{(x-1)(x-2)(x-4)}dx$ hisoblansin.

<u>**Yechish.**</u> $\frac{x^2+2x+6}{(x-1)(x-2)(x-4)}dx=\frac{A}{x-1}+\frac{B}{x-2}+\frac{C}{x-4}$ *dan:*

$x^2+2x+6=A(x-2)(x-4)+B(x-1)(x-4)+C(x-1)(x-2),$

$x^2+2x+6=A(x^2-6x+8)+B(x^2-5x+4)+C(x^2-3x+2),$

$x^2+2x+6=(A+B+C)x^2+(-6A-5B-3C)x+(8A+4B+2C).$

Bir xil darajali $x$ larning koeffitsientlarini taqqoslab, quyidagi tenglamalar sistemasini hosil qilamiz va uni echamiz:

$$\begin{cases}A+B+C=1,\\-6A-5B-3C=2,\\8A+4B+2C=6,\end{cases}\qquad A=3,\ B=-7,\ C=5.$$

Demak

$$\frac{x^2+2x+6}{(x-1)(x-2)(x-4)}=\frac{3}{x-1}-\frac{7}{x-2}+\frac{5}{x-4}.$$

Natijada $\int\frac{x^2+2x+6}{(x-1)(x-2)(x-4)}dx=3\int\frac{dx}{x-1}-7\int\frac{dx}{x-2}+5\int\frac{dx}{x-4}=$

$$=3\ln|x-1|-7\ln|x-2|+5\ln|x-4|+C=\ln\left|\frac{(x-1)^3(x-4)^5}{(x-2)^7}\right|+C.$$

**15-Misol.** $\int\frac{dx}{x^2(x-1)(x^2+x+1)}$ hisoblansin.

<u>**Yechish.**</u> $\frac{1}{x^2(x-1)(x^2+x+1)}=\frac{A}{x^2}+\frac{B}{x}+\frac{C}{x-1}+\frac{Dx+E}{x^2+x+1}$ dan:

$1=A(x-1)(x^2+x+1)+Bx(x-1)(x^2+x+1)+Cx^2(x^2+x+1)+(Dx+E)x^2(x-1),$

yoki $1=A(x^3-1)+B(x^4-x)+C(x^4+x^3+x^2)+Dx^4+Ex^3-Dx^3-Ex^2.$

0 va 1 sonlar maxrajning ildizlari bo'lgani uchun $x=0$ da $1=-A$, ya'ni $A=-1$; $x=1$ da $1=3C$, ya'ni $C=1/3$.

$x^4, x^3, x^2$ larning koeffitsientlarini taqqoslab, tenglamalar sistemasini tuzamiz:

$$\begin{cases} B+C+D=0, \\ A+C+E-D=0, \\ C-E=0 \end{cases}$$

Sistemani yechsak, B=0, D=-1/3, E=1/3.

Demak, $$\frac{1}{x^2(x-1)(x^2+x+1)} = -\frac{1}{x^2} + \frac{1}{3(x-1)} - \frac{x-1}{3(x^2+x+1)}$$

Natijada $$\int \frac{dx}{x^2(x-1)(x^2+x+1)} = -\int \frac{dx}{x^2} + \frac{1}{3}\int \frac{dx}{x-1} - \frac{1}{3}\int \frac{x-1}{x^2+x+1}dx =$$

$$= \frac{1}{x} + \frac{1}{3}\ln|x-1| - \frac{1}{6}\int \frac{2x+1-3}{x^2+x+1}dx = \frac{1}{x} + \frac{1}{3}\ln|x^2+x+1| + \frac{1}{2}\int \frac{dx}{(x+\frac{1}{2})^2+\frac{3}{4}} =$$

$$= \frac{1}{x} + \frac{1}{6}\ln\frac{(x-1)^2}{x^2+x+1} + \frac{1}{\sqrt{3}}arctg\frac{2x+1}{\sqrt{3}} + C.$$

## MASHQLAR

Quyidagi integrallar hisoblansin.

| | | | | | |
|---|---|---|---|---|---|
| 1 | $\int \frac{3x^2+2x-3}{x^3-x}dx$ | 8 | $\int \frac{3x+2}{2x^2+x-3}dx$ | 15 | $\int \frac{(2x+3)}{x^4-16}dx$ |
| 2 | $\int \frac{2x^2-5x+1}{x^3-2x^2+x}dx$ | 9 | $\int \frac{dx}{x^3-8}$ | 16 | $\int \frac{(x^3+x+2)dx}{(x-3)(x-4)}$ |
| 3 | $\int \frac{5x-1}{x^3-3x-2}dx$ | 10 | $\int \frac{xdx}{(x^2+2x+2)^2}$ | 17 | $\int \frac{(x^2+x+5)dx}{x(x+3)(x-2)}$ |
| 4 | $\int \frac{2x^2+x+4}{x^3+x^2+4x+4}dx$ | 11 | $\int \frac{x^2+1}{(x-1)^2(x+3)}dx$ | 18 | $\int \frac{(x^2+x-1)dx}{x^3(x-2)^2}$ |
| 5 | $\int \frac{dx}{x^3+8}$ | 12 | $\int \frac{x^3+3x^2+5x+7}{x^2+2}dx$ | 19 | $\int \frac{(5x-7)dx}{x^3+x^2+4x+4}$ |
| 6 | $\int \frac{x+1}{x^4+4x^2+4}dx$ | 13 | $\int \frac{xdx}{(2x+1)(x-3)}$ | 20 | $\int \frac{x^2dx}{x^4+x^2-2}$ |
| 7 | $\int \frac{dx}{(x^2-9)(x^2+2)}$ | 14 | $\int \frac{xdx}{x^3+1}$ | | |

## 1.5. Trigonometrik funksiyalar qatnashgan ifodalarni integrallash

1. $\int Sinmx Cosnx\, dx$, $\int Cosmx Cosnx\, dx$ va $\int Sinmx Sinnx\, dx$ kabi integrallarni hisoblash uchun trigonometrik funksiyalarning ko'paytmalarini yig'indiga keltirish formulalaridan foydalaniladi:

$$\int Sinmx Cosnx dx = \frac{1}{2}\int [Sin(m+n)x + Sin(m-n)x]dx.$$

$$\int Cosmx Cosnx dx = \frac{1}{2}\int [Cos(m+n)x + Cos(m-n)x]dx,$$

$$\int Sinmx Sinnx dx = \frac{1}{2}\int [Cos(m-n)x - Cos(m+n)x]dx.$$

Quyidagi integrallar hisoblansin:

1. $\int Sin8xCos7xdx$,  3. $\int Sin3xSin7xdx$,

2. $\int Cos4xCos9xdx$,  4. $\int Sin2xCos5xSin9xdx$.

**Yechimlari.**

1. $\int Sin8xCos7xdx = \frac{1}{2}\int Sin15xdx + \frac{1}{2}\int Sinxdx = \frac{1}{2}\left(-\frac{1}{15}Cos15x\right) +$

$+\frac{1}{2}(-Cosx) + C = -\frac{Cos15x}{30} - \frac{Cosx}{2} + C.$

2. $\int Cos4xCos9xdx = \frac{1}{2}\int Cos13xdx + \frac{1}{2}\int Cos5xdx = \frac{Sin13x}{26} + \frac{Sin5x}{10} + C.$

3. $\int Sin3xSin7xdx = -\frac{1}{2}\int Cos4xdx - \frac{1}{2}\int Cos10xdx = -\frac{Sin4x}{8} - \frac{Sin10x}{20} + C.$

4. $\int Sin2xCos5xSin9xdx = \frac{1}{2}\int[Sin7x - Sin3x]Sin9x =$

$= \frac{1}{2}\int Sin7xSin9xdx - \frac{1}{2}\int Sin3xSin9xdx =$

$= \frac{1}{4}\int Cos2xdx - \frac{1}{4}\int Cos16xdx - \frac{1}{4}\int Cos6xdx + \frac{1}{12}\int cos12xdx =$

$= \frac{1}{8}Sin2x - \frac{1}{64}Sin16x - \frac{1}{24}Sin6x + \frac{1}{144}Sin12x + C.$

2. $\int Sin^m xCos^n xdx$ ko'rinishdagi integrallarni hisoblash uchun quyidagi 3 holni qarab o'tamiz.

**1-hol.** Agar $m=2k+1$ kabi butun musbat toq son bo'lsa,

$$\int Sin^{2k+1}xCos^n xdx = \int Sin^{2k}xCos^n xSinxdx = -\int (1-Cos^2x)^k Cos^n xd(Cosx),$$

ya'ni integral darajali funksiyalar yig'indisini integrallashga keltirildi.

Agar kosinusning darajasi ham butun musbat toq son bo'lsa, yuqoridagiga o'xshash usuldan foydalaniladi.

**Misollar:**

1) $\int Sin^5xdx = \int Sin^4xSinxdx = -\int(1-Cos^2x)^2 d(Cosx) =$

$= -\int(1-2Cos^2x + Cos^4x)d(Cosx) = -\int d(Cosx) + 2\int Cos^2xd(Cosx) - \int Cos^4xd(Cosx) =$

$= -Cosx + \frac{2}{3}Cos^3x - \frac{1}{5}Cos^5x + C.$

$$2)\int Cos^7xdx=\int(1-Sin^2x)^3d(Sinx)=$$
$$=\int d(Sinx)-3\int Sin^2xd(Sinx)+3\int Sin^4xd(Sinx)-\int Sin^6xd(Sinx)=Sinx-Sin^3x+\frac{3}{5}Sin^5x-$$
$$-\frac{1}{7}Sin^7x+C.$$
$$3)\int Sin^4xCos^3xdx=\int Sin^4xCos^2xCosxdx=\int Sin^4x(1-Sin^2x)d(Sinx)=$$
$$=\int Sin^4xd(Sinx)-\int Sin^6xd(Sinx)=\frac{1}{5}Sin^5x-\frac{1}{7}Sin^7x+C.$$
$$4)\int Sin^5xCos^2xdx=\int Sin^4xCos^2xSinxdx=-\int(1-Cos^2x)^2Cos^2xd(Cosx)=$$
$$=-\int Cos^2xd(Cosx)+2\int Cos^4xd(Cosx)-\int Cos^6xd(Cosx)=$$
$$=-\frac{1}{3}Cos^3x+\frac{2}{5}Cos^5x-\frac{1}{7}Cos^7x+C.$$
$$5)\int\frac{Cos^3xdx}{Sin^6x}=\int\frac{Cos^2xCosxdx}{Sin^6x}=\int\frac{(1-Sin^2x)d(Sinx)}{Sin^6x}=\int\frac{d(Sinx)}{Sin^6x}-\int\frac{d(Sinx)}{Sin^4x}=$$
$$=-\frac{1}{5Sin^5x}+\frac{1}{3Sin^3x}+C.$$
$$6)\int\frac{Sin^5xdx}{Cos^4x}=\int\frac{Sin^4xSinxdx}{Cos^4x}=-\int\frac{(1-Cos^2x)^2d(Cosx)}{Cos^4x}=$$
$$=-\int\frac{(1-2Cos^2x+Cos^4x)d(Cosx)}{Cos^4x}=-\int Cos^{-4}xd(Cosx)+2\int Cos^{-2}xd(Cosx)-\int d(Cosx)=$$
$$=\frac{1}{3Cos^3x}-\frac{1}{2Cosx}-Cosx+C.$$
$$7)\int\frac{Sin^3xdx}{Cos^5x}=\int\frac{Sin^2xSinxdx}{Cos^5x}=-\int\frac{(1-Cos^2x)d(Cosx)}{Cos^5x}=-\int\frac{d(Cosx)}{Cos^5x}+\int\frac{d(Cosx)}{Cos^3x}=$$
$$=\frac{1}{4Cos^4x}-\frac{1}{2Cos^2x}+C.$$

<u>**2-hol.**</u> Agar *m+n=-2k (k>0 va butun son)* bo‘lsa, tgx=z yoki ctgx=z kabi almashtirishlar orqali qaralayotgan integral darajali funksiyalarni integrallashga keltiriladi.

Bu yerda tgx=z bo‘lsa, $dx=\frac{dz}{1+z^2}$, $Sinx=\frac{z}{\sqrt{1+z^2}}$,

$Cosx=\frac{1}{\sqrt{1+z^2}}$ ва $ctgx=z$ bo‘lganda $dx=-\frac{dz}{1+z^2}$, $Sinx=\frac{1}{\sqrt{1+z^2}}$, $Cosx=\frac{z}{\sqrt{1+z^2}}$

larni inobatga olish lozim.

**Misollar:**

$1)\int\frac{Sin^4xdx}{Cos^8x}$ да $m=4, n=-8$ ва $m+n=-4$; $tgx=z$ алмаштиришдан фойдаланамиз:

$$\int\frac{Sin^4xdx}{Cos^8x}=\int\frac{z^4dz}{\left(\frac{1}{\sqrt{1+z^2}}\right)^4(1+z^2)}=\int z^4(1+z^2)dz=\frac{z^5}{5}+\frac{z^7}{7}+C=\frac{1}{5}tg^5x+\frac{1}{7}tg^7x+C.$$

$$2)\int\frac{Cos^2xdx}{Sin^8x}=\int ctg^2x\frac{dx}{\sin^6x}=|ctgx=z|=-\int z^2\frac{1}{\left(\frac{1}{\sqrt{1+z^2}}\right)^6}\frac{dz}{1+z^2}=-\int z^2(1+z^2)^2dz=$$

$$=-\int z^2dz-2\int z^4dz-\int z^6dz=-\frac{z^3}{3}-\frac{2}{5}z^5-\frac{z^7}{7}+C=-\frac{1}{3}ctg^3x-\frac{2}{5}ctg^5x-\frac{1}{7}ctg^7x+C.$$

$$3)\int\frac{Cos^3xdx}{Sin^9x}=\int ctg^3x\frac{dx}{Sin^6x}=|ctgx=z|=\int z^3\frac{-\frac{dz}{1+z^2}}{\left(\frac{1}{\sqrt{1+z^2}}\right)^6}=-\int z^3(1+z^2)^2dz=$$

$$=-\int z^3dz-2\int z^5dz-\int z^7dz=-\frac{z^4}{4}-\frac{1}{3}z^6-\frac{z^8}{8}+C=-\frac{1}{4}ctg^4x-\frac{1}{3}ctg^6x-\frac{1}{8}ctg^8x+C.$$

$$4)\int\frac{dx}{Sin^8x}=|ctgx=z|=\int\frac{-\frac{dz}{1+z^2}}{\left(\frac{1}{\sqrt{1+z^2}}\right)^8}=-\int(1+z^2)^3dz=-\int dz-3\int z^2dz-3\int z^4dz-\int z^6dz=$$

$$=-ctgx-ctg^3x-\frac{3}{5}ctg^5x-\frac{1}{7}ctg^7x+C.$$

$$5)\int\frac{dx}{Cos^6x}=|tgx=z|=\int\frac{(\sqrt{1+z^2})^6dz}{(1+z^2)}=\int(1+z^2)^2dz=\int dz+2\int z^2dz+\int z^4dz=$$

$$=z+\frac{2}{3}z^3+\frac{1}{5}z^5+C=tgx+\frac{2}{3}tg^3x+\frac{1}{5}tg^5x+C.$$

$$6)\int\frac{dx}{Sin^3x\cos^5x}=|tgx=z|=\int\frac{\frac{dz}{1+z^2}}{\left(\frac{z}{\sqrt{1+z^2}}\right)^3\left(\frac{1}{\sqrt{1+z^2}}\right)^5}=\int\frac{(1+z^2)^4dz}{z^3(1+z^2)}=\int\frac{(1+z^2)^3dz}{z^3}=$$

$$=\int\frac{dz}{z^3}+3\int\frac{dz}{z}+3\int zdz+\int z^3dz=-\frac{1}{2z^2}+3\ln|z|+\frac{3}{2}z^2+\frac{1}{4}z^4+C=-\frac{1}{2tg^2x}+3\ln|tgx|+$$

$$+\frac{3}{2}tg^2x+\frac{1}{4}tg^4x+C.$$

<u>3-hol.</u> Agar $\int Sin^mxCos^nxdx$ da $m+n=0$ bo'lsa (m va n lar butun sonlar), mazkur integral, yoki $\int\frac{Sin^mx}{Cos^mx}dx=\int tg^mxdx$ (m>0), yoki $\int\frac{Cos^nxdx}{Sin^nx}=\int ctg^nxdx$ (n>0) kabi integrallarni integrallashga keltiriladiki, ular mos ravishda tgxqz yoki ctgx=z almashtirishlar orqali hisoblanadi.

**Misollar:**

$$1)\int tg^4xdx=|tgx=z|=\int z^4\frac{dz}{1+z^2}=\int(z^2-1+\frac{1}{1+z^2})dz=\frac{z^3}{3}-z+arctgzC=\frac{1}{3}tg^3x-tgx+$$

$$+arctg(tgx)+C=\frac{tg^3x}{3}-tgx+x+C.$$

$$2)\int ctg^3xdx=\left|ctgx=z\right|=-\int\left(z-\frac{z}{1+z^2}\right)dz=-\int zdz+\int\frac{zdz}{1+z^2}=-\frac{z^2}{2}+\frac{1}{2}\ln\left|1+z^2\right|+C=$$
$$=-\frac{1}{2}ctg^2x+\frac{1}{2}\ln\left|1+ctg^2x\right|+C=-\frac{1}{2}ctg^2x+\frac{1}{2}\ln\left|\frac{1}{Sin^2x}\right|+C=-\frac{1}{2}ctg^2x-\ln\left|Sinx\right|+C.$$

**4.** Agar $\int Sin^{2n}xdx$ va $\int Cos^{2n}xdx$ ($n>0$ – butun son) yoki $\int Sin^{2m}xCos^{2n}xdx$ ($m>0$. $n>0$ – butun sonlar) kabi integrallarni hisoblash lozim bo'lsa, $Sin^2x=\frac{1-Cos2x}{2}$, $Cos^2x=\frac{1+Cos2x}{2}$ ko'rinishdagi formulalar qo'llaniladi.

Mazkur formulalarni qo'llash natijasida integral ishorasi ostidagi trigonometrik funksiyalarning darajalari birmuncha pasayadi.

**Misollar:**

$$1)\int Cos^4xdx=\int(Cos^2x)^2dx=\int\left(\frac{1+Cos2x}{2}\right)^2dx=\frac{1}{4}\int dx+\frac{1}{2}\int Cos2xdx+\frac{1}{4}\int Cos^2 2xdx=$$
$$=\frac{x}{4}+\frac{1}{4}Sin2x+\frac{1}{4}\int\frac{1+Cos4x}{2}dx=\frac{1}{4}x+\frac{1}{4}Sin2x+\frac{1}{8}x+\frac{1}{16}Sin4x+C=\frac{3}{8}x+\frac{1}{4}Sin2x+$$
$$+\frac{1}{16}Sin4x+C.$$

$$2)\int Sin^2xdx=\int\frac{1-Cos2x}{2}dx=\frac{1}{2}\int dx-\frac{1}{2}\int Cos2xdx=\frac{1}{2}-\frac{1}{4}Sin2x+C.$$

$$3)\int Sin^2xCos^2xdx=\frac{1}{4}\int Sin^2 2xdx=$$
$$=\frac{1}{4}\cdot\frac{1}{2}\int(1-Cos4x)dx=\frac{1}{8}\int dx-\frac{1}{8}\int Cos4xdx=\frac{x}{8}-\frac{1}{32}Sin4x+C.$$

$$4)\int Cos^4xSin^2xdx=\int\sin^2xCos^2xCos^2xdx=\frac{1}{4}\int\sin^2 2xCos^2x=$$
$$=\frac{1}{4}\int\frac{1-Cos4x}{2}\cdot\frac{1+Cos2x}{2}dx=\frac{1}{16}\int(1+Cos2x-Cos4x-Cos2xCos4x)dx=$$
$$=\frac{1}{16}x+\frac{1}{32}\sin2x-\frac{1}{64}Sin4x-\frac{1}{16}\int\frac{1}{16}(Cos6x+Cos2x)dx=$$
$$=\frac{1}{16}x+\frac{1}{32}\sin2x-\frac{1}{64}Sin4x-\frac{1}{32}\cdot\frac{1}{6}Sin6x-\frac{1}{32}\cdot\frac{1}{2}Sin2x+C=$$
$$=\frac{1}{16}x+\frac{1}{64}\sin2x-\frac{1}{64}Sin4x-\frac{1}{192}Sin6x+C.$$

**4.** Agar faqat trigonometrik funksiyalarga ratsional ravishda bog'liq bo'lgan ifoda, ya'ni *Sinx* va *Cosx* ning ratsional funksiyasi bo'lgan R (Sinx, Cosx) ning aniqmas integrali $\int R(Sinx,Cosx)dx$ ni hisoblash lozim bo'lsa, unda **universal trigonometrik almashtirish** deb ataluvchi $tg\frac{x}{2}=z$ dan foydalaniladi. Chunki, bu almashtirish yordamida R (Sinx, Cosx) funksiya har doim *z* ga nisbatan ratsional funksiya ko'rinishiga keltiriladi.

Bu yerda, $Sinx=\dfrac{2tg\frac{x}{2}}{1+tg^2\frac{x}{2}}=\dfrac{2z}{1+z^2}$, $Cosx=\dfrac{1-tg^2\frac{x}{2}}{1+tg^2\frac{x}{2}}=\dfrac{1-z^2}{1-z^2}$, $\dfrac{x}{2}=arctgz$

yoki x=2arctgz dan $dx=\dfrac{2dz}{1+z^2}$ ni inobatga olsak,

$\int R(Sinx,Cosx)dx=\int R\left(\dfrac{2z}{1+z^2},\dfrac{1-z^2}{1+z^2}\right)\dfrac{2dz}{1+z^2}=\int R_1(z)dz$ ni hosil qilamiz. Bu yerda, $R_1(z)$ - o'zgaruvchi z ga nisbatan ratsional funksiya.

Ta'kidlash lozimki, amaliyotda bu almashtirish ko'pincha ancha murakkab ratsional funksiyalarga olib keladi. Shu sababli, ba'zan undan foydalanmasdan ancha sodda o'rniga qo'yish usullardan foydalaniladi. quyida biz 3 holni ko'rib o'tamizki, u yerda universal trigonometrik almashtirishlarsiz ish yuritish mumkin.

**1-hol.** Agar R (*Sinx, Cosx*) funksiya *Sinx* ga nisbatan toq bo'lsa, ya'ni R (*-Sinx, Cosx*)= - R(*Sinx, Cosx*) bo'lsa, u holda z=Cosx, dz=-Sinxdx kabi o'rniga quyish bu funksiyani ratsionallashtiradi.

**2-hol.** Agar R(*Sinx, Cosx*) funksiya Cosx ga nisbatan toq bo'lsa, ya'ni R(*Sinx, -Cosx*)= -R(*Sinx, Cosx*) bo'lsa, u holda z=Sinx, dz=Cosxdx kabi o'rniga quyish bu funksiyani ratsionallashtiradi.

**3-hol.** Agar R(*Sinx, Cosx*) funksiya Sinx va Cosx ga nisbatan juft bo'lsa, ya'ni R(*-Sinx,- Cosx*)= R(*Sinx, Cosx*) bo'lsa, u holda z=tgx, $dx=\dfrac{dz}{1+z^2}$ kabi o'rniga qo'yish bu funksiyani ratsionallashtiradi. Bu yerda, $Sin^2x=\dfrac{tg^2x}{1+tg^2x}=\dfrac{z^2}{1+z^2}$ ni va $Cos^2x=\dfrac{1}{1+tg^2x}=\dfrac{1}{1+z^2}$ larni inobatga olish lozim.

**Misollar.**

**1.** $\int\dfrac{dx}{Sin^5x}$ ni hisoblashda $tg\dfrac{x}{2}=z$ almashtirishdan foydalanamiz.

$Sin^5x=\left(\dfrac{2z}{1+z^2}\right)^5=\dfrac{32z^5}{(1+z^2)^5}$ ni va $dx=\dfrac{2dz}{1+z^2}$ ni inobatga olsak,

$$\int\frac{dx}{Sin^5x}=\frac{1}{16}\int\frac{(1+z^2)^4}{z^5}dz=\frac{1}{16}\int\frac{1+4z^2+6z^4+4z^6+z^8}{z^5}dz=$$

$$=\frac{1}{16}\int\frac{dz}{z^5}+\frac{1}{4}\int\frac{dz}{z^3}+\frac{3}{8}\int\frac{dz}{z}+\frac{1}{4}\int zdz+\frac{1}{16}\int z^3dz=-\frac{1}{64z^4}-\frac{1}{8z^2}+\frac{3}{8}\ln|z|+\frac{z^2}{8}+\frac{z^4}{64}+C.$$

yoki z ni $tg\dfrac{x}{2}$ bilan almashtirsak,

$$\int\frac{dx}{Sin^5x}=-\frac{1}{16}\left(\frac{1}{4}ctg^4\frac{x}{2}+2ctg^2\frac{x}{2}-6\ln\left|tg\frac{x}{2}\right|-2tg^2\frac{x}{2}-\frac{1}{4}tg^4\frac{x}{2}\right)+C.$$

**2.** $\int\frac{(2+Sinx)dx}{Sinx(4+3Cosx)}$ ni hisoblashda ham $tg\frac{x}{2}=z$ almashtirishni qo'llaymiz:

$$\int\frac{(2+\sin x)}{\sin x(4+3\cos x)}=\int\frac{2+\frac{2z}{1+z^2}}{(4+\frac{3(1-z^2)}{1+z^2})\frac{2z}{1+z^2}}\frac{2dz}{1+z^2}=2\int\frac{(z^2+z+1)dz}{z(z^2+7)};\frac{z^2+z+1}{z(z^2+7)}=\frac{A}{z}+\frac{Bz+D}{z^2+7}$$

dan, A= 1/7, B= 6/7 va D=1 larni topamiz. U holda:

$$\int\frac{(2+Sinx)dx}{(4+3Cosx)Sinx}=\frac{2}{7}\int\frac{dz}{z}+\frac{6}{7}\int\frac{2zdz}{z^2+7}+2\int\frac{dz}{7+z^2}=\frac{2}{7}\ln|z|+\frac{6}{7}\ln(z^2+7)+$$

$$+\frac{2}{\sqrt{7}}arctg\frac{z}{\sqrt{7}}+C=\frac{2}{7}\ln\left|tg\frac{x}{2}\right|+\frac{6}{7}\ln(7+tg^2\frac{x}{2})+\frac{2}{\sqrt{7}}arctg(\frac{tg\frac{x}{2}}{\sqrt{7}})+C.$$

**3.** $\int\frac{dx}{Sin^3xCos^2x}$ ni hisoblashda Cosx=z almashtirishni qo'llaymiz, chunki $(-Sinx)^3=-Sin^3x$. $Sinx=\sqrt{1-Cos^2x}=\sqrt{1-z^2}$ ni va $-Sinxdx=dz$ dan $dx=-\frac{dz}{\sqrt{1-z^2}}$ ekanliklarini inobatga olsak, quyidagiga ega bo'lamiz:

$$\int\frac{dx}{Sin^3xCos^2x}=\int\frac{-1}{(\sqrt{1-z^2})^3z^2}\frac{dz}{\sqrt{1-z^2}}=-\int\frac{dz}{(1-z^2)^2z^2} \text{ va}$$

$$\int\frac{dx}{Sin^3xCos^2x}=\frac{1}{Cosx}-\frac{Cosx}{2Sin^2x}+\frac{3}{2}\ln\left|tg\frac{x}{2}\right|+C.$$

ni hosil qilamiz.

**4.** $\int\frac{Cos^3xdx}{Sin^4x}$ ni hisobllashda $Sinx=z$ almashtirishdan foydalanamiz.

$$\int\frac{Cos^3xdx}{Sin^4x}=\int\frac{Cos^2xCosxdx}{Sin^4x}=\int\frac{(1-z^2)dz}{z^4}=\int\frac{dz}{z^4}-\int\frac{dz}{z^2}=-\frac{1}{3z^3}+\frac{1}{z}+C=-\frac{1}{3Sin^3x}+\frac{1}{Sinx}+C.$$

**5.** $\int\frac{ax}{(1+Sin^2x)Cos^2x}$ ni hisoblashda $tgx=z$ almashtirishni qo'llaymiz, chunki integral belgisi ostidagi funksiya $Sinx$ va $Cosx$ ga nisbatan juft funksiyadir.

$$\int\frac{dx}{(1+Sin^2x)Cos^2x}=\int\frac{\frac{dx}{Cos^2x}}{1+Sin^2x}=\int\frac{dz}{1+\frac{z^2}{1+z^2}}=\int\frac{(1+z^2)dz}{1+2z^2}=\frac{1}{2}\int dz+\frac{1}{2}\int\frac{dz}{1+2z^2}=$$

$$=\frac{1}{2}z+\frac{1}{4}\int\frac{dz}{\frac{1}{2}+z^2}=\frac{z}{2}+\frac{1}{4}\frac{1}{\frac{1}{\sqrt{2}}}arctg\frac{z}{\frac{1}{\sqrt{2}}}+C=\frac{1}{2}tgx+\frac{\sqrt{2}}{4}arctg(\sqrt{2}tgx)+C.$$

## MASHQLAR

1. $\int Sin6xSin7xdx$

2. $\int Cos3xCos9xdx$

3. $\int Sin2xSin5xdx$

4. $\int Sin\frac{3x}{4}Cos\frac{x}{4}dx$

5. $\int SinxCos2xCos3xdx$

6. $\int CosxCos2xCos5xdx$

7. $\int Sin5xSin\frac{x}{2}dx$

8. $\int Sin\frac{x}{6}Sin\frac{5x}{6}dx$

9. $\int Cos\frac{x}{2}Cos\frac{x}{3}dx$

10. $\int Sin^3xdx$

11. $\int Cos^5xdx$

12. $\int Sin^7xdx$

13. $\int Cos^9xdx$

14. $\int Cos^2xSin^3xdx$

15. $\int Sin^4xCos^7xdx$

16. $\int Sin^2xCos^5xdx$

17. $\int Sin^2xCos^3xdx$

18. $\int Sin^7xCos^6xdx$

19. $\int\frac{Sin^3xdx}{Cos^2x}$

20. $\int\frac{Sin^3xdx}{Cosx}$

21. $\int\frac{Sinxdx}{Cos^3x}$

22. $\int\frac{Cos^3xdx}{Sinx}$

23. $\int\frac{Cosxdx}{Sin^3x}$

24. $\int\frac{Cos^7xdx}{Sin^4x}$

25. $\int\frac{Cos^3xdx}{\sqrt{Sinx}}$

26. $\int\frac{Cos^5xdx}{\sqrt[3]{Sin^2x}}$

27. $\int\frac{Sin^3xdx}{\sqrt{Cosx}}$

28. $\int Cos^5x\sqrt[3]{Sin^2x}dx$

29. $\int\frac{Cos^3xdx}{Sin^5x}$

30. $\int\frac{dx}{Sin^4x}$

31. $\int\frac{dx}{Cos^4x}$

32. $\int\frac{dx}{Sin^6x}$

33. $\int\frac{dx}{Cos^8x}$

34. $\int\frac{Sin^7xdx}{Cos^{13}x}$

35. $\int\frac{Cos^4xdx}{Sin^6x}$

36. $\int\frac{Cos^4xdx}{Sin^4x}$

37. $\int\frac{dx}{Siin^4xCos^6x}$

38. $\int\frac{dx}{Sin^3xCosx}$

39. $\int\frac{dx}{SinxCos^3x}$

41. $\int\frac{dx}{CosxSin^3x}$

42. $\int tg^6xdx$

43. $\int tg^5xdx$

44. $\int ctg^4xdx$

45. $\int ctg^5xdx$

46. $\int Sin^4xdx$

47. $\int Cos^2xSin^4xdx$

48. $\int Sin^6xdx$

49. $\int Cos^4xSin^4xdx$

51. $\int\frac{dx}{Cos^3x}$

52. $\int\frac{dx}{Sin^3x}$

53. $\int\frac{dx}{Cos^5x}$

54. $\int\frac{dx}{Sin^7x}$

55. $\int\frac{(5+9Sinx)dx}{Cosx(2+3Sinx)}$

56. $\int\frac{dx}{Cosx(1-Sinx)}$

57. $\int\frac{dx}{Cos^2x(1+Cosx)}$

58. $\int\frac{Cos^5xdx}{Sin^4x}$

59. $\int\frac{dx}{Sin^2xCos^3x}$

40. $\int \frac{dx}{Sin^3 x Cos^3 x}$ 50. $\int Sin^4 x Cos^6 x dx$ 60. $\int \frac{dx}{(1+Cos^2 x)Sin^2 x}$

## 1.6. Algebraik irratsionalliklarni integrallash

**1)** $\int R(x^{\alpha}, x^{\beta}, ..., x^{\gamma})dx$ kabi integrallarni hisoblash lozim bo'lsin. Bu yerda: $\alpha, \beta, ..., \gamma$ - kasr ratsional sonlar bo'lib, $\int R(x^{\alpha}, x^{\beta}, ..., x^{\gamma})$ esa, o'z argumentlariga nisbatan ratsional funksiyadir. Bu xildagi irratsionallik $x=z^n$, $dx=nz^{n-1}dz$ dek almashtirish orqali ratsionallashtiriladi. Bundagi $n$, $\alpha, \beta, ..., \gamma$ kasr sonlar maxrajlarining eng kichik umumiy karralisidir.

2) $\int R[x, (\frac{ax+b}{cx+d})^{\alpha}, (\frac{ax+b}{cx+d})^{\beta}, ..., (\frac{ax+b}{cx+d})^{\gamma}]dx$ turdagi integralni ratsionallashtirish uchun (bu yerda ham $\alpha, \beta, ..., \gamma$ lar kasr ratsional sonlar va $R$, o'z argumentlariga nisbatan ratsional funksiya) $\frac{ax+b}{cx+d} = z^n$ almashtirishdan foydalanamiz. Xususan, $\frac{ax+b}{cx+d}$ ning o'rnida $ax+b$ qatnashganda ham $ax+b=z^n$ almashtirish qo'llaniladi ($n$- $\alpha, \beta, ..., \gamma$ sonlar maxrajlarining eng kichik umumiy karralisidir).

**1-Misol.** $\int \frac{\sqrt[3]{x}dx}{\sqrt{x} - \sqrt[3]{x^2}}$

<u>**Yechish.**</u>

$$\int \frac{\sqrt[3]{x}dx}{\sqrt{x} - \sqrt[3]{x^2}} = \int \frac{x^{\frac{1}{3}}dx}{x^{\frac{1}{2}} - x^{\frac{2}{3}}} = \left| \begin{matrix} x = z^6 \\ dx = 6z^5dz \\ z = \sqrt[6]{x} \end{matrix} \right| = \int \frac{z^2 6z^5 dz}{z^3 - z^4} = 6\int \frac{z^7 dz}{z^3(1-z)} =$$

$$= -6\int \frac{z^4 dz}{z-1} = -6\int (z^3 + z^2 + z + 1 + \frac{1}{z-1})dz = -6\left[\frac{z^4}{4} + \frac{z^3}{3} + \frac{z^2}{2} + z + \ln|z-1|\right] + C =$$

$$= -6\left[\frac{1}{4}\sqrt[3]{x^2} + \frac{1}{3}\sqrt{x} + \frac{1}{2}\sqrt[3]{x} + \sqrt[6]{x} + \ln\left|\sqrt[6]{x} - 1\right|\right] + C.$$

**2-Misol.** $\int \frac{dx}{\sqrt[3]{x^2}(\sqrt{x} + \sqrt[3]{x})}$

<u>**Yechish.**</u>

$$\int \frac{dx}{\sqrt[3]{x^2}(\sqrt{x} + \sqrt[3]{x})} = \left| \begin{matrix} x = z^6 \\ z = \sqrt[6]{x} \\ dx = 6z^5dz \end{matrix} \right| = \int \frac{6z^5 dz}{z^4(z^3 + z^2)} = 6\int \frac{dz}{z(z+1)} =$$

$$=6\int\frac{(1+z-z)dz}{z(1+z)}=6\int\frac{dz}{z}-6\int\frac{dz}{1+z}=6\ln|z|-6\ln|z+1|+C=6\ln\left|\frac{z}{z+1}\right|+C=6\ln\left|\frac{\sqrt[6]{x}}{\sqrt[6]{x}+1}\right|+C.$$

**3-Misol.** $\int\frac{x^2dx}{\sqrt{x-1}}$

**Yechish.** $$\int\frac{x^2dx}{\sqrt{x-1}}=\left|\begin{array}{l}x-1=z^2\\ x=z^2+1\\ dx=2zdz\\ z=\sqrt{x-1}\end{array}\right|=\int\frac{(z^2+1)^2 2zdz}{z}=2\int(z^4+2z^2+1)dz=$$

$$=2(\frac{z^5}{5}+\frac{2}{3}z^3+z)+C=\frac{2}{5}(x-1)^2\sqrt{x-1}+\frac{4}{3}(x-1)\sqrt{x-1}+C=$$

$$=2\sqrt{x-1}\left[\frac{(x-1)2}{5}+\frac{4}{3}(x-1)+1\right]+C.$$

**4-Misol.** $\int\frac{\sqrt{2x-3}}{x}dx$

**Yechish.** $$\int\frac{\sqrt{2x-3}}{x}dx=\left|\begin{array}{l}\sqrt{2x-3}=z\\ 2x-3=z^2\\ x=\frac{1}{2}(3+z^2)\\ dx=zdz\end{array}\right|=2\int\frac{z^2dz}{3+z^2}=2\int\frac{z^2+3-3}{3+z^2}dz=$$

$$=2\int dz-6\int\frac{dz}{(\sqrt{3})^2+z^2}=2z-\frac{6}{\sqrt{3}}arctg\frac{z}{\sqrt{3}}+C=2\sqrt{2x-3}-\frac{6}{\sqrt{3}}arctg\frac{\sqrt{2x-3}}{\sqrt{3}}+C.$$

**5-Misol.** $\int\frac{dx}{\sqrt[3]{(1+x)^2}-\sqrt{1+x}}$

**Yechish.** $$\int\frac{dx}{\sqrt[3]{(1+x)^2}-\sqrt{1+x}}=\left|\begin{array}{l}\sqrt[6]{1+x}=z\\ 1+x=z^6\\ dx=6z^5dz\end{array}\right|=\int\frac{6z^5dz}{z^4-z^3}=6\int\frac{z^2dz}{z-1}=$$

$$=6\int\frac{(z^2-1+1)dz}{z-1}=6\int(z+1+\frac{1}{z-1})dz=6\left(\frac{z^2}{2}+z+\ln|z-1|\right)+C=3\sqrt[3]{1+x}+6\sqrt[6]{1+x}+$$

$$+6\ln\left|\sqrt[6]{1+x}-1\right|+C.$$

**6-Misol.** $\int\frac{1}{x}\sqrt{\frac{x-1}{x+1}}dx$

**Yechish.** $$\int\frac{1}{x}\sqrt{\frac{x-1}{x+1}}dx=\left|\begin{array}{l}\frac{x-1}{x+1}=z^2\\ x=\frac{1+z^2}{1-z^2}\\ dx=\frac{4zdz}{(1-z^2)^2}\end{array}\right|=\int\frac{1-z^2}{1+z^2}z\frac{4zdz}{(1-z^2)^2}=-4\int\frac{z^2dz}{(z^2+1)(z^2-1)}=$$

$$=-4\int\frac{z^2dz}{(z-1)(z+1)(z^2-1)}.$$

Integral belgisi ostidagi funksiyani eng sodda kasrlarga ajratamiz:

$$\frac{z^2}{(z-1)(z+1)(z^2+1)}=\frac{A}{z-1}+\frac{B}{z+1}+\frac{Cz+D}{z^2+1} \text{ dan}$$

$$z=Az^3+Az^2+Az+A+Bz^3-Bz^2-B+Cz^3+Dz^2-Cz-D$$

$$\left.\begin{array}{l} z^3: A+B+C=0, \\ z^2: A-BD=1, \\ z: A+B-C=0, \\ z^0: A-B-D=0 \end{array}\right\} \Rightarrow \begin{array}{l} A=\frac{1}{4}, \\ B=-\frac{1}{4}, \\ C=0, \\ D=\frac{1}{2} \end{array}$$

Demak, $$\int\frac{1}{x}\sqrt{\frac{x-1}{x+1}}dx=-\int\frac{dz}{z-1}+\frac{dz}{z+1}-\int\frac{2dz}{z^2+1}=$$

$$=\ln|z+1|-\ln|z-1|-2arctgz+C=\ln\left|\frac{z+1}{z-1}\right|-2arctgz+C=$$

$$=\ln\left|\frac{\sqrt{\frac{x-1}{x+1}}+1}{\sqrt{\frac{x-1}{x+1}}-1}\right|-2arctg\sqrt{\frac{x-1}{x+1}}+C=\ln\left|\frac{\sqrt{x-1}+\sqrt{x+1}}{\sqrt{x-1}-\sqrt{x+1}}\right|-2arctg\sqrt{\frac{x-1}{x+1}}+C=$$

$$=\ln\left|x+\sqrt{x^2-1}\right|-2arctg\sqrt{\frac{x-1}{x+1}}+C.$$

3) $\int R(x,\sqrt{ax^2+bx+c})dx$ kabi integrallarni hisoblashning ayrim hollarini ko'rib o'tamiz.

a) Agar $\int\frac{dx}{\sqrt{ax^2+bx+c}}$ ko'rinishdagi integralni hisoblash lozim bo'lsa, u yerda kvadrat uchhaddan to'la kvadrat ajratilgandan so'ng, mazkur integral jadval integraliga keltiriladi.

**7-Misol.** $\int\frac{dx}{\sqrt{x^2+6x+25}}$

<u>**Yechish.**</u> $\int\frac{dx}{\sqrt{x^2+6x+25}}=\int\frac{dx}{\sqrt{(x+3)^2+16}}=\ln\left|(x+3)+\sqrt{x^2+6x+25}\right|+C.$

**8-Misol.** $\int\frac{dx}{\sqrt{5-x^2-4x}}$

<u>**Yechish.**</u> $\int\frac{dx}{\sqrt{5-x^2-4x}}=\int\frac{dx}{\sqrt{-(x^2+4x-5)}}=\int\frac{dx}{\sqrt{-[(x+2)^2-9]}}=\int\frac{dx}{\sqrt{9-(x+2)^2}}=$

$=\arcsin\frac{x+2}{3}+C.$

b) Ushbu $\int\frac{Ax+B}{\sqrt{ax^2+bx+c}}dx$ ko'rinishdagi integralni hisoblash uchun avvalo, suratda kvadrat uchhadning hosilasi $2ax+b$ ni ajratiladi, keyin u ikkita integral yig'indisiga ajraladi. Ulardan biri, darajali funksiyadan olingan integral bo'lib, ikkinchisi a) bandda qaralgan integraldir.

**9-Misol.** $\frac{(3x+8)dx}{\sqrt{3x^2+x+9}}$

**Yechish.** $\int\frac{(3x+8)dx}{\sqrt{3x^2+x+9}}=\int\frac{(6x+1)\frac{1}{2}+\frac{15}{2}}{\sqrt{3x^2+x+9}}dx=\frac{1}{2}\int\frac{d(3x^2+x+9)}{\sqrt{3x^2+x+9}}+$

$+\frac{15}{2}\int\frac{dx}{\sqrt{3\left[(x+\frac{1}{6})^2+\frac{107}{36}\right]}}=\sqrt{3x^2+x+9}+\frac{15}{2}\frac{1}{\sqrt{3}}\ln\left|(x+\frac{1}{6})+\sqrt{(x+\frac{1}{6})^2+\frac{107}{36}}\right|+$

$+C=\sqrt{3x^2+x+9}+\frac{5\sqrt{3}}{2}\ln\left|(x+\frac{1}{6})+\sqrt{x^2+\frac{1}{3}x+3}\right|+C\cdot$

**10-Misol.** $\int\frac{(5x-3)dx}{\sqrt{1-13x-5x^2}}$

**Yechish.** $\int\frac{(5x-3)dx}{\sqrt{1-13x-5x^2}}=\int\frac{(-10x-13)(-\frac{1}{2})-\frac{19}{2}}{\sqrt{1-13x-5x^2}}dx=$

$=-\frac{1}{2}\int\frac{d(1-13x-5x^2)}{\sqrt{1-13x-5x^2}}-\frac{19}{2}\int\frac{dx}{\sqrt{-5(x^2+\frac{13}{5}x-\frac{1}{5})}}=$

$=-\sqrt{1-13x-5x^2}-\frac{19}{2\sqrt{5}}\int\frac{dx}{\sqrt{\frac{189}{100}-(x+\frac{13}{10})^2}}=-\sqrt{1-13x-5x^2}-\frac{19\sqrt{5}}{10}\arcsin\frac{x+\frac{13}{10}}{\frac{\sqrt{189}}{10}}+$

$+C=-\sqrt{1-13x-5x^2}-\frac{19\sqrt{5}}{10}\arcsin\frac{10x+13}{\sqrt{189}}+C\cdot$

c) Quyidagi $\int\frac{dx}{(x-\alpha)^k\sqrt{ax^2+bx+c}}$ integralni hisoblash uchun $\frac{1}{x-\alpha}=z$ almashtirish orqali uni (a) bandda ko'rilgan integralga keltiriladi.

**11-Misol.** $\int\frac{dx}{(x-1)\sqrt{1+x-x^2}}$

**Yechish.** Uni hisoblashda $\frac{1}{x-1}=z$ yoki $x-1=\frac{1}{z}$ dan foydalanamiz. Agar $x=1+\frac{1}{z}$, $dx=-\frac{dz}{z^2}$, $\sqrt{1+x-x^2}=\sqrt{1+1+\frac{1}{z}-1-\frac{2}{z}-\frac{1}{z^2}}=\frac{\sqrt{z^2-z-1}}{z}$ ekanliklarini inobatga olsak, quyidagini hosil qilamiz:

$$\int\frac{dx}{(x-1)\sqrt{1+x-x^2}}=-\int z\frac{zdz}{z^2\sqrt{z^2-z-1}}=-\int\frac{dz}{\sqrt{z^2-z-1}}=-\int\frac{d(z-\frac{1}{2})}{\sqrt{(z-\frac{1}{2})^2-\frac{5}{4}}}=$$

$$=-\ln\left|(z-\frac{1}{2})+\sqrt{(z-\frac{1}{2})^2-\frac{5}{4}}\right|+C=-\ln\left|\frac{1}{x-1}-\frac{1}{2}+\sqrt{(\frac{1}{x-1})^2-\frac{1}{x-1}-1}\right|+C=$$

$$=-\ln\left|\frac{3-x+\sqrt{1+x-x^2}}{2(x-1)}\right|+C\cdot$$

**12-Misol.** $\int\frac{dx}{x^2\sqrt{15+3x^2}}$

<u>Yechish.</u> $$\int\frac{dx}{x^2\sqrt{15+3x^2}}=\left|\begin{array}{l} x=\frac{1}{z}, dx=-\frac{dz}{z^2}, \\ 15+3x^2=15+\frac{3}{z^2} \\ \sqrt{15+3x^2}=\frac{\sqrt{15z^2+3}}{z}\end{array}\right|=\int\frac{-\frac{dz}{z^2}}{\frac{1}{z^2}\frac{\sqrt{3(5z^2+1)}}{z}}=$$

$$=-\frac{1}{\sqrt{3}}\int\frac{zdz}{\sqrt{5z^2+1}}=-\frac{1}{3}\frac{1}{10}\int\frac{d(5z^2+1)}{\sqrt{5z^2+1}}=-\frac{1}{3}\frac{1}{10}2\sqrt{5z^2+1}+C=-\frac{1}{5\sqrt{3}}\sqrt{\frac{5}{x^2}+1}+C=$$

$$=-\frac{\sqrt{5+x^2}}{5\sqrt{3}x}+C=-\frac{1}{15}\frac{\sqrt{15+3x^2}}{x}+C.$$

4) Endi, $\sqrt{ax^2+bx+c}$ ga ratsional bog'liq bo'lgan umumiy ko'rinishdagi $\int R(x,\sqrt{ax^2+bx+c})dx$ ni qarab chiqamiz. Kvadrat uchhaddan to'la kvadrat ajratilgandan so'ng, ya'ni, $ax^2+bx+c=a(x+\frac{b}{2a})^2-\frac{b^2-4ac}{4a}$ da $x+\frac{b}{2a}=z,\ dx=dz$ deb belgilashlarni kiritamiz. Natijada, dastlabki integral $a$ va $(b^2$-$4ac)$ ning ishoralariga bog'liq holda integrallardan birini topishga keltiriladi:

**a)** Agar $a>0$ va $b^2$-$4ac<0$ bo'lsa, u holda $\int R(x,\sqrt{ax^2+bx+c})dx=\int R_1(z,\sqrt{m^2+n^2z^2})dz$, bu yerda $a=n^2$, $-\frac{b^2-4ac}{4a}=m^2$.

Mazkur integral $z=\frac{m}{n}tgt$ almashtirish orqali $\int R_1(Sint;Cost)dt$ ga keltiriladi.

**b)** Agar $a>0$, $b^2$-$4ac>0$ bo'lsa, u holda $\int R(x,\sqrt{ax^2+bx+c})dx=\int R_1(z,\sqrt{n^2z^2-m^2})dz$ kabi bo'lib (bu yerda: $a$=n$^2$, $m^2=\frac{b^2-4ac}{4ac}$), uni $\int R_2(Sint,Cost)dt$ ga keltirish uchun $z=\frac{m}{n}\sec t$ almashtirishdan foydalaniladi.

c) Agar $a<0$, $b^2$-$4ac>0$ bo'lsa, u holda $\int R(x,\sqrt{ax^2+bx+c})dx =$
$= \int R_1(z,\sqrt{m^2-n^2z^2})dz$ da $z=\frac{m}{n}\sin t$ (yoki $z=\frac{m}{n}Cost$ ) almashtirish qo'llanadi. Bu yerda : $n^2=-a$, $m^2=-\frac{b^2-4ac}{4a}>0$.

Yuqorida bayon etilgan almashtirishlar, odatda, irratsionalliklarni integrallashda trigonometrik almashtirishlar deb yuritiladi.

**13-Misol.** $\int\frac{dx}{\sqrt{(4+x^2)^3}}$

Yechish. Uni hisoblashda $x=2tgt$ almashtirishni qo'llaymiz. U holda

$$dx=\frac{2dt}{Cos^2t},\ 4+x^2=\frac{4}{Cos^2t}.$$

$$\int\frac{dx}{\sqrt{(4+x^2)^3}}=\int\frac{Cos^3t2dt}{Cos^2t8}=\frac{1}{4}\int Costdt=\frac{1}{4}S\text{int}+C=\frac{1}{4}Sin(arctg\frac{x}{2})+C.$$

Agar $Sin(arctg\frac{x}{2})=\frac{x}{\sqrt{4+x^2}}$ ekanligini nazarda tutsak, natijada

$$\int\frac{dx}{\sqrt{(4+x^2)^3}}=\frac{1}{4}\frac{x}{\sqrt{4+x^2}}+C.$$

**14-Misol.** $\int\frac{dx}{(4-x^2)\sqrt{4-x^2}}$

Yechish. $\int\frac{dx}{(4-x^2)\sqrt{4-x^2}}=\left|\begin{matrix}x=2S\text{int}\\ dx=2Cosdt\end{matrix}\right|=\int\frac{2Cosdt}{4Cos^2t2Cost}=\frac{1}{4}\int\frac{dt}{Cos^2t}=\frac{1}{4}tgt+C.$

Agar $S\text{int}=\frac{x}{2}$ va $Cost=\sqrt{1-Sin^2t}=\sqrt{1-\frac{x^2}{4}}=\frac{\sqrt{1-x^2}}{2}$ ekanligini inobatga olsak, $tgt=\frac{x}{\sqrt{4-x^2}}$. Demak, $\int\frac{dx}{(4-x^2)\sqrt{4-x^2}}=\frac{1}{4}\frac{x}{\sqrt{4-x^2}}+C$.

**15-Misol.** $\int\frac{dx}{(x^2-10)\sqrt{x^2-10}}$

Yechish. Bu integralda $x=\sqrt{10}Sect$ yoki $x=\frac{\sqrt{10}}{Cost}$ almashtirish bajariladi. Agar $dx=\frac{\sqrt{10}Sint}{Cos^2t}$ va $x^2-10=\frac{10Sin^2t}{Cos^2t}$ larni o'rniga qo'yilsa, quyidagini hosil qilamiz:

$$\int\frac{dx}{(x^2-10)\sqrt{x^2-10}}=\frac{1}{10}\int\frac{Cosdt}{Sin^2t}=\frac{1}{10}\int\frac{d(Sint)}{Sin^2t}=-\frac{1}{10Sint}+C=-\frac{1}{10}\frac{x}{\sqrt{x^2-10}}+C.$$

**16-Misol.** $\int\frac{dx}{\sqrt{(x^2+2x+17)^3}}$

Yechish. $\int\frac{dx}{\sqrt{(x^2+2x+17)^3}}=\int\frac{dx}{\sqrt{[(x+1)^2+16]^3}}=\left|\begin{matrix}x+1=4tgt\\ dx=\frac{4dt}{Cos^2t}\end{matrix}\right|=$

c) Agar $a<0$, $b^2$-$4ac>0$ bo'lsa, u holda $\int R(x,\sqrt{ax^2+bx+c})dx =$ $=\int R_3(z,\sqrt{m^2-n^2z^2})dz$ da $z=\frac{m}{n}\sin t$ (yoki $z=\frac{m}{n}Cost$ ) almashtirish qo'llanadi. Bu yerda : $n^2=-a$, $m^2=-\frac{b^2-4ac}{4a}>0$.

Yuqorida bayon etilgan almashtirishlar, odatda, irratsionalliklarni integrallashda trigonometrik almashtirishlar deb yuritiladi.

**13-Misol.** $\int\frac{dx}{\sqrt{(4+x^2)^3}}$

**Yechish.** Uni hisoblashda $x=2tgt$ almashtirishni qo'llaymiz. U holda

$$dx=\frac{2dt}{Cos^2t},\ 4+x^2=\frac{4}{Cos^2t}.$$

$$\int\frac{dx}{\sqrt{(4-x^2)^3}}=\int\frac{Cos^3t2dt}{Cos^2t8}=\frac{1}{4}\int Costdt=\frac{1}{4}S\text{int}+C=\frac{1}{4}Sin(arctg\frac{x}{2})+C.$$

Agar $Sin(arctg\frac{x}{2})=\frac{x}{\sqrt{4+x^2}}$ ekanligini nazarda tutsak, natijada

$$\int\frac{dx}{\sqrt{(4+x^2)^3}}=\frac{1}{4}\frac{x}{\sqrt{4+x^2}}+C.$$

**14-Misol.** $\int\frac{dx}{(4-x^2)\sqrt{4-x^2}}$

**Yechish.** $\int\frac{dx}{(4-x^2)\sqrt{4-x^2}}=\left|\begin{matrix}x=2S\text{int}\\dx=2Cosdt\end{matrix}\right|=\int\frac{2Cosdt}{4Cos^2t2Cost}=\frac{1}{4}\int\frac{dt}{Cos^2t}=\frac{1}{4}tgt+C.$

Agar $S\text{int}=\frac{x}{2}$ va $Cost=\sqrt{1-Sin^2t}=\sqrt{1-\frac{x^2}{4}}=\frac{\sqrt{1-x^2}}{2}$ ekanligini inobatga olsak, $tgt=\frac{x}{\sqrt{4-x^2}}$. Demak, $\int\frac{dx}{(4-x^2)\sqrt{4-x^2}}=\frac{1}{4}\frac{x}{\sqrt{4-x^2}}+C$.

**15-Misol.** $\int\frac{dx}{(x^2-10)\sqrt{x^2-10}}$

**Yechish.** Bu integralda $x=\sqrt{10}Sect$ yoki $x=\frac{\sqrt{10}}{Cost}$ almashtirish bajariladi. Agar $dx=\frac{\sqrt{10}Sint}{Cos^2t}$ va $x^2-10=\frac{10Sin^2t}{Cos^2t}$ larni o'rniga qo'yilsa, quyidagini hosil qilamiz:

$$\int\frac{dx}{(x^2-10)\sqrt{x^2-10}}=\frac{1}{10}\int\frac{Cosdt}{Sin^2t}=\frac{1}{10}\int\frac{d(Sint)}{Sin^2t}=-\frac{1}{10Sint}+C=-\frac{1}{10}\frac{x}{\sqrt{x^2-10}}+C.$$

**16-Misol.** $\int\frac{dx}{\sqrt{(x^2+2x+17)^3}}$

**Yechish.** $\int\frac{dx}{\sqrt{(x^2+2x+17)^3}}=\int\frac{dx}{\sqrt{[(x+1)^2+16]^3}}=\left|\begin{matrix}x+1=4tgt\\dx=\frac{4dt}{Cos^2t}\end{matrix}\right|=$

$$=\int\frac{\frac{4dt}{Cos^2t}}{\frac{64}{Cos^3t}}=\frac{1}{16}\int Costdt=\frac{1}{16}Sint+C=\frac{1}{16}Sint+C=\frac{1}{16}\frac{tgt}{\sqrt{1+tg^2t}}+C=$$

$$=\frac{1}{16}\frac{\frac{x+1}{4}}{\sqrt{1+(\frac{x+1}{4})^2}}+C=\frac{1}{16}\frac{x+1}{\sqrt{x^2+2x+17}}+C.$$

**17-Misol.** $\int\sqrt{-x^2+6x+7}dx$

<u>**Yechish.**</u> $\int\sqrt{-x^2+6x+7}dx=\int\sqrt{-\left[(x-3)^2-16\right]}dx=\int\sqrt{16-(x-3)^2}dx=$

$$=\begin{vmatrix}x-3=4Sint\\ dx=4Cosdt\end{vmatrix}=16\int Cos^2tdt=8\int(1+Cos2t)dt=8t+4Sin2t+C=$$

$$=\begin{vmatrix}Sint=\frac{x-3}{4}\\ Cost=\frac{\sqrt{7+6x-x^2}}{4}\\ t=arcSin\frac{x-3}{4}\end{vmatrix}=8arcSin\frac{x-3}{4}+2\frac{x-3}{4}\frac{\sqrt{7+6x-x^2}}{4}+C=$$

$$=8arcSin\frac{x-3}{4}+\frac{(x-3)\sqrt{7+6x-x^2}}{8}+C.$$

**18-Misol.** $\int\frac{(x+4)dx}{\sqrt{x^2+2x-8}}$

<u>**Yechish.**</u> $\int\frac{(x+4)dx}{\sqrt{x^2+2x-8}}=\int\frac{(x+4)dx}{\sqrt{(x+1)^2-9}}=\begin{vmatrix}x+1=\frac{3}{Cost}\\ dx=\frac{3Sintdt}{Cos^2t}\end{vmatrix}=$

$$=\int\frac{(\frac{3}{Cost}+3)\frac{3Sintdt}{Cos^2t}}{\frac{3Sint}{Cost}}=3\int\frac{(1+Cost)dt}{Cos^2t}=3\int\frac{dt}{Cos^2t}+3\int\frac{dt}{Cost}=3tgt+3\ln\left|tg(\frac{\pi}{4}+\frac{t}{2})\right|+C.$$

Bu yerda, $Cost=\frac{3}{x+1}, Sint=\frac{\sqrt{x^2+2x-8}}{x+1}, tg(\frac{\pi}{4}+\frac{t}{2})=\frac{1+Sint}{Cost}=\frac{1+\frac{\sqrt{x^2+2x-8}}{x+1}}{\frac{3}{x+1}}=$

$=\frac{x+1+\sqrt{x^2+2x-8}}{3}$ larni e'tiborga olsak. mazkur integral oxirigacha hisoblanadi.

$$\int\frac{(x+4)dx}{\sqrt{x^2+2x-8}}=\sqrt{x^2+2x-8}+3\ln\left|\frac{(x+1)+\sqrt{x^2+2x-8}}{3}\right|+C.$$

➤ <u>**Eslatma.**</u> Biz bu yerda algebraik irratsionalliklarni integrallashning Eyler almashtirishlari, differensial binomlarni integrallash, giperbolik almashtirishlar va boshqa usullarni yoritmadik.

Kitobxon ushbu va boshqa usullarni oliy matematikaga bag'ishlangan ko'pgina adabiyotlardan mustaqil ravishda o'rganishi mumkin.

## MASHQLAR

1. $\int \frac{\sqrt{x}dx}{\sqrt[3]{x^2}-\sqrt{x}}$
2. $\int \frac{\sqrt[3]{x}dx}{x(\sqrt{x}+\sqrt[3]{x})}$
3. $\int \frac{dx}{\sqrt{x}+\sqrt[3]{x}}$
4. $\int \frac{\sqrt{x}dx}{\sqrt[4]{x^3}+1}$
5. $\int \frac{dx}{\sqrt{x}(1+\sqrt[4]{x})^2}$
6. $\int \frac{(1-\sqrt[4]{x})^3 dx}{\sqrt[3]{x}}$
7. $\int \frac{(\sqrt{x}+\sqrt[3]{x}+\sqrt[4]{x})dx}{\sqrt[3]{x^2}+\sqrt[6]{x^5}}$
8. $\int \frac{\sqrt[6]{x}dx}{x(\sqrt[3]{x}+\sqrt[4]{x})}$
9. $\int \frac{\sqrt[6]{x}dx}{1+\sqrt[3]{x}}$
10. $\int \frac{(\sqrt{x}+\sqrt[3]{x})dx}{\sqrt[4]{x^5}-\sqrt[6]{x^7}}$
11. $\int \frac{(x+\sqrt[3]{x^2}+\sqrt[6]{x})dx}{x(1+\sqrt[3]{x})}$
12. $\int \frac{(\sqrt{x}-1)dx}{\sqrt[3]{x}+1}$
13. $\int \frac{xdx}{1+\sqrt{2x+1}}$
14. $\int \frac{dx}{2+\sqrt{x-8}}$
15. $\int \frac{dx}{\sqrt{1-2x}-\sqrt[4]{1-2x}}$
16. $\int \frac{\sqrt{x-5}dx}{x^3}$
17. $\int \frac{\sqrt{3x+4}dx}{x^2}$
18. $\int \frac{\sqrt{x+2}+3}{\sqrt{x+2}-4}$
19. $\int \frac{dx}{\sqrt[3]{(3+2x)^2}-\sqrt{3+2x}}$
20. $\int \frac{\sqrt{x+1}+1}{\sqrt[3]{x+1}-1}dx$
21. $\int \sqrt{\frac{2+x}{2-x}}dx$
22. $\int \sqrt{\frac{3-4x}{9-5x}}dx$
23. $\int \sqrt{(\frac{1-x}{1+x})^3}$
24. $\int (x-2)\sqrt{\frac{1+x}{1-x}}dx$
25. $\int \frac{(2+\sqrt{x+1})dx}{(x+1)^2-\sqrt{x+1}}$
26. $\int \frac{(3+\sqrt{x+4})dx}{(x+4)^2-\sqrt{x+4}}$
27. $\int \frac{dx}{\sqrt[4]{(1+x)^3(1-x)}}$
28. $\int \frac{dx}{\sqrt[3]{(x+1)^2(x-2)^4}}$
29. $\int \sqrt[3]{\frac{x+1}{x-1}}dx$
30. $\int x\sqrt{\frac{x-1}{x+1}}dx$
31. $\int \frac{dx}{\sqrt{7x^2+5x+4}}$
32. $\int \frac{dx}{\sqrt{14x^2+9x+1}}$
33. $\int \frac{dx}{\sqrt{5+13x+8x^2}}$
34. $\int \frac{dx}{\sqrt{11+5x+6x^2}}$
35. $\int \frac{dx}{\sqrt{5+3x-x^2}}$
36. $\int \frac{dx}{\sqrt{9-x-x^2}}$
37. $\int \frac{dx}{\sqrt{11+9x-7x^2}}$
38. $\int \frac{dx}{\sqrt{3+2x-5x^2}}$
39. $\int \frac{xdx}{\sqrt{2x^2+5x+3}}$
40. $\int \frac{(2-3x)dx}{\sqrt{4x^2-x-7}}$
41. $\int \frac{xdx}{\sqrt{8-3x-2x^2}}$
42. $\int \frac{(2x+5)dx}{\sqrt{7+8x-11x^2}}$

43. $\int \frac{(3x-7)dx}{\sqrt{5x^2+8x+1}}$

44. $\int \frac{dx}{x\sqrt{5+x^2}}$

45. $\int \frac{dx}{x^2\sqrt{7-x^2}}$

46. $\int \frac{dx}{(x-2)^3\sqrt{3x^2-8x+5}}$

47. $\int \frac{(3x+2)dx}{\sqrt{x^2+x+2}}$

48. $\int \frac{dx}{x\sqrt{2x^2+2x+1}}$

49. $\int \frac{dx}{9x+1)^3\sqrt{x^2+2x}}$

50. $\int \frac{dx}{(x+1)^3\sqrt{x^2+3x+2}}$

51. $\int \frac{dx}{(x^2-14)\sqrt{x^2-14}}$

52. $\int \frac{dx}{(3-x^2)\sqrt{3-x^2}}$

53. $\int \frac{dx}{\sqrt{1-x^2}\,(4+x^2)}$

54. $\int \frac{dx}{\sqrt{(16+x^2)^3}}$

55. $\int \frac{x^2dx}{\sqrt{(4+x^2)^3}}$

56. $\int \frac{\sqrt{16-x^2}}{x^2}dx$

57. $\int \frac{(5x+4)dx}{\sqrt{x^2+2x+5}}$

58. $\int \sqrt{x^2-2x+2}\,dx$

59. $\int \sqrt{x^2-4}dx$

60. $\int \frac{dx}{(x-1)\sqrt{x^2-3x+2}}$

## 1.7 Giperbolik funksiyalarni integrallash

Giperbolik funksiyalarga nisbatan ratsional bog'lanishdagi funksiyalarni integrallash ham aynan trigonometrik funksiyalarni integrallashga o'xshash olib boriladi. Bu yerda, quyidagilarni inobatga olish maqsadga muvofiqdir:

$shx = \frac{e^x - e^{-x}}{2}$ (giperbolik sinus),

$chx = \frac{e^x + e^{-x}}{2}$ (giperbolik kosinus),

$thx = \frac{shx}{chx} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$ (giperbolik tangens),

$cthx = \frac{chx}{shx} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$ (giperbolik kotangens)

$$ch^2x - sh^2x = 1,\ ch^2x + sh^2x = ch2x,$$

$$sh^2x = \frac{1}{2}(ch2x-1),\ ch^2x = \frac{1}{2}(ch2x+1),\ Sh2x = 2shxchx$$

Shuningdek,

Agar $th\frac{x}{2} = t$ bo'lsa, $shx = \frac{2t}{1-t^2}$; $chx = \frac{1+t^2}{1-t^2}$ va $dx = \frac{2dt}{1+t^2}$ dir.

**1-Misol.** $\int ch^2xdx = \frac{1}{2}\int(ch2x+1)dx = \frac{1}{2}\int dx + \frac{1}{2}\frac{1}{2}\int ch2xd(2x) = \frac{1}{2}x + \frac{1}{4}sh2x + C$.

**2-Misol.**

$$\int sh^3xdx=\int sh^2xshxdx=\int(ch^2x-1)d(chx)=\int ch^2xd(chx)-\int d(chx)=\frac{ch^3x}{3}-chx+C.$$

**3-Misol.** $\int sh^2xch^2xdx=\frac{1}{4}\int sh^22xdx=\frac{1}{4}\frac{1}{2}\int(ch4x-1)dx=\frac{1}{8}\frac{1}{4}sh4x-\frac{1}{8}x+C=$

$=\frac{1}{8}(\frac{1}{4}sh4x-x)+C.$

**4-Misol.** $\int\frac{dx}{sh^2xch^2x}=\int\frac{(ch^2x-sh^2x)dx}{sh^2xch^2x}=\int\frac{dx}{sh^2x}-\int\frac{dx}{ch^2x}=-cthx-thx+C.$

**5-Misol.** $\int th^3xdx=\int\frac{sh^3x}{ch^3x}dx=\int\frac{sh^2xshxdx}{ch^3x}=\int\frac{(ch^2x-1)shxdx}{ch^3x}=$

$=\int\frac{shxdx}{chx}-\int\frac{shxdx}{ch^3x}=\int\frac{d(chx)}{chx}-\int ch^{-3}xd(chx)=\ln|chx|+\frac{1}{2ch^2x}+C.$

**6-Misol.** $\int\frac{dx}{shx+2chx}$ ni hisoblashda, $th\frac{x}{2}=t$ almashtirishni qo'llaymiz. Agar $shx+2chx=\frac{2t}{1-t^2}+\frac{2+2t^2}{1-t^2}=\frac{2(t^2+t+1)}{1-t^2}$ ni inobatga olsak, natijada:

$$\int\frac{dx}{shx+2chx}=\int\frac{dt}{t^2+t+1}=\int\frac{dx}{\frac{\sqrt3}{4}+(t+\frac{1}{2})^2}=\frac{2}{\sqrt3}arctg\frac{2t+1}{\sqrt3}+C=\frac{2}{\sqrt3}arctg\frac{2th\frac{x}{2}+1}{\sqrt3}+C$$

## MASHQLAR

1. $\int ch^3xdx$
2. $\int ch^2xdx$
3. $\int ch^4xdx$
4. $\int ch^3xchxdx$
5. $\int\frac{dx}{shxch^2x}$
6. $\int cth^3xdx$
7. $\int th^4xdx$
8. $\int cth^4xdx$
9. $\int\frac{dx}{sh^4x+ch^2x}$
10. $\int\frac{dx}{2shx+3chx}$
11. $\int\frac{shxdx}{\sqrt{ch2x}}$
12. $\int\frac{shxchxdx}{sh^2x+ch^2x}$
13. $\int sh^3xch^3xdx$
14. $\int\frac{sh\sqrt{1-x}}{\sqrt{1-x}}$
15. $\int\frac{dx}{chxsh^2x}$
16. $\int\frac{dx}{3shx+4chx}$
17. $\int\frac{dx}{6-5chx}$
18. $\int xthx^2dx$
19. $\int(10x+7)sh(5x^2+7x+9)dx$
20. $\int\frac{x^2dx}{sh2(4x^3+9)}$

## 2.1. Aniq integral tushunchasi. Aniq integralning asosiy xossalari va uni hisoblash

1. Aytaylik, $y=f(x)$ funksiya [a,b] kesmada aniqlangan va uzluksiz bo'lsin. SHu kesmani $a=x_0,x_1,x_2,...,x_{n-1},x_n=b(x_0<x_1<...<x_n)$ nuqtalar orqali uzunligi $\Delta x_k=x_k-x_{k-1}\,(k=\overline{1,n})$ bo'lgan $n$ ta bo'laklarga ixtiyoriy ravishda bo'lib chiqamiz. Har bir $[x_{k-1};x_k]$ kesmada $\xi_k$ nuqtani ixtiyoriy tanlab, funksiyaning [a,b] kesma bo'yicha n-integral yig'indisi deb ataluvchi

$$S_n=\sum_{k=1}^{n}f(\xi_k)\Delta x_k$$

ni tuzamiz.

**Ta'rif.** *Agar* $[x_{k-1};x_k](k=\overline{1,n})$ *kesmalar ichidagi eng katta uzunlikka ega bo'lgan kesmaning uzunligi nolga intilganda,* $S_n$ *integral yig'indining chekli limiti mavjud bo'lib, u limit [a,b] kesmaning bo'laklarga bo'linish usuliga va* $\xi_k$ *nuqtalarning tanlanilishiga bog'liq bo'lmasa, u holda u limit o'q* $f(x)$ *funksiyaning [a,b] kesma bo'yicha aniq integrali deb ataladi va* $\int_a^b f(x)dx$ *kabi belgilanadi, ya'ni ta'rifga ko'ra,*

$$\int_a^b f(x)dx=\lim_{\max\Delta x_k\to 0}\sum_{k=1}^{n}f(\xi_k)\Delta x_k$$

Bu yerda [$a$;b] – integrallash kesmasi, $a$ va $b$ lar mos ravishda aniq integralning quyi va yuqori chegaralari, $x$ esa integrallash o'zgaruvchisi.

2. Endi $f(x)$ va $\varphi(x)$ funksiyalarni qaralayotgan [a;b] kesmada integrallanuvchi funksiyalar deb faraz qilib, aniq integralning asosiy xossalarini sanab o'tamiz:

1. $\int_a^b Cf(x)dx=C\int_a^b f(x)dx$ $(C=\text{const})$, ya'ni o'zgarmas ko'paytuvchini aniq integral belgisidan tashqariga chiqarib yoziladi.

2. $\int_a^b[f(x)\pm\varphi(x)]dx=\int_a^b f(x)\pm\int_a^b\varphi(x)dx$, ya'ni ikkita funksiya algebraik yig'indisining aniq integrali, qo'shiluvchilar integraiining algebraik yig'indisiga teng.

3. $\int_a^a f(x)dx=0$. Agar aniq integralning quyi va yuqori chegaralari o'zaro teng bo'lsalar, aniq integralning qiymati nolga teng.

4. $\int_a^b f(x)dx = -\int_b^a f(x)dx$ , aniq integral chegaralarining o'rinlari almashtirilsa, uning qiymati teskari ishoraga o'zgaradi.

5. Agar $c, [a;b]$ dagi ixtiyoriy nuqta bo'lsa, u holda quyidagi tenglik har doim o'rinlidir (aniq integralning additivlik xossasi):

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$

6. Aniq integralning qiymati integrallash o'zgaruvchisining ko'rinishiga bog'liq emas $\int_a^b f(x)dx = \int_a^b f(y)dy = \int_a^b f(t)dt = ... = \int_a^b f(\alpha)d\alpha$

7. Agar [a;b] kesmada $\varphi(x) \leq f(x)$ bo'lib, a<b bo'lsa, u holda har doim $\int_a^b \varphi(x)dx \leq \int_a^b f(x)dx$ dir.

8. Agar [a;b] kesmada $f(x) \geq 0$ bo'lib, a<b bo'lsa, u holda $\int_a^b f(x)dx \geq 0$ dir.

9. Agar $f(x)$ funksiya [a;b] da uzluksiz bo'lib, a<b bo'lsa, u holda, $m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$. Bu yerda, m va M lar $f(x)$ funksiyaning [a;b] dagi mos ravishda eng kichik va eng katta qiymatlaridir.

10. Agar $f(x)$ funksiya [a;b] kesmada uzluksiz bo'lsa, u holda shu kesmada hech bo'lmaganda shunday bitta $x=c$ nuqta mavjudki ($a \leq c \leq b$), har doim quyidagi tenglik o'rinli bo'ladi:

$\int_a^b f(x)dx = f(c)(b-a)$ (o'rta qiymat haqidagi teorema).

3. Agar F(x) funksiya $[a;b]$ da $f(x)$ funksiyaning boshlang'ich funksiyasi bo'lsa, u holda aniq integral, Nyuton-Leybnits formulasi deb ataladigan quyidagi formula orqali hisoblanadi.:

$$\int_a^b f(x)dx = F(x)\Big|_a^b = F(b) - F(a)$$

Endi aniq integralning xossalari hamda Nyuton-Leybnits formulasining qo'llanishiga doir ayrim misollarning yechilishini keltiramiz.

**1-Misol.** $\int_0^{\frac{\pi}{2}} Sin2xdx = -\frac{1}{2}Cos2x\Big|_0^{\frac{\pi}{2}} = -\frac{1}{2}(Cos\pi - Cos0) = -\frac{1}{2}(-1-1) = 1$.

**2-Misol.** $\int_0^2 \frac{dx}{\sqrt{16-x^2}} = arcSin\frac{x}{4}\Big|_0^2 = arcSin\frac{1}{2} - arcSin0 = \frac{\pi}{6}$.

**3-Misol.** Agar $f(x)=\begin{cases} x^2 \text{ ra}, 0\le x\le 1 \\ \sqrt{x} \text{ ra}, 1\le x\le 2\end{cases}$ kabi bo'lsa, $\int\limits_0^2 f(x)dx$ hisoblansin.

<u>**Yechish.**</u> Aniq integralning additivlik xossasiga binoan,

$$\int\limits_0^2 f(x)dx=\int\limits_0^1 f(x)dx+\int\limits_1^2 f(x)dx=\int\limits_0^1 x^2dx+\int\limits_1^2\sqrt{x}dx=\frac{x^3}{3}\Big|_0^1+\frac{2}{3}x^{\frac{3}{2}}\Big|_1^2=\frac{1}{3}+\frac{4\sqrt{2}}{3}-\frac{2}{3}=\frac{1}{3}(4\sqrt{2}-1).$$

**4-Misol.** $\int\limits_0^{\sqrt{3}}\frac{dx}{1+x^2}=arctgx\Big|_0^{\sqrt{3}}=arctg\sqrt{3}-arctg0=\frac{\pi}{3}.$

**5-Misol.** $\int\limits_0^1\frac{dx}{1+x}=\int\limits_0^1\frac{d(1+x)}{1+x}=\ln|1+x|\Big|_0^1=\ln 2-\ln 1=\ln 2.$

**6-Misol.** $\int\limits_0^1\sqrt{1+x^4}dx$ baholansin.

<u>**Yechish.**</u> O'rta qiymat haqidagi teoremaga binoan,

$$\int\limits_0^1\sqrt{1+x^4}dx=\sqrt{1+\zeta^4},\quad 0\le\zeta\le 1$$

Ammo $1<\sqrt{1+\zeta^4}<\sqrt{2}$ ligi uchun $1<\int\limits_0^1\sqrt{1+x^4}dx\le\sqrt{2}\cong 1.414$

## MASHQLAR

1. $\int\limits_{-2}^{-1}\frac{dx}{(11+5x)^3}$
2. $\int\limits_{-\pi}^{\pi}Sin^2\frac{x}{2}dx$
3. $\int\limits_0^{\frac{\pi}{4}}\frac{x^2dx}{1+x^2}$
4. $\int\limits_e^{e^2}\frac{dx}{x\ln x}$
5. $\int\limits_0^1\frac{e^xdx}{1+e^{2x}}$
6. $\int\limits_0^1\frac{x^3dx}{1+x^8}$
7. $\int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sqrt{Cosx-Cos^3x}dx$
8. $\int\limits_1^2(x^2+\frac{1}{x^4})dx$
9. $\int\limits_0^{\frac{\pi}{2}}Sin^3xdx$
10. $\int\limits_{\frac{1}{\pi}}^{\frac{2}{\pi}}\frac{Sin(\frac{1}{x})dx}{x^2}$
11. $\int\limits_3^8\frac{dx}{3\sqrt{x+1}}$
12. $\int\limits_{-2}^1\frac{dx}{x^2+4x+5}$
13. $\int\limits_{-3}^{-2}\frac{dx}{x^2-1}$
14. $\int\limits_2^6\sqrt{x-2}dx$
15. $\int\limits_0^8(\sqrt[3]{x}+\sqrt{2x})dx$
16. $\int\limits_0^{\frac{\pi}{4}}Cos^2xdx$
17. $\int\limits_1^e\frac{Sin(\ln x)}{x}dx$
18. $\int\limits_3^4\frac{dx}{x^2-3x+2}$
19. $\int\limits_2^{3.5}\frac{dx}{\sqrt{5+4x-x^2}}$
20. $\int\limits_0^1 chxdx$
21. $\int\limits_0^{\pi}sh^2xdx$
22. $\int\limits_{\ln 2}^{\ln 3}\frac{dx}{ch^2x}$
23. $\int\limits_1^4\frac{1+\sqrt{x}}{x^2}$
24. $\int\limits_0^1\frac{x^2dx}{\sqrt{x^6+4}}$

**Yechish.** Uni hisoblashda $tg\frac{x}{2}=z$ almashtirishdan foydalanamiz. Bu yerda : $dx=\frac{2dz}{1+z^2}, Cosx=\frac{1-z^2}{1+z^2}$, x=0 da z=0 va $x=\pi/2$ da z=1. Natijada.:

$$\int_0^{\pi/2}\frac{dx}{3+2Cosx}=2\int_0^1\frac{dz}{5+z^2}=\frac{2}{\sqrt5}arctg\frac{z}{\sqrt5}\Big|_0^1=\frac{2}{\sqrt5}arctg\frac{1}{\sqrt5}.$$

**1.** Aytaylik, u=u(x) va v=v(x) funksiyalar [a;b] kesmada uzluksiz diferentsiallanuvchi funksiyalar bo'lsin.

Ushbu $\int_a^b udv=uv\Big|_a^b-\int_a^b vdu$ formulani bo'laklab integrallash formulasi deb yuritiladi va uning qo'llanilishi aniqmas integraldagi mos formulaga o'xshashdir.

**6-Misol.** $\int_0^{\sqrt3}x^3\sqrt{1+x^2}dx$

**Yechish.** $\int_0^1 \arccos xdx=\left|\begin{array}{l|l} u=\arccos x & dv=dx \\ du=-\frac{dx}{\sqrt{1-x^2}} & v=x \end{array}\right|=x\arccos x\Big|_0^1+\int_0^1\frac{xdx}{\sqrt{1-x^2}}=$

$$=1\cdot\arccos1-0\cdot\arccos0-\sqrt{1-x^2}\Big|_0^1=1.$$

**7-Misol.** $\int_1^e \ln^2 xdx$

**Yechish.** $\int_1^e \ln^2 xdx=\left|\begin{array}{l} u=\ln^2 x \\ du=2\ln x\frac{dx}{x} \\ dv=dx, v=x \end{array}\right|=x\ln^2 x\Big|_1^e-2\int_1^e \ln xdx=e\ln^2 e-2\int_1^e \ln xdx=\left|\begin{array}{l} u=\ln x \\ du=\frac{dx}{x} \\ v=x \end{array}\right|=$

$$=e-2\left[x\ln x\Big|_1^e-\int_1^e dx\right]=e-2\left[e\ln e-1\ln1-x\Big|_1^e\right]=e-2(e-e+1)=e-2.$$

**8-Misol.** $\int_0^{\pi/2}xCosxdx$

**Yechish.** $\int_0^{\pi/2}xCosxdx=\left|\begin{array}{l} u=x, du=dx \\ dv=Cosxdx \\ v=Sinx \end{array}\right|=xSinx\Big|_0^{\pi/2}-\int_0^{\pi/2}Sinxdx=\frac{\pi}{2}+Cosx\Big|_0^{\pi/2}=\frac{\pi}{2}-1$

**9-Misol.** $\int_0^1 x^2e^xdx$

**Yechish.** $\int_0^1 x^2e^xdx=\left|\begin{array}{l} u=x^2; du=2xdx \\ dv=e^xdx \\ v=e^x \end{array}\right|=x^2e^x\Big|_0^1-2\int_0^1 xe^xdx=\left|\begin{array}{l} u=x, du=dx \\ dv=e^xdx \\ v=e^x \end{array}\right|=$

$$= e-2\left[xe^x\Big|_0^1-\int_0^1 e^x dx\right]=e-2\left(e-e^x\Big|_0^1\right)=e-2(e-e+1)=e-2.$$

**10-Misol.** $\int\limits_0^1 x\,arctgx\,dx$

**Yechish.**

$$\int\limits_0^1 x\,arctgx\,dx=\left|\begin{matrix} u=arctgx \\ du=\dfrac{dx}{1+x^2} \\ dv=xdx \\ v=\dfrac{x^2}{2}\end{matrix}\right|=\frac{x^2}{2}arctgx\Big|_0^1-\frac{1}{2}\int\limits_0^1\frac{x^2dx}{1+x^2}=\frac{1}{2}arctg1-\frac{1}{2}x\Big|_0^1+\frac{1}{2}arctg\Big|_0^1=$$

$$=\frac{1}{2}\cdot\frac{\pi}{4}-\frac{1}{2}+\frac{1}{2}\cdot\frac{\pi}{4}=\frac{\pi}{4}-\frac{1}{2}=\frac{\pi-2}{4}.$$

## MASHQLAR

1. $\int\limits_0^1\sqrt{\frac{1-x}{1+x}}\,dx$
2. $\int\limits_0^1\frac{xdx}{1-\sqrt{x}}$
3. $\int\limits_0^{1/4}\frac{dx}{(4x+1)\sqrt{x}}$
4. $\int\limits_0^1\frac{dx}{1+\sqrt{x}}$
5. $\int\limits_0^1\frac{dx}{2x+\sqrt{3x+1}}$
6. $\int\limits_{-1}^{3}\frac{dx}{\sqrt{7-2x}}$
7. $\int\limits_{-1}^{1}\frac{xdx}{\sqrt{3-2x}}$
8. $\int\limits_0^{\pi/2}\frac{dx}{1-Sinx}$
9. $\int\limits_{-1}^{1}\frac{dx}{(1+x^2)^2}$
10. $\int\limits_0^1\sqrt{2x-x^2}\,dx$
11. $\int\limits_0^3\frac{dx}{\sqrt{x+1}+\sqrt{(x+1)^3}}$
12. $\int\limits_0^{2\pi}\frac{dx}{5-2Cosx}$
13. $\int\limits_1^e\ln^3 xdx$
14. $\int\limits_0^{\pi^2/4}Sin\sqrt{x}dx$
15. $\int\limits_0^1 arctg\sqrt{x}dx$
16. $\int\limits_{\pi/4}^{\pi/3}\frac{xdx}{Sin^2x}$
17. $\int\limits_0^1 x\ln(1+x^2)dx$
18. $\int\limits_0^1(x-1)e^{-x}dx$
19. $\int\limits_1^2\frac{\ln xdx}{x^2}$
20. $\int\limits_0^{\pi/4}xtg^2xdx$
21. $\int\limits_0^1 arcSinxdx$
22. $\int\limits_1^e x^2\ln xdx$
23. $\int\limits_0^{\pi/2}x^2Sinxdx$
24. $\int\limits_0^1\frac{\arcsin xdx}{\sqrt{1+x}}$
25. $\int\limits_0^1(arcSinx)^2dx$
26. $\int\limits_0^{\pi/3}\frac{xSinxdx}{Cos^2x}$
27. $\int\limits_1^e x\ln^2 xdx$
28. $\int\limits_0^2(3-2x)e^{-x}dx$
29. $\int\limits_0^{\pi^2}Cos\sqrt{x}dx$
30. $\int\limits_0^1 x^3Cos3xdx$

## 2.Quyida aniq integrallarni hisoblashni yengillashtiradigan ba'zi bir omillarni keltiramiz

a) Agar *f(x)* funksiya [-*a*; *a*] kesmada juft bo'lsa, u holda

$$\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$$

b) Agar *f(x)* funksiya [-a;a] kesmada toq bo'lsa, u holda

$$\int_{-a}^{a} f(x)dx = 0$$

**1-Misol.** $\int_{-1}^{1} |x|\,dx$ hisoblansin.

**Yechish.** Integral belgisi ostidagi funksiya $f(x)=|x|$- juft funksiya bo'lganligi sababli, $\int_{-1}^{1} |x|\,dx = 2\int_{0}^{1} |x|\,dx = 2\int_{0}^{1} x dx = x^2\Big|_{0}^{1} = 1.$

**2-Misol.** $\int_{-5}^{5} \frac{x^5 Sin^2 x dx}{x^4 + 2x^2 1}$ ning qiymati nolga teng, chunki integral belgisi ostida toq funksiya ishtirok etmoqda.

**3-Misol.** $\int_{-1/2}^{1/2} Cosx \cdot \ln(\frac{1+x}{1-x})dx$

**Yechish.** Bu yerda Cosx juft funksiya, $f(x) = \ln(\frac{1+x}{1-x})$ esa toq funksiyadir.

Haqiqatan ham: $f(-x) = \ln(\frac{1-x}{1+x}) = \ln(\frac{1+x}{1-x})^{-1} = -\ln(\frac{1+x}{1-x}) = -f(x)$

Demak, integral belgisi ostida juft va toq funksiyalarning ko'paytmasi – toq funksiya shtirok etmoqda. Shu sababli, mazkur integralning qiymati nolga teng bo'ladi.

**4-Misol.** $\int_{-\sqrt{2}}^{\sqrt{2}} \frac{2x^5 + 3x^4 - x^3 - 6x^2 + x}{x^2 - 2} dx$

**Yechish.** Integralni hisoblash uchun uni shunday ikkita integral yig'indisi ko'rinishida fodalaymizki, ulardan biri toq funksiyaning, ikkinchisi esa juft funksiyaning integrali bo'lsin, ya'ni:

$$\int_{-\sqrt{2}}^{\sqrt{2}} \frac{2x^5 + 3x^4 - x - 6x^2 + x}{x^2 - 2} dx = \int_{-\sqrt{2}}^{\sqrt{2}} \frac{2x^5 - x^3 + x}{x^2 - 2} dx + \int_{-\sqrt{2}}^{\sqrt{2}} \frac{3x^4 - 6x^2}{x^2 - 2} dx =$$

$$= 0 + 2\int_{0}^{\sqrt{2}} \frac{3x^2(x^2 - 2)}{x^2 - 2} dx = 6\int_{0}^{\sqrt{2}} x^2 dx = 6 \cdot \frac{x^3}{3}\Big|_{0}^{\sqrt{2}} = 2(\sqrt{2})^3 = 4\sqrt{3}.$$

## MASHQLAR

1. Agar $f(x)$–juft funksiya bo‘lsa, $\int_{-\pi}^{\pi} f(x)Cosnxdx$ bilan $\int_{-\pi}^{\pi} f(x)Sinnxdx$ larning qiymatlari nimaga teng?

2. Agar $f(x)$–toq funksiya bo‘lsa, $\int_{-\pi}^{\pi} f(x)Cosnxdx$ bilan $\int_{-\pi}^{\pi} f(x)Sinnxdx$ larning qiymatlari nimaga teng?

3. $\int_{-a}^{a} Cosf(x^2)dx = 2\int_{0}^{a} Cosf(x^2)dx$ ni isbotlang.

4. $\int_{-\pi/4}^{\pi/4} \frac{x^8 + Sin^9 x}{Sin^7 x}dx$ ni hisoblang.

5. $\int_{-1/2}^{1/2} e^{Cosx}dx = 2\int_{0}^{1/2} e^{Cosx}dx$ ni isbotlang.

6. $\int_{-\pi}^{\pi} SinnxCosmxdx$ ni hisoblang.

7. $\int_{-\pi}^{\pi} Sinxf(Cosx)dx$ ni hisoblang.

8. $\int_{-1}^{1} \frac{x^4 Sinx}{x^2 + 4}dx$ ni hisoblang.

### 2.3. Aniq integrallarni taqribiy hisoblash usullari

Ma'lumki, nazariy jihatdan qaralganda har qanday uzluksiz funksiyaning boshlang‘ich funksiyasi mavjud bo‘lar edi. Ammo, amalda esa, ko‘pincha uzluksiz funksiya uchun uning boshlang‘ich funksiyasini elementar funksiyalar orqali ifodalash mumkin bo‘lavermaydi. Bunday hollarda aniq integralni hisoblash uchun Nьyuton-Leybnits formulasini qo‘llash mumkin bo‘lmaganligi sababli, uni hisoblashda taqribiy hisoblash usullaridan foydalaniladi.

#### 1.To‘g‘ri to‘rtburchaklar usuli

Mazkur usulda integrallash kesmasi [a;b]ni $x_0=a<x_1<x_2<\ldots<x_{n-1}<x_n=b$ nuqtalar orqali n ta teng bo‘laklarga bo‘linadi. Har bir bo‘lakning uzunligi $h=\frac{b-a}{n}$ ga teng bo‘lib, uni integrallash qadami deb yuritiladi.

Har bir bo‘linish nuqtasida integrallanuvchi funksiya $f(x)$ ning qiymatlari $f(x_0), f(x_1), \ldots, f(x_{n-1})$ hisoblanadi.

$\int_a^b f(x)dx$ ni taqribiy hisoblash uchun quyidagi formuladan foydalaniladi

$$\int_a^b f(x)dx \cong \frac{b-a}{n}[f(x_0)+f(x_1)+f(x_2)+...+f(x_{n-1})]$$

Mazkur formula yordamida aniq integralni taqribiy hisoblash usulini to'g'ri to'rtburchaklar usuli deb ataladi.

## 2.Trapetsiyalar usuli

Bu holda ham [a:b] oraliqni n ta teng bo'laklarga bo'linadi va taqribiy integrallash formulasi quyidagicha yoziladi:

$$\int_a^b f(x)dx \cong \frac{b-a}{2n}\{f(x_0)+f(x_n)+2[f(x_1)+f(x_2)+..+f(x_{n-1})]\}.$$

## 3. Parabolalar yoki Simpson usuli.

Ushbu usulda integrallash kesmasini $x_0=a<x_1<x_2<\ldots<x_{2n-2}<x_{2n-1}=b$ kabi nuqtalar orqali 2n juft son miqdoridagi teng bo'laklarga bo'linadi. Agar integrallanuvchi funksiyaning bu nuqtalardagi qiymatlarini $y_0$, $y_1$, $y_2$,........, $y_{2n-2}$,$y_{2n-1}$,$y_{2n}$ deb belgilasak, aniq integralni taqribiy integrallash formulasi quyidagi Simpson formulasi deb ataluvchi formula bilan amalga oshiriladi:

$$\int_a^b f(x)dx \cong \frac{b-a}{2n}[(y_0+y_{2n})+4(y_1+y_3+...+y_{2n-1})+2(y_2+y_4+..+y_{2n-2})].$$

Ta'kidlash lozimki, mazkur formula avvalgi formulalarga nisbatan yuqorirok darajadagi aniqlikni ta'minlaydi.

**1 Misol.** $\int_0^1 \sqrt{1+10x}dx$ ni yuqorida bayon qilingan har uchchala usulda ham taqribiy hisoblansin. Bu yerda, integrallash kesmasi 10 qismga bo'linsin.

<u>**Yechish.**</u> a) To'g'ri to'rtburchaklar usuli bilan echamiz. $h=\frac{b-a}{n}=\frac{1-0}{10}=0.1$ bo'lganligidan, bo'linish nuqtalari:

$x_0=0$; $x_1=0.1$; $x_2=0.2$; $x_3=0.3$; $x_4=0.4$; $x_5=0.5$; $x_6=0.6$; $x_7=0.7$; $x_8=0.8$; $x_9=0.9$

U holda

$$f(x_0)=\sqrt{1+10\cdot 0}=1$$

$$f(x_1)=\sqrt{1+10\cdot 0.1}=\sqrt{2}=1.414$$

$$f(x_2)=\sqrt{1+10\cdot 0.2}=\sqrt{3}=1.732$$

$$f(x_3) = \sqrt{1+10\cdot 0.3} = \sqrt{4} = 2$$
$$f(x_4) = \sqrt{1+10\cdot 0.4} = \sqrt{5} = 2.236$$
$$f(x_5) = \sqrt{1+10\cdot 0.5} = \sqrt{6} = 2.449$$
$$f(x_6) = \sqrt{1+10\cdot 0.6} = \sqrt{7} = 2.646$$
$$f(x_7) = \sqrt{1+10\cdot 0.7} = \sqrt{8} = 2.828$$
$$f(x_8) = \sqrt{1+10\cdot 0.8} = \sqrt{9} = 3$$
$$f(x_9) = \sqrt{1+10\cdot 0.9} = \sqrt{10} = 3.162$$

---

$$\sum = 22.467$$

Natijada, $\int_0^1 \sqrt{1+10x}dx \cong 0.1\cdot 22.467 = 2.2467$ hosil bo'ladi.

b) Trapetsiyalar formulasini qo'llab echamiz.

Bu yerda, $\frac{b-a}{2n} = \frac{1}{20} = 0.05$

$$f(x_0) = f(0) = \sqrt{1+10\cdot 0} = 1$$
$$f(x_{10}) = f(1) = \sqrt{1+10\cdot 1} = \sqrt{11} = 3.317$$

---

$$\sum = 4.317$$
$$f(x_1) + f(x_2) + \ldots + f(x_9) = 21.467$$

U holda $\int_0^1 \sqrt{1+10x}dx \cong 0.05(4.317 + 2\cdot 21.467) = 0.05(4.317 + 42.934) =$

$$= 0.05\cdot 47.251 = 2.36225$$

c) Endi Simpson formulasidan foydalanamiz.

Bu yerda, 2n=10 yoki n=5

$$h = \frac{b-a}{6n} = \frac{1}{30}$$
$$y_0 = \sqrt{1+10\cdot 0} = 1$$
$$y_{10} = \sqrt{1+10\cdot 1} = \sqrt{11} = 3.37$$

---

$$\sum = 4.317$$

| | |
|---|---|
| $y_1 = \sqrt{1+10\cdot 0.1} = 1.414$ | $y_2 = \sqrt{1+10\cdot 0.2} = 1.732$ |
| $y_3 = \sqrt{1+10\cdot 0.3} = 2$ | $y_4 = \sqrt{1+10\cdot 0.4} = 2.236$ |
| $y_5 = \sqrt{1+10\cdot 0.5} = 2.449$ | $y_6 = \sqrt{1+10\cdot 0.6} = 2.646$ |
| $y_7 = \sqrt{1+10\cdot 0.7} = 2.828$ | $y_8 = \sqrt{1+10\cdot 0.8} = 3$ |
| $\sum = 11.853$ | $\sum = 9.614$ |

U holda $\int_0^1 \sqrt{1+10x}dx \cong \frac{1}{30}(4.317+4\cdot 11.853+2\cdot 9.614)=\int_0^1 \sqrt{1+10x}dx=$

$$=\frac{1}{10}\int_0^1 (1-10x)^{\frac{1}{2}}d(1+10x)=0.1\cdot\frac{(1+10x)^{\frac{3}{2}}}{\frac{3}{2}}\Bigg|_0^1=0.2\cdot\frac{1}{3}\left[11^{\frac{3}{2}}-1\right]=0.2\cdot\frac{1}{3}\left(11\sqrt{11}-1\right)=$$

$$=\frac{1}{30}(4.317+47.412+19.228)=\frac{1}{30}\cdot 70.957=2.3652$$

Agar yuqoridagi integralning aniq yechimi 2.3658 ekanligini inobatga olsak, Simpson usuli bilan hosil qilingan taqribiy qiymatning juda yaxshi aniqlikda hisoblanganligini ko'ramiz.

## MASHQLAR

Quyidagi integrallar har uchchala taqribiy usulda ham hisoblansin (integrallash kesmasi n=10 teng bo'lakka bo'linsin)

1. $\int_1^2 \frac{dx}{x}$
2. $\int_0^{0.8} Cosxdx$
3. $\int_0^1 \frac{arctgxdx}{x}$
4. $\int_0^1 e^{-x^2}dx$
5. $\int_0^{\pi/2} \frac{Sinx}{x}dx$
6. $\int_0^1 \frac{dx}{1+x}$
7. $\int_0^4 \sqrt{2+x^3}dx$
8. $\int_1^5 \frac{dx}{\sqrt{8+x^3}}$
9. $\int_0^3 \frac{dx}{\sqrt{7x+3}}$
10. $\int_0^3 \frac{dx}{9+x^2}$
11. $\int_0^1 \frac{dx}{1+x}$
12. $\int_0^1 \frac{dx}{1+x^2}$
13. $\int_0^4 e^x dx$
14. $\int_1^7 \frac{dx}{2+\lg x}$

## 2.4. Xosmas integrallar

Ma'lumki, $\int_a^b f(x)dx$ aniq integral tushunchasi kiritilganda, integrallash kesmasi *[a;b]* ning chekli ekanligi va integrallanuvchi funksiya *f(x)* ning mazkur kesmada uzluksizligi faraz qilingan edi. Agar ushbu shartlardan birortasi bajarilmasa, «odatdagidek» aniq integral tushunchasini kiritib bo'lmaydi. Shu ma'noda qaralganda bu holdagi integral «odatdagidek bo'lmagan» ya'ni xosmas integral deb yuritiladi.

Quyida ularning ikki xilini ko'rib chiqamiz.

### 2.4.1. Chegaralari cheksiz xosmas integrallar

Aytaylik, $f(x)$ funksiya $[a; +\infty)$ da uzluksiz bo'lsin. U holda mazkur funksiyaning $[a; +\infty)$ dagi xosmas integrali $\int_a^{+\infty} f(x)dx$, quyidagi tenglik bilan aniqlanadi:

$$\int_a^{+\infty} f(x)dx = \lim_{b \to +\infty} \int_a^b f(x)dx$$

Agar ushbu tenglikning o'ng tomonidagi limit mavjud bo'lsa, u holda xosmas integral yaqinlashuvchi deyiladi va limit integralning qiymati sifatida qabul qilinadi.

Agarda ko'rsatilgan limit mavjud bo'lmasa, xosmas integral uzoqlashuvchi deyiladi.

$\int_{-\infty}^{a} f(x)dx$ kabi xosmas integral ham aynan yuqoridagiga o'xshash aniqlanadi.

$$\int_{-\infty}^{b} f(x)dx = \lim_{a \to -\infty} \int_a^b f(x)dx$$

Shuningdek, agar $f(x)$ funksiya $(-\infty;+\infty)$ da uzluksiz bo'lsa, u holda umumlashgan xosmas integral quyidagicha aniqlanadi:

$$\int_{-\infty}^{+\infty} f(x)dx = \int_{-\infty}^{a} f(x)dx + \int_a^{+\infty} f(x)dx.$$

Ushbu tenglikning o'ng tomonidagi har ikkala xosmas integral ham yaqinlashuvchi bo'lsa, u holda chap tomondagi xosmas integral ham yaqinlashuvchi bo'ladi. Aksincha, agar o'ng tomondagi xosmas integrallardan hech bo'lmaganda birortasi uzoqlashuvchi bo'lsa, o'ng tomondagi xosmas integral ham uzoqlashuvchi bo'ladi.

**1-Misol.** $\int_1^{+\infty} \frac{dx}{x^2}$

**Yechish.** $\int_1^{+\infty} \frac{dx}{x^2} = \lim_{b \to +\infty} \int_1^b \frac{dx}{x^2} = -\lim_{b \to +\infty} \frac{1}{x}\Big|_1^b = -\lim_{b \to +\infty} (\frac{1}{b} - 1) = 1 - \frac{1}{\infty} = 1 - 0 = 1.$

Demak, mazkur xosmas integral yaqinlashuvchi va qiymati 1 ga teng ekan.

**2-Misol.** $\int_{-\infty}^{-1} \frac{dx}{x^2 + 2x + 5}$

**Yechish.** $\int_{-\infty}^{-1} \frac{dx}{x^2 + 2x + 5} = \lim_{a \to -\infty} \int_a^{-1} \frac{dx}{(x+1)^2 + 4} = \lim_{a \to -\infty} \frac{1}{2} arctg \frac{x+1}{2}\Big|_a^{-1} =$

$= \frac{1}{2} \lim_{a \to -\infty} (arctg 0 - arctg \frac{a+1}{2}) = \frac{1}{2} arctg \infty = \frac{1}{2} \cdot \frac{\pi}{2} = \frac{\pi}{4}.$

**3-Misol.** $\int\limits_0^{+\infty} Cosxdx$

<u>**Yechish.**</u> $\int\limits_0^{+\infty} Cosxdx = \lim\limits_{b\to+\infty}\int\limits_0^b Cosxdx = \lim\limits_{b\to+\infty} Sinx\Big|_0^b = \lim\limits_{b\to+\infty}(Sinb - Sin0) = \lim\limits_{b\to+\infty} Sinb.$

Ammo, oxirgi limit mavjud emas. SHuning uchun mazkur xosmas integral uzoqlashuvchidir.

**4-Misol.** $\int\limits_{-\infty}^{+\infty} xe^{-x^2}dx$ hisoblansin.

<u>**Yechish.**</u> Ta'rifga asosan,

$$\int\limits_{-\infty}^{+\infty} xe^{-x^2}dx = \int\limits_{-\infty}^{0} xe^{-x^2}dx + \int\limits_{0}^{+\infty} xe^{-x^2}dx = \lim\limits_{a\to-\infty}\int\limits_a^0 xe^{-x^2}dx + \lim\limits_{b\to+\infty}\int\limits_0^b xe^{-x^2}dx = -\frac{1}{2}\lim\limits_{a\to-\infty} e^{-x^2}\Big|_a^0 - \frac{1}{2}\lim\limits_{a\to+\infty} e$$

$$= -\frac{1}{2}\left[\lim\limits_{a\to-\infty}(1-e^{-a^2}) + \lim\limits_{b\to+\infty}(e^{-b^2}-1)\right] = -\frac{1}{2}(1-e^{-\infty}+e^{-\infty}-1) = 0.$$

### 2.4.2 Chegaralanmagan (cheksiz) funksiyalarning xosmas integrallari

Agar $f(x)$ funksiya (s; b] yarim ochiq oraliqda uzluksiz bo'lib, x=c chap chegaraviy nuqtada aniqlanmagan yoki 2-tur uzilishga ega bo'lsa, mazkur funksiyaning xosmas integrali $\int\limits_c^b f(x)dx$ kabi belgilanib, u quyidagi tenglik bilan aniqlanadi:

$$\int\limits_c^b f(x)dx = \lim\limits_{a\to c-0}\int\limits_a^b f(x)dx$$

Ushbu tenglikning o'ng tomonidagi limit mavjud bo'lsa, u holda xosmas integral yaqinlashuvchi deyilib, aks holda esa uzoqlashuvchi deyiladi.

Agar $f(x)$ funksiya [a;c) yarim ochiq oraliqda uzluksiz bo'lib, x=c o'ng chegaraviy nuqtada aniqlanmagan yoki 2-tur uzilishga ega bo'lsa, uning xosmas integrali ham yuqoridagidek aniqlanadi.

$$\int\limits_a^c f(x)dx = \lim\limits_{b\to c-0}\int\limits_a^b f(x)dx$$

Umuman, [a; b] kesmaning biron bir x=c oraliq nuqtasida aniqlanmagan yoki cheksiz uzilishga ega bo'lgan $f(x)$ funksiyaning xosmas integrali quyidagicha aniqlanadi:

$$\int\limits_a^b f(x)dx = \int\limits_a^c f(x)dx + \int\limits_c^b f(x)dx$$

Bu xosmas integralning yaqinlashuvchi bo'lishi uchun tenglikning o'ng tomonidagi har ikkala xosmas integrallar ham yaqinlashuvchi bo'lishlari lozim, aks holda, ya'ni ulardan ikkala bittasi uzoqlashuvchi bo'lsa ham mazkur xosmas integral uzoqlashuvchi bo'ladi.

**Eslatma:** Odatda, 1-bandda qaralgan xosmas integrallarni I-tur xosmas integrallar va 2-bandda qaralganlarni II-tur xosmas integrallar ham deb yuritiladi.

Quyidagi misollarda berilgan 2-tur xosmas integrallar tekshirilsin.

**1-Misol.** $\int_0^2 \frac{dx}{\sqrt[3]{x^2}}$

**Yechish.** $\int_0^2 \frac{dx}{\sqrt[3]{x^2}} = \lim_{a\to+0}\int_a^2 \frac{dx}{x^{2/3}} = 3\lim_{a\to+0}\sqrt[3]{x}\Big|_a^2 = 3\lim_{a\to+0}(\sqrt[3]{2}-\sqrt[3]{a}) = 3\sqrt[3]{2}.$

integral yaqinlashuvchidir.

**2-Misol.** $\int_1^3 \frac{dx}{3-x}$

**Yechish.**

$$\int_1^3 \frac{dx}{3-x} = \lim_{b\to 3-0}\int_1^b \frac{dx}{3-x} = -\lim_{b\to 3-0}\ln|3-x|\Big|_1^b = -\lim_{b\to 3-0}\left[\ln|3-b|-\ln 2\right] = \ln 2-\ln 0 = \ln 2+\infty = \infty.$$

integral uzoqlashuvchi ekan.

**3-Misol.** $\int_{-\pi/2}^{\pi/2} \frac{dx}{Sinx}$

**Yechish.** $\int_{-\pi/2}^{\pi/2} \frac{dx}{Sinx}$ kabi 2-tur xosmas integraldagi integrallanuvchi funksiya x=0 da aniqlanmagan. Shuning uchun uni ikkita xosmas integrallarning yig'indisi shaklida ifodalab olamiz:

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{dx}{\sin x} = \int_{-\frac{\pi}{2}}^{0} \frac{dx}{\sin x} + \int_{0}^{\frac{\pi}{2}} \frac{dx}{\sin x};$$

o'ng tomondagi xosmas integrallarni tekshiramiz:

$$\int_{-\frac{\pi}{2}}^{0} \frac{dx}{\sin x} = \lim_{a\to-\frac{\pi}{2}+0}\int_a^0 \frac{dx}{\sin x} = \lim_{a\to-\frac{\pi}{2}+0} \ln\left|tg\frac{x}{2}\right|\Big|_a^0 = \lim_{a\to-\frac{\pi}{2}+0}\left[\ln 0-\ln\left|tg\frac{a}{2}\right|\right] = -\infty.$$

Ikkinchi xosmas integralni tekshirishning hojati yo'q, chunki birinchi qo'shiluvchi xosmas integral uzoqlashuvchi ekan. Demak, mazkur xosmas integral ham uzoqlashuvchidir.

**4-Misol.**

$$\int_0^3 \frac{dx}{\sqrt{4-x^2}} = \int_0^2 \frac{dx}{\sqrt{4-x^2}}9 + \int_2^3 \frac{dx}{\sqrt{4-x^2}} = \lim_{b\to2-0}\int_0^b \frac{dx}{\sqrt{4-x^2}} + \lim_{a\to2+0}\int_a^3 \frac{dx}{\sqrt{4-x^2}} =$$

$$= \lim_{b\to2-0}\arcsin\frac{x}{2}\Big|_0^b + \lim_{a\to2+0}\arcsin\frac{x}{2}\Big|_a^3 = \lim_{b\to2-0}\left(\arcsin\frac{b}{2}-\arcsin 0\right)+$$

$$+\lim_{a\to 2+0}\left(\arcsin\frac{3}{2}-\arcsin\frac{a}{2}\right)=\arcsin\frac{2-0}{2}+\arcsin\frac{3}{2}-\arcsin\frac{2+0}{2}=\arcsin\frac{3}{2}.$$

Qaralayotgan xosmas integral uzoqlashuvchi, chunki arcsin1,5 mavjud emas.

## MASHQLAR

1. $\int\limits_{e^2}^{\infty}\frac{dx}{x\ln^2 x}$; 6. $\int\limits_{-\infty}^{+\infty}\frac{dx}{4+x^2}$; 11. $\int\limits_{0}^{1}\frac{dx}{\sqrt{x}}$; 16. $\int\limits_{0}^{\frac{\pi}{2}}\frac{dx}{\cos x}$;

2. $\int\limits_{-\infty}^{+\infty}\frac{dx}{x^2-6x+10}$; 7. $\int\limits_{1}^{\infty}\frac{xdx}{3+x^2}$; 12. $\int\limits_{0}^{4}\frac{dx}{\sqrt{4-x}}$; 17. $\int\limits_{1}^{e}\frac{dx}{x\sqrt[3]{\ln x}}$;

3. $\int\limits_{0}^{\infty}x\sin x dx$; 8. $\int\limits_{2}^{\infty}\frac{x^2dx}{x^2+9}$; 13. $\int\limits_{0}^{1}\frac{dx}{\sqrt{1-x^2}}$; 18. $\int\limits_{2}^{4}\frac{dx}{\sqrt[3]{(4-x)^2}}$;

4. $\int\limits_{1}^{\infty}\frac{dx}{x(1+x^2)}$; 9. $\int\limits_{0}^{\infty}\frac{xdx}{(x+1)^3}$; 14. $\int\limits_{1}^{2}\frac{dx}{x\ln x}$; 19. $\int\limits_{-3}^{3}\frac{dx}{\sqrt{9-x^2}}$;

5. $\int\limits_{0}^{\infty}\frac{xdx}{\sqrt{(x^2+1)^3}}$; 10. $\int\limits_{1}^{\infty}\frac{\sqrt{x}dx}{(1+x)^2}$; 15. $\int\limits_{0}^{1}x\ln x dx$; 20. $\int\limits_{0}^{2}\frac{xdx}{\sqrt{(x^2-4)^3}}$;

### Taqqoslash teoremalari.

Agar *f(x)* funksiyaga boshlang'ich funksiyani topish qiyin bo'lsa, yoki umuman topish mumkin bo'lmasa, u holda xosmas integrallarni tekshirish, taqqoslash teoremalari orqali hal qilinishi mumkin.

1. Agar *f(x)* va φ(x) funksiyalar $[a;+\infty)$da $0\le f(x)\le\varphi(x)$ shartni qanoatlantirib, $\int\limits_{a}^{+\infty}\varphi(x)dx$ yaqinlashuvchi bo'lsa, $\int\limits_{a}^{+\infty}f(x)dx$ ham yaqinlashuvchi bo'ladi, aksincha, agar $\int\limits_{a}^{+\infty}f(x)dx$ uzoqlashuvchi bo'lsa, $\int\limits_{a}^{+\infty}\varphi(x)dx$ ham uzoqlashuvchi bo'ladi.

2. Agar *f(x)* funksiyaning $[a;+\infty)$ da ishorasi o'zgaruvchi bo'lsa, u holda $\int\limits_{a}^{+\infty}f(x)dx$ ning yaqinlashuvchi bo'lishi uchun $\int\limits_{a}^{+\infty}|f(x)|dx$ yaqinlashuvchi bo'lishi lozim. Bunda $\int\limits_{a}^{+\infty}f(x)dx$ ni mutloq yaqinlashuvchi

deb yuritiladi. Agar $\int_a^{+\infty} f(x)dx$ yaqinlashuvchi bo'lib, $\int_a^{+\infty} |f(x)|dx$ uzoqlashuvchi bo'lsa, $\int_a^{+\infty} f(x)dx$ ni shartli yaqinlashuvchi deb ataladi.

Yuqorida bayon etilgan teoremalar $\int_{-\infty}^{b} f(x)dx$, va umuman $\int_{-\infty}^{+\infty} f(x)dx$ kabi integrallar uchun ham o'rinlidir. Shuningdek 2-tur xosmas integrallarni tekshirish uchun ham mazkur taqqoslash teoremalaridan foydalaniladi.

Amaliyotda taqqoslash funksiyalari sifatida ko'pincha ko'rsatkichli $e^{-\alpha}$ va darajali $x^{-m}$ funksiyalar qo'llaniladi.

**1-Misol.** $\int_0^{+\infty} \frac{Cosxdx}{x^2}$ tekshirilsin.

__Yechish.__ $\left|\frac{Cosx}{x^2}\right| \le \frac{1}{x^2}$ bo'lganligi va $\int_a^{+\infty} \frac{dx}{x^2}$ yaqinlashuvchi bo'lganligi uchun mazkur xosmas integral nafaqat yaqinlashuvchi, balki mutloq yaqinlashuvchidir.

**2-Misol.** $\int_0^{+\infty} \frac{x^2dx}{4+x^2}$

__Yechish.__ $\int_0^{+\infty} \frac{x^2dx}{4+x^2}$ ni tekshirish uchun $\int_0^{+\infty} \frac{xdx}{4+x^2}$ ni tekshiramiz.

$$\frac{x^2}{4+x^2} \ge \frac{x}{4+x^2}$$

bo'lganligidan va $\int_0^{+\infty} \frac{xdx}{4+x^2}$ ning uzoqlashuvchanligidan, mazkur xosmas integral ham uzoqlashuvchidir.

**Quyidagi xosmas integrallar tekshirilsin.**

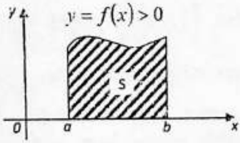
1. $\int_0^{+\infty} \frac{Cosxdx}{1+x^2}$
2. $\int_0^{+\infty} \frac{Sinxdx}{1+x^2}$
3. $\int_0^{+\infty} \frac{Sinx}{x}dx$
4. $\int_1^{+\infty} \frac{Cosx}{x}dx$
5. $\int_1^{+\infty} \frac{\sqrt{x}dx}{1+x}$

## 2.5. Aniq integralning tadbiqlari

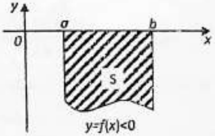
### 2.5.1. Yassi figuralarning yuzlarini hisoblash

1. Ma'lumki, agar [a; b] kesmada $f(x) \ge 0$ bo'lsa, mazkur funksiyaning mazkur kesma bo'yicha aniq integrali geometrik jihatdan

$x=a$ va $x=b$ to'g'ri chiziqlar, $Ox$ o'qi hamda $y=f(x)$ egri chiziqlar bilan chegaralangan egri chiziqli trapetsiyaning yuzini ifodalar edi.

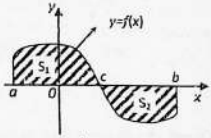



Agar [a; b] kesmada $f(x)\le 0$ bo'lsa, mazkur egri chiziqli trapetsiya $Ox$ o'qidan pastda joylashadi va uning yuzi $S=\int_a^b |f(x)|dx$ formula orqali



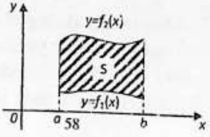

hisoblanadi.

Agar $y=f(x)$ egri chiziq, $Ox$ o'qi va $x=a$, $x=b$ to'g'ri chiziqlar bilan chegaralangan figura $Ox$ o'qidan yuqorida va pastda joylashgan bo'lsa,



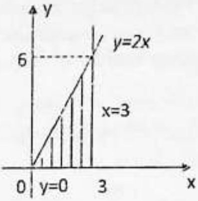

uning yuzi $S=S_1+S_2=\int_a^c f(x)dx+\int_c^b |f(x)|dx$ formula bilan hisoblanadi.

Agar $x=a$, $x=b$ to'g'ri chiziqlar, $y=f_1(x)$ va $y=f_2(x)$ egri chiziqlar $(f_1(x)\le f_2(x))$ bilan chegaralangan yuzani hisoblash lozim bo'lsa, uni ushbu $S=\int_a^b [f_2(x)-f_1(x)]dx$ formula bilan hisoblanadi.

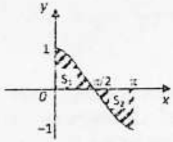


**1-Misol.** Tenglamalari $y=2x$, $y=0$ va $x=3$ bo'lgan to'g'ri chiziqlar bilan chegaralangan figuraning yuzi hisoblansin.

<u>**Yechish.**</u> $S=\int_0^3 2xdx=2\int_0^3 xdx=x^2\Big|_0^3=9$ кв. б.




**2-Misol.** $y=cosx$, $y=0$ chiziqlar bilan chegaralangan figuraning yuzi hisoblansin, bunda $x\in[0;\pi]$.

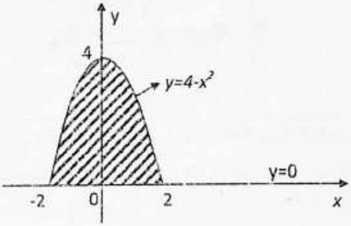



<u>**Yechish.**</u>

$$S=S_1+S_2=\int_0^{\pi/2}\cos xdx+\int_{\pi/2}^{\pi}|\cos x|dx=\sin x\Big|_0^{\pi/2}+|\sin x|\Big|_{\pi/2}^{\pi}=1+1=2\text{ кв. б.}$$

**3-Misol.** $y=0$ va $y=-x^2+4$ chiziqlar bilan chegaralangan figuraning yuzi hisoblansin.

<u>Yechish.</u> Bu yerda hisoblanishi lozim bo'lgan yuza $y=4-x^2$ parabola bilan $Ox$ o'q



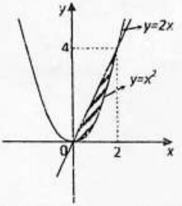
orasida joylashgan. Agar $y=0$ bo'lsa, $x=\pm 2$.

$$S=\int_{-2}^{2}(4-x^2)dx=2\int_{0}^{2}(4-x^2)dx=2\left[4x-\frac{x^3}{3}\right]_0^2=2\left(8-\frac{8}{3}\right)=\frac{32}{3}=10\frac{2}{3}\text{ кв. б.}$$

**4-Misol.** Tenglamalari $y=x^2$ va $y=2x$ bo'lgan parabola va to'g'ri chiziq orasida joylashgan yuza hisoblansin.

<u>**Yechish.**</u> $y=x^2$ va $y=2x$ tenglamalarni birgalikda yechib, parabola bilan to'g'ri chiziqning kesishish nuqtalarining abstsissalari $x_1=0$ va $x_2=2$ larni topamiz.

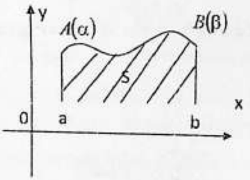
$$S=\int_{0}^{2}(2x-x^2)dx=\left(x^2-\frac{x^3}{3}\right)\Big|_0^2=4-\frac{8}{3}=\frac{4}{3}=1\frac{1}{3}\text{ кв. б.}$$

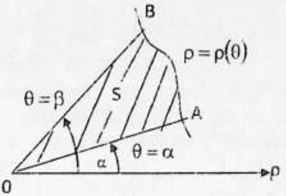



a) Agar egri chiziqli trapetsiyaning yuzi $x=x(t)$, $y=y(t)$ kabi parametrik shaklda berilgan egri chiziq, $x=a$ va $x=b$ to'g'ri chiziqlar hamda $Ox$ o'q bilan chegaralangan bo'lsa, u yuza quyidagi formula bilan hisoblanadi

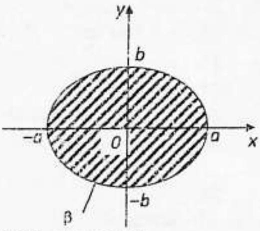
$$S=\int_{\alpha}^{\beta}y(t)x'(t)dt$$

bunda $\alpha\le t\le\beta$ va $x(\alpha)=a,\ x(\beta)=b$






b) Agar $(\rho;\theta)$ qutb koordinatlari sistemasida biror uzluksiz egri chiziq o'zining $\rho=\rho(\theta)$ kabi tenglamasi orqali berilgan bo'lsa, u holda $\theta=\alpha,\ \theta=\beta$ qutb radiuslari hamda egri chiziqning $AB$ yoyi bilan




chegaralangan $AOB$ sektor yuzi

$$S=\frac{1}{2}\int_{\alpha}^{\beta}\rho^2 d\theta$$

formula bilan hisoblanadi.

**5-Misol.** $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ ellips bilan chegaralangan yuza topilsin.

__Yechish.__ Ellipsning parametrik tenglamasini yozib olamiz $x=a\cos t,\ y=b\sin t$ . U holda

$$S=\int_0^{2\pi}b\sin t(-a\sin t)dt=-ab\int_0^{2\pi}\sin^2 t dt=\frac{ab}{2}\int_0^{2\pi}(\cos 2t-1)dt=\frac{ab}{2}\left[\frac{\sin 2t}{2}\Big|_0^{2\pi}-t\Big|_0^{2\pi}\right]=$$

$=-\pi ab.$

$$S=|-\pi ab|=\pi ab \text{ kv.b}$$

**6-Misol.** $\rho^2=2\cos 2\theta$ lemniskata bilan chegaralangan yuza topilsin.

__Yechish.__ Izlanayotgan yuzaning to'rtdan bir qismiga $0\le\theta\le\frac{\pi}{4}$ burchak mos keladi. Shuning uchun

$$S=\frac{1}{2}\int_0^{\pi/4}2\cos 2\theta d\theta=2\sin 2\theta\Big|_0^{\pi/4}=2 \text{ кв. б.}$$

## MASHQLAR

Quyidagi chiziqlar bilan chegaralangan yuzalar hisoblansin.

1. $y=0, y=3-2x+x^2;$
2. $x=e, y=0, y=\ln x;$
3. $y=0; x\pm 1, y=x^2-2x;$
4. $y=x+4, y=x^2+4x$
5. $y=0; y=4x-x^2;$
6. $x=0, y=8, y=x^3;$
7. $y=-x; y=2x-x^2;$
8. $x=1, y=e^{-x}, y=e^{x};$

. $x+y=3, y=1+x^2;$

10. $x=4, y=0, y=3x^2-6x;$
11. $y=0, y=x^2+6x+5;$
12. $x=1, x=2, y=2x^2;$
13. $x+y=4, xy=3;$
14. $y=3x, y^2=9x;$
15. $y=0, y=x^3-4x;$
16. $x+2y-=0, y=0, x=-3, x=2;$
17. $x=2, x=3, y=0, y=x^2;$
18. $y=x, y\geq 0, x=1, x=4;$
19. $y=\sin x, y=0, x=0, x=\pi;$
20. $y=-6x, y=0, x=4;$
21. $\rho=2\cos\varphi;$
22. $\rho=3\cos 2\varphi;$
23. $\rho=\cos 3\varphi;$
24. $\rho=2\cos 4\varphi;$
25. $\rho=2, \rho=2(1-\cos\theta);$
26. $\rho=3\sqrt{2}\cos\theta, \rho=3\sin\theta;$
27. $x=\frac{t}{3}(6-t), y=\frac{t^2}{8}(6-t)$ egri chiziq sirtmog'i
28. $x=t^2, y=t-\frac{t^3}{3};$
29. $x=t^2, y=t-\frac{t^3}{3};$
30. $Ox$ o'qi va $x=2(t-\sin t), y=2(1-\cos t)$ sikloidaning bir arkasi.

### 2.5.2. Egri chiziq yoyining uzunligini hisoblash

1. Agar tekis egri chiziq uzining $y=f(x)$ tenglamasi bilan berilgan bo'lib, $y'=f'(x)$ hosila uzluksiz bo'lsa, u holda egri chiziqning $[a;b]$ kesmaga mos keluvchi yoyining uzunligi

$$l=\int_a^b\sqrt{1+y'^2}\,dx$$

formula bilan hisoblanadi.

2. Egri chiziq o'zining $x=x(t)$ va $y=y(t)$ kabi parametrik shakldagi tenglamasi bilan berilgan bo'lsin. Agar $x'(t)$ va $y'(t)$ hosilalar $[\alpha;\beta]$ kesmada uzluksiz bo'lsalar, mazkur egri chiziqning $[\alpha;\beta]$ kesmaga mos keluvchi yoyining uzunligi

$$l=\int_\alpha^\beta\sqrt{x'^2(t)+y'^2(t)}\,dt$$

formula orqali hisoblanadi. Bu yerda $\alpha\le t\le\beta$.

3. Aytaylik, egri chiziqning tenglamasi qutb koordinatalarida sistemasida $\rho=\rho(\theta)$ tenglama bilan berilgan bo'lsin. U holda uning biror $AB$ yoyining uzunligi

$$l=\int_\alpha^\beta\sqrt{\rho^2+\rho'^2}\,d\theta$$

formula bilan hisoblanadi. Bu yerdagi $\alpha$ va $\beta$ lar qutb burchaklarining $AB$ yoy uchlariga mos keluvchi qiymatlaridir.

**1-Misol.** $y=\frac14x^2-\frac12\ln x$ funksiya bilan berilgan egri chiziqning abstsissalari $x=1$ va $x=2$ bo'lgan nuqtalari orasidagi yoyining uzunligi hisoblansin.

__Yechish.__ $y'=\frac12x-\frac1{2x}$ va $(y')^2=\frac14x^2-\frac12+\frac1{4x^2}$ bo'lganligidan, $1+y'^2=\left(\frac x2+\frac1{2x}\right)^2$ dan iboratdir.

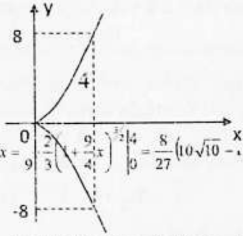
U holda

$$l=\int_1^2\sqrt{1+y'^2}\,dx=\int_1^2\left(\frac x2+\frac1{2x}\right)dx=\left(\frac{x^2}4+\frac12\ln x\right)\Bigg|_1^2=1+\frac12\ln2-\frac14=\frac34+\frac12\ln2$$

**2-Misol.** Tenglamasi $y^2=x^3$ bo'lgan yarim kubik parabolaning (0; 0) va (4;8) nuqtalar orasidagi yoyining uzunligi hisoblansin.

__Yechish.__ Berilgan nuqtalar I-chorakda joylashganliklari uchun $y=\sqrt{x^3}$,

$$y'=\frac32\sqrt x \text{ ва } \sqrt{1+y'^2}=\sqrt{1+\frac94x}\text{ дир.}$$




$$l=\int_0^4\sqrt{1+\frac94x}\,dx=\frac49\cdot\frac23\left(1+\frac94x\right)^{3/2}\Bigg|_0^4=\frac8{27}\left(10\sqrt{10}-1\right)$$

**3-Misol.** $\rho=2(1+\cos\theta)$ kardioidaning $0\le\theta\le\pi$ ga mos keluvchi yoyining uzunligi hisoblansin.

**Yechish.** $\rho' = -2\sin\theta$ ligidan,

$$l = \int_0^{\pi} \sqrt{4(1+\cos\theta)^2 + 4\sin^2\theta}\,d\theta = \int_0^{\pi} \sqrt{4\left[1+2\cos\theta+\cos^2\theta+\sin^2\theta\right]}\,d\theta =$$

$$= 2\int_0^{\pi} \sqrt{2(1+\cos\theta)}\,d\theta = 4\int_0^{\pi} \cos\frac{\theta}{2}\,d\theta = 8\sin\frac{\theta}{2}\Big|_0^{\pi} = 8.$$

## MASHQLAR

Quyidagi tenglamalar orqali berilgan egri chiziqlarning ko'rsatilgan yoylarining uzunliklari hisoblansin.

1. $y = 2\sqrt{x}$; $x = 0$ дан $x = 1$ гача;
2. $y = \ln x$; $x = \sqrt{3}$ дан $x = \sqrt{8}$ гача;
3. $y = 4 - x^2$; $Ox$ o'q bilan kesishish nuqtalari orasidagi qismi;
4. $x^{2/3} + y^{2/3} = \sqrt[3]{4}$ sikloidaning butun uzunligi;
5. $x = 4(t - \sin t)$, $y = 4(1 - \cos t)$ sikloidaning bir arkasi;
6. $y = 1 - \ln\cos x$; $x = 0$ дан $x = \dfrac{\pi}{4}$ гача.
7. $\rho = 2\sin^3\dfrac{\rho}{3}$ ning butun uzunligi;
8. $x = \dfrac{1}{6}t^6$ ва $y = 2 - \dfrac{1}{4}t^4$ larning Ox o'q bilan kesishish nuqtalari orasidagi bo'lagi;
9. $x = (t^2 - 2)\sin t + 2t\cos t$, $y = (2 - t^2)\cos t + 2t\sin t$ нинг $t_1 = 0$ ва $t_2 = \pi$ гач bo'lagi;
10. $\rho = 3\theta$ Arximed spiralining birinchi o'rami.

### 2.5.3. Aylanma sirtlarning yuzi va aylanma jismlarning hajmlarini hisoblash

1. Aytaylik, egri chiziq $y=f(x)$ tenglama orqali berilgan bo'lsin. Agar bu funksiya [a;b] kesmada uzluksiz differensiallanuvchi bo'lib, egri chiziqning [a;b] ga mos keluvchi $AB$ yoyi $Ox$ o'q atrofida aylantirilsa, u holda hosil bo'ladigan aylanma sirtning yuzi

$$S_x = 2\pi\int_a^b f(x)\sqrt{1 + f'^2(x)}\,dx$$

formula bilan hisoblanadi.

Agar egri chiziq boshqa xildagi tenglamalari bilan berilgan bo'lsa, u holda mazkur yuzani hisoblash uchun yuqoridagi formuladagi har bir holga mos keluvchi almashishlarni bajarish yetarlidir.

2. a) Egri chiziqli trapetsiya $y=f(x)$ egri chiziq $x=a$ va $x=b$ vertikal to'g'ri chiziqlar hamda $Ox$ o'q bilan chegaralangan bo'lsin. U holda uni $Ox$ va $Oy$ o'qlar atrofida aylantirishdan hosil bo'ladigan aylanma jismlarning hajmlari mos ravishda quyidagi formulalar bilan hisoblanadi.

$$V_x = \pi\int_a^b y^2dx;\ V_y = 2\pi\int_a^b xydx.$$

Bu yerda ham, agar egri chiziq o'zining $y=f(x)$ kabi tenglamasidan boshqa xildagi tenglamalari bilan berilgan bo'lsa, yuqoridagi formulada kerakli almashtirishlar bajariladi.

b) $y_1 = f_1(x)$ va $y_2 = f_2(x)(f_1(x)\le f_2(x))$ egri chiziqlar, $x=a$ va $x=b$ to'g'ri chiziqlar bilan chegaralangan figuraning $Ox$ va $Oy$ o'qlari atrofida aylanishidan hosil bo'lgan aylanma jismlarning hajmlari mos ravishda

$$V_x = \pi\int_a^b \left(y_1^2 - y_2^2\right)dx$$

$$V_y = 2\pi\int_a^b x(y_2 - y_1)dx$$

kabi formulalar bilan hisoblanadi.

**1-Misol.** Tenglamasi $y=\frac{1}{2}\sqrt{4x-1}$ bo'lgan egri chiziq yoyining $x_1=1$ dan $x_2=9$ gacha bo'lgan qismi $Ox$ atrofida aylanishidan hosil bo'lgan aylanma sirt yuzi hisoblansin.

**Yechish.** $y'=\frac{1}{2}\cdot\frac{4}{2\sqrt{4x-1}}=\frac{1}{\sqrt{4x-1}}$ va $\sqrt{1+y'^2}=\sqrt{1+\frac{1}{4x-1}}=\frac{2\sqrt{x}}{\sqrt{4x-1}}$ larni inobatga olsak

$$S_x = 2\pi\int_1^9 \frac{1}{2}\sqrt{4x-1}\cdot\frac{2\sqrt{x}}{\sqrt{4x-1}}dx = 2\pi\int_1^9\sqrt{x}dx = 2\pi\frac{2}{3}x^{\frac{3}{2}}\Big|_1^9 = \frac{104\pi}{3}.$$

**2-Misol.** $y^2=4+x$ parabola yoyining abtsissalari $x_1=-4$ va $x_2=2$ bo'lgan nuqtalari orasida joylashgan bo'lagining Ox o'qi atrofida aylanishidan hosil bo'lagi aylanma sirtning yuzi aniqlansin.

**Yechish.**

$$y=\sqrt{4+x},\ y'=\frac{1}{2\sqrt{4+x}},\ y'^2=\frac{1}{4(4+x)}$$

larga asosan

$$S_x = 2\pi\int_{-4}^{2}\sqrt{4+x}\sqrt{1+\frac{1}{4(4+x)}}dx = 2\pi\int_{-4}^{2}\sqrt{4+x+\frac{1}{4}}dx = 2\pi\int_{-4}^{2}\sqrt{x+\frac{17}{4}}dx =$$

$$= 2\pi\int_{1/2}^{5/2}2t^2dt = \frac{4\pi}{3}t^3\Big|_{1/2}^{5/2} = \frac{4\pi}{3}\left(\frac{125}{8}-\frac{1}{8}\right) = \frac{4\pi\cdot 124}{3\cdot 8} = \frac{62\pi}{3} \text{ кв. б.}$$

Bu integralni hisoblashda $x+\frac{17}{4}=t^2, dx=2tdt, \frac{1}{2}\le t\le\frac{5}{2}$ almashtiris bajarildi.

**3-Misol.** $y=\frac{x^2}{2}$ parabolaning $y=\frac{3}{2}$ to‘g‘ri chiziq bilan kesilga bo‘lagining *Oy* o‘q atrofida aylanishidan hosil bo‘lgan sirt yuz hisoblansin.

__**Eslatma.**__ Agarda $x=\varphi(y)$ silliq egri chiziqnng yoyi Oy o‘q atrofida aylansa, aylanma sirtning yuzi

$$S_y = 2\pi\int_c^d x\sqrt{1+x'^2}dy$$

formuladan topiladi.

__**Yechish.**__ $x=\sqrt{2y}, x'=\frac{1}{\sqrt{2y}}, (x')^2=\frac{1}{2y}, 0\le y\le\frac{3}{2}$ largi ko‘ra,

$$S_y = 2\pi\int_0^{3/2}\sqrt{2y}\sqrt{1+\frac{1}{2y}}dy = 2\pi\int_0^{3/2}\sqrt{2y+1}dy = 2\pi\int_1^2 t^2dt = \frac{2\pi}{3}t^3\Big|_1^2 = \frac{2\pi}{3}(8-1) = \frac{14\pi}{3}$$

bu integralni hisoblashda $2y+1=t^2, dy=tdt, 1\le\tau\le 2$ almashtirish bajarildi.

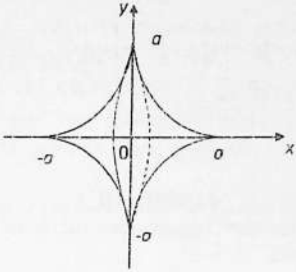
**4-Misol.** $x=a\cos^3 t, y=a\sin^3 t$ astroidaning Ox o‘qi atrofida aylanishidan hosil bo‘lgan aylanma sirtning yuzi aniqlansin.

__**Yechish.**__ $x'=-3a\cos^2 t\sin t, y'=3a\sin^2 t\cos t.$

$$S_x = 2\cdot 2\pi\int_0^{\pi/2}a\sin^3 t\sqrt{9a^2\cos^4 t\sin^2 t+9a^2\sin^4 t\cos^2 t}dt =$$

$$= 4\pi a\int_0^{\pi/2}\sin^3 t\sqrt{9a^2\cos^2 t\sin^2 t(\cos^2 t+\sin^2 t)}dt = 12\pi a^2\int_0^{\pi/2}\sin^4 t\cos t dt =$$

$$= 12\pi a^2\int_0^{\pi/2}\sin^4 t d\sin t = \frac{12}{5}\pi a^2\sin^5 t\Big|_0^{\pi/2} = 2,4\pi a^2 \text{ кв б.}$$




**5-Misol.** $\rho=\sqrt{\cos 2\theta}$ lemniskataning $\left(0\le\theta\le\frac{\pi}{4}\right)$ qutb o'qi atrofida aylanishidan hosil bo'lgan aylanma sirtning yuzi hisoblansin.

<u>**Yechish.**</u> $y=\rho\sin\theta$ bo'lganligidan, $y=\sin\theta\sqrt{\cos\theta}$ . $dl=\sqrt{1+y'^2}=\sqrt{\rho^2+\rho'^2}$ dan esa,

$$dl=\sqrt{\cos 2\theta+\left(\frac{-\sin 2\theta}{\sqrt{\cos 2\theta}}\right)^2}=\frac{d\theta}{\sqrt{\cos 2\theta}}$$

ni inobatga olsak, u holda:

$$S_x=2\pi\int_0^{\pi/4}\sin\theta\sqrt{\cos\theta}\,dl=2\pi\int_0^{\pi/4}\sin\theta d\theta=2\pi(-\cos\theta)\Big|_0^{\pi/4}=2\pi\left(-\frac{\sqrt{2}}{2}+1\right)=\pi\left(2-\sqrt{2}\right).$$

<u>**6-Misol.**</u> $\frac{x^2}{9}+\frac{y^2}{4}=1$ ellipsning $Ox$ o'qi atrofida aylanishidan hosil bo'lgan jismning hajmi aniqlansin.

<u>**Yechish.**</u> Ellips tenglamasini $y^2$ ga nisbatan echamiz va y=0 va $x^2=9$, $x_{1,2}=\pm3$ bo'lgani uchun

$$V_x=\pi\int_{-3}^{3}\frac{4}{9}\left(9-x^2\right)dx=\frac{4}{9}\pi\left[9\int_{-3}^{3}dx-\int_{-3}^{3}x^2dx\right]=\frac{9}{4}\pi\left[9x\Big|_{-3}^{3}-\frac{x^3}{3}\Big|_{-3}^{3}\right]=$$

$$=\frac{4}{9}\pi(27+27-9-9)=\frac{4}{9}\pi 36=16\pi \text{ кв.б.}$$

**7-Misol.** $x^2-y^2=4$, $y=\pm2$ chiziqlar bilan chegaralangan yassi figura Oy o'q atrofida aylanadi. Aylanma jismning hajmi aniqlansin.

<u>**Yechish.**</u> $x^2=4+y^2$, $y_1=-2$ ва $y_2=2$ lardan

$$V_y=\pi\int_{-2}^{2}\left(4+y^2\right)dy=\pi\left[4\int_{-2}^{2}dy+\int_{-2}^{2}y^2dy\right]=\pi\left[4y\Big|_{-2}^{2}+\frac{y^3}{3}\Big|_{-2}^{2}\right]=$$

$$=\pi\left(8+8+\frac{8}{3}+\frac{8}{3}\right)=\frac{64}{3}\text{куб.б.}$$

## MASHQLAR

Quyidagi egri chiziqlarning aylanishidan hosil bo'lgan aylanma sirtning yuzi hisoblansin.

1. $y=\frac{x^3}{3}$ kubik parabolaning [-2;2] kesmaga mos yoyi $Ox$ o'qi atrofida aylanadi.
2. $x=t^2, y=\frac{1}{3}t\left(t^2-3\right)$ egri chiziq sirtmog'i $Ox$ o'qi atrofida aylanadi.
3. $y=\cos x$ ning bitta yarim to'lqini $Oy$ o'q atrofida aylanadi.
4. $y=\sin x$ ning bitta yarim to'lqini $Ox$ o'q atrofida aylanadi.
5. $y=2x$ to'g'ri chiziqning *xq0* va *xq2* gacha bo'lgan oraliqdagi kesmasi $Oy$ o'q atrofida aylanadi.
6. $\rho=2(1+\cos\theta)$ kardioida qutb o'qi atrofida aylanadi.
7. $4x^2+y^2=4$ ellips $Oy$ o'q atrofida aylanadi.
8. Tenglamasi $x=e^t\sin t, y=e^t\cos t$ bo'lgan egri chiziqning *tq0* dan $t=\frac{\pi}{2}$ gacha bo'lgan bo'lagi $Ox$ o'q atrofida aylanadi.
9. $\rho=4\sin\theta$ egri chiziq qutb o'qi atrofida aylanadi.
10. $x=2(t-\sin t), y=2(1-\cos t)$ sikloidaning bitta arkasi $(0\le t\le 2\pi)$ $Ox$ o'q atrofida aylanadi.

Quyidagi chiziqlar bilan chegaralangan yassi figuralarning aylanishidan hosil bo'lgan aylanma jismlarning hajmlari aniqlansin.

1. $xy=4, x=1, x=4, y=0$. $Ox$ o'q atrofida;
2. $(y-a)^2=ax, x=0, y=2a$. $Ox$ o'q atrofida;
3. $y^2=4-x, x=0$. $Oy$ o'q atrofida;
4. $y=x^3, x=0, y=8$. $Oy$ o'q atrofida;
5. $y=\frac{64}{x^2+16}, x^2=8y$. $Ox$ o'q atrofida;
6. $y^2=(x-1)^3, x=2$. $Oy$ o'q atrofida;
7. $y=2x-x^2, y-0$. $Ox$ o'q atrofida;
8. $x^2-y^2=4, y=\pm2$. $Oy$ o'q atrofida;

9. $y=x^2,\ y=\sqrt{x}$. $Ox$ o'q atrofida;

10. $x=2\cos^3 t,\ y=2\sin^3 t$. $Oy$ o'q atrofida;

11. $y=3-x^2,\ y=1+x^2$. $Ox$ o'q atrofida;

12. $y=x^2+3,\ x=4,\ y=0,\ x=0$. $Ox$ o'q atrofida;

13. $xy=9,\ y=3,\ y=9,\ x=0$. $Oy$ o'q atrofida;

14. $y=10-x^2,\ 2x+y-4=0$. $Ox$ o'q atrofida.

### 2.5.4. Aniq integral yordamida fizika va mexanika masalalarini yechish

1. a) Agar $v=f(t)$ funksiya moddiy nuqtaning biror chiziq bo'ylab harakatining tezligini ifodalasa, u holda $[t_1;t_2]$ vaqt mobaynida bosib o'tilgan yo'l

$$S=\int_{t_1}^{t_2} f(t)dt$$

formulasi orqali ifodalanadi.

**1-Misol.** Moddiy nuqtaning harakat tezligi $v=(6t^2+4)\,{}^{м}\!/_{сек}$ bo'lsa, harakat boshlanishidan boshlab 5 sek. mobaynida bosib o'tilgan yo'l hisoblansin.

<u>Yechish.</u> Shartga ko'ra, $f(t)=6t^2+4,\ t_1=0$ va $t_2=5$.

$$S=\int_0^5(6t^2+4)dt=(2t^3+3t)\Big|_0^5=2\cdot5^3+4\cdot5=250+20=270\,м$$

**2-Misol.** Jismning harakat tezligi $v=(18t-3t^2)\,{}^{м}\!/_{сек}$ bo'lsa, uning harakat boshlanishidan to harakat tugagunga qadar bosib o'tgan yo'li hisoblansin.

<u>Yechish.</u> Jismning harakat boshlanishi va tugashi paytidagi tezligi nolga teng. Harakat qaysi paytda tugashini aniqlaymiz, uning uchun $18t-3t^2=0$ tenglamani echamiz. Bundan: $3t(6-t)=0$ va $t_1=0, t_2=6$.

$$S=\int_0^6(18t-3t^2)dt=(9t^2-t^3)\Big|_0^6=9\cdot6^2-6^3=36\cdot3=108\,м.$$

**3-Misol.** Agar jism er sirtining yuzasidan vertikal holatda yuqoriga tomon $v=(29{,}4-9{,}8t)\,{}^{м}\!/_{сек}$ tezlik bilan otilgan bo'lsa, u holda jism eng ko'pi bilan necha metr balandlikka ko'tariladi?

<u>Yechish.</u> Jism eng katta balandlikka $t$ vaqtning shunday bir paytida erishadiki, o'sha paytda $v=0$ bo'ladi.

Demak, $24{,}9-9{,}8t=0$ dan $t=3сек$.

$$S=\int_0^3(29{,}4-9{,}8t)dt=\left(29{,}4t-4{,}9t^2\right)\Big|_0^3=44{,}1м.$$

b) Aytaylik, moddiy nuqta o'zgaruvchan $F(x)$ ta'sirida $Ox$ o' bo'ylab to'g'ri chiziqli harakat qilayotgan bo'lsin. Moddiy nuqta $x=$ holatdan $x=b$ holatga ko'chganda, ushbu kuchning bajargan is $A=\int_a^b F(x)dx$ formula bilan hisoblanadi.

__Eslatma.__ Kuchning bajargan ishini hisoblashga doir masalalar yechishda, ko'pincha Guk qonunining formulasi $F=kx$ dan foydalanila (k-proportsionallik koeffitsienti).

**4-Misol.** Agar prujina 60 H kuch ostida 0,02m cho'ziladigan bo'lsa uni 0,12m cho'zish uchun qancha ish bajarilishi kerak bo'ladi?

__Yechish.__ Guk qonuniga ko'ra, prujinani $x$ m ga cho'zuvchi kuc $F=kx$. Agar x=0,02 m bo'lsa, F=60H. Demak,

$r=\frac{60}{0{,}02}=3000$ va $F=3000x$. Natijada

$$A=\int_0^{0{,}12}3000\,xdx=1500\,x^2\Big|_0^{0{,}12}=1500\cdot0{,}0144=21{,}6(Ж).$$

**5-Misol.** Agar prujinaning dastlabki uzunligi 0,1 m ga teng bo'lib prujinani 0,01 m ga cho'zish uchun 20N kuch kerak bo'lsa, uni 0,12 m dan 0,14 m ga cho'zish uchun qancha ish bajarish kerak bo'ladi?

__Yechish.__ $\kappa=\frac{20}{0{,}01}=2000$ va $F=2000x;a=0{,}12-0{,}1=0{,}02$ v $b=0{,}14-0{,}1=0{,}04$.

$$A=\int_{0{,}02}^{0{,}04}2000\,xdx=1000\,x^2\Big|_{0{,}02}^{0{,}04}=1000\,(0{,}0016-0{,}0004)=1{,}2(Ж).$$

2. a) Ma'lumki, biror $l$ o'qdan $r$ masofada bo'lgan $m$ massa moddiy nuqtaning $l$ o'qiga nisbatan statik momenti deb, $M_l=m$ miqdorga aytilar edi. Faraz qilaylik, $xOy$ koordinatalar tekisligid tenglamasi $y=f(x)$ bo'lgan moddiy egri chiziqning biror $AB$ yoy $a\le x\le b$ qaralayotgan bo'lsin. Uning har bir nuqtasida zichlik es $\gamma=\gamma(x)$ kabi funksiya bilan ifodalansin. U holda $AB$ yoyning $Ox$ va $O$ o'qlarga nisbatan statik momentlari mos holda quyidagi formula bila hisoblanadi:

$$M_y=\int_a^b\gamma(x)\cdot x\cdot\sqrt{1+y'^2}\,dx,$$

$$M_x=\int_a^b \gamma(x)\cdot y\cdot\sqrt{1+y'^2}\,dx,$$

Bu yerda, $dl=\sqrt{1+y'^2}\,dx$ - yoy uzunligining elementi.

Xususan agar $\gamma(x)=\gamma$ o'zgarmas son bo'lsa (egri chiziq bir jinsli bo'lganda), yuqoridagi formulalar quyidagicha ko'rinishida yoziladilar

$$M_x=\gamma\int_a^b y\,dl,\ M_y=\gamma\int_a^b x\,dl.$$

b) Shuningdek egri chiziqning AB yoyi og'irlik markazi $P(x_0;y_0)$ nuqtaning koordinatalari quyidagi formula bilan topiladi:

$$x_0=\int_a^b\frac{\gamma(x)\cdot x\,dl}{\int_a^b\gamma(x)dl},\quad y_0=\int_a^b\frac{\gamma(x)\cdot y\,dl}{\int_a^b\gamma(x)dl}$$

yoki agar $\gamma(x)=\gamma$ o'zgarmas son bo'lsa

$$x_0=\int_a^b\frac{x\,dl}{\int_a^b dl},\quad y_0=\int_a^b\frac{y\,dl}{\int_a^b dl}$$

c) Aytaylik, $xOy$ koordinatalar tekisligida $y=f(x)$ egri chiziq, $Ox$ o'qi va $x=a$, $x=b$ vertikal to'g'ri chiziqlar bilan chegaralangan egri chiziqli trapetsiya qaralayotgan bo'lib, uning zichligi ham yana $\gamma=\gamma(x)$ kabi uzluksiz funksiya bo'lsin.

U holda, ushbu egri chiziqli trapetsiyaning $Ox$ va $Oy$ o'qlarga nisbatan statik momentlari quyidagi formulalardan aniqlanadi

$$M_x=\frac{1}{2}\int_a^b\gamma(x)y^2dx,\ M_y=\int_a^b\gamma(x)\cdot xy\,dx$$

Agar $\gamma=const\neq0$ bo'lsa, ya'ni egri chiziqli trapetsiya birjinsli bo'lsa, yuqoridagi formulalar quyidagi ko'rinishda yoziladi

$$M_x=\frac{\gamma}{2}\int_a^b y^2dx,\ M_y=\gamma\int_a^b xy\,dx.$$

d) Yuqorida qaralgan egri chiziqli trapetsiyaning og'irlik markazi $P(x_0;y_0)$ nuqtaning koordinatalari quyidagicha hisoblanadi:

$$x_0=\int_a^b\frac{\gamma(x)xy\,dx}{\int_a^b\gamma(x)y\,dx},\ y_0=\frac{1}{2}\int_a^b\frac{\gamma(x)y^2dx}{\int_a^b\gamma(x)y\,dx}$$

yoki agar $\gamma=const\neq0$ bo'lsa,

$$x_0=\int_a^b\frac{xy\,dx}{\int_a^b y\,dx},\ y_0=\frac{1}{2}\int_a^b\frac{y^2dx}{\int_a^b y\,dx}$$

**6-Misol.** Tenglamasi $y=\sqrt{x}$ bo'lgan parabolaning abtsissalari $x=$ va $x=4$ bo'lgan nuqtalarga mos keluvchi yoyning koordinata o'qlari nisbatan statik momentlari hisoblansin.

**Yechish.** $\gamma=1$ deb olamiz $y'=\frac{1}{2\sqrt{x}}$ bo'lganligi uchun

$$\sqrt{1+y'^2}=\sqrt{1+\frac{1}{4x}}=\frac{\sqrt{4x+1}}{2\sqrt{x}};$$

$$M_x=\int_0^4\sqrt{x}\cdot\frac{\sqrt{4x+1}}{2\sqrt{x}}dx=\frac{1}{2}\int_0^4\sqrt{4x+1}dx=\frac{1}{3}\cdot\frac{2}{3}\cdot\sqrt{(4x+1)^3}\Big|_0^4=\frac{1}{3}\left(17\sqrt{17}-1\right)$$

$$M_y=\int_0^4 x\frac{\sqrt{4x+1}}{2\sqrt{x}}dx\frac{1}{2}\int_0^4\sqrt{4x^2+x}dx=\int_0^4\sqrt{x^2+\frac{x}{4}+\frac{1}{64}-\frac{1}{64}}dx=$$

$$\int_0^4\sqrt{\left(x+\frac{1}{8}\right)^2-\frac{1}{64}}dx=\left[\frac{x+\frac{1}{8}}{2}\sqrt{x^2+\frac{x}{4}}-2\ln\left|x+\frac{1}{8}+\sqrt{x^2+\frac{x}{4}}\right|\right]_0^4=$$

$$=\frac{33}{16}\sqrt{17}-2\ln\left|\frac{33}{8}+\sqrt{17}\right|+2\ln\frac{1}{8}=\frac{33\sqrt{17}}{16}+\ln\left|\frac{1}{8\left(\frac{33}{8}+\sqrt{17}\right)}\right|^2=$$

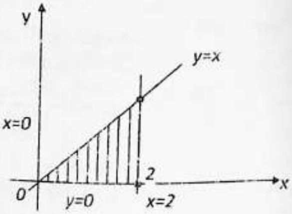
$$=\frac{33\sqrt{17}}{16}-2\ln\left(33+8\sqrt{17}\right);$$

**7-Misol.** $y=x, x=2, y=0$ to'g'ri chiziqlar bilan chegaralangan uchburchakning koordinata o'qlariga nisbatan statik momentlari hisoblansin.

**Yechish.** $\gamma=1$ deb olamiz.

$$M_x=\frac{1}{2}\int_0^2 y^2dx=\frac{1}{2}\int_0^2 x^2dx=\frac{1}{2}\cdot\frac{x^3}{3}\Big|_0^2=\frac{4}{3};$$

$$M_y=\int_0^2 xydx=\int_0^2 x^2dx=\frac{x^3}{3}\Big|_0^2=\frac{8}{3};$$




**8-Misol.** Tenglamasi $x^2+y^2=4$ bo'lgan aylananing yuqori yarim pallasi og'irlik markazining koordinatalari hisoblansin.

**Yechish.** Bu yerda ham $\gamma = 1$ deb olamiz. $y = \sqrt{4 - x^2}$ dan

$$y' = -\frac{x}{\sqrt{4 - x^2}} \text{ va } M = \sqrt{1 + y'^2} = \sqrt{1 + \frac{x^2}{4 - x^2}} = \frac{2}{\sqrt{4 - x^2}}.$$

$$x_0 = \frac{\int_{-2}^{2} \frac{2x dx}{\sqrt{4 - x^2}}}{\int_{-2}^{2} \frac{2dx}{\sqrt{4 - x^2}}} = \frac{-2\sqrt{4 - x^2}\Big|_{-2}^{2}}{2 \arcsin \frac{x}{2}\Big|_{-2}^{2}} = 0;$$

$$y_0 = \frac{\int_{-2}^{2} 2dx}{\int_{-2}^{2} \frac{2dx}{\sqrt{4 - x^2}}} = \frac{4}{2 \arcsin 1} = \frac{4}{\pi};$$

**9-Misol.** 7-misoldagi uchburchakning og'irlik markazi topilsin.

**Yechish.**

$$x_0 = \frac{\int_0^2 x^2 dx}{\int_{-2}^{2} x dx} = \frac{\frac{x^3}{3}\Big|_0^2}{\frac{x^2}{2}\Big|_0^2} = \frac{\frac{8}{3}}{2} = \frac{4}{3}; \quad y_0 = \frac{\frac{1}{2}\int_0^2 x^2 dx}{\int_{-2}^{2} x dx} = \frac{\frac{x^3}{6}\Big|_0^2}{\frac{x^2}{2}\Big|_0^2} = \frac{\frac{8}{6}}{2} = \frac{2}{3};$$

## MASHQLAR

(9-Masaladan boshlab, $\gamma = 1$ deb olinsin)

1) Jismning harakat tezligi $v = (3t^2 + 2t - 1) \text{ м}/_{\text{сек.}}$ qonun bilan o'zgaradi. Uning harakat boshlangandan so'ng 10 sek. mobaynida bosib o'tgan yo'li hisoblansin.

2) Nuqtaning harakat tezligi $v = (2t + 8e^{-2}) \text{ м}/_{\text{сек.}}$ bo'lsa, uning 2sek. da bosib o'tgan yo'li hisoblansin.

3) Agar jismning harakat tezligi $v = (24t - 3t^2) \text{ м}/_{\text{сек.}}$ bo'lsa, uning harakat boshlanishidan to to'xtaguncha bo'lgan vaqt mobaynida bosib o'tgan yo'li hisoblansin.

4) Jism er sirtining yuzasidan vertikal holatda yuqoriga tomon $v = (39{,}2 - 9{,}8t) \text{ м}/_{\text{сек.}}$ tezlik bilan otildi. Jism eng ko'pi bilan necha metr balandlikka ko'tarildi.

5) Prujinani 0,05 metrga cho'zishi uchun 80N kuch lozim. Agar prujinaning uzunligi 0,15 metrga teng bo'lsa, uni 0,2 metrgacha cho'zish uchun qanday ish bajarish kerak bo'ladi.

6) Prujinani 0,05 metrga cho'zish uchun 30J ish bajarish kerak bo'ladi. Uni 0,08 metrga cho'zish uchun esa qancha ish bajarish lozim.

7) Agar prujinani 0,02 metrga cho'zish uchun 16J ish bajarish lozim bo'lsa, 100J ish bajarilganda prujina qancha uzunlikka cho'ziladi.

8) Prujinaning uzunligi 0,1 metrga teng bo'lib, uni 20N kuch 0,01 metrga cho'za oladi. Prujinani 0,12 metrdan 0,14 metrgacha cho'zish uchun qanday ish bajarish kerak bo'ladi?

9) $y=2-x$ to'g'ri chiziqning koordinata o'qlari orasidagi kesmasining $Ox$ va $Oy$ o'qlarga nisbatan statik momentlari hisoblansin.

10) $y=\cos x$ ning $\left[-\frac{\pi}{2};\frac{\pi}{2}\right]$ kesmaga mos keluvchi yoyning $Ox$ o'qqa nisbatan statik momenti topilsin.

11) $y=x^2$ va $y=\sqrt{x}$ parabolalar hosil qilgan figuraning $Ox$ o'qqa nisbatan statik momenti topilsin.

12) $y^2=2x$ parabolaning $x=0$ dan $x=2(y>0)$ gacha bo'lgan oraliqda yoyning $Ox$ va $Oy$ o'qlarga nisbatan statik momentlari hisoblansin.

13) $y^2=4x$ parabola, $x=4$ to'g'ri chiziq va $Ox$ o'qlar bilan chegaralangan figuraning og'irlik markazining koordinatalari hisoblansin.

14) $y=2x-x^2$ bilan $Ox$ o'q chegaralagan figuraning og'irlik markazi topilsin.

15) $\frac{x^2}{9}+\frac{y^2}{4}=1$ ellipsning I-chorakdagi qismining og'irlik markazi topilsin.

16) $y=6-x^2$ parabola, $y=2$ to'g'ri chiziqlar chegaralagan figuraning og'irlik markazi hisoblansin.

17) $y=x$ to'g'ri chiziq bilan $y=x^2-2x$ parabola chegaralagan figuraning og'irlik markazi hisoblansin.

18) $y=\sin x$ sinusoidaning $\left[0;\frac{\pi}{2}\right]$ kesmaga mos keluvchi bo'lagining og'irlik markazining koordinatalari topilsin.

19) Tenglamasi $x=2(t-\sin t), y=2(1-\cos t)$ bo'lgan sikloidaning birinchi arkasiga mos keluvchi yoyi $(0\le t\le 2\pi)$ ning og'irlik markazi topilsin.

20) Tenglamasi $9x^2+16y^2=144$ bo'lgan ellips, $Ox$ va $Oy$ koordinata o'qlari bilan chegaralangan figura $(x\ge 0; y\ge 0)$ ning og'irlik markazi hisoblan

21) sin.

## 3. KO'P O'ZGARUVCHILI FUNKSIYALAR

### 3.1. Ko'p o'zgaruvchili funksiyalarga doir asosiy tushunchalar

*1. Ta'rif. Agar biror D to'plamning har bir juft (x;y) haqiqiy sonlariga biror qonun yoki qoida yordamida boshqa bir E to'plamdan yagona z haqiqiy son mos qilib quyilgan bo'lsa, u holda D to'plamda ikki haqiqiy x va y o'zgaruvchilarning funksiyasi aniqlangan deb ataladi. Bu yerda x va y erkli o'zgaruvchilar yoki argumentlar, z esa erksiz o'zgaruvchi yoki funksiya deb yuritiladi.*

*Ikki o'zgaruvchining funksiyasi* $z = f(x;y), z = z(x;y)$ *va hokazo ko'rinishlarda belgilanadi. Shuningdek, D va E larni mos ravishda ikki o'zgaruvchili funksiyaning aniqlanish va o'zgarish sohalari deb ataladi. Ikki argumentli funksiyaning aniqlanish sohasi D, xOy tekislikning biror chiziqlar bilan chegaralanagan qismi sifatida ifoda etiladi hamda agar chiziqlar D sohada yotsa uni yopiq soha deb atalib* $\overline{D}$ *bilan belgilanadi, aksincha D ni ochiq soha deb yuritiladi. Biror* $z = f(x;y)$ *funksiyaning grafigi deyilganda fazodagi shunday* $M(x;y;z)$ *nuqtalar to'plamidan tashkil topgan sirt tushuniladiki, u sirtdagi nuqtalarning koordinatalari* $z = f(x;y)$ *tenglamani qanoatlantiradi.*

Ikki argumentli funksiyaning ta'rifini uch va undan ortiq haqiqiy o'zgaruvchilarning funksiyalari uchun ham osonlikcha umumlashtirishi mumkin.

**Ko'p argumentli funksiyalarga bundan buyongi tushunchalarni faqat ikki argumentli funksiyalarga moslab bayon etamiz, chunki ularni uch va undan ortiq o'zgaruvchili funksiyalarga ko'chirish hech qanday qiyinchiliklarsiz amalga oshiriladi.**

2. ***Ta'rif.*** *Agar oldindan berilgan istalgancha kichik* $\varepsilon > 0$ *son uchun shunday bir kichik* $\delta > 0$ *son mavjud bo'lib,* $|x - x_0| < \delta$ *va* $|y - y_0 < \delta|$ *tengsizliklarni qanoatlantiruvchi barcha x va y lar uchun* $|f(x) - A| < \varepsilon$ *tengsizlik o'rinli bo'lsa, u holda A sonini* $z = f(x;y)$ *funksiyaning tayinli* $M_0(x_0;y_0)$ *nuqtadagi limiti deb ataladi va uni quyidagicha yoziladi:*

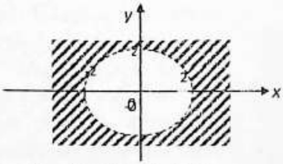
$$A = \lim_{\substack{x \to x_0 \\ y \to y_0}} f(x;y)$$

***Ta'rif.*** *Agar* $z = f(x;y)$ *funksiya biror* $M_0(x_0;y_0)$ *nuqtada va uning atrofida aniqlangan bo'lib,* $A = \lim_{\substack{x \to x_0 \\ y \to y_0}} f(x;y) = f(x_0;y_0)$ *bo'lsa, u holda* *mazkur funksiyani $M_0(x_0;y_0)$ nuqtada uzluksiz funksiya deb ataladi. Biror D soxaning barcha nuqtalarida uzluksiz bo'lgan funksiyani u sohada uzluksiz deyiladi. $z=f(x;y)$ funksiya uchun uzluksizlik shartlari bajarilmagan nuqtalarni uzilish nuqtalari deb atalib, funksiyani esa mazkur nuqtalarda uzilishga ega deyiladi.*

*Eslatib o'tamizki, ikki o'zgaruvchining funksiyalari ayrim nuqtalarda yoki butun bir chiziqlarda uzilishga ega bo'lishlari mumkin.*

**1-Misol.** $z=\ln(x^2+y^2-4)$ funksiyaning aniqlanish sohasi topilsin.

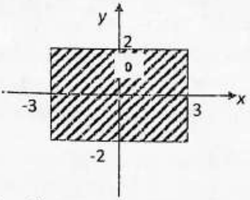
__Yechish.__ Mazkur funksiyaning aniqlanish sohasi shunday *(x; y)* juft sonlardan iboratki, ular uchun $x^2+y^2-4>0$ yoki $x^2+y^2>4$ shart bajarilishi darkor. Bu esa geometrik jihatdan markazi *(0, 0)* nuqtada bo'lib, radiusi 2 dan iborat bo'lgan doiradan tashqarida yotuvchi *xOy* tekislikdagi nuqtalar to'plamini ifoda etadi.




**2-Misol.** $z=\sqrt{9-x^2}+\sqrt{4-y^2}$ funksiyaning aniqlanish sohasi topilsin.

__Yechish.__ Bu funksiyaning aniqlanish sohasi $9-x^2\geq 0$ va $4-y^2\geq 0$ tengsizliklarni qanoatlantiruvchi barcha *(x,y)* sonlar juftliklaridan iborat, ya'ni: $-3\leq x\leq 3, -2\leq y\leq 2$.

Demak, funksiya $x=\pm 3, y=\pm 2$ lar bilan chegaralangan to'g'ri to'rtburchakda aniqlangan ekan.




**3-Misol.** $A=\lim\limits_{\substack{x\to 1\\ y\to 0}}\frac{x^2+y^2-1}{\sqrt{x^2+y^2}-1}$ hisoblansin.

**Yechish.**

$$A=\lim_{\substack{x\to1\\y\to0}}\frac{x^2+y^2-1}{\sqrt{x^2+y^2}-1}=\lim_{\substack{x\to1\\y\to0}}\frac{\left(\sqrt{x^2+y^2}-1\right)\left(\sqrt{x^2+y^2}+1\right)}{\sqrt{x^2+y^2}-1}=\lim_{\substack{x\to1\\y\to0}}\left(\sqrt{x^2+y^2}+1\right)=2.$$

**4-Misol.** $z=\dfrac{1}{x^2+3y^2}$ funksiya uzluksizlikka tekshirilsin.

**Yechish.** Ushbu funksiya tekislikning O(0,0) nuqtasidan tashqari barcha nuqtalarda uzluksiz bo'lib, O(0,0) da uzilishga ega (chunki funksiya, O(0,0) da aniqlanmagan).

## MASHQLAR

Quyidagi funksiyalarning aniqlanish sohalari topilsin.

1. $z=\sqrt{x^2-4y+4}.$
2. $z=\sqrt{x+y}.$
3. $z=\sqrt{4-x^2-y^2}.$
4. $z=\ln(3-xy).$
5. $z=\sqrt{xy}+\sqrt{x-y}.$
6. $z=\arcsin\dfrac{x}{y}.$

Quyidagi limitlar hisoblansin.

1. $\lim\limits_{\substack{x\to0\\y\to0}}\dfrac{xy}{1-\sqrt{xy+1}}$
2. $\lim\limits_{\substack{x\to0\\y\to0}}\dfrac{\sin xy}{x}$
3. $\lim\limits_{\substack{x\to0\\y\to0}}(1+xy)\dfrac{1}{x}.$
4. $\lim\limits_{\substack{x\to0\\y\to0}}(x^2+y^2)\sin\dfrac{1}{x^2+y^2}.$

Quyidagi funksiyalarni uzluksizlikka tekshirilsin.

1. $z=\dfrac{1}{(x-1)^2+(y+1)^2}.$
2. $z=\dfrac{1}{\sin^2\pi x+\sin^2\pi y}.$
3. $z=\ln(1-x^2-y^2).$
4. $z=\dfrac{x^2+y^2}{(x+y)(y^2-x)}.$
5. $z=\dfrac{1}{x^2+y^2-4}.$
6. $z=\dfrac{1}{x+y}.$

## 3.2. Xususiy hosilalar va xususiy differensiallar

1. Agar $z=f(x,y)$ funksiyadagi $y$ ni o'zgarishsiz qoldirib, o'zgaruvchi $x$ ga biror $\Delta x$ orttirma berilsa, u holda funksiya $x$ o'zgaruvchiga nisbatan xususiy orttirma deb ataluvchi $\Delta_x z$ orttirmaga ega bo'ladi

$$\Delta_x z=f(x+\Delta x,y)-f(x,y).$$

Aynan yuqoridagidek, $x$ o'zgarishsiz qoldirilib $y$ ga $\Delta y$ orttirma berilsa, u holda funksiyaning $y$ ga nisbatan xususiy orttirmasi

$$\Delta_y z = f(x, y + \Delta x) - f(x, y)$$

kabi yoziladi.

Agar

$$\lim_{\Delta x \to 0} \frac{\Delta_x z}{\Delta x} \equiv \frac{\partial z}{\partial x} \equiv z'_x \equiv f'_x(x, y)$$

$$\lim_{\Delta y \to 0} \frac{\Delta_y z}{\Delta y} \equiv \frac{\partial z}{\partial y} \equiv z'_y \equiv f'_y(x, y)$$

dek chekli limitlar mavjud bo'lsa, ularni mos ravishda $z = f(x, y)$ funksiyaning $x$ va $y$ o'zgaruvchilarga nisbatan xususiy hosilalari deb ataladi.

**Uch va undan ortiq argumentli funksiyalarning xususiy hosilalari ham aynan yuqoridagidek aniqlanadi.**

**Ta'kidlash joizki, bir argumentli funksiyaning asosiy differensiallash qoidalari va formulalarining barchasi ikki va undan ortiq o'zgaruvchili funksiyalarning xususiy hosilalarini hisoblash uchun ham o'rinli bo'laveradi.**

**1-Misol.** $z = x^3 + 3x^2y - 4xy^2 + 5y^3 + 2x - 3y + 1$. $\frac{\partial z}{\partial x}$ va $\frac{\partial z}{\partial y}$ lar topilsin.

<u>**Yechish.**</u> $y$ ni o'zgarmas deb,

$$\frac{\partial z}{\partial x} = 3x^2 + 6xy - 4y^2 + 2.$$

Endi $x$ ni o'zgarmas hisoblab, topamiz.

$$\frac{\partial z}{\partial y} = 3x^2 + 8xy - 15y^2 - 3.$$

**2-Misol.** $z = xe^{-xy}$; $\frac{\partial z}{\partial x}$ va $\frac{\partial z}{\partial y}$ lar topilsin.

<u>**Yechish.**</u>

$$\frac{\partial z}{\partial x} = e^{-xy} + xe^{-xy}(-y) = e^{-xy}(1 - xy).$$

$$\frac{\partial z}{\partial x} = xe^{-xy}(-x) = -x^2e^{-xy}.$$

**3-Misol.** $z = \ln(x^2 + y^2)$ funksiyaning xususiy hosilalari topilsin.

<u>**Yechish.**</u> $\frac{\partial z}{\partial x} = \frac{2x}{x^2 + y^2}$, $\frac{\partial z}{\partial y} = \frac{2y}{x^2 + y^2}$.

**4-Misol.** $z = \sin^2(x + y) - \sin^2 x - \sin^2 y$. $\frac{\partial z}{\partial x}$ va $\frac{\partial z}{\partial y}$ lar topilsin.

$$\frac{\partial z}{\partial x} = 2\sin(x + y)\cos(x + y) - 2\sin x\cos x = \sin 2(x + y) - \sin 2x = 2\sin y\cos(2x + y)$$

$$\frac{\partial z}{\partial y}=2\sin(x+y)\cos(x+y)-2\sin y\cos y=\sin 2(x+y)-\sin 2y=2\sin y\cos(x+2y);$$

**5-Misol.** $u=\sqrt{x^2+y^2+z^2}-xyz$ ning xususiy hosilalari hisoblansin.

<u>**Yechish.**</u>

$$\frac{\partial u}{\partial x}=\frac{x}{\sqrt{x^2+y^2+z^2}}-yz,$$

$$\frac{\partial u}{\partial y}=\frac{y}{\sqrt{x^2+y^2+z^2}}-xz,$$

$$\frac{\partial u}{\partial z}=\frac{z}{\sqrt{x^2+y^2+z^2}}-xy.$$

2. Agar $z=f(x,y)$ funksiyaning erkli o'zgaruvchilaridan birini o'zgarishsiz qoldirib, ikkinchisi bo'yicha undan differensial hisoblanadigan bo'lsa, u differensiallar odatda funksiyaning o'zgaruvchilariga nisbatan xususiy differensiallari deb ataladi:

$$d_x z=f'_x(x;y)dx,$$
$$d_y z=f'_y(x;y)dy.$$

Bu yerda, $dx=\Delta x$ va $dy=\Delta y$ lar erkli o'zgaruvchilarning ixtiyoriy orttirmalari bo'lib, ular erkli o'zgaruvchilarning differensiallari deb yuritiladi.

**6-Misol.** $z=\sqrt[3]{(x^2-y^3)^2}$ ning xususiy differensiallari topilsin.

<u>**Yechish.**</u>

$$z'_x=\frac{2}{3}(x^2-y^3)^{-\frac{1}{3}}\cdot 2x=\frac{4x}{3\sqrt[3]{x^2-y^3}} \text{ va}$$

$$z'_y=\frac{2}{3}(x^2-y^3)^{-\frac{1}{3}}\cdot(-3y^2)=-\frac{2y^2}{\sqrt[3]{x^2-y^3}}$$

bo'lgani uchun

$$d_x z=\frac{4x\,dx}{3\sqrt[3]{x^2-y^3}},\quad d_y z=-\frac{2y^2dy}{\sqrt[3]{x^2-y^3}};$$

**7-Misol.** $u=\arcsin\sqrt{xy^2z^3}$ berilgan. $d_x u, d_y u$ ва $d_z u$ lar topilsin.

<u>**Yechish.**</u>

$$d_x u=\frac{\frac{y^2z^3}{2\sqrt{xy^2z^3}}}{\sqrt{1-x^2y^4z^6}}dx=\frac{y\sqrt{z^3}dx}{2\sqrt{x}\sqrt{1-x^2y^4z^6}};$$

$$d_y u=\frac{\frac{2xyz^3}{2\sqrt{xy^2z^3}}dy}{\sqrt{1-x^2y^4z^6}}=\frac{\sqrt{xz^3}dy}{\sqrt{1-x^2y^4z^6}};$$

$$d_z u = \frac{\dfrac{3xy^2z^2}{2\sqrt{xy^2z^3}}dz}{\sqrt{1-x^2y^4z^5}} = \frac{3\sqrt{xz}\,ydz}{2\sqrt{1-x^2y^4z^6}};$$

## MASHQLAR

Quyidagi funksiyalarning xususiy hosilalari hisoblansin.

1. $z = \arcsin\frac{x}{y};$
2. $z = 3^{\sin\frac{y}{x}};$
3. $z = x^y;$
4. $z = tg^2(x-y);$
5. $z = \ln(xy + \ln xy);$
6. $z = \ln(x-2y);$
7. $z = e^{\frac{x}{y}} + e^{-\frac{y}{x}};$
8. $z = (x^3+y^3)^4;$
9. $z = \cos^2(x+y);$
10. $z = xy + \frac{y}{x};$
11. $z = \sqrt{x^2 - y^2};$
12. $z = \frac{x-y}{x+y};$
13. $u = z^{xy};$
14. $u = (xy)^z;$
15. $z = arctg\frac{y}{x};$
16. $u = \frac{z}{\sqrt{x^2+y^2}};$
17. $u = \ln\left(\sin\frac{x+y}{z}\right);$
18. $u = tg^2(x - y^2 + z^3);$
19. $u = arctg\left(\frac{xy}{z}\right);$
20. $u = \ln\sqrt{\frac{(x^2+y^2)}{(x^2+z^2)}};$

### 3.3. To'la orttirma va to'l differensial. Murakkab va oshkormas funksiyalarni differensiallash

1. $z = f(x;y)$ funksiyaning biror *(x; y)* nuqtadagi to'la orttirmasi deb, $\Delta z = f(x+\Delta x; y+\Delta y) - f(x;y)$ ifodaga aytiladi.

Agar $z = f(x,y)$ funksiya *(x,y)* nuqtada uzluksiz xususiy hosilalarga ega bo'lsa, uning to'la orttirmasini

$$\Delta z = \frac{\partial z}{\partial x}\Delta x + \frac{\partial x}{\partial y}\Delta y + \varepsilon\rho$$

ko'rinishda yozish mumkin, bu yerda, agar $\rho = \sqrt{|\Delta x|^2 + |\Delta y|^2} \to 0$ da $\varepsilon \to 0$. Ushbu tenglikdagi $\frac{\partial z}{\partial x}\Delta x + \frac{\partial z}{\partial y}\Delta y$ ni $z = f(x,y)$ funksiyaning *(x;y)* nuqtadagi to'la differensialli deb ataladi va $dz$ deb belgilanadi. Agar $\Delta x = dx, \Delta y = dy$ ekanliklarini nazarda tutsak, u holda:

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy.$$

Taqribiy hisoblashlarda $\Delta z \approx dz$ deb olinadi. Demak

$$f(x+\Delta x; y+\Delta y) \approx f(x,y) + dz$$

ya'ni

$$f(x+\Delta x; y+\Delta y) \approx f(x,y) + \frac{\partial z}{\partial x}\Delta x + \frac{\partial z}{\partial y}\Delta y$$

ekan.

2. Agar $u = \varphi(x;y)$ va $v = \phi(x;y)$ lar berilgan bo'lsalar, u holda $z = f(u;v)$ ni $x$ va $y$ o'zgaruvchilarga nisbatan murakkab funksiya deb ataladi va uning xususiy hosilalari quyidagi formulalar yordamida hisoblanadi:

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x},$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y},$$

Xususan, agar $u = \varphi(x), v = \varphi(x)$ bo'lsalar, yuqoridagi formulalardan birinchisi $\frac{dz}{dx} = \frac{\partial z}{\partial u}\frac{du}{dx} + \frac{\partial z}{\partial v}\frac{dv}{dx}$ dek yozilib, ikkinchisi esa aynan 0 ga teng bo'ladi.

Oxirgi formulada $u = x, \quad v = y = \varphi(x)$ bo'lsalar, u holda

$$\frac{dz}{dx} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\frac{dy}{dx}.$$

Ushbu ifodani ikki o'zgaruvchili to'liq funksiyaning hosilasi deb ataladi.

3. Agar $F(x;y) = 0$ tenglama, biror $y = y(x)$ funksiyani oshkormas shaklda ifodalaydigan bo'lib, $F'_y(x;y) \neq 0$ bo'lsa, u holda

$$\frac{dy}{dx} = -\frac{F'_x(x;y)}{F'_y(x;y)}.$$

Agar $F(x;y) = 0$ tenglama, $z = z(x;y)$ funksiyani oshkormas ko'rinishda ifodalab, $F'_z(x;y;z) \neq 0$ bo'lsa, u holda

$$\frac{\partial y}{\partial x} = -\frac{F'_x(x;y;z)}{F'_z(x;y;z)}, \qquad \frac{\partial z}{\partial y} = -\frac{F'_y(x;y;z)}{F'_z(x;y;z)}.$$

**1-Misol.** $z = arctg\frac{x+y}{x-y}$, $dz$ hisoblansin.

**Yechish.** $\frac{\partial z}{\partial x} = \frac{1}{1+\left(\frac{x+y}{x-y}\right)^2}\cdot\frac{-2y}{(x-y)^2} = -\frac{y}{x^2+y^2},$

$$\frac{\partial z}{\partial y}=\frac{1}{1+\left(\frac{x+y}{x-y}\right)^2}\cdot\frac{2x}{(x-y)^2}=\frac{x}{x^2+y^2}.$$

u holda $\quad dz=\frac{xdy-ydx}{x^2+y^2}.$

**2-Misol.** $z=\ln^2\left(x^2+y^2\right)$, $dz$ hisoblansin.

**Yechish.**

$$\frac{\partial z}{\partial x}=2\ln\left(x^2+y^2\right)\cdot\frac{2x}{x^2+y^2},$$

$$\frac{\partial z}{\partial y}=2\ln\left(x^2+y^2\right)\cdot\frac{2y}{x^2+y^2},$$

$$dz=4\ln\left(x^2+y^2\right)\cdot\frac{(xdx+ydy)}{x^2+y^2}.$$

**3-Misol.** $\sqrt{5e^{0,02}+2,03^2}$ ni taqribiy hisoblansin.

**Yechish.** Izlanayotgan sonni $\sqrt{5e^x+y^2}$ funksiyaning $x=0$, $y=2$ nuqtalaridagi qiymatining orttirmasi deb qaraymiz.

$$\Delta x=0,02, \Delta y=0,03, z\Big|_{\substack{x=0\\y=2}}=\sqrt{5e^0+2^2}=\sqrt{9}=3.$$

Funksiyaning to'la orttirmasini topamiz:

$$\Delta z\approx dz=\frac{\partial z}{\partial x}\Delta x+\frac{\partial z}{\partial y}\Delta y=\frac{5e^x}{2\sqrt{5e^x+y^2}}\Delta x+\frac{2y}{2\sqrt{5e^x+y^2}}\Delta y=$$

$$=\frac{5\cdot 0,02+2\cdot 2\cdot 0,03}{2\cdot 3}=\frac{0,1+0,12}{6}=\frac{0,22}{6}=0,037.$$

Demak, $\sqrt{5e^{0,02}+2,03^2}\approx 3+0,037=3,037$. Bu misolni yechishda

$$f(x+\Delta x;y+\Delta y)\approx f(x,y)+\frac{\partial z}{\partial x}\Delta x+\frac{\partial z}{\partial y}\Delta y$$

formuladan foydalandik.

**4-Misol.** $(1,02)^{3,01}$ ni taqribiy hisoblansin.

**Yechish.** Izlanayotgan sonni $z=x^y$ funksiyaning to'la orttirmasidan foydalanib hisoblaymiz.

$x=1$ va $y=3$ da

$$z=1^3=1, \Delta x=1,02-1=0,02, \Delta y=3,01-3=0,01.$$

$$dz=yx^{y-1}\Delta x+x^y\ln x\cdot\Delta y$$

ekanligidan,

$$dz=3\cdot 1^2\cdot 0,02+1^3\cdot\ln 1\cdot 0,01=0,06.$$

U holda

$$z=(1,02)^{3,01}\approx z+dz+1+0,06=1,06.$$

**5-Misol.** Agar $u=3x-2y, v=xy$ bo'lsa, $z=\cos(u\cdot v)$ ning xususiy hosilalari hisoblansin.

$$\frac{\partial z}{\partial x}=-v\sin(u\cdot v)\cdot 3-u\sin(u\cdot v)\cdot y=-3xy\cdot\sin\left(3x^2y-2xy^2\right)-3xy-2y^2\cdot$$
$$\cdot\sin\left(3x^2y-2xy^2\right)=-\sin\left(3x^2y-2xy^2\right)\cdot\left(6xy-2y^2\right)$$
$$\frac{\partial z}{\partial y}=-v\sin(u\cdot v)\cdot(-2)-u\sin(u\cdot v)\cdot x=2xy\cdot\sin\left(3x^2y-2xy^2\right)-\left(3x^2-2xy\right)\cdot$$
$$\cdot\sin\left(3x^2y-2xy^2\right)=\sin\left(3x^2y-2xy^2\right)\cdot\left(4xy-3x^2\right)$$

**6-Misol.** Agar $z=x+y^2$ bo'lib, $y=\sin x$ bo'lsa, $\frac{dz}{\partial x}$ hisoblansin.

<u>**Yechish.**</u>

$$\frac{dz}{dx}=\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}\cdot\frac{dy}{dx}=1+2y\cdot\cos x=1+2\sin x\cos x=1+\sin 2x.$$

**7-Misol.** Agar $x^2-y^2+e^{x/y}+3=0$, bo'lsa, $\frac{dy}{dx}$ hisoblansin.

<u>**Yechish.**</u>

$$\left(x^2-y^3+e^{x/y}+3\right)'_x=2x+\frac{1}{y}\cdot e^{x/y}$$
$$\left(x^2-y^3+e^{x/y}+3\right)'_y=-3y^2-\frac{x}{y^2}\cdot e^{x/y}$$

lardan

$$\frac{dy}{dx}=-\frac{2x+\frac{1}{y}\cdot e^{x/y}}{-3y^2-\frac{x}{y^2}\cdot e^{x/y}}=\frac{\left(2xy-e^{x/y}\right)\cdot y}{3y^4+xe^{x/y}}.$$

**8-Misol.** Agar $\left(xyz-x^2+y^2-z^2\right)=0$ bo'lsa, $\frac{\partial z}{\partial x}$ va $\frac{\partial z}{\partial y}$ lar hisoblansin.

<u>**Yechish.**</u> Agar

$$\left(xyz-x^2+y^2-z^2\right)'_x=yz-2x,$$
$$\left(xyz-x^2+y^2-z^2\right)'_y=xz+2y,$$
$$\left(xyz-x^2+y^2-z^2\right)'_z=xy-2z$$

ekanligini inobatga olsak,

$$\frac{\partial z}{\partial x}=-\frac{yz-2x}{xy-2z},$$
$$\frac{\partial z}{\partial y}=-\frac{xz+2y}{xy-2z}.$$

## MASHQLAR

Quyidagi berilgan funksiyalarning to'la diferentsiallari hisoblansin.

1. $z=\sqrt{x^2+y^2}$
2. $z=e^{y/x}$
3. $z=\frac{y}{x}-\frac{x}{y}$.
4. $z=x\ln y$.
5. $z=\sin\left(x^2+y^2\right)$
6. $z=\ln\left(tg\frac{x}{y}\right)$.
7. $z=e^x(\cos y+x\sin y)$
8. $z=\frac{y}{\sqrt{x^2+y^2}}$.
9. $z=x\ln\frac{y}{x}$.
10. $z=arcctg\frac{x}{x+1}$.
11. $z=x^3+3x^2y-y^3$.
12. $z=(x+y)e^{xy}$.

Quyidagi ifodalar taqribiy hisoblansin

13. $1{,}02^{4{,}03}$.
14. $\ln\left(0{,}09^3+0{,}99^3\right)$
15. $\sqrt[3]{1{,}02^2+0{,}05^2}$.
16. $arctg\frac{1{,}02}{0{,}95}$.
17. $\sqrt[3]{tg1{,}05+\ln 1{,}02}$.
18. $\sqrt{\sin^2 1{,}55+8e^{0{,}01}}$.
19. $\sqrt{(4{,}05)^2+(2{,}93)^2}$.
20. $(1{,}02)^3\cdot(0{,}97)^4$.

21. Agar $u=x\sin y, v=\cos x, z=\sqrt{u^2+v^2}$ berilgan bo'lsa, $\frac{\partial z}{\partial x}$ va $\frac{\partial z}{\partial y}$ lar hisoblansin.

22. Agar $u=xy, v=\frac{x}{y}, z=\ln\left(u^3+v^3\right)$ berilgan bo'lsa, $\frac{\partial z}{\partial x}$ va $\frac{\partial z}{\partial y}$ lar hisoblansin.

23. Agar $u=\frac{y}{x}, v=x^2+y^2, z=u^2\ln v$ berilgan bo'lsa, $\frac{\partial z}{\partial x}$ va $\frac{\partial z}{\partial y}$ lar hisoblansin.

24. Agar $u=\ln\left(x^2+y\right), v=y^2x, z=\sqrt{u\cdot v}$ berilgan bo'lsa, $\frac{\partial z}{\partial x}$ va $\frac{\partial z}{\partial y}$ lar hisoblansin.

25. Agar $y=\sin\sqrt{x}$ va $z=tg^2\left(x^2-y^2\right)$ bo'lsa, $\frac{\partial z}{\partial x}$ hisoblansin.

26. Agar $y=e^{-x}$ va $z=arctg\sqrt{x^2+y^2}$ bo'lsa, $\frac{\partial z}{\partial x}$ hisoblansin.

27. Agar $\sin xy-x^2-y^2=5$ bulsa, $\frac{\partial z}{\partial x}$ hisoblansin.

28. Agar $x^2y^2z^2+7y^4-8xz^3+z^4=10$ bo'lsa, $\frac{\partial z}{\partial x}$ va $\frac{\partial z}{\partial y}$ lar hisoblansin.

29. Agar $\sin^2(xy^2z^2)+4xyz=0$ bo'lsa, $\frac{\partial z}{\partial x}$ va $\frac{\partial z}{\partial y}$ lar hisoblansin.

30. Agar $x^3+y^3+z^3-xyz=2$ bo'lsa, $\frac{\partial z}{\partial x}$ va $\frac{\partial z}{\partial y}$ larning $M_0(1;1;1)$ nuqtadagi qiymatari hisoblansin.

### 3.4. Yuqori tartibli xususiy hosilalar va to'la differensiallar

1. $z=f(x,y)$ funksiyadan olingan ikkinchi tartibli xususiy hosilalar deb, uning birinchi tartibli xususiy hosilalaridan har ikkala o'zgaruvchilar bo'yicha olingan xususiy hosilalarga aytiladi va ular quyidagicha bo'ladilar

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial^2 z}{\partial x^2}=f''_{xx}(x,y);\qquad \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial^2 z}{\partial x\partial y}=f''_{xy}(x,y);$$

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial^2 z}{\partial y\partial x}=f''_{yx}(x,y);\qquad \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial^2 z}{\partial y^2}=f''_{yy}(x,y).$$

Uchinchi va undan yuqori tartibli xususiy hosilalar ham yuqoridagiga o'xshash aniqlanadi va belgilanadi, masalan:

$$\frac{\partial}{\partial x}\left(\frac{\partial^2 z}{\partial x^2}\right)=\frac{\partial^3 z}{\partial x^3}=f'''_{xxx}(x,y);\ \frac{\partial}{\partial y}\left(\frac{\partial^2 z}{\partial x^2}\right)=\frac{\partial^3 z}{\partial x^2\partial y}=f'''_{xxy}(x,y)\ \text{va hokazo}$$

Faqat hosilasining tartibi bilan farq qiladigan xususiy hosilalarni aralash xususiy hosilalar deyiladi. Aralash xususiy hosilalar uzluksiz funksiyalar bo'lsalar, u holda ular o'zaro teng bo'ladilar: $\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial^2 z}{\partial y\partial x}$; $\frac{\partial^3 z}{\partial x^2\partial y}=\frac{\partial^3 z}{\partial x\partial y\partial x}=\frac{\partial^3 z}{\partial y\partial x^2}$ va hokazo.

2. YUqori tartibli to'la differensiallar quyidagicha hisoblanadilar va belgilanadilar

$$d^2z=\frac{\partial^2 z}{\partial x^2}dx^2+2\frac{\partial^2 z}{\partial x\partial y}dxdy+\frac{\partial^2 z}{\partial y^2}dy^2;$$

$$d^3z=\frac{\partial^3 z}{\partial x^3}dx^3+3\frac{\partial^3 z}{\partial x^2 dy}dx^2dy+3\frac{\partial^3 z}{\partial x\partial^2 y}dxdy^2+\frac{\partial^3 z}{\partial y^3}dy^3\ \text{va hokazo.}$$

Umuman olganda, $d^n z=\left(\frac{\partial}{\partial x}dx+\frac{\partial}{\partial y}dy\right)^n z$ -simvolik formula o'rinli bo'lib, u binominal qonun bo'yicha ochiladi.

**1-Misol.** $z=x^3+x^2y+y^3$ funksiyaning uchinchi tartibli xususiy hosilalari topilsin.

**Yechish.**

$$\frac{\partial z}{\partial x}=3x^2+2xy,\ \frac{\partial z}{\partial y}=x^2+3y^2,\ \frac{\partial^2 z}{\partial x^2}=6x+2y,\ \frac{\partial^2 z}{\partial x\partial y}=2x,$$

$$\frac{\partial^2 z}{\partial y^2}=6y,\ \frac{\partial^2 z}{\partial y\partial x}=2x,\ \frac{\partial^3 z}{\partial x^3}=6,\ \frac{\partial^3 z}{\partial x^2\partial y}=2,\ \frac{\partial^3 z}{\partial x\partial y\partial x}=2,\ \frac{\partial^3 z}{\partial x\partial y^2}=0,\ \frac{\partial^3 z}{\partial y^2\partial x}=0,$$

$$\frac{\partial^3 z}{\partial y^3}=6,\ \frac{\partial^3 z}{\partial y\partial x^2}=2,\ \frac{\partial^3 z}{\partial y\partial x\partial y}=0.$$

**2-Misol.** $z=arctg\frac{y}{x}$ funksiyaning ikkinchi tartibli xususiy hosilalari topilsin.

**Yechish.**

$$\frac{\partial z}{\partial x}=-\frac{y}{x^2+y^2},\ \frac{\partial z}{\partial y}=\frac{x}{x^2+y^2},$$

$$\frac{\partial^2 z}{\partial x^2}=\frac{2xy}{\left(x^2+y^2\right)^2},\ \frac{\partial^2 z}{\partial x\partial y}=\frac{\partial^2 z}{\partial y\partial x}=\frac{y^2-x^2}{\left(x^2+y^2\right)^2},\ \frac{\partial^2 z}{\partial y^2}=-\frac{2xy}{\left(x^2+y^2\right)^2}.$$

**3-Misol.** $z=\sin x\cos y$ funksiyaning ikkinchi tartibli to'la differensiali topilsin.

**Yechish.** $\frac{\partial z}{\partial x}=\cos x\cos y,\ \frac{\partial z}{\partial y}=-\sin x\sin y,$

$$\frac{\partial^2 z}{\partial x^2}=-\sin x\cos y,\ \frac{\partial^2 z}{\partial x\partial y}=-\cos x\sin y,\ \frac{\partial^2 z}{\partial y^2=-\sin x\cos y},$$

$$d^2z=-\sin x\cos y dx^2-2\cos x\sin y dxdy-\sin x\cos y dy^2.$$

**4-Misol.** $z=x^2y$ funksiyaning ikkinchi tartibli to'la differensiali topilsin.

**Yechish.** $\frac{\partial z}{\partial x}=2xy,\ \frac{\partial z}{\partial y}=x^2,\ \frac{\partial^2 z}{\partial x^2=2y},\ \frac{\partial^2 z}{\partial x\partial y}=2x,\ \frac{\partial^2 z}{\partial y^2}=0,$

$$d^2z=2ydx^2+4xydxdy.$$

## MASHQLAR

1. $z=x^4+3x^2y^2-2y^2$ funksiyaning 4-tartibli xususiy hosilalari topilsin.
2. $z=xy+\sin(x+y)$ funksiyaning 3-tartibli xususiy hosilalari topilsin.
3. $z=x^2\ln(x+y)$ funksiyaning 2-tartibli xususiy hosilalari topilsin.
4. $z=x\ln\frac{y}{x}$ funksiyaning 2-tartibli to'liq differensiali topilsin.
5. $z=e^{xy}$ funksiyaning 2-tartibli to'liq differensiali topilsin.

6. $z=\cos(x+y)$ funksiyaning 2-tartibli to'liq differensiali topilsin.

7. $z=e^{x}(x\cos y-y\sin y)$ funksiyaning $\dfrac{\partial^2 z}{\partial x^2}+\dfrac{\partial^2 z}{\partial y^2}=0$ tenglamani qanoatlantirishi isbotlansin.

8. $z=\ln\left(x^2+y^2\right)$ funksiyaning 2-tartibli xususiy hosilalari topilsin.

9. $z=arctg\dfrac{y}{x}$ funksiyaning $\dfrac{\partial^2 z}{\partial x^2}+\dfrac{\partial^2 z}{\partial y^2}=0$ tenglamani qanoatlantirishi isbotlansin.

10. Agar $z=e^{x^2y}$ funksiya berilgan bo'lsa, $z'''_{xxy}(0;1)$ va $z'''_{xyy}(0;1)$ lar hisoblansin.

### 3.5. Sirtga urinma tekislik va normalning tenglamalari

**_Ta'rif._** *Sirtning $M_0(x_0,y_0,z_0)$ nuqtasidan uning hamma egri chiziqlariga o'tkazilgan urinma to'g'ri chiziqlar yotadigan tekislikni sirtning $M_0$ nuqtasidagi urinma tekisligi deb ataladi.*

$M_0$ nuqtada urinma tekkislikka o'tkazilgan perpendikulyar to'g'ri chiziqni sirtning $M_0$ nuqtasidagi normali deb ataladi.

Agarda sirtning tenglamasi $F(x,y,z)=0$ berilgan bo'lib, $M_0(x_0,y_0,z_0)$ nuqtada $\left.\dfrac{\partial F}{\partial x}\right|_{M_0}, \left.\dfrac{\partial F}{\partial y}\right|_{M_0}, \left.\dfrac{\partial F}{\partial z}\right|_{M_0}$ xususiy hosilalar chekli va bir vaqtning o'zida nolga aylanmasa, sirtning $M_0$ nuqtasidagi urinma tekislikning tenglamasi

$$\left(\frac{\partial F}{\partial x}\right)_{M_0}(x-x_0)+\left(\frac{\partial F}{\partial y}\right)_{M_0}(y-y_0)+\left(\frac{\partial F}{\partial z}\right)_{M_0}(z-z_0)=0$$

ko'rinishda, normalining tenglamasi esa

$$\frac{x-x_0}{\left(\dfrac{\partial F}{\partial x}\right)_{M_0}}=\frac{y-y_0}{\left(\dfrac{\partial F}{\partial y}\right)_{M_0}}=\frac{z-z_0}{\left(\dfrac{\partial F}{\partial z}\right)_{M_0}}$$

ko'rinishda yoziladi.

Agar sirtning tenglamasi $z=f(x,y)$ ko'rinishda berilgan bo'lsa, urinma tekislikning tenglamasi

$$z-z_0=\left(\frac{\partial z}{\partial x}\right)_{M_0}(x-x_0)+\left(\frac{\partial z}{\partial y}\right)_{M_0}(y-y_0)$$

ko'rinishda, normalining tenglamasi esa

$$\frac{x-x_0}{\left(\frac{\partial z}{\partial x}\right)_{M_0}}=\frac{y-y_0}{\left(\frac{\partial z}{\partial y}\right)_{M_0}}=\frac{z-z_0}{-1}$$

ko'rinishda bo'ladi.

**1-Misol.** $z=x^2+2y^2$ sirtning $M_0(1;1;3)$ nuqtasidagi urinma tekisligi va normalining tenglamalari yozilsin.

**Yechish.**

$$\left(\frac{\partial z}{\partial x}\right)_{M_0}=2x\Big|_{x=1}=2,\left(\frac{\partial z}{\partial y}\right)_{M_0}=4y\Big|_{y=1}=4$$

larga ko'ra, urinma tekislikning tenglamasi $z-3=2(x-1)+4(y-1)$ yoki $2x+4y-z-3=0$ ko'rinishda bo'ladi. Normalining tenglamasi esa, $\frac{x-1}{2}=\frac{y-1}{4}=\frac{z-3}{-1}$ dir.

**2-Misol.** $x^2+4y^2+z^2=36$ sirtning $x+y-z=0$ yoki slikka parallel bo'lgan urinma tekisligi topilsin.

**Yechish.**

Sirtning tenglamasini $F(x,y,z)=x^2+4y^2+z^2-36=0$ ko'rinishda yozib olamiz. $\frac{\partial F}{\partial x}2x, \frac{\partial F}{\partial y}=8y, \frac{\partial F}{\partial z}=2z$. Urinma tekislik berilgan tekislikka parallel bo'lgani uchun ularning normallarining proektsiyalari o'zaro proportsional bo'ladi, ya'ni $\frac{2x}{1}=\frac{8y}{1}=\frac{2z}{-1}$ yoki $\frac{x}{1}=\frac{4y}{1}=\frac{z}{-1}$ to'g'ri chiziq berilgan sirtni ikkita nuqtada kesib o'tadi. Bu nuqtalarni topish uchun $x^2+4y^2+z^2=36$ va $\frac{x}{1}=\frac{4y}{1}=\frac{z}{-1}$ tenglamalarni birgalikda echamiz: $z=-x, z=-4y$.

$z^2+4\frac{z^2}{16}+z^2=36,$ yoki $z^2+\frac{z^2}{4}+z^2=36, z^2=16, z_{1,2}=\pm 4,$ $x_{1,2}=\mp 4, y_{1,2}=\mp 1$. Demak, urinma tekisliklar $M_1(-4;-1;4)$ va $M_2(4;1;-4)$ nuqtalardan o'tadi

a) $(x+4)+1\cdot(y+1)-1\cdot(z-4)=0$ yoki $x+y-z+9=0$,

b) $(x-4)+1\cdot(y-1)-1\cdot(z+4)=0$ yoki $x+y-z-9=0$

Javob. Urinma tekislik ikkita ekan: $x+y-z=\pm 9$.

## MASHQLAR

1. $z=x^2-4xy+y^2-x+4y$ sirtning $M_0(1,1,1)$ nuqtadagi urinma tekisligi va normali topilsin.

2. $x^2+2y^2+3z^2=11$ sirtning $x+y+z=1$ tekislikka parallel bo'lgan urinma tekisligi topilsin.

3. $z=1+x^2+y^2$ sirtning $M_0(1;1;3)$ nuqtadagi urinma tekisligi va normali topilsin.

4. $x^2+2y^2+3z^2=21$ sirtning $x+4y+6z=0$ tekislikka parallel bo'lgan urinma tekisliklari topilsin.

5. $z=\sin x\cos y$ sirtning $M_0\left(\frac{\pi}{4};\frac{\pi}{4};\frac{1}{2}\right)$ nuqtadagi urinma tekisligi va normali topilsin.

## 3.6. Ikki o'zgaruvchili funksiyaning ekstremumi hamda eng kichik va eng katta qiymatlari

1. $z=f(x;y)$ ikki o'zgaruvchili funksiya va $M_0(x_0;y_0)$ nuqta berilgan bo'lsin.

Agarda $M_0(x_0;y_0)$ nuqtaning yetarlicha kichik atrofida $f(x_0;y_0)>f(x;y)$ shart bajarilsa, $f(x;y)$ funksiya $M_0$ nuqtada maksimumga ega deb atalib, agarda $f(x_0;y_0)<f(x;y)$ shart bajarilsa, $f(x;y)$ funksiya $M_0$ nuqtada minimumga ega deb ataladi.

Funksiyaning maksimum va minimumini uning ekstremumi deyiladi.

Agarda differensiallanuvchi $z=f(x;y)$ funksiya $M_0(x_0;y_0)$ nuqtada ekstremumga erishsa, u holda bu nuqtada $\frac{\partial f(x_0,y_0)}{\partial x}=0$, $\frac{\partial f(x_0,y_0)}{\partial y}=0$ yoki ulardan hech bo'lmaganda birortasi mavjud bo'lmaydi (ekstremumning zaruriylik sharti).

Funksiyaning xususiy hosilalarini nolga aylantiradigan yoki mavjud bo'lmaydigan hamda uning aniqlanish sohasida yotadigan nuqtalarni kritik yoki statsionar nuqtalar deyiladi.

$M_0(x_0,y_0)$ nuqtaning atrofida $z=f(x,y)$ funksiya ikki marta differensiallanuvchi bo'lib, bu nuqtada $\frac{\partial f(x_0,y_0)}{\partial x}=0$, $\frac{\partial f(x_0,y_0)}{\partial y}=0$ shartlar bajarilsin.

Quyidagicha belgilashlarni kiritamiz:

$$A=\frac{\partial^2 f(x_0,y_0)}{\partial x^2},\ B=\frac{\partial^2 f(x_0,y_0)}{\partial x\partial y},\ C=\frac{\partial^2 f(x_0,y_0)}{\partial y^2}.$$

U holda:

1. $AC-B^2>0$ va $A<0$ shartlar bajarilsa, $f(x,y)$ funksiya $M_0$ nuqtada maksimumga erishadi;

2. $AC-B^2>0$ va $A>0$ shartlar bajarilsa, $f(x,y)$ funksiya $M_0$ nuqtada minimumga erishadi,

3. $AC-B^2<0$ bo'lsa $f(x,y)$ funksiya ekstremumga erishmaydi;

4. $AC-B^2=0$ bo'lsa $f(x,y)$ funksiya ekstremumga erishishi ham mumkin, erishmasligi ham mumkin (qo'shimcha tekshirishlar talab etiladigan hol).

**1-Misol.** $z=x^2-xy+y^2+9x-6y+20$ funksiyaning ekstremumi topilsin.

__Yechish.__ Birinchi tartibli xususiy hosilalarini topamiz:

$\frac{\partial z}{\partial x}=2x-y+9, \frac{\partial z}{\partial y}=-x+2y-6.$ Endi kritik nuqtalarini topamiz.

$\begin{cases}2x-y+9=0\\ x-2y+6=0\end{cases}$ tenglamalar sistemasidan $x=-4, y=1; M_0(-4;1)$ yagona kritik nuqta. Ikkinchi tartibli xususiy hosilalarning $M_0$ nuqtadagi qiymatlarini topamiz:

$$A=\frac{\partial^2 z}{\partial x^2}=2, B=\frac{\partial^2 z}{\partial x\partial y}=-1, C=\frac{\partial^2 z}{\partial y^2}=2.$$

$AC-B^2=2\cdot 2-(-1)^2=3>0, A>0$ lar bajarilganligi uchun berilgan funksiya $M_0(-4;1)$ nuqtada minimumga erishadi: $z_{min}=-1$.

**2-Misol.** $z=y\sqrt{x}-y^2-x+6y$ funksiyaning ekstremumi topilsin.

__Yechish.__ $\frac{\partial z}{\partial x}=\frac{y}{2\sqrt{x}}-1, \frac{\partial z}{\partial y}=\sqrt{x}-2y+6.$ $\begin{cases}\frac{y}{2\sqrt{x}}-1=0,\\ \sqrt{x}-2y+6=0,\end{cases}$

sistemani yechib, $M_0(4;4)$ kritik nuqta topiladi.

$A=\left.\frac{\partial^2 z}{\partial x^2}\right|_{M_0}=\left.-\frac{y}{4\sqrt{x^3}}\right|_{M_0}=-\frac{1}{8}, B=\left.\frac{\partial^2 z}{\partial x\partial y}\right|_{M_0}=\left.\frac{1}{2\sqrt{x}}\right|_{M_0}=\frac{1}{4}, C=\frac{\partial^2 z}{\partial z^2}=-2.$

$AC-B=\frac{3}{16}>0$ va $A=-\frac{1}{8}<0$ larga asosan berilgan funksiya $M_0(4;4)$ nuqtada maksimumga erishadi: $z_{max}=12$

## MASHQLAR

Quyidagi funksiyalarning ekstremumlari topilsin:

1. $z=x^2+8y^3-6xy+1.$
2. $z=2xy-4x-2y.$
6. $z=(x-5)^2+y^2-1.$
7. $z=xy6-x-y.$

3. $z = xy^2(1 - x - y)$.

4. $z = 3x^2 - 2x\sqrt{y}y - 8x + 8$.

5. $z = 1 + 6x - x^2 - xy - y^2$.

8. $z = 2xy - 2x^2 - 4y^2$.

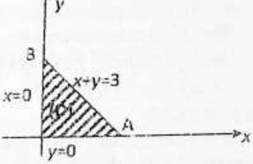
9. $z = x^3 + 8y^3 - 6xy + 1$.

10. $z = x^3 + y^3 - 3xy$.

2. Agar $z = f(x;y)$ funksiya chegaralangan yopiq $\overline{D}$ sohada differensiallanuvchi bo'lsa, o'zining eng kichik va eng katta qiymatlariga yoki $\overline{D}$ soha ichida yotuvchi kritik nuqtalarda yoki uning chegarasida erishishi mumkin.

Umuman, $z = f(x;y)$ funksiyaning chegaralanagan yopiq $\overline{D}$ sohadagi eng kichik va eng katta qiymatlarini topish uchun $\overline{D}$ soha ichida yotuvchi barcha kritik nuqtalarda funksiyaning qiymatlari hisoblanib, keyin esa $\overline{D}$ sohaning chegarasida funksiyaning eng kichik va eng katta qiymatlari hisoblanadi. Topilgan barcha qiymatlar o'zaro taqqoslanib, ular ichidan eng kichigi va eng kattasi tanlaniladi.

**1-Misol.** $f = x^2 - 2y^2 + 4xy - 6x + 5$ funksiyaning $x = 0, y = 0,\ \ x + y = 3$ to'g'ri chiziqlar bilan chegaralangan sohadagi eng kichik va eng katta qiymatlari hisoblansin.




<u>**Yechish.**</u> $\begin{cases} \dfrac{\partial z}{\partial x} = 2x + 4y - 6 = 0, \\ \dfrac{\partial z}{\partial x} = -4y + 4x = 0 \end{cases}$

dan $x = 1, y = 1$ larni aniqlaymiz.

Demak, $(1,1)$ nuqta kritik nuqta ekan, $z_1(1;1) = 2$. Qaralayotgan funksiyani $D$ sohaning chegarasida tekshiramiz.

$OA$ kesmada $y = 0$ bo'lgani uchun $z = x^2 - 6x + 5$.

Bu funksiyaning $[0;3]$ kesmadagi eng kichik va eng katta qiymatlarni topamiz: $z_1' = 2x - 6 = 0, x = 3$ ва $z_2(3;0) = -4$.

$OB$ kesmada: $x = 0$ ва $z = -2y^2 + 5; z_1' = -4y = 0$ dan $y = 0, z_3(0;0) = 5$.

$AB$ kesmada: $x + y = 3$. U holda $z = -5x^2 + 18x - 13$,

$z_x' = -10x + 18 = 0$ dan $x = \dfrac{9}{5}$ va $z_4\left(\dfrac{9}{5}\right) = \dfrac{16}{5}$.

Barcha topilgan qiymatlarni o'zaro taqqoslab eng kichik $z(3;0) = -5$ va eng katta $z(0;0) = 5$ larni hosil qilamiz.

## MASHQLAR

Quyidagi funksiyalarning ko‘rsatilgan sohalardagi, eng kichik va eng katta qiymatlari hisoblansin.

1. $z = 3x + y - xy,$ $\quad \overline{D}: y = x, x = 0, y = 4.$
2. $z = x^2 + 2xy - y^2 - 4x,$ $\quad \overline{D}: x = 3, x - y = 1.$
3. $z = x^2y(4 - x - y),$ $\quad \overline{D}: x = 0, y = 0, x + y = 6.$
4. $z = 4 - 2x^2 - y^2,$ $\quad \overline{D}: y = x, y = \sqrt{1 - x^2}.$
5. $z = x^2 + 2xy + 4x - y^2,$ $\quad \overline{D}: x = 0, y = 0, x + y = -2.$
6. $z = x^3 + y^3 - 3xy,$ $\quad \overline{D}: x = 0, x = 2, y = -1, y = 2.$
7. $z = xy - 3x - 2y,$ $\quad \overline{D}: x = 0, x = 4, y = 0, y = 4.$
8. $z = x^2 + xy - 2,$ $\quad \overline{D}: y = 0, y = 4x^2 - 4.$
9. $z = x^3 + y^3 - 3xy,$ $\quad \overline{D}: x = 0, x = 2, y = -1, y = 2.$
10. $z = x^2 - y^2.$ $\quad \overline{D}: x^2 + y^2 \le 1.$

### 3. Shartli ekstremum.

***Ta’rif.*** $z = f(x, y)$ *funksiyaning shartli ekstremumi deb,* $\varphi(x, y) = 0$ *shartni hisobga olib topilgan ekstremumiga aytiladi.*

Shartli ekstremumni topish uchun Lagranj funksiyasi deb ataladigan quyidagi yordamchi funksiya tuziladi:

$U(x, y, \lambda) = f(x, y) + \lambda\varphi(x, y),$ $\lambda$ -topilishi kerak bo‘lgan o‘zgarmas ko‘paytuvchi.

Lagranj funksiyasining ekstremumga erishishining zaruriy shartlari quyidagi ko‘rinishda bo‘ladi.

$$\begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial f}{\partial x} + \lambda \dfrac{\partial \varphi}{\partial x} = 0, \\ \dfrac{\partial u}{\partial y} = \dfrac{\partial f}{\partial y} + \lambda \dfrac{\partial \varphi}{\partial y} = 0, \\ \qquad \varphi(x, y) = 0. \end{cases}$$

Bu uchta tenglamalar sistemasini yechib noma’lumlar $x, y$ va $\lambda$ lar topiladi.

**1-Misol.** Hajmi $v$ ga teng bo‘lgan to‘g‘ri burchakli ochiq basseynning sirti eng kichik bo‘lishi uchun uning o‘lchovlari qanday bo‘lishi kerak?

**Yechish.** Basseynning bo‘yi, eni va balandligi $x$, $y$, $z$, lar orqali belgilaymiz. Masala, $S = xy + xz + 2yz$ funksiyaning $xyz = v$ shart bajarilganda minimumini topishga olib kelinadi. Lagranj funksiyasini topamiz $u(x, y, z\lambda) = xy + 2xz + 2yz + \lambda(xyz - v).$

Uning xususiy hosilalarini topamiz va ularni nolga tenglaymiz.

$$\begin{cases} \dfrac{\partial u}{\partial x} = y + 2z + \lambda yz = 0, \\ \dfrac{\partial u}{\partial y} = x + 2z + \lambda xz = 0, \\ \dfrac{\partial u}{\partial z} = 2y + 2x + \lambda xy = 0, \\ \qquad xyz - v = 0. \end{cases}$$

Bu sistemani yechish uchun birinchi tenglamani $x$ ga, ikkinchisini $y$ ga uchinchisini $z$ ga ko'paytirib qo'shamiz:

$$\begin{cases} xy + 2xz + \lambda xyz = 0, \\ xy + 2yz + \lambda xyz = 0, \\ 2yz + 2xz + \lambda xyz = 0. \end{cases}$$

Endi birinchi tenglamadan ikkinchi tenglamani ayiramiz:

$2xz - 2yz = 0$ yoki $(x-y)z = 0, z > 0$ bo'lgani uchun $x - y = 0$, bundan $x = y$.

Xuddi shunga o'xshab, ikkinchi tenglamadan uchinchisini ayiramiz:

$xy - 2xz = 0$ yoki $(y - 2z)x = 0, x > 0$ bo'lgani uchun $y - 2z = 0$, bundan $z = \dfrac{y}{2}$. $x$ va $z$ larning qiymatlarini sistemadagi to'rtinchi tenglamaga qo'yib topamiz

$$y \cdot y \cdot \frac{y}{2} = v,\ y^3 = 2v,\ y = \sqrt[3]{2v},\ z = 0.5\sqrt[3]{2v}.$$

Demak basseynning sirti eng kichik bo'lishi uchun bo'yi bilan eni teng bo'lishi, balandligi esa bo'yi yoki enining yarmiga teng bo'lishi kerak ekan. Savol tug'iladi, nima uchun aynan $x$, $y,z$ larning topilgan qiymatlarida sirt eng kichik bo'ladi? Boshqacha aytganda $S$ funksiya minimumga erishadi? Bu savolga javob berish uchun geometrik nuqtai nazardan mulohaza yuritamiz. Faraz qilaylik, agarda $x \to 0$ yoki $y \to 0$ bajarilsa $z \to \infty$, bundan ko'rinadiki sirt nihoyatda kattalashib ketadi. Aksincha, agarda $z \to 0$ bajarilsa, u holda x yoki y lar cheksizlikga intiladi, demak sirt yana kattalashadi. SHunday qilib sirt

$$x = y = \sqrt[3]{2v}, z = 0.5\sqrt[3]{2v}$$

qiymatlardagina eng kichik bo'ladi.

**2-Misol.** Yuzi $S$ ga teng bo'lgan hamma to'g'ri burchakli uchburchaklardan shunaqasi topilsinki, uning gipotenuzasi eng kichik qiymatga ega bo'lsin.

**<u>Yechish.</u>** Izlanayotgan uchburchakning katetlari $x$ va $y$ gipotenuzasi esa $z$ bo'lsin. $z^2 = x^2 + y^2$ bo'lgani uchun masala $x^2 + y^2$ funksiyaning

$\frac{xy}{2}=S$ shart bajarilgandagi eng kichik qiymatini topishga keltiriladi.
Lagranj funksiyasi

$$u(x,y,\lambda)=x^2+y^2+\lambda(xy-2S)$$

ni tuzamiz va uning xususiy hosilalarini topamiz.

$$\frac{\partial u}{\partial z}=2x+\lambda y,\ \frac{\partial u}{\partial z}=2y+\lambda x.$$

Endi
$$\begin{cases} 2x+\lambda y=0, \\ 2y+\lambda x=0, \\ \frac{xy}{2}=S. \end{cases}$$

sistemani yechib $z$, $y$, $\lambda$ larni topamiz.

Buning uchun $x>0,\ y>0$ larni olib sistemaning birinchi tenglamasini $x$ ga, ikkinchisini esa $y$ ga ko'paytirib ayiramiz

$2x^2+\lambda xy-2y^2-\lambda xy=0, 2x^2-2y^2=0, x^2=y^2$, bundan $x=y, \lambda=-2$.

Shunday qilib, $x=y=\sqrt{2S}$, ya'ni uchburchakning katetlari o'zaro teng bo'lsa gipotenuza eng kichik qiymatga ega bo'ladi.

## MASHQLAR

1. To'la sirt $S$ ga teng bo'lgan tsilindrning hajmi eng katta bo'lishi uchun uning o'lchovlari (radius va balandligi) qanday bo'lishi kerak.

2. $z=x^2+y^2$ funksiyaning $\frac{x}{4}+\frac{y}{3}=1$ shartdagi ekstremumi topilsin.

3. Doira ichiga chizilgan barcha uchburchaklarning shundayini topinki, uning yuzi eng katta bo'lsin.

4. $z=x+2y$ funksiyaning $x^2+y^2=5$ shartdagi ekstremumi topilsin.

5. $z=xy^2$ funksiyaning $x+2y=1$ shartdagi ekstremumi topilsin.

# 4. BIRINCHI TARTIBLI ODDIY DIFFERENSIAL TENGLAMALAR

## 4.1. Differensial tenglamalarga doir asosiy tushunchalar

1. Erkli o'zgaruvchilar va shu erkli o'zgaruvchilarga bog'liq bo'lgan noma'lum funksiya hamda uning turli tartibdagi hosilalari (yoki differensiallari)ni bog'lovchi munosabat differensial tenglama deb yuritiladi.

Agar differensial tenglama tarkibidagi noma'lum funksiya bitta erkli o'zgaruvchigagina bog'liq bo'lsa, unday differensial tenglamani oddiy differensial tenglama deb atalib, agar differensial tenglama tarkibidagi noma'lum funksiya bittadan ortiq erkli o'zgaruvchilarga bog'liq bo'lsa, u differensial tenglamani xususiy hosilali differensial tenglama deb yuritiladi. Biz bundan buyon faqat oddiy differensial tenglamalarnigina o'rganamiz.

Diferentsial tenglamaning tartibi deyilganda uning tarkibidagi hosilalarning eng yuqori tartibi tushuniladi.

Masalan, $x^2y''+5xy'-y^3=0$ – ikkinchi tartibli, $x(1+y^2)dx+y(1-x^2)dy=0$ esa, birinchi tartibli differensial tenglamalardir va hokazo.

Umuman, birinchi tartibli differensial tenglamaning umumiy ko'rinishi $F(x;y,y')=0$ kabi yoziladi (bu yerdagi F o'zining argumentlariga nisbatan uzluksiz funksiyadir).

Agar uni $y'$ga nisbatan yechish mumkin bo'lsa, yechib, $y'=f(x;y)$ ni hosil qilamiz.

Mazkur tenglama hosilaga nisbatan yechilgan birinchi tartibli differensial tenglama deb yuritiladi. Shuningdek,

$$P(x;y)dx+Q(x;y)dy=0$$

ni differensial shakldagi birinchi tartibli differensial tenglama deb ataladi.

Birinchi tartibli differensial tenglamaning yechimi yoki integrali deb, uni ayniyatga aylantiradigan har qanday differensialanuvchi $y=\varphi(x)$ kabi funksiyaga aytiladi va u yechimning grafigini integral egri chiziq deyiladi.

Birinchi tartibli differensial tenglamaning umumiy yechimi deb, shunday bir $y=\varphi(x;C)$ funksiyaga aytiladiki (bunda S-ixtiyoriy o'zgarmas son), u quyidagi shartlarga bo'ysunadi:

1. u ixtiyoriy o'zgarmas S ning har qanday qiymatida ham tenglamani qanoatlantiradi;

2. boshlang'ich shartlar deb ataluvchi $x=x_0$ bo'lganda $y=y_0$ bo'ladigan qo'shimcha shartlar qanday bo'lmasin S ning shunday muayyan $S_0$ qiymatini topish mumkinki, $y=\varphi(x;C_0)$ funksiya berilgan boshlang'ich shartni qanoatlantiradi, ya'ni $y_0=\varphi(x_0;C_0)$.

Umumiy yechimdan ixtiyoriy o'zgarmas S ning mumkin bo'lgan qiymatlarida hosil qilinadigan yechimlar xususiy yechimlar deyiladi.

Agar umumiy yoki xususiy yechimlar oshkormas funksiyalar holida berilsalar, ularni mos ravishda umumiy yoki xususiy integrallar deb yuritiladi.

Umumiy yechim (yoki umumiy integral ) geometrik jihatdan bitta S parametrga bog'liq integral egri chiziqlar oilasi bilan tasvirlanadi. Xususiy yechim (yoki xususiy integral) bu oilaning integral egri chiziqlaridan biridir.

Birinchi tartibli $y'=f(x;y)$ differensial tenglamaning berilgan $y(x_0)=y_0$ kabi boshlang'ich shartlarni qanoatlantiruvchi xususiy yechimini topish masalasi, odatda Koshi masalasi deb yuritiladi.

**1-Misol.** $xy'=2y$ tenglamaning yechimi $y=5x^2$ ekanligi ko'rsatilsin.

**Yechish.** $y'=10x$ bo'lganligidan, $x\cdot 10x=2\cdot 5x^2$ yoki $10x^2=10x^2$

**2-Misol.** $y=Ce^x$ egri chiziqlar oilasining differensial tenglamasi tuzilsin.

**Yechish.** $y=Ce^x$ bo'lganligidan, $\frac{y'}{y}=1$ yoki $y'=y$ ni hosil qilamiz. Mazkur tenglamani $y=Ce^x$ qanoatlantirishini tekshirish qiyin emas.

**3-Misol.** $x^2-y^2=C$ kabi egri chiziqlar oilasidan $y(0)=5$ boshlang'ich shartlarni qanoatlantiruvchi egri chiziq topilsin.

**Yechish.** $x=0$ bo'lganda $y=5$ bo'lganligidan, $0-25=C$ va $C=-25$. U holda, $y^2-x^2=25$ hosil bo'ladi.

Quyida berilgan differensial tenglamalar uchun ko'rsatilgan funksiyalar yechim bo'la olishligi ko'rsatilsin.

1. $y'=\frac{y}{x}$, y = 10x;
2. $y'+y=0$, $y=3e^{-x}$
3. $xy'+y=0$, $y=\frac{6}{x}$;
4. $yy'+xy-3x^3=0$, $y=x^2$
5. $(x+y)y'-2y=0$, $y=x$;

Quyidagi tenglamalar bilan berilgan egri chiziqlar oilasining differensial tenglamalari tuzilsin.

1. $y = Cx^3$
2. $y = Sin(x + C)$
3. $x^2 + Cy^2 = 2y$
4. $y^2 + Cx = x^3$
5. $x^2 + y^2 = C^2$

## 4.2. O'zgaruvchilari ajraladigan differensial tenglamalar

1. Ushbu

$$M(x)dx + N(y)dy = 0$$

tenglama, o'zgaruvchilari ajralgan differensial tenglama deb yuritiladi. Mazkur tenglamaning umumiy integralini topish uchun uni hadlab integrallaymiz:

$$\int M(x)dx + \int N(y)dy = C$$

Bu yerdagi S ni berilgan tenglama uchun qulay bo'lgan istalgan ko'rinishda tanlanadi.

**1-Misol.** $xdx - \frac{ydy}{1+y^2} = 0$ differensial tenglamaani yeching.

<u>**Yechish.**</u>U o'zgaruvchilari ajralgan tenglamadir. Tenglamani hadlab integrallasak,

$$\frac{x^2}{2} - \frac{1}{2}\ln(1 + y^2) = C$$

ni hosil qilamiz. Yuqorida aytilgan mulohazaga asosan, S ning o'rniga $\frac{1}{2}\ln C$ i tanlab $x^2 = \ln C(1 + y^2)$ni topamiz.

2. Agar hosilaga nisbatan yechilgan birinchi tartibli $y' = f(x;y)$ kabi tenglamaning o'ng tomonidagi funksiya faqat bitta x yoki y o'zgaruvchiga bog'liq bo'lgan funksiyalarning ko'paytmasi (yoki nisbati) ko'rinishida berilgan bo'lsa, uni o'zgaruvchilari ajraladigan tenglama deb ataladi, ya'ni:

$$y' = \varphi(x) \cdot q(y)$$

Bu yerda, $y' = \frac{dy}{dx}$ va $q(y) \neq 0$ ekanliklarini e'tiborga olsak, $\frac{dy}{q(y)} = \varphi(x)dx$ ni hosil qilamiz. Bu esa, o'zgaruvchilari ajralgan tenglamadir. Uni integrallab,

$$\int \frac{dy}{q(y)} = \int \varphi(x)dx + C$$

ko'rinishdagi umumiy integral topiladi.

Quyidagi

$$f_1(x)\cdot f_2(y)dx + \varphi_1(x)\cdot\varphi_2(y)dy = 0$$

tenglama ham o'zgaruvchilari ajraladigan tenglamadir. Tenglikning har ikkala tomonini $\varphi_1(x)\cdot f_2(y) \neq 0$ ga hadlab bo'lib yuborib, o'zgaruvchilari ajralgan tenglama

$$\frac{f_1(x)}{\varphi_1(x)}dx + \frac{\varphi_2(y)}{f_2(y)}dy = 0$$

ni hosil qilamiz. Uni integrallab, umumiy integralini topamiz.

**2-Misol.** $x(1+y^2)dx + (1+x^2)ydy = 0$ tenglama yechilsin.

<u>**Yechish.**</u> O'zgaruvchilarni ajratish uchun tenglikning har ikkala tomonini $(1+x^2)(1+y^2) \neq 0$ ga bo'lib yuboramiz:

$$\frac{xdx}{1+x^2} + \frac{ydy}{1+y^2} = 0.$$

Mazkur o'zgaruvchilari ajralgan tenglamani hadlab integrallab, $\frac{1}{2}\ln(1+x^2) + \frac{1}{2}\ln(1+y^2) = C$ ni yoki $\ln[(1+x^2)(1+y^2)] = \ln C$ (bu yerda, 2S=lnC deb olinadi) ni hosil qilamiz. Bundan

$$(1+x^2)(1+y^2) = C$$

umumiy integralni hosil qilamiz..

**3-Misol.** $xyy' = 1 - x^2$ tenglama yechilsin.

<u>**Yechish.**</u> Tenglamani $y'$ga nisbatan echamiz

$$y' = \frac{1-x^2}{xy}.$$

Bu esa, o'zgaruvchilari ajraladigan tenglamadir. $y' = \frac{dy}{dx}$ ekanligi e'tiborga olinsa, $ydy = \frac{1-x^2}{x}dx$ hosil bo'ladi. U holda, $\frac{y^2}{2} = \ln|x| - \frac{x^2}{2} + C$ yoki $x^2 + y^2 = \ln(Cx^2)$ kabi umumiy integral topiladi (bu yerda 2S=lnC deb olinadi).

**4-Misol.** $y'tgx = y$ tenglamaning $y\left(\frac{\pi}{4}\right) = \sqrt{2}$ boshlang'ich shartni qanoatlantiruvchi xususiy yechimi topilsin.

<u>**Yechish.**</u> $\frac{dy}{dx} = \frac{y}{tgx}$ yoki $\frac{dy}{y} = ctgxdx$ dan $\ln|y| = \ln|Sinx| + \ln C$ ni va bundan $y = CSinx$ kabi umumiy yechimni topamiz.

$x=\frac{\pi}{4}$ bo'lganda $y=\sqrt{2}$ bo'lganligidan, $\sqrt{2}=CSin\frac{\pi}{4}$ yoki $\frac{\sqrt{2}}{2}C=\sqrt{2}$. Bundan esa, S=2. Demak, xususiy yechim $y=2Sinx$ ekan.

3. $y'=f(ax+by+c)$ ko'rinishdagi tenglamani qaraymiz. Ushbu tenglamada $ax+by+c=z$ almashtirish kiritamiz. Bundan, $a+by'=z'$ yoki $y'=\frac{z'-a}{b}$ bo'lganligi sababli, $\frac{1}{b}(z'-a)=f(z)$ yoki $z'=a+bf(z)$ ni hosil qilamiz. Bu esa, o'zgaruvchilari ajraladigan tenglamadir. Uni yechib, yechimning qiymatini $ax+by+c=z$ ga qo'ysak, dastlabki tenglamaning umumiy yechimi yoki umumiy integrali aniqlanadi.

**5-Misol.** $y'=\frac{1}{x+2y}$ tenglama yechilsin.

**Yechish.** $x+2y=z$, $1+2y'=z'$, $y'=\frac{1}{2}(z'-1)$ larni inobatga olsak, $\frac{1}{2}(z'-1)=\frac{1}{z}$ yoki $z'=\frac{2}{z}+1$ ni hosil qilamiz. $z'=\frac{dz}{dx}$ ligidan

$$\frac{dz}{dx}=1+\frac{2}{z}.$$

Bu yerda o'zgaruvchilarni ajratib integrallaymiz:

$\frac{zdz}{z+2}=dx,\ \int dz-2\int\frac{dz}{z+2}=\int dx+C$ yoki $z-2\ln|z+2|=x+C$

z ni x+2y bilan almashtirib, berilgan tenglamaning

$$y-\ln|x+2y+2|=C$$

kabi umumiy integralini topamiz.

## MASHQLAR

Quyidagi differensial tenglamalar yechilsin

1. $y'x^3=2y$
2. $(x^2+x)y'=2y+1$
3. $(1+x^2)y'=1+y^2$
4. $(1-y)dx+(1+x)dy=0$
5. $y'=2\sqrt{y}\ln x$
6. $2xyy'=1-y^2$
7. $3^{x-y}dx-4^{x+y}dy=0$
8. $ytgxdx+dy=0$
9. $\sqrt{1+y^2}dx-xdy=0$
10. $xy'=y^2-y$
11. $y'=\sqrt{\frac{y}{x}}$
12. $y'+y^2=1$
13. $y'=2x^25x+12$
14. $y'-y^2-3y+4=0$

21. $y'=\frac{1}{3x+y}$
22. $y'(x+y)=1$
23. $y'=2^{2y+3x}$
24. $y'=4x+3y$

15. $xydx = (1 + x^2)dy$

16. $y^2dx + (x-2)dy = 0$

17. $(1 + y^2)dx - \sqrt{x}dy = 0$

18. $\sqrt{1-x^2}dy - x\sqrt{1-y^2}dx = 0$

19. $(x^2 - yx^2)dy + (y^2 + xy^2)dx = 0$

20. $x^2dy - (2xy + 3y)dx = 0$

25. $y' = \sqrt{2x + y + 4}$

26. $y' = 3 - \dfrac{2}{x + 2y}$

27. $z' = 10^{x+z}$

28. $y' = Cos(y - x)$

29. $y' = \sqrt{4x + 2y - 1}$

30. $y' - y = 2x - 3$

Quyidagi differensial tenglamalarning berilgan boshlang'ich shartlarga binoan xususiy yechimlari yoki xususiy integrallari topilsin.

31. $(x^2 - 1)y' + 2xy^2 = 0;\ y(0) = 1$

32. $y'ctgx + y = 2;\ y(0) = -1$

33. $(1 + e^{2x})yy' = e^x;\ y(0) = 1$

34. $\dfrac{dy}{x^2} = \dfrac{dx}{y^2};\ y(0) = 2$

35. $(1 + x)ydx + (1 - y)xdy = 0;$ $y(1) = 1$

36. $y^2dx - e^xdy = 0;\ y(0) = 1$

37. $y' = 3\sqrt[3]{y^2};\ y(2) = 0$

38. $(x + 2y)y' = 1;\ y(0) = -1$

39. $(2)dy + (x - 1)dx = 0;\ y(0) = 4$

40. $(xy^2 + x)dx + (x^2y - y)dy = 0;$ $y(0) = 1$

### 4.3. Bir jinsli differensial tenglama va unga keltiriladigan diferentsial tenglamalar

1. **1- Ta'rif.** *Agar* $\lambda \neq 0$ *uchun* $f(\lambda x; \lambda y) = \lambda^m f(x; y)$ *kabi munosabat o'rinli bo'lsa,* $f(x; y)$*funksiyani o'z argumentlariga nisbatan m-o'lchovli bir jinsli funksiya deb ataladi (m-natural son).*

**2-Ta'rif.** *Agar* $y' = f(x; y)$ *tenglamadagi* $f(x; y)$ *funksiya o'z argumentlariga nisbatan 0 – o'lchovli bir jinsli funksiya bo'lsa, uni bir jinsli diferentsial tenglama deyiladi.*

Differensial shaklda berilgan $P(x; y)dx + Q(x; y)dy = 0$ tenglama bir jinsli bo'lishi uchun, ham $P(x; y)$, ham $Q(x; y)$ lar bir xil o'lchovli bir jinsli funksiyalar bo'lishlari kerak. Bir jinsli tenglamada $\frac{y}{x} = u$ (ёки $y = u \cdot x$) almashtirish kiritilib, uni yangi $u = u(x)$ noma'lum funksiyaga nisbatan o'zgaruvchilari ajraladigan tenglamaga keltiriladi.

**1-Misol.** $x^2 + y^2 - 2xyy' = 0$ tenglama yechilsin.

**Yechish.** Berilgan tenglamani $(x^2 + y^2)dx - 2xydx = 0$ ko'rinishda yozsak, $P(x; y) = x^2 + y^2$, $Q(x; y) = -2xy$ funksiyalarning ikki o'lchovli bir jinsli funksiyalar ekanligi ko'rinadi. Shuning uchun $y=ux,\ dy=xdu+udx$ almashtirishdan keyin berilgan tenglama

$$(x^2+u^2y^2)dx-2ux^2(xdu+udx)=0$$

yoki

$$(1-u^2)dx-2xudu=0$$

ko'rinishga keladi. O'zgaruvchilarni ajratib integrallasak,

$$\frac{dx}{x}-\frac{2udu}{1-u^2}=0$$

$$\int\frac{dx}{x}-\int\frac{2udu}{1-u^2}=\ln C$$

$$\ln|x|+\ln|1-u^2|=\ln C$$

ni yoki $x(1-u^2)=C$ ni hosil qilamiz. $u=\frac{y}{x}$ ni oxirgi tenglikka qo'yib,

$$x(1-\frac{y^2}{x^2})=C \text{ yoki } x^2-y^2=Cx$$

kabi umumiy integralni topamiz.

**2-Misol.** $y'=\frac{y}{x}+Sin\frac{y}{x}$ tenglamaning $y(1)=\frac{\pi}{2}$ kabi boshlang'ich shartdagi xususiy yechimi topilsin.

<u>**Yechish.**</u> $\frac{y}{x}=u,\ y=ux,\ y'=u+xu',\ u+xu'=u+Sinu$ yoki

$$x\frac{du}{dx}=Sinu,\ \frac{du}{Sinu}=\frac{dx}{x},$$

$$\int\frac{du}{Sinu}=\int\frac{dx}{x}+\ln C,$$

$$\ln\left|tg\left(\frac{u}{2}\right)\right|=\ln|x|+\ln C,$$

$$tg(\frac{u}{2})=Cx,\qquad \frac{u}{2}=arctg(Cx)\ .$$

yoki dastlabki o'zgaruvchiga qaytsak

$$y=2xarctg(Cx)$$

umumiy yechimini topamiz. Bu yechimdagi x, y larning o'rniga x=1, $y=\frac{\pi}{2}$ qiymatlarni qo'yib S=1 ni topamiz. Natijada, $y=2xarctgx$ - xususiy yechimdir.

**3-Misol.** $xy'=y(1+\ln\frac{y}{x})$ tenglama yechilsin.

<u>**Yechish.**</u> O'zgaruvchilarni almashtirib integrallasak

$$\frac{dy}{dx}=\frac{y}{x}(1+\ln\frac{y}{x}),$$

$$y=ux,\ \frac{dy}{dx}=u+x\frac{du}{dx},\ u+x\frac{du}{dx}=u(1+\ln u),\ x\frac{du}{dx}=u\ln u,$$

$$\frac{du}{u\ln u}=\frac{dx}{x},\ \int\frac{du}{u\ln u}=\int\frac{dx}{x}+\ln C,$$

$$\ln|\ln u|=\ln|x|+\ln C,$$

$$\ln u=Cx,\ u=e^{Cx},$$

Bu yerdan $\frac{y}{x}=e^{Cx}$, $y=xe^{Cx}$ kabi umumiy yechimni hosil qilamiz.

2. Agar $y'=f\left(\frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}\right)$ ko'rinishdagi tenglamada, $\Delta=\begin{vmatrix}a_1 & b_1\\ a_2 & b_2\end{vmatrix}\neq 0$ shart bajarilsa, $x=u+\alpha,\ y=v+\beta$ almashtirishlar orqali u va v yangi o'zgaruvchilarga nisbatan bir jinsli tenglama hosil qilinadi. Bu yerdagi α va β o'zgarmas sonlar quyidagi $\begin{cases}a_1\alpha+b_1\beta+c_1=0\\ a_2\alpha+b_2\beta+c_2=0\end{cases}$ tenglamalar sistemasidan aniqlanadi.

Agar $\Delta=0$ bo'lsa, qaralayotgan tenglamada $a_1x+b_1y=z$ kabi almashtirish kiritib, o'zgaruvchilari ajraladigan tenglamani hosil qilamiz.

**4-Misol.** $y'=\frac{x+3y+1}{3x+y+3}$ tenglama yechilsin.

__Yechish.__ $\Delta=\begin{vmatrix}1 & 3\\ 3 & 1\end{vmatrix}=-8\neq 0$ $x=u+\alpha$ va $y=v+\beta$ almashtirishlardan foydalanamiz.

$v'=\frac{u+\alpha+3v+3\beta+1}{3u+3\alpha+v+\beta+3}$ yoki $v'=\frac{u+3v+(\alpha+3\beta+1)}{3u+v+(3\alpha+\beta+3)}$

$\begin{cases}\alpha+3\beta+1=0\\ 3\alpha+\beta+3=0\end{cases}$ ni yechib, α= -1 va β=0 larni topamiz. U holda

$$\frac{dv}{du}=\frac{u+3v}{3u+v}$$

da $v=zu$ va $v'=z'u+z$ kabi almashtirishlarni kiritamiz: $z'u+z=\frac{u(1+3z)}{u(3+z)}$, $z'u=\frac{1+3z}{3+z}-z$, $\frac{dz}{du}u=\frac{1+3z-3z-z^2}{3+z}$ yoki

$$\frac{du}{u}=\frac{(z-3)dz}{1-z^2}.$$

Uni integrallab, $-\frac{1}{2}\ln\left|1-z^2\right|+\frac{3}{2}\left|\frac{1+z}{1-z}\right|=\ln|u|+\ln C$ ni topamiz. Oxirgi ifodani soddalashtirib, $\frac{1+z}{(1-z)^2}=Cu$ ni va bu yerda z ni $\frac{v}{u}$ bilan almashtirsak, $\frac{u+v}{(u-v)^2}=C$ ni hosil qilamiz. Oxirgi tenglikdan, $u=x+1$ va $v=y$ larga ko'ra,

$$x+y+1=C(x-y+1)^2$$

kabi umumiy integralni aniqlaymiz.

**5-Misol.** $y'=\frac{x+2y+1}{2x+4y+3}$ tenglama yechilsin.

**Yechish.** $\Delta=\begin{vmatrix}1 & 2\\ 2 & 4\end{vmatrix}=0$ bo'lganligi uchun $x+2y=z$ almashtirish kiritamiz. Bundan $1+2y'=z'$ yoki $y'=\frac{1}{2}(z'-1)$. U holda berilgan tenglama

$$\frac{1}{2}(z'-1)=\frac{z+1}{2z+3}$$

yoki

$$z'=\frac{4z+5}{2z+3}$$

ko'rinishga keladi. Bu yerda, o'zgaruvchilarni ajratgandan so'ng integrallab

$$z+\frac{1}{4}\ln|4z+5|=2x+C$$

ni topamiz. z ni x+2y bilan almashtirsak, $2y+\frac{1}{4}\ln|4x+8y+5|=x+C$ yoki

$$\ln|4x+8y+5|+8y-4x=C$$

ni hosil qilamiz. Bu esa, berilgan differensial tenglamaning umumiy integralidir.

## MASHQLAR

Quyidagi differensial tenglamalarning umumiy yechimlari (yoki umumiy integrallari) topilsin.

1. $y'=-\frac{x+y}{x}$
2. $y'=\frac{x+y}{x-y}$

14. $y'=\frac{x-y-1}{x-y-2}$
15. $y'=\frac{2x+3y-5}{x+4y}$

3. $y'=\dfrac{y}{x-2\sqrt{xy}}$

4. $y'=\dfrac{xy+y^2}{x^2}$

5. $y'=\dfrac{x(x+y)}{y^2}$

6. $(x^2+y^2)y'=2xy$

7. $xy'-y=xtg\dfrac{y}{x}$

8. $2x^3y'=y(2x^2-y^2)$

9. $y^2+x^2y'=xyy'$

10. $xy'-y=(x+y)\ln\dfrac{x+y}{x}$

11. $xy'=y-xe^{\frac{y}{x}}$

12. $xy'=yCos(\ln\dfrac{y}{x})$

13. $xy'-y=\sqrt{x^2-y^2}$

16. $(2x-4y+6)dx+(x+y-3)dy=0$

17. $(2x+y+1)dx-(4x+2y-3)dy=0$

18. $y'=\dfrac{y+2}{2x+y-4}$

19. $xy'=xSin\dfrac{y}{x}+y$

20. $y'=e^{\frac{y}{x}}+\dfrac{y}{x}+1$

21. $y'=\dfrac{-(3x+3y-1)}{x+y+1}$

22. $y'=\dfrac{2y-x-5}{2x-y+4}$

23. $(x^2-3y^2)dx+2xydy=0$

24. $y'=\dfrac{y}{x}(1+\ln y-\ln x)$

25. $(8y+10x)dx+(5y+7x)dy=0$

Quyidagi differensial tenglamalarning berilgan boshlang'ich shartlarga nisbatan xususiy yechimlari (yoki xususiy integrallari)topilsin.

26. $y'=\dfrac{x^3+y^3}{xy^2},\ y(1)=3$

27. $y'=\dfrac{x^2+y^2}{xy},\ y(0)=0$

28. $y'=\dfrac{y}{x}+Sin\dfrac{y}{x},\ y(1)=\dfrac{\pi}{2}$

29. $2xyy'+x^2-y^2=0,\ y(0)=0$

30. $y'=\dfrac{y^2}{x^2}-\dfrac{y}{x},\ y(-1)=1$

## 4.4. Chiziqli tenglamalar. Bernulli tenglamasi

**1.** Noma'lum funksiya va uning hosilasiga nisbatan chiziqli bo'lgan differensial tenglama birinchi tartibli chiziqli differensial tenglama deb ataladi. Uning umumiy ko'rinishi quyidagicha yoziladi:

$$y'+P(x)y=Q(x)$$

Bu yerdagi P(x) va Q(x) funksiyalar, o'z argumentlariga nisbatan uzluksiz funksiyalardir (xususan, o'zgarmas sonlar ham bo'lishlari mumkin).

$$y'+P(x)y=0$$

ni qaralayotgan tenglamaga mos bo'lgan chiziqli bir jinsli differensial tenglama deb ataladi.

Chiziqli bir jinsli bo'lmagan tenglamani yechishning ikki xil usuli mavjud bo'lib, ulardan biri «ixtiyoriy o'zgarmasni variatsiyalash» usulidir (ikkinchisi o'rniga qo'yish usuli deb yuritiladi).

Biz bu yerda birinchi usulni bayon etish bilan cheklanamiz (ikkinchi usul tafsilotini har bir kitobxon mavjud adabiyotlardan mustaqil o'rganishlari mumkin).

Mazkur usulga binoan, avvalo, bir jinsli tenglamaning umumiy yechimi $y=Ce^{-\int P(x)dx}$ ni aniqlab, bundagi o'zgarmas S ni vaqtinchalik o'zgaruvchi (S=S(x)) deb faraz qilamiz. So'ngra, $y$ bilan $y'$ larning qiymatlarini bir jinsli bo'lmagan tenglamaga qo'yib, C(x) ning qiymatini aniqlaymiz va uning qiymatini $y=Ce^{-\int P(x)dx}$ ga qo'yamiz. Natijada, berilgan chiziqli bir jinsli bo'lmagan tenglamaning umumiy yechimi aniqlanadi.

**1-Misol.** $y'-tgxy=Sinx$ tenglama yechilsin.

<u>Yechish.</u> $y'-tgxy=0$ dan, $\frac{dy}{y}=tgxdx$ ni va uni integrallab,

$$y=C\frac{1}{Cosx}$$

ni topamiz. C=C(x) deb faraz qilsak, u holda

$$y'=\frac{C'}{Cosx}+\frac{CSinx}{Cos^2x}$$

hosil bo'ladi. $y$ bilan $y'$ larni keltirib tenglamaga qo'yamiz:

$$\frac{C'}{Cosx}+\frac{CSinx}{Cos^2x}-\frac{Sinx}{Cosx}\cdot\frac{C}{Cosx}=Sinx$$

yoki $C'=SinxCosx$.

$C'=\frac{dC}{dx}$ ekanligini nazarda tutgan holda, $C(x)=-\frac{1}{4}Cos2x+C$ ni aniqlaymiz. Natijada,

$$y=(-\frac{1}{4}Cos2x+C)\frac{1}{Cosx}$$

kabi umumiy yechim hosil bo'ladi.

**2-Misol.** $y'+y=e^{2x}$ tenglamaning $y(0)=1$ boshlang'ich shartni qanoatlantiruvchi xususiy yechimi aniqlansin.

**Yechish.** $y'+y=0$ dan, $\frac{dy}{dx}=-y$ ni yoki

$$\ln|y|=-x+\ln C$$

ni hosil qilamiz. Potentsirlab, $y=C(x)e^{-x}$ ni topamiz.

$$y'=C'e^{-x}-Ce^{-x}$$

bo'lganligi sababli, $C'e^{-x}-Ce^{-x}+Ce^{-x}=e^{2x}$ va $C'=e^{3x}$ dir. Bundan,

$$C(x)=\frac{1}{3}e^{3x}+C_1$$

ni topib, $y=(\frac{1}{3}e^{3x}+C_1)e^{-x}$ ni yoki

$$y=\frac{1}{3}e^{2x}+Ce^{-x}$$

ni aniqlaymiz.

Endi boshlang'ich shartga ko'ra, $1=\frac{1}{3}e^{0}+Ce^{0}$ va $C=\frac{2}{3}$. Natijada

$$y=\frac{1}{3}e^{2x}+\frac{2}{3}e^{-x}.$$

**2.** Ushbu $y'+P(x)y=Q(x)y^{n}$ ko'rinishdagi tenglamada $n\neq 0$ va $n\neq 1$ bo'lsa, uni Bernulli tenglamasi deb ataladi. Agar bu yerda $n=0$ bo'lsa, chiziqli va agar $n=1$ bo'lsa, o'zgaruvchilari ajraladigan tenglamalar hosil bo'ladi.

Bernulli tenglamasini yechish uchun uning har bir hadini $y^{-n}$ ga ko'paytirib, so'ngra $y^{1-n}=z$ almashtirish orqali uni noma'lum z=z(x) funksiyaga nisbatan chiziqli tenglamaga keltiriladi. Chiziqli tenglama yechilgandan keyin, u yerda dastlabki o'zgaruvchiga qaytiladi.

Ta'kidlash joizki, Bernulli tenglamasini $y=u(x)v(x)$ almashtirish orqali ham yechish mumkin.

**3-Misol.** $y'+\frac{y}{x}=\frac{\ln x}{x}y^{2}$ kabi Bernulli tenglamasi yechilsin.

**Yechish.** Tenglamaning har bir hadini $y^{-2}$ ga hadlab ko'paytirib,

$$y^{-2}y'+\frac{1}{x}y^{-1}=\frac{\ln x}{x}$$

ni hosil qilamiz. $y^{-1}=z$ almashtirish kiritamiz. U holda: $-y^{-2}y'=z'$ yoki $y^{-2}y'=-z'$. Bularni tenglamaga qo'ysak, $-z'+\frac{z}{x}=\frac{\ln x}{x}$ yoki

$$z'-\frac{z}{x}=-\frac{\ln x}{x}$$

ko'rinishdagi chiziqli tenglama hosil bo'ladi. Bu tenglamani yuqorida keltirilgan usul bilan echamiz:

$$z' - \frac{z}{x} = 0; \quad \frac{dz}{z} = \frac{dx}{x}; \quad \ln|z| = \ln|x| + \ln C;$$

$$z = C(x)x; \quad z' = C'x + C; \quad C'x + C - \frac{Cx}{x} = -\frac{\ln x}{x};$$

$$C'(x) = -\frac{\ln x}{x^2}$$

Oxirgi tenglamani yechib, $C = \frac{1}{x}(\ln x + 1) + C_1$ ni topamiz. bu qiymatni z=Cx ga qo'ysak, $z = \ln x + 1 + C_1 x$ hosil bo'ladi.

Bernulli tenglamasining umumiy yechimi esa, $y^{-1} = Cx + \ln x + 1$ dan topiladi, ya'ni

$$y = \frac{1}{Cx + \ln x + 1}.$$

## MASHQLAR

Quyida keltirilgan chiziqli yoki Bernulli tenglamalari yechilsin (agar boshlang'ich shartlar ko'rsatilgan bo'lsa, xususiy yechimlar yoki xususiy integrallar aniqlansin).

1. $y' - \frac{2}{x+1}y = (x+1)^3$
2. $y' - y\,ctgx = Sinx$
3. $xy' - 2y = 2x^4$
4. $(2x+1)y' = 4x + 2y$
5. $y' + 2y = e^x y^2$
6. $xy^2 y' = x^2 + y^3$
7. $xy' - 2x^2\sqrt{y} = 4y$
8. $xy' + (x+1)y = 3x^2 e^{-x}$
9. $y' + \frac{2}{x}y = x^3$
10. $y' + \frac{2}{x}y = x^3$
11. $y' + xy^2 = -\frac{y}{x}$
12. $y' - \frac{2x-1}{x^2}y - 1 = 0$
13. $y' + y = e^{-x}$
16. $y' - \frac{3}{x}y = x$
17. $y' + \frac{y}{2x+1} = \frac{x}{2x+1}$
18. $y'Cosx - ySinx = Sin2x$
19. $y' + xy = xy^3$
20. $(1-x^2)y' - xy = xy^2$
21. $y' - \frac{y}{1-x^2} = x + 1, \; y(0) = 0$
22. $y' - y\,ctgx = \frac{1}{Cosx}, \; y(0) = 0$
23. $y' - \frac{xy}{1+x^2} = \frac{2}{1+x^2}, \; y(0) = 3$
24. $y' - \frac{3}{x}y = x^3 e^x, \; y(1) = e$
25. $y' + \frac{2}{x}y = \frac{1}{x^2}, \; y(2) = 1$
26. $y'Cos^2 x = tgx - y, \; y(0) = 0$
27. $y' - \frac{y}{x+2} = x^2 + 4x + 5, \; y(-1) = \frac{3}{2}$
28. $y' - 2xy = 1 - 2x^2, \; y(0) = 2$

14. $y' + xy = x^3y^3$

15. $y' + yCosx = \frac{1}{2}Sin2x$

29. $xy' + y = x^2 + 3x + 2,\ y(1) = \frac{29}{6}$

30. $xy' - y^2 \ln x + y = 0,\ y(1) = \frac{1}{2}$

## 4.5. To'la differensialli differensial tenglamalar

Agar $P(x;y)dx + Q(x;y)dy = 0$ tenglamaning chap tomoni biror $u(x;y)$ funksiyaning to'la differensialini ifoda etsa ya'ni,

$$P(x;y)dx + Q(x;y)dy = du(x;y)$$

bo'lsa, mazkur tenglamani to'la differensialli differensial tenglama deb ataladi.

Agar berilgan tenglama to'la differensialli differensial tenglama bo'lsa, $du(x;y) = 0$ deb yozish mumkinligidan uning umumiy integrali $u(x;y) = C$ tenglik bilan aniqlanadi. Agar $P(x;y)$ , $Q(x;y)$ funksiyalar o'zlarining $\frac{\partial P(x;y)}{\partial y}$ va $\frac{\partial Q(x;y)}{\partial x}$ kabi xususiy hosilalari bilan birgalikda biror bir bog'lamli D sohadagi uzluksiz funksiyalar bo'lsa, u holda mazkur tenglama to'la defferentsialli differensial tenglama bo'lishi uchun u sohada har doim $\frac{\partial P(x;y)}{\partial y} = \frac{\partial Q(x;y)}{\partial x}$ kabi tenglikning bajarilishi zarur va yetarli shartdir. Umuman $(x_0;y_0) \in D$ bo'lsa, to'la differensialli differensial tenglamaning umumiy yechimi

$$u = \int_{x_0}^{x} P(x;y)dx + \int_{y_0}^{y} Q(x;y)dy$$

formula bilan topiladi.

Agarda $\frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x}$ bo'lsa, u holda shunday bir $\mu(x;y)$ funksiyani topish mumkinki, uni berilgan tenglamaga ko'paytirsak, u tenglama to'la differensialli differensial tenglamaga aylanadi. $\mu(x;y)$ funksiyani integrallovchi ko'paytuvchi deb ataladi va uni quyidagi hollarda topish osonlashadi:

a) $\dfrac{\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}}{Q} = \varphi(x)$ - funksiya faqat $x$ ga bog'liq bo'lsa, $\ln \mu = \int \varphi(x)dx$ ;

b) $\dfrac{\dfrac{\partial Q}{\partial x}-\dfrac{\partial P}{\partial y}}{P}=\psi(y)$ - funksiya faqat $y$ ga bog'liq bo'lsa, $\ln\mu=\int\psi(y)dy$

**1-Misol.** $\dfrac{2x}{y^3}dx+\dfrac{y^2-3x^2}{y^4}dy=0$ tenglama yechilsin.

**Yechish.**

$$P=\frac{2x}{y^3},\ Q=\frac{y^2-3x^2}{y^4};$$

$$\frac{\partial P}{\partial y}=-\frac{6x}{y^4},\ \frac{\partial Q}{\partial x}=-\frac{6x}{y^4}$$

Demak, $\dfrac{\partial P}{\partial y}=\dfrac{\partial Q}{\partial x}$ bajarilganligi uchun berilgan tenglamaning chap tomoni qandaydir noma'lum $u(x;y)$ funksiyaning to'la differensiali ekan. SHu funksiyani topamiz.

$\dfrac{\partial u}{\partial x}=\dfrac{2x}{y^3}$ ekanligidan

$$u=\int\frac{2x}{y^3}dx+\varphi(y)=\frac{x^2}{y^3}+\varphi(y)$$

funksiyaga ega bo'lamiz, bu yerdagi $\varphi(y)$ funksiya hozircha noma'lum bo'lgan $y$ ning funksiyasidir. Endi bu funksiyaning $u$ bo'yicha hosilasini topib, uni Q ga tenglashtiramiz:

$$\frac{\partial u}{\partial y}=Q=\frac{y^2-3x^2}{y^4},\ \frac{\partial u}{\partial y}=[\frac{x^2}{y^3}+\varphi(y)]$$

$$\frac{\partial u}{\partial y}=-\frac{3x^2}{y^4}+\varphi'(y)=\frac{y^2-3x^2}{y^4}$$

bundan $\varphi'(y)=\dfrac{1}{y^2},\ \varphi(y)=\int\dfrac{dy}{y^2}=-\dfrac{1}{y}+C_1,$

$$u(x,y)=\frac{x^2}{y^3}-\frac{1}{y}+C_1$$

SHunday qilib, $\dfrac{x^2}{y^3}-\dfrac{1}{y}=C$ berilgan tenglamaning umumiy integrali bo'ladi.

**2-Misol.** $(y+xy)^2dx-xdy=0$ tenglama yechilsin.

**Yechish.** Bu yerda

$$P=y+xy^2,\ Q=-x;\ \frac{\partial P}{\partial y}=1+2xy,\ \ \frac{\partial Q}{\partial x}=-1,\ \frac{\partial P}{\partial y}\neq\frac{\partial Q}{\partial x}$$

Demak, berilgan tenglamaning chap tomoni to'la differensial emas.

Integrallovchi ko'paytuvchini topamiz.

$$\frac{\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}}{P}=\frac{-1-1-2xy}{y+xy^2}=-\frac{2}{y}=\psi(y)$$

ekanligidan, yuqorida keltirilgan qoidaga ko'ra

$$\ln\mu=-\int\frac{2dy}{y}=-2\ln|y|,$$

bundan $\mu=\frac{1}{y^2}$. Berilgan tenglamani topilgan $\mu$ ning qiymatiga

ko'paytirib $(\frac{1}{y}+x)dx-\frac{x}{y^2}dy=0$

to'la differensialli differensial tenglamani hosil qilamiz. Bu tenglamani yechib, uning $\frac{x}{y}+\frac{x^2}{2}+C=0$ - umumiy integralini yoki umumiy

$$y=-\frac{2x}{x^2+2C}$$

yechimini topamiz.

**3-Misol.** $2xCos^2ydx+(2y-x^2Sin2y)dy=0$ tenglama yechilsin.

**Yechish.** Bu yerda: $P(x,y)=2xCos^2y$, $Q(x,y)=2y-x^2Sin2y$,

$$\frac{\partial P}{\partial y}=2x2Cosy(-Siny)=-2xSin2y,\ \frac{\partial Q}{\partial x}=-2xSin2y$$

ya'ni, berilgan tenglama to'la defferentsialli differensial tenglama ekan. Uning yechimini

$$\int_{x_0}^{x}P(x;y)dx+\int_{y_0}^{y}Q(x;y)dy=C$$

formuladan topamiz. $x_0=0,\ y_0=0$ deb olishimiz mumkin:

$$\int_0^x 2xCos^2ydx+\int_0^y(2y-x^2\cdot Sin2y)dy=x^2Cos^2y\Big|_0^x+y^2\Big|_0^y=x^2Cos^2y+y^2=C$$

Demak,

$$x^2Cos^2y+y^2=C$$

umumiy integraldir.

## MASHQLAR

Quyidagi differensial tenglamalar yechilsin.

1. $(3x^2+2y)dx+(2x-3)dy=0$
2. $3x^2e^ydx+(x^3e^y-1)dy=0$
3. $(xCos2y+1)dx-x^2Sin2ydy=0$
4. $(4\frac{y^2}{x^2})dx+\frac{2y}{x}dy=0$
9. $(x^2+y^2+2x)dx+2xydy=0$
10. $(x^3-3xy^2+2)dx-(3x^2y-y^2)dy=0$
11. $(x+y^2)dx-2xydy=0$
12. $y(1+xy)dx-xdy=0$

5. $(x^2-y)dx+xdy=0$

6. $y^2dx+(yx-1)dy=0$

7. $2xtgydx+(x^2-2Siny)dy=0$

8. $(Sinx+e^y)dx+Cosxdy=0$

13. $(x^2+y)dx+(x-2y)dy=0$

14. $(y-3x^2)dx-(4y-x)dy=0$

15. $(y^3-x)y'=y$

## 4.6. Lagranj va Klero tenglamalari

1. Lagranj tenglamasi deb, $x$ va $y$ o'zgaruvchilarga nisbatan chiziqli bo'lgan

$$y=\varphi(y')x+\psi(y')$$

kabi birinchi tartibli differensial tenglamaga aytiladi.

Agar $y'=p$ deb olsak, u holda Lagranj tenglamasi quyidagicha yoziladi

$$y=\varphi(p)x+\psi(p)$$

Ushbu tenglamani $x$ bo'yicha differensiallab, u yerda $y'$ ni $p$ bilan almashtirib va $\frac{dy}{dp}$ ga ko'paytirilgandan so'ng

$$\frac{dx}{dp}-\frac{\varphi'(p)}{p-\varphi(p)}x=\frac{\psi'(p)}{p-\varphi(p)}$$

ko'rinishdagi $x(p)$ va $x'(p)$ larga nisbatan chiziqli bo'lgan tenglamani hosil qilamiz. Uning umumiy integrali $\Phi(x;p;C)=0$ bilan $y=\varphi(p)x+\psi(p)$ tenglamadan parametr $p$ ni yo'qotsak, Lagranj tenglamasining umumiy integrali aniqlanadi.

***__Eslatma.__*** *Biz bu yerda, $p-\varphi(p)\neq 0$ deb faraz qilib umumiy integralni aniqladik. Agar $p-\varphi(p)=0$ bo'lsa, bu tenglama $p=p_i\,(i=\overline{1,k})$ ildizlarga ega bo'ladi. Ushbu ildizlar $y=x\varphi(p_i)+\psi(p_i)$ kabi yechimlarni beradiki, ular Lagranj tenglamasining maxsus yechimlari ham bo'lishi mumkin.*

**1-Misol**. $y=2y'x+y'^2$ tenglama yechilsin.

**__Yechish.__** $y'=p$ desak, $y=2px+p^2$ hosil bo'ladi. Uni differensiallab $y'=2p'x+2p+2pp'$ni hosil qilamiz. Bundan esa, $pdx=2(x+p)dp+2pdx$ ni yoki

$$\frac{dx}{dp}-\frac{2}{p}x=2$$

ni aniqlaymiz. Ushbu tenglamani yechib, $x=Cp^2-2p$ ni topamiz.

Endi $\begin{cases}x=Cp^2-2p,\\ y=2px+p^2\end{cases}$ dan $p$ ni yo'qotib, berilgan tenglamaning umumiy integrali $\quad y-\frac{y+x^2}{Cx+1}=2x\sqrt{\frac{y+x^2}{cx+1}}$

ni aniqlaymiz. Bu yerda, $y=0$ trivial yechim bo'ladi.

1. Lagranj tenglamasida $\varphi(y') \equiv y'$ bo'lsa, undan Klero tenglamasi deb ataluvchi quyidagi tenglama hosil qilinadi.

$$y = xy' + \psi(y')$$

Bu yerda ham $y' = p$ almashtirish kiritib, $y = xp + \psi(p)$ tenglamaga kelamiz. Uni differensiallab, $y' = p + xp' + \psi'(p) \cdot p'$ ni yoki $p'[x + \psi'(p)] = 0$ ni hosil qilamiz. Bundan $p' = 0$ va $x + \psi'(p) = 0$ yoki $p = C$ va $x = -\psi'(p)$ lar hosil bo'ladi.

Natijada, $y = Cx + \psi(c)$ kabi umumiy yechimni va $\begin{cases} x = -\psi'(p) \\ y = -p\psi'(p) + \psi(p) \end{cases}$ kabi parametrik shakldagi umumiy integralni aniqlaymiz. Ushbu integral Klero tenglamasining maxsus integralini ifoda etadi.

**2-Misol.** $y = xy' + \frac{1}{y'}$ ko'rinishdagi Klero tenglamasi yechilsin.

**Yechish.** $y' = p$ dan $y = xp + \frac{1}{p}$ ni va uni differensiallab, $y' = p + xp' - \frac{p'}{p^2}$ ni aniqlaymiz. Bundan esa, $p'(x - \frac{1}{p^2}) = 0$ ni va undan $p' = 0$ hamda $x = \frac{1}{p^2}$ larni aniqlaymiz.

Natijada, $y = Cx + \frac{1}{c}$ kabi umumiy yechim hamda $x = \frac{1}{p^2}$ bilan $y = \frac{2}{p}$ shakldagi maxsus yechimga ega bo'lamiz. Agar parametr $p$ dan holos bo'linsa,

$$y^2 = 4x$$

ko'rinishdagi maxsus yechim hosil bo'ladi.

## MASHQLAR

Quyida berilgan Lagranj va Klero tenglamalari yechilsin, agar maxsus yechimlari mavjud bo'lsa, ular ham aniqlansin.

1. $y = xy' - y'^2$
2. $y = 2xy' - 4y'^3$
3. $y = xy' - \frac{1}{3}y'^3$
4. $y = xy' - \ln y'$
5. $2yy'^2 - 2xy'^3 - 1 = 0$
6. $y + xy' = 4\sqrt{y'}$
7. $xy' - (2 + y') = y$
8. $y = xy'^2 - 2y'^3$
9. $y = 2xy' + xy'^2$
10. $2xy' - \ln y' = y$
11. $y = x(1 + y') + y'^2$
12. $yy'^2 + 2xy' = y$
13. $y = y' + \sqrt{1 - y'^2}$
14. $y = xy' - \frac{1}{y'^2}$
15. $y - xy' = \sqrt{y'^3}$

# 5. YUQORI TARTIBLI DIFFERENSIAL TENGLAMALAR

## 5.1. Umumiy tushunchalar

Agar u yoki bu differensial tenglama tarkibida noma'lum funksiyaning n – tartibli hosilasi qatnashsa, uni n – tartibli differensial tenglama deb ataladi.

$$F(x;y;y';y'';...;y^{(n-1)})=0$$

kabi tenglamani u=u(x) noma'lum funksiyaga nisbatan n – tartibli differensial tenglamaning umumiy ko'rinishi deb yuritiladi. Agar mazkur tenglamani $u^{(n)}$ ga nisbatan yechish mumkin bo'lsa, yechib

$$y^{(n)}=f(x;y;y';y'';...;y^{(n-1)}) \tag{1}$$

ni hosil qilamiz. Mazkur tenglamani yuqori tartibli hosilaga nisbatan yechilgan differensial tenglama deb ataladi.

Ushbu tenglamaning umumiy yechimi tarkibida ixtiyoriy $S_1,S_2,...,S_n$ lar kabi n ta o'zgarmaslar ishtirok etadi. Ularning muayyan qiymatlarini aniqlash uchun boshlang'ich shartlar deb ataluvchi biron-bir qo'shimcha shartlar oldindan berilgan bo'ladi. Umuman, (1) tenglamaning boshlang'ich shartlari deb ataluvchi

$$y(x_0)=y_0, \qquad y'(x_0)=y_0', \qquad y''(x_0)=y_0'' \quad ..., \quad y^{(n-1)}(x_0)=y_0^{(n-1)}$$

kabi berilgan shartlarni qanoatlantiruvchi yechimini topish masalasi Koshi masalasi deb atalib, Koshi masalasining yechimini esa, (1)-tenglamaning xususiy yechimi deb ataladi.

Bu yerda ham umumiy yoki xususiy yechimlarni noma'lum *u* ga nisbatan yechish mumkin bo'lmasa, ularni mos ravishda umumiy yoki xususiy integrallar deb yuritiladi.

$y^{(n)}=f(x)$ **tenglama.**

Ushbu tenglamaning umumiy yechimini aniqlash uchun n marta integrallash usulidan foydalanamiz. Tenglamaning har ikkala tomonini dx ga ko'paytirib integrallasak,

$$y^{(n-1)}=\int f(x)dx=\varphi_1(x)+C_1$$

ni hosil qilamiz.

Mazkur usulni yana bir marta takrorlasak,

$$y^{(n-2)}=\int[\varphi_1(x)+C_1]dx=\int\varphi_1(x)dx+C_1\int dx+C_2=\varphi_2(x)+C_1x+C_2$$

hosil bo'ladi.

Shu yo'sinda davom eta borib oxirida

$$y=\varphi_n(x)-C_1\frac{x^{n-1}}{n-1}+C_2\frac{x^{n-2}}{n-2}+...+C_{n-1}x+C_n$$

kabi umumiy yechimni topamiz.

**1 - Misol.** $y''' = x\sin 3x$ ning umumiy yechimi topilsin.

**Yechish.** $y'' = \int e^{2x}dx + C_1 = \frac{1}{2}e^{2x} + C_1,$

$$y' = \frac{1}{2}\int e^{2x}dx + C_1\int dx + C_2 = \frac{1}{4}e^{2x} + C_1x + C_2,$$

$$y = \frac{1}{4}\int e^{2x}dx + C_1\int xdx + C_2\int dx + C_3 = \frac{1}{8}e^{2x} + C_1\frac{x^2}{2} + C_2x + C_3.$$

**2 – Misol.** $y'' = x\sin 3x$ ning umumiy yechimi topilsin.

**Yechish.**

$$y' = \int x\sin 3xdx + C_1 = \begin{vmatrix} u = x, du = dx \\ dv = \sin 3xdx \\ v = -\frac{1}{3}\cos 3x \end{vmatrix} = -\frac{x}{3}\cos 3x + \frac{1}{3}\int \cos 3xdx + C_1 =$$

$$= -\frac{1}{3}x\cos 3x + \frac{1}{9}\sin 3x + C_1,$$

$$y = -\frac{1}{3}\int x\cos 3xdx + \frac{1}{9}\int \sin 3xdx + C_1\int dx + C_2 = \begin{vmatrix} u = x, du = dx \\ dv = \cos 3xdx \\ v = \frac{1}{3}\sin 3x \end{vmatrix} =$$

$$= -\frac{1}{3}\left[\frac{1}{3}x\sin 3x - \frac{1}{3}\int \sin 3xdx\right] - \frac{1}{27}\cos 3x + C_1x + C_2$$

$$y = -\frac{1}{3}x\sin 3x - \frac{2}{27}\cos 3x + C_1x + C_2.$$

**3-Misol.** $y^{IV} = \cos 2x$ нинг $x = 0$ да $y = 0,\ y' = 0,$ $y'' = -1,\ y''' = 2$ bo'ladigan boshlang'ich shartlarni qanoatlantiruvchi xususiy yechimi topilsin.

**Yechish** Avvalo tenglamaning umumiy yechimini topamiz, ya'ni:

$$y''' = \frac{1}{2}\sin 2x + C_1, \quad y'' = -\frac{1}{4}\cos 2x + C_1x + C_2,$$

$$y' = -\frac{1}{8}\sin 2x + C_1\frac{x^2}{2} + C_2x + C_3,$$

$$y = \frac{1}{16}\cos 2x + C_1\frac{x^3}{6} + C_2\frac{x^2}{2} + C_3x + C_4.$$

Endi boshlang'ich shartlardan foydalanib, $C_4 = -\frac{1}{16},\ C_3 = 0,$ $C_2 = -\frac{3}{4}$ ва $C_1 = 2$ larni topamiz.

Xususiy yechim esa,

$$y = \frac{1}{16}\cos 2x + \frac{x^3}{3} - \frac{3x^2}{8} - \frac{1}{16}.$$

## 5.2. Tartiblari pasaytirilib yechiladigan ayrim yuqori tartibli differensial tenglamalar

Tartibini pasaytirish yo'li bilan yechiladigan yuqori tartibli differensial tenglamalarning quyidagi ikki xilini qarab o'tamiz.

1. Aytaylik, n – tartibli differensial tenglama tarkibida noma'lum funksiya va uning ($k$-1) – tartibgacha bo'lgan ($1 \le k \le n$) hosilalari ishtirok etmasin, ya'ni:

$$F(x, y^{(k)}, y^{(k+1)}, \ldots, y^{(n)}) = 0 \tag{2}$$

YAngi z(x) noma'lum funksiyani $y^{(k)} = z$ formula orqali kiritib, $y^{(k+1)} = z'$, $y^{(k+2)} = z''$,..., $y^{(n)} = z^{(n-k)}$ larni inobatga olsak, z(x) ga nisbatan,

$$F(x, z, z', z'', \ldots, z^{(n-k)}) = 0 \tag{3}$$

kabi (n-k) tartibli tenglamani hosil qilamiz. Agar (3) – tenglamaning biror z = φ(x, $C_1$, $C_2$, … ,$C_{n-k}$) kabi umumiy yechimini topa olsak, u holda

$y^{(k)} = z = \varphi(x, C_1, C_2, \ldots, C_{n-k})$ ni $k$ marta integrallab, qaralayotgan (2)-tenglamaning umumiy yechimini aniqlagan bo'lamiz.

Xususan, agar n=2 bo'lsa, (3) tenglama birinchi tartibli tenglamadir.

**1-misol.** $y'' - \frac{y'}{x} = 0$ ni yechilsin.

**Yechish.** $y' = z$ deb belgilash kiritsak, $z' - \frac{z}{x} = 0$ ni hosil qilamiz.

Bundan $z' = \frac{z}{x}$ ёки $\frac{dz}{z} = \frac{dx}{x}$ va $\ln|z| = \ln|x| + \ln C_1$, $z = C_1 x$.

Endi, $y' = C_1 x$ дан $y = C_1\frac{x^2}{2} + C_2$ va

$$y = C_1\frac{x^3}{6} + C_2 x + C_3$$

ni topamiz.

2 – Misol. $y''' = (y'')^2$ ni yechilsin.

**Yechish.** $y'' = z$ дан $z' = z^2$ ва $-\frac{1}{z} = x + C_1$ ni hamda $z = -\frac{1}{x + C_1}$ ni hosil qilamiz.

$y'' = -\frac{1}{x + C_1}$ ni ketma–ket integrallab, avval $y' = -\ln(x + C_1) + C_2$ ni, keyin esa

$$y=-(x+C_1)\ln(x+C_1)+x+C_1+C_2x+C_3=-(x+C_1)\ln(x+C_1)+C_2(x+1)+C_1+C_3$$
$$y=-(x+C_1)\ln(x+C_1)+C_2x+C_3$$

ni aniqlaymiz. Bu yerda, $\int\ln u du=u\ln u-u+C$ dan foydalandik. Shuningdek, $C_2+1$ ni $C_2$ bilan, $C_1+C_3$ ni esa, $C_3$ bilan almashtirdik.

**3-Misol.** $y''x\ln x=y'$ ning umumiy yechimi topilsin.

__Yechish.__ $y'=z$ dan $\frac{dz}{z}=\frac{dx}{x\ln x}$ va $\ln|z|=\ln|\ln x|+C_1$ ni, undan esa,
$$z=C_1|\ln x|$$
ni topamiz. $y'=C_1\ln x$ дан $y=C_1x(\ln x-1)+C_2$ ni aniqlaymiz.

2. Aytaylik, n – tartibli differensial tenglama tarkibida argument $x$ oshkor tarzda ishtirok etmasin, ya'ni:
$$F(y,y',y'',...,y^{(n)}=0)\tag{4}$$
Agar ushbu tenglamani yechish lozim bo'lsa, har doim uning tartibini $y'=z(y)$ almashtirish yordamida bittaga pasaytirish mumkin. Buning uchun $y'',y''',...,y^{(n)}$ larning har birini z(y) ning hosilalari orqali ifodalanadi:
$$y''=\frac{dz}{dx}=\frac{dz}{dy}\cdot\frac{dy}{dx}=z\frac{dz}{dy},$$
$$y'''=\frac{d}{dx}\left(z\frac{dz}{dy}\right)=\frac{dz}{dx}\cdot\frac{dz}{dy}+z\cdot\frac{d^2z}{dxdy}=z\left(\frac{dz}{dy}\right)^2+z^2\frac{d^2z}{dy^2}$$
va hokazo. Bu holni quyidagi misollar orqali batafsil tushuntirib o'tamiz.

**4-Misol.** $yy''=y'^2$

__Yechish.__ $y'=z(y)$, $y''=z\frac{dz}{dy}$ дан, $yz\frac{dz}{dy}=z^2$ ni hosil qilamiz.
$$z\left(y\frac{dz}{dy}-z\right)=0$$
dan $z=0$ ёки $y=C$ kabi yechimini topamiz. Endi $y\frac{dz}{dy}-z=0$ dan umumiy yechimini aniqlaymiz.
$$y\frac{dz}{dy}=z,\quad \frac{dz}{z}=\frac{dy}{y},\qquad \ln|z|=\ln|y|+\ln C_1\quad ёки\quad z=C_1y.\quad y'=z(y)$$

ga ko'ra , $y'=C_1y$ ни ва бундан $\ln|y|=C_1x+C_2$ ni yoki
$$y=C_2e^{C_1x}$$
kabi umumiy yechimni hosil qilamiz.

**5-Misol.** $y''=\frac{(1+(y')^2)}{2y}$ ning umumiy yechimi topilsin.

**Yechish.** $z\dfrac{dz}{dy}=\dfrac{(1+(y')^2)}{2y}$

dan

$$\frac{zdz}{1+z^2}=\frac{dy}{2y} \text{ ни ёки } \frac{1}{2}\ln\left|1+z^2\right|=\frac{1}{2}\ln y+C_1$$

ni topamiz.

Potensirlasak, $1+z^2=C_1y$ ёки $z=\pm\sqrt{C_1y-1}$ ni hosil qilamiz.

$y'=\pm\sqrt{C_1y-1}$ dan esa

$$\frac{dy}{\sqrt{C_1y-1}}=\pm dx$$

ni va uni integrallab,

$$2\sqrt{C_1y-1}=\pm x+C_2$$

kabi umumiy integralni topamiz.

**6-Misol.** $2yy''=(y')^2$ ни $y(-1)=4$ ва $y'(-1)=1$ kabi boshlang'ich shartlarda yechilsin.

**Yechish.** $2y\, z\dfrac{dz}{dy}=z^2$, $\quad z\left(2y\dfrac{dz}{dy}-z\right)=0,$

va bu yerdan

$$z=0 \text{ ва } 2y\frac{dz}{dy}-z=0$$

larni hosil qilamiz. Ikkinchi tenglamadan

$$2y\frac{dz}{dy}=z, \quad \frac{dz}{z}=\frac{1}{2}\frac{dy}{y}$$

$$z=C_1\sqrt{y}$$

Endi z ning o'rniga qo'yib

$$y'=C_1\sqrt{y} \quad 2\sqrt{y}=C_1x+C_2$$

$$y=\frac{1}{4}(C_1x+C_2)^2$$

kabi umumiy yechimni hosil qilamiz.

Boshlang'ich shartlarga ko'ra $C_2-C_1=4$ ва $2C_1=1$ larni va $C_1=\dfrac{1}{2},\ C_2=\dfrac{9}{2}$ larni aniqlaymiz.

U holda, xususiy yechim $y=\dfrac{1}{16}(x+9)^2$ kabi bo'ladi.

## MASHQLAR

Quyidagi tenglamalarning umumiy va xususiy yechimlari (agar boshlang'ich shartlar berilgan bo'lsa) yoki umumiy va xususiy integrallari topilsin.

| | | | |
|---|---|---|---|
| 1. | $y''' = \frac{6}{x^3}$ | 21. | $y'' = y'e^y$ |
| 2. | $y'' = 4Cos2x,$ | 22. | $y'' = \frac{1}{y^3},$ |
| 3. | $y''' = \frac{2\cos x}{\sin^3 x},$ | 23. | $yy'' = (y')^2 - (y')^3,$ |
| 4. | $y''' = x^2 - \sin x,$ | 24. | $y^3 y'' = 1,$ |
| 5. | $xy^{IV} = y'''$ | 25. | $y(1-\ln y)y'' + (1+\ln y)(y')^2 = 0,$ |
| 6. | $x^2 y''' = (y'')^2$ | 26. | $2(y')^2 = (y-1)y'',$ |
| 7. | $xy'' - y' = x^2 e^x,$ | 27. | $2y'' = 3y^2,$ |
| 8. | $xy'' + y' = y'^2$ | 28. | $yy'' - (y')^2 = y^2 \ln y,$ |
| 9. | $x^3 y'' + x^2 y' = 1$ | 29. | $y'' = 2 - y,$ |
| 10. | $xy'' + y' - x - 1 = 0,$ | 30. | $y'' + \frac{2}{1-y}(y')^2 = 0,$ |
| 11. | $(1+x^2)y'' + (y')^2 + 1 = 0,$ | 31. | $y'' = \frac{1}{1+x^2},\ y(0) = y'(0) = 0,$ |
| 12. | $2xy'' = y',$ | 32. | $y'' = \cos^2 x,\ \ y(0) = 1,\ \ y' = -\frac{1}{8},$ |
| 13. | $xy'' = y' + x^2,$ | 33. | $y'' = arctgx,\ \ y(0) = y'(0) = 0,$ |
| 14. | $xy'' + y' = \ln x,$ | 34. | $y'' = \frac{\sin x}{\cos^3 x},\ \ y(0) = \frac{1}{2},\ \ y'(0) = 0,$ |
| 15. | $y''tgx + y - 1 = 0,$ | 35. | $y'' + 4y' = 2x^2,\ \ y(0) = 0,\ \ y'(0) = 0,$ |
| 16. | $y'' - 2x(y')^2 = 0,$ | 36. | $y'' + y'tgx = \sin 2x,\ \ y(0) = -1,\ \ y'(0) = 0,$ |
| 17. | $y''' + y''tgx = \sec x,$ | 37. | $(1+x^2)y'' - 2xy' = 0,\ \ y(0) = 0,\ \ y'(0) = 0$ |
| 18. | $y'''x \ln x = y'',$ | 38. | $xy'' = y',\ \ y(0) = y'(0) = 0$ |
| 19. | $(1+x^2)y'' = 2xy,$ | 39. | $yy'' + (y')^2 = 1,\ y(0) = y'(0) = 1,$ |
| 20. | $xy'' - y' = 2x^2 e^x$ | 40. | $y'' + y = (y')^2\ \ y(1) = -\frac{1}{4},\ \ y'(1) = \frac{1}{2}$ |

## 5.3. Chiziqli differensial tenglamalar

1. Quyidagi

$$y^{(n)} + a_1(x)y^{(n-1)} + a_2(x)y^{(n-2)} + ... + a_{n-1}(x)y' + a_n(x)y = f(x) \quad (1)$$

kabi tenglama, noma'lum $y = y(x)$ funksiyaga nisbatan $n$-tartibli chiziqli bir jinsli bo'lmagan differensial tenglama deb yuritiladi.

Bu yerdagi, $a_k(x)$ $(k = \overline{1,n})$ lar bilan $f(x)$, biror D sohada aniqlangan funksiyalardir. Agar $f(x) = 0$ bo'lsa,

U holda,

$$y^{(n)}+a_1(x)y^{(n-1)}+a_2(x)y^{(n-2)}+...+a_{n-1}(x)y'+a_n(x)y=0 \qquad (2)$$

ni (1) ga mos bo'lgan bir jinsli tenglama deb ataladi.

Agar barchasi nol bo'lmagan shunday $S_1, S_2, \dots, S_n$ o'zgarmas sonlar mavjud bo'lib, $C_1y_1+C_2y_2+...+C_ny_n=0$ kabi tenglik $x\in(a,b)$ ning barcha qiymatlari uchun o'rinli bo'lsa, $y=y_1(x), y=y_2(x), ..., y=y_n(x)$ larni $(a,b)$ da o'zaro chiziqli bog'lanishdagi funksiyalar deb atalib, aks holda ularni o'zaro chiziqli bog'lanmagan funksiyalar deb yuritiladi.

Agar (2) tenglamaning biror oraliqdagi o'zaro chiziqli bog'lanmagan xususiy yechimlari (yechimlarning fundamental sistemasi) $y_1, y_2, ..., y_n$ lar ma'lum bo'lsa, u holda uning umumiy yechimi quyidagicha yoziladi.

$$y=C_1y_1+C_2y_2+...+C_ny_n \qquad (3)$$

SHuningdek, (1)-tenglamaning umumiy yechimining ko'rinishi quyidagicha yoziladi:

$$y=\tilde{y}+y^* \qquad (4)$$

Bu yerda: $\tilde{y}$ – (2) ning umumiy yechimi bo'lib, $u^*$ esa, (1) ning biron-bir xususiy yechimidir.

Agar (2) tenglama yechimlarining fundamental sistemasi $y_1, y_2, ..., y_n$ lar ma'lum bo'lsa, u holda (1) tenglamaning umumiy yechimini

$$y=C_1(x)y_1+C_2(x)y_2+...+C_n(x)y_n \qquad (5)$$

kabi formula bilan aniqlash mumkin. Bu yerdagi $C_k(x)$ $(k=\overline{1,n})$ noma'lum koeffitsientlarni quyidagi tenglamalar sistemasidan aniqlanadi:

$$\begin{cases} C_1'(x)y_1+C_2'(x)y_2+...+C_n'(x)y_n=0, \\ C_1'(x)y_1'+C_2'(x)y_2'+...+C_n'(x)y_n'=0, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ C_1'(x)y_1^{(n-2)}+C_2'(x)y_2^{(n-2)}+...+C_n'(x)y_n^{(n-2)}=0, \\ C_1'(x)y_1^{(n-1)}+C_2'(x)y_2^{(n-1)}+...+C_n'(x)y_n^{(n-1)}=f(x). \end{cases} \qquad (6)$$

Unga o'zgarmasni variatsiyalash usuli deb ataladi.

1 **Misol.** $xy''-xy'=3x$ ni o'zgarmasni variatsiyalash usulida echamiz.

<u>**Yechish.**</u> Tenglamani quyidagicha yozib olamiz:

$$y''-\frac{y'}{x}=3x.$$

Bir jinsli $y''-\frac{y'}{x}=0$ tenglamadan $y=C_1\frac{x^2}{2}+C_2$ ni topamiz. $y_1=\frac{x^2}{2}$ va $y_2=1$ deb olsak, (6) sistemani quyidagicha yoziladi:

$$\begin{cases} C_1'(x)\cdot\frac{x^2}{2}+C_2'(x)\cdot 1=0 \\ C_1'(x)\cdot x+C_2'(x)\cdot 0=3x \end{cases}$$

Ushbu sistemani yechib, $C_1'(x)=3$ ва $C_2'(x)=-\frac{3x^2}{2}$ ni, undan esa,

$C_1(x)=3x+A$ ва $C_2(x)=-\frac{x^3}{2}+B$

larni topamiz.

Natijada.

$$y=(3x+A)\cdot\frac{x^2}{2}-\frac{x^3}{2}+B=x^3+\frac{A}{2}x^2+B$$

kabi umumiy yechim hosil bo'ladi.

**Eslatma.** Umuman, (2) – tenglamaning fundamental yechimlari sistemasini topishning umumiy usuli bo'lmaganligi uchun (1) tenglamaning xususiy yechimini ham (shu jumladan umumiy yechimini ham) topib bo'lmaydi. Faqat xususiy holdagina, ya'ni (1) tenglamadagi barcha $a_k(x)$ koeffitsientlar o'zgarmas sonlar bo'lsagina, xususiy yechimlarning fundamental sistemasini topish usuli hamda u orqali umumiy yechimni topish usuli mavjud. Shu boisdan quyida biz koeffitsientlari o'zgarmas bo'lgan chiziqli differensial tenglamalarni o'rganamiz.

2. Avvalo koeffitsientlari o'zgarmas bo'lgan 2 – tartibli chiziqli bir jinsli bo'lgan differensial tenglama

$$y''+py'+qy=0 \tag{7}$$

ni qaraymiz. Ushbu tenglama uchun yechimlarning fundamental sistemasini tashkil etadigan xususiy yechimlarni aniqlash maqsadida xususiy yechimni $y=e^{kx}$ kabi axtaramiz va noma'lum $k$ ni aniqlash uchun (7) ga mos bo'lgan harakteristik tenglama deb ataluvchi

$$k^2+pk+q=0 \tag{8}$$

ni tuzib olamiz.

a) Agar xarakteristik tenglamaning ildizlari $k_1\neq k_2$ bo'lgan xaqiqiy sonlardan iborat bo'lsa, o'zaro chiziqli bog'lanishda bo'lmagan xususiy yechimlar $y_1=e^{k_1x}$ ва $y_2=e^{k_2x}$ kabi yoziladi. Natijada, umumiy yechim

$$y=C_1e^{k_1x}+C_2e^{k_2x}$$

kabi bo'ladi.

b) Agar xarakteristik tenglamaning yechimlari $k_1=k_2=r$ (haqiqiy son) bo'lsa, yechimlarining fundamental sistemasi $y_1=e^{rx}$ ва $y_2=xe^{rx}$ kabi bo'lib, umumiy yechimning ko'rinishi

$$y=C_1e^{rx}+C_2xe^{rx}$$

kabi bo'ladi.

c) Agar (8) tenglamaning ildizlari $k_1=\alpha+i\beta$ va $k_2=\alpha-i\beta$ kabi o'zaro qo'shma kompleks sonlar bo'lsa, fundamental sistemani tashkil etuvchi xususiy yechimlar $y_1=e^{\alpha x}\cos\beta x$ va $y_2=e^{\alpha x}\sin\beta x$ kabi bo'ladi. Natijada, umumiy yechimning ko'rinishi

$$y=e^{\alpha x}\left(C_1\cos\beta x+C_2\sin\beta x\right)$$

dek yoziladi.

**Misollar.**

1. $y''-5y'-6y=0,$
2. $y''-8y'+16y=0,$
3. $y''-4y'+5y=0,$
4. $y''+9y=0.$

kabi tenglamalarning umumiy yechimlari topilsin.

<u>**Yechimlari**</u> 1) Xarakteristik tenglama

$k^2-5k-6=0$, ni yechib, $k_1=6, k_2=-1$ larni topamiz.

U holda umumiy yechim

$$y=C_1e^{6x}+C_2e^{-x}.$$

2) $k^2-8k+16=0$ dan $k_1=k_2=4$.

Demak umumiy yechim

$$y=C_1e^{4x}+C_2xe^{4x}.$$

3) $k^2-4k+5=0$ xarakteristik tenglamadan $k_1=2-i, k_2=2+i$ larni topamiz. Natijada, umumiy yechimning ko'rinishi

$$y=e^{2x}\left(C_1\cos x+C_2\sin x\right)$$

kabi bo'ladi.

3 ) Ushbu tenglamaga mos xarakteristik tenglama $k^2+9=0$ bo'lganligi uchun $k_{1,2}=\pm 3i$ va

$$y=C_1\cos 3x+C_2\sin 3x$$

3. Endi koeffitsientlari o'zgarmas bo'lgan ikkinchi tartibli chiziqli birjinsli bo'lmagan differensial tenglama

$$y''+py'+qy=f(x) \tag{9}$$

ni qaraymiz.

Ushbu tenglamaga mos bir jinsli tenglamaning umumiy yechimi orqali, (9) tenglamaning xususiy yechimi va u orqali umumiy yechimni o'zgarmasni variatsiyalash usuli bilan aniqlash mumkin ekanligini eslatib o'tmoqchimiz. Biroq, agar (9) tenglamaning o'ng tomonidagi $f(x)$ funksiya maxsus ko'rinishlarga ega bo'lganda xususiy yechimni tanlash usuli deb ataluvchi oddiy usuldan foydalanish maqsadga muvofiqdir.

Quyidagi hollarni ko'rib o'tamiz.

I. Aytaylik $f(x)=P_n(x)e^{\alpha x}$ bo'lsin.

Bu yerda: $P_n(x)$ n-darajali ko'phad, $\alpha$ esa biror xaqiqiy son.

a) Agar $\alpha$ soni xarakteristik tenglama (8) ning ildizi bo'lmasa, xususiy yechimni

$$y^* = Q_n(x)e^{\alpha x}$$

kabi tanlanadi.

b) Agar $\alpha$ soni xarakteristik tenglama (8) ning ildizlaridan biriga teng bo'lsa, xususiy yechim

$$y^* = xQ_n(x)e^{\alpha x}$$

kabi bo'ladi.

c) Agar $\alpha$ soni (8) ning har ikkala ildiziga ham teng bo'lsa, xususiy yechim

$$y^* = x^2Q_n(x)e^{\alpha x}$$

kabi tanlanadi.

Yuqoridagi $Q_n(x)$ koeffitsientlari noma'lum bo'lgan n-darajali ko'pxad bo'lib, uning koeffitsientlarini noma'lum koeffitsientlar usuli deb ataluvchi usul bilan aniqlanadi.

II. Aytaylik, $f(x)=P(x)e^{\alpha x}\cos\beta x+Q(x)e^{\alpha x}\sin\beta x$ kabi bo'lsin. Bu yerda, P(x) bilan Q(x) lar ko'pxadlardir.

a) Agar $\alpha+i\beta$ kompleks son (8) tenglamaning ildizi bo'lmasa, xususiy yechimni quyidagicha tanlash lozim

$$y^* = U(x)e^{\alpha x}\cos\beta x+V(x)e^{\alpha x}\sin\beta x \tag{10}$$

bu yerda, U(x) bilan V(x) lar koeffitsientlari noma'lum bo'lgan hamda darajalari R(x) bilan Q(x) larning eng katta darajalariga teng bo'lgan ko'pxadlardir.

b) Agar $\alpha+i\beta$ kompleks son (8) ning ildizi bo'lsa, u holda

$$y^* = x\left(U(x)e^{\alpha x}\cos\beta x+V(x)e^{\alpha x}\sin\beta x\right) \tag{11}$$

Eslatib o'tamizki, agar (9) tenglamaning o'ng tomonidagi f(x) funksiyaning ko'rinishi yoki faqat $f(x)=P(x)e^{\alpha x}\cos\beta x$ yoki faqat $f(x)=Q(x)e^{\alpha x}\sin\beta x$ kabi bo'lganda ham $u^*$ xususiy yechimni yoki (10) yoki (11) ko'rinishda tanlanaveradi.

**1-Misol.** $y''-5y'-6y=x+1$ning xususiy yechimi topilsin.

**Yechish.** $y''-5y'-6y=0$ ga mos $k^2-5k-6=0$. ni yechib, $k_1=6,\ k_2=-1$ larni topamiz.

Xususiy yechimni, $y^*=Ax+B$ kabi tanlaymiz. CHunki, bu yerda $\alpha=0$ soni $-1$ ga ham, 6 ga ham teng emas. $y^{*\prime}=A,\ y^{*\prime\prime}=0$ bo'lganligidan,

$$-5A-6Ax-6B=x+1.$$

Noma'lum koeffitsientlar usulini qo'llab, $-6A=1$ va $-5A-6B=1$ ni hosil qilamiz. Undan $A=-\frac{1}{6}$, $B=-\frac{1}{36}$ natijada

$$y^{*}=-\frac{1}{6}x-\frac{1}{36}.$$

**2 – Misol.** $y''-8y'+16y=x^2e^{4x}$

__Yechish.__ Xarakteristik tenglama ildizlari $k_1=k_2=4$.

Bu yerdagi $\alpha=4$ soni, 2 karrali ildiz bo'lgani uchun

$$y^*=x^2\left(Ax^2+Bx+C\right)e^{4x}=\left(Ax^4+Bx^3+Cx^2\right)e^{4x}$$

$$y^{*\prime}=\left(4Ax^3+3Bx^2+2Cx\right)e^{4x}+4\left(Ax^4+Bx^3+Cx^2\right)e^{4x}$$

$$y^{*\prime\prime}=\left(12Ax^2+6Bx+2C\right)e^{4x}+8\left(4Ax^3+3Bx^2+2Cx\right)e^{4x}+16\left(Ax^4+Bx^3+Cx^2\right)e^{4x}$$

$y^*, y^{*\prime}$ va $y^{*\prime\prime}$ larni tenglamaga qo'yib, $e^{4x}$ ga bo'lib yuborsak,

$$12Ax^2+6Bx+2C+32Ax^3+24Bx^2+16C+16Ax^4+16Bx^3+16C-32Ax^3-24Bx^2-16C-$$

$$-32Ax^4-32Bx^3-32C+16Ax^4+16Bx^3+16C=x^2$$

hosil bo'ladi.

Bundan $12Ax^2+6Bx+2C=x^2$ va $A=\frac{1}{12}$, $B=C=0$ larni aniqlaymiz. Demak

$$y^{*}=\frac{1}{12}x^4e^{4x}.$$

**3 – Misol.** $y''-4y'+5y=3xCos\,2x.$

__Yechish.__ Xarakteristik tenglama $k^2-4k+5=0$ ning ildizlari $k_1=2-i,\ k_2=2+i$ edi. Bu yerda $f(x)=3xCos2x$ va $\alpha=0,\beta=2$ son xarakteristik tenglama ildizlaridan birortasiga ham teng emas. SHu boisdan,

$$y^*=(Ax+B)\cos2x+(Cx+D)\sin2x.$$

Noma'lum koeffitsientlarni aniqlaymiz.

$$y^{*\prime}=A\cos2x+C\sin2x-2(Ax+B)\sin2x+2(Cx+D)\cos2x,$$

$$y^{*\prime\prime}=-4A\sin2x+4C\cos2x-4(Ax+B)\cos2x-4(Cx+D)\sin2x.$$

$y^* y^{*\prime}$ va $y^{*\prime\prime}$ larni tenglamaga qo'yamiz:

$$-4A\sin2x+4C\cos2x-4(Ax+B)\cos2x-4(Cx+D)\sin2x-$$

$$-4\left(A\cos2x+C\sin2x-2(Ax+B)\sin2x+2(Cx+D)\cos2x\right)+$$

$$+5\left((Ax+B)\cos2x+(Cx+D)\sin2x\right)=3x\cos2x$$

yoki

$$\left[(Ax-B)-8(Cx+D)-4A\right]\cos2x+\left[8(Ax+B)+(Cx+D)-4A\right]\sin2x=3x\cos2x$$

Bu yerdan

$$\begin{cases} A-8C=3, \\ B-4A-8D=0, \\ 8A+C=0, \\ 8B-4A+D=0. \end{cases}$$

Ushbu sistemasini yechib, $A=\frac{3}{65}$, $B=\frac{108}{4225}$, $C=-\frac{24}{65}$ *va* $D=-\frac{84}{4225}$ larni aniqlaymiz. U holda

$$y^{*}=\left(\frac{3}{65}x+\frac{108}{4225}\right)Cos\,2x+\left(-\frac{24}{65}x-\frac{84}{4225}\right)\sin 2x.$$

**4-Misol.** $y''+9y=4\sin 4x+5\cos 4x$ ning umumiy yechimi aniqlansin.

__Yechish__. $y''+9y=0$ ga mos xarakteristik tenglama $k^2+9=0$ ning ildizlari orqali, $y''+9y=0$ ning umumiy yechimi

$$\tilde{y}=C_1\cos 3x+C_2\sin 3x$$

ni topamiz. Bu yerda, $\alpha+i\beta=4i$ mavhum son $k_{1,2}=\pm 3i$ teng bo'lmaganligi sababli,

$$y^{*}=A\cos 4x+B\sin 4x$$

Noma'lum koeffitsientlarni hisoblaymiz

$$y^{*\prime}=-4A\sin 4x+4B\cos 4x$$
$$y^{*\prime\prime}=-16A\cos 4x-16B\sin 4x.$$

U holda

$$-16A\cos 4x-16B\sin 4x+9A\cos 4x+9B\sin 4x=4\sin 4x+5\cos 4x$$

$$\begin{cases}-16A+9A=5\\-16B+9B=4\end{cases}$$

Bu yerdan $A=-\frac{5}{7}$, $B=-\frac{4}{7}$, mos ravishda xususiy yechim

$$y^{*}=-\frac{5}{7}Cos\,4x-\frac{4}{7}Sin\,4x.$$

Umumiy yechim esa

$$y=\tilde{y}+y^{*}=C_1Cos\,3x+C_2\sin 3x-\frac{1}{7}(5Cos\,4x+4\sin 4x).$$

**5-Misol.** $y''-2y'=xe^{2x}$ ning $y(0)=0$, $y'(0)=3$ kabi boshlang'ich shartlarni qanoatlantiruvchi xususiy yechimi topilsin.

__Yechish.__ 1) Avvalo, $y''-2y'=0$ ning umumiy yechimini aniqlaymiz.

Xarakteristik tenglama $k^2-2k=0$ ning ildizlari $k_1=0, k_2=2$ bo'lganligi bois,

$$\tilde{y}=C_1+C_2e^{2x}.$$

2) $f(x)=xe^{2x}$ dan $\alpha=2$ ning $k_2=2$ ga tengligini inobatga olib, xususiy yechimni

$$y^{*}=x(Ax+B)e^{2x}=\left(Ax^2+Bx\right)e^{2x}$$

kabi tanlaymiz.

$$y^{*\prime}=(2Ax+B)e^{2x}+2(Ax^2+Bx)e^{2x}$$
$$y^{*\prime\prime}=2Ae^{2x}+3(2Ax+B)e^{2x}+4(Ax^2+Bx)e^{2x}$$

larni tenglamaga qo'ysak,

$$2A+3(2Ax+B)+4(Ax^2+Bx)-2(2Ax+B)-4(Ax^2+Bx)=x$$

yoki

$$\begin{cases} 2A+B=0 \\ 2A=1 \end{cases}$$

ni hosil qilamiz.

Bundan esa, $A=\frac{1}{2}$ ва $B=-1$ larni topamiz. Xususiy yechim,

$$y^*=\left(\frac{1}{2}x^2-x\right)e^{2x}$$

kabi bo'lib, umumiy yechim esa,

$$y=C_1+C_2e^{2x}+\left(\frac{1}{2}x^2-x\right)e^{2x}$$

dan iborat bo'ladi.

3) Endi boshlang'ich shartlardan foydalanamiz.

$$y'=2C_2e^{2x}+(x-1)e^{2x}+2\left(\frac{1}{2}x^2-x\right)e^{2x}$$

va $y'$ lardan, $C_1+C_2=0$ va $2C_2=4$ larni hamda $C_1=-2$ va $C_2=2$ larni topamiz. Natijada, xususiy yechim quyidagicha yoziladi

$$y=-2+\left(\frac{x^2}{2}-x+2\right)e^{2x}.$$

**Eslatma.** Agar (9) tenglamaning o'ng tomonidagi funksiya $f(x)=f_1(x)+f_2(x)$ kabi bo'lganda, ham $y''+py'+qy=f_1(x)$ ga, ham $y''+py'+qy=f_2(x)$ ga mos bo'lgan $y_1{}^*$ va $y_2{}^*$ xususiy yechimlarni aniqlab, keyin ularning algebraik yig'indisi $y=y_1{}^*+y_2{}^*$ ni aniqlanadi.

4. Koeffitsientlari o'zgarmas bo'lgan n – tartibli chiziqli birjinsli differensial tenglama

$$y^{(n)}+a_1y^{(n-1)}+a_2y^{(n-2)}+...+a_{n-1}y'+a_ny=0$$

ni yechish ham 2 – tartibli tenglamaga o'xshash bajariladi.

$$k^n+a_1k^{n-1}+a_2y^{n-2}+...+a_{n-1}k+a_n=0$$

berilgan tenglamaga mos bo'lgan xarakteristik tenglama bo'ladi va differensial tenglamaning o'zaro chiziqli bog'lanishda bo'lmagan xususiy yechimlarining ko'rinishi ham xarakteristik tenglama ildizlarining xillariga bog'liq bo'ladi. Ularni biz quyidagi jadval orqali tushuntiramiz.

| № | Xarakteristik tenglama ildizlarining xillari | Differensial tenglamaning xususiy yechimlari. |
|---|---|---|
| 1 | Agar r, oddiy xaqiqiy ildiz bo'lsa | $e^{rx}$ |
| 2 | Agar r soni, k – karrali xaqiqiy ildiz bo'lsa | $e^{rx}, xe^{rx}, x^2e^{rx}, ..., x^{k-1}e^{rx}$ |

| | | |
|---|---|---|
| 3 | Agar α±iβ kabi oddiy o‘zaro qo‘shma kompleks son ildizlari bo‘lsa | $e^{\alpha x}Cos\beta x, e^{\alpha x}Sin\beta x$ |
| 4 | Agar α±iβ lar o‘zaro qo‘shma bo‘lgan k –karrali kompleks son ildizlar bo‘lsa | $e^{\alpha x}Cos\beta x, xe^{\alpha x}Cos\beta x, ..., x^{k-1}e^{\alpha x}Cos\beta x$ $e^{\alpha x}Sin\beta x, xe^{\alpha x}Sin\beta x, ..., x^{k-1}e^{\alpha x}Sin\beta x$ |

Umumiy yechim esa

$$y = C_1 y_1 + C_2 y_2 + ... + C_n y_n$$

kabi aniqlanadi.

Koeffitsientlari o‘zgarmas bo‘lgan n- tartibli chiziqli bir jinsli bo‘lmagan differensial tenglamaning yechilish sxemasi ham ikkinchi tartibli tenglamaga o‘xshash bajariladi.

**1 – Misol.** $y''''' - 9y''' = 0$ ning umumiy yechimini topilsin.

**Yechish.** Xarakteristik tenglama $k^5 - 9k^3 = 0$ ning ildizlarini topamiz: $k_1 = k_2 = k_3 = 0$ va $k_4 = -3, k_5 = 3$.

Xususiy yechimlarining fundamental sistemasi

$$y_1 = 1, y_2 = x, y_3 = x^2, y_4 = e^{-3x}, y_5 = e^{3x}.$$

U holda umumiy yechim

$$y = C_1 + C_2 x + C_3 x^2 + C_4 e^{-3x} + C_5 e^{3x}.$$

**2 – Misol.** $y''' + 16y' = 0$ ning umumiy yechimi aniqlansin.

**Yechish.** Xarakteristik tenglama $k^3 + 16k = 0$ ning ildizlari $k_1 = 0, k_2 = -4i, k_3 = 4i$. Umumiy yechim esa

$$y = C_1 + C_2 \cos 4x + C_3 \sin 4x.$$

**3–Misol.** $y''' - 3y' - 2y = 0$ ning $y(0) = 0,\ y'(0) = 1\ \ y''(0) = 0$ kabi boshlang‘ich shartlarni qanoatlantiruvchi xususiy yechimi topilsin.

**Yechish.** $k^3 - 3k - 2 = 0$ ning ildizlari $k_1 = k_2 = -1, k_3 = 2$.

Ulardan foydalanib umumiy yechimni yozamiz

$$y = (C_1 + C_2 x)e^{-x} + C_3 e^{2x}$$

$$y' = C_2 e^{-x} - (C_1 + C_2 x)e^{-x} + 2C_3 e^{2x},$$

$$y'' = -2C_2 e^{-x} + (C_1 + C_2 x)e^{-x} + 4C_3 e^{2x}.$$

Boshlang‘ich shartlardan:

$$C_1 + C_3 = 0,$$

$$C_2 - C_1 + 2C_3 = 1,$$

$$2C_2 + C_1 + 4C_3 = 0$$

sistemani hosil qilamiz, ulardan esa $S_1$, $S_2$, va $S_3$ larni aniqlaymiz

$$C_1=-\frac{2}{9},\ C_2=\frac{1}{3},\ C_3=\frac{2}{9}.$$

U holda, xususiy yechim

$$y=\frac{1}{3}\left(x-\frac{2}{3}\right)e^{-x}+\frac{2}{9}e^{2x}.$$

**4 – Misol.** $y''''-2y'''+y''=e^x$ ning umumiy yechimi topilsin.

**Yechish.**

$k^4-2k^3+k^2=0$ dan $k^2(k^2-2k+1)=0$, bu yerdan esa, $k_1=k_2=0, k_3=k_4=1$ kabi ildizlarni aniqlaymiz.

Ularga ko'ra umumiy yechim

$$\tilde{y}=C_1+C_2x+(C_3+C_4x)e^x.$$

Bu yerda, $\alpha=1$ soni xarakteristik tenglamaning ikki karrali ildizi bo'lmoqda. Shuning uchun, xususiy yechim quyidagicha tanlanadi

$$y^*=Ax^2e^x$$

Bu funksiyaning hosilalarini topamiz.

$$y'^*=2Axe^x+Ax^2e^x,\quad y''^*=2Ae^x+4Axe^x+Ax^2e^x,\quad y'''^*=6Ae^x+6Axe^x+Ax^2e^x,$$
$$y''''^*=12Ae^x+8Axe^x+Ax^2e^x.$$

Ushbu topilganlarni tenglamaga qo'yib, $e^x$ ga qisqartirsak,

$$12A+8Ax+Ax^2-12A-12Ax-2Ax^2+2A+4Ax+Ax^2=1$$

Undan esa $A=\frac{1}{2}$ ni topamiz.

Umumiy yechim

$$\tilde{y}=C_1+C_2x+(C_3+C_4x)e^x+\frac{1}{2}x^2e^x.$$

## MASHQLAR

Quyidagi tenglamalarni o'zgarmasni variatsiyalash usulida yechilsin.

| | | |
|---|---|---|
| 1. $y''+y=tgx,$ | 3. $y''+y'=\sec x,$ | 5. $y''+y'=\frac{1}{Cosx},$ |
| 2. $y''+y=ctgx,$ | 4. $y''-2y'+y=\frac{e^x}{x},$ | 6. $y''+y=\frac{1}{\sin x}$ |

Quyidagi tenglamalarning umumiy yechimlari topilsin.

| | |
|---|---|
| 1. $y''-5y'+6y=0,$ | 11. $y''-8y'+16y=0$ |
| 2. $y''-9y=0,$ | 12. $y''-4y'+13y=0$ |
| 3. $y''-2y'+2y=0$ | 13. $y'''-5y''+16y'-12=0$ |
| 4. $y''+4y'+13y=0,$ | 14. $y'''-8y''+7y=0,$ |
| 5. $y''-4y'+2y=0,$ | 15. $y''-6y'+25y=0,$ |

6. $y'' + 5y' + 6y = 0,$
7. $y'' - 2y' - 4y = 0,$
8. $y'' + 6y' + 9y = 0,$
9. $y'' - 6y' + 18y = 0,$
10. $3y'' - 2y' - 8y = 0$
16. $y''' - 3y'' + 3y' - y = 0.$
17. $y^{V} - 6y^{IV} + 9y''' = 0$
18. $y^{VI} - 3y^{V} + 3y^{IV} = 0,$
19. $y^{IV} - 8y'' + 16y = 0,$
20. $y''' + 2y'' + 10y' = 0.$

Quyidagi bir jinsli bo'lmagan differensial tenglamalar uchun xususiy yechimlarning ko'rinishi tanlansin.

1. $y'' - 4y = (x+1)e^{2x},$
2. $y'' + 9y = 3Cos3x,$
3. $y'' + 2y' + 2y = e^{2x}\sin x,$
4. $y'' - y = 3e^{2x},$
5. $y''' - 4y'' + 4y = (2x^2 + 3)e^{2x},$
6. $y^{V} - 4y''' = 5x\sin 2x,$
7. $y^{IV} - 16y = 2xe^{3x},$
8. $y^{IV} - 5y'' + 4y = 4e^{x},$
9. $y'' - 7y' + 6y = 3\sin 2x,$
10. $y''' - 4y'' + 5y' - 2y = 2x + 3.$

Quyidagi bir jinsli bo'lmagan differensial tenglamalarning umumiy yoki xususiy yechimlari (agar boshlang'ich shartlar ko'rsatilgan bo'lsa) topilsin.

1. $y'' - 2y' + y = x^3,$
2. $y'' + 2y = 2 + x^2,$
3. $y'' - 3y' + 2y = 3x + 5\sin 2x,$
4. $2y'' - 5y' = 4x^2 + 3x + 1,$
5. $y'' - 9y = e^{3x}\cos x,$
6. $y'' + 4y = x\sin 2x,$
7. $y'' + 7y' + 12y = x,$
8. $y'' + y' - 2y = 8\sin 2x,$
9. $y'' - 25y = 9 - 2x,$
10. $y'' - 3y' + 2y = e^{-x}(4x^2 - 5),$
11. $y^{IV} + y''' = x^2 - 1,$
12. $y''' - 3y' = 4(2 + x^2),$
13. $y'' + y' - 6y = xe^{2x},$
14. $y'' + 9y = 2x\sin x + xe^{2x},$
15. $y'' + 9y = 2x\sin x + xe^{2x},$
16. $y'' - 2y' + 2y = 4e^{x}\sin x,$
17. $y^{IV} + y'' = x^2 + 2x,$
18. $y''' + y'' = 6x + e^{-x},$
19. $4y'' - y = x^3 - 24x,$
20. $y''' + y'' = x^2 + 1 + 3xe^{x},$
21. $y''' + y' = \cos 2x,$
22. $y^{IV} + y''' = \cos 2x,$
23. $y'' + 2y' = 0, \quad y(0) = 1, \; y'(0) = 0$
24. $y'' - 4y' + 3y = 0, \quad y(0) = 0, \quad y'(0) = 2$
25. $y'' - 3y' + 2y = (x^2 + x)e^{3x}, \; y(0) = 1, \; y'(0) = -2$
26. $4y'' - 8y' + 5y = 5\cos x, \quad y(0) = 0, \; y'(0) = -\frac{1}{3}$
27. $y'' + 6y' + 13y = 26x - 1, \quad y(0) = 1, \quad y'(0) = 1$
28. $y'' + 3y' - 10y = xe^{-2x}, \quad y(0) = y'(0) = 0.$
29. $y'' + 3y' - 4y = 8\sin x, \; y(0) = 4, \; y'(0) = 0$
30. $y'' + 6y' = 0, \quad y(0) = 1, \quad y'(0) = -6.$

## 5.4. Differensial tenglamalar sistemasi

1. Ko'pgina masalalarni yechishda shunday $y_1 = y_1(x), y_2 = y_2(x),..., y_n = y_n(x)$ kabi noma'lum funksiyalarni aniqlashga to'g'ri keladiki, ular argument $x$, noma'lum funksiyalar va ularning hosilalariga nisbatan oddiy differensial tenglamalar sistemasini qanoatlantiradilar.

Quyidagicha birinchi tartibli oddiy differensial tenglamalar sistemasini qaraymiz:

$$\begin{cases} \frac{dy_1}{dx} = f_1(x, y_1, y_2, ..., y_n), \\ \frac{dy_2}{dx} = f_2(x, y_1, y_2, ..., y_n), \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ \frac{dy_n}{dx} = f_n(x, y_1, y_2, ..., y_n), \end{cases} \qquad (1)$$

bu yerda, $x$ – argument $y_1, y_2, ..., y_n$ lar noma'lum funksiyalardir. Bu sistemani odatda normal shaklda berilgan sistema deb yuritiladi.

Mazkur sistema uchun Koshi masalasi quyidagicha ta'riflanadi: Berilgan sistemaning shunday bir $y_1 = y_1(x), y_2 = y_2(x), ..., y_n = y_n(x)$ kabi yechimini topish lozimki, ular berilgan $y_1(x_0) = y_{10}, y_2(x_0) = y_{20}, ..., y_n(x_0) = y_{n0}$ kabi boshlang'ich shartlarni qanoatlantiradigan bo'lsin. Bu yerda $y_{10}, y_{20}, ..., y_{n0}$ .

Qaralayotgan sistemani yechish uchun ko'pincha "noma'lumlarni yo'qotish usuli" deb ataluvchi usuldan foydalaniladi. Mazkur usulni quyidagi misollar orqali bayon etamiz.

**1 –Misol.** $\begin{cases} \frac{dy_1}{dx} = x + y_1 + y_2, \\ \frac{dy_2}{dx} = 2x - 4y_1 - 3y_2. \end{cases}$

**Yechish.** Birinchi tenglamani $x$ bo'yicha differensiallaymiz

$$\frac{d^2y_1}{dx^2} = 1 + \frac{dy_1}{dx} + \frac{dy_2}{dx}.$$

$\frac{dy_1}{dx}$ va $\frac{dy_2}{dx}$ larning sistemadagi qiymatlarini bu erga qo'ysak,

$\frac{d^2y_1}{dx^2} = -3y_1 - 2y_2 + 3x + 1$ ni hosil qilamiz.

Sistemaning birinchi tenglamasidan $y_1 = \frac{dy_1}{dx} - x - y_2$ ni keltirib yuqoridagi tenglamaga qo'yamiz:

$$\frac{d^2y_1}{dx^2} + 2\frac{dy_1}{dx} + y_1 = 5x + 1,$$

bu esa $y_1 = y_1(x)$ga nisbatan 2 – tartibli chiziqli bir jinsli bo'lmagan tenglamadir.

a) $k^2 + 2k + 1 = 0$ dan, $k_1 = k_2 = -1$ va

$$\widetilde{y}_1 = (C_1 + C_2x)e^{-x}$$

b) Bir jinsli bo'lmagan tenglamaning xususiy yechimi uchun $y_1^* = Ax + B, \quad y_1^{*\prime} = A, \quad y_1^{*\prime\prime} = 0$ dan

$2A + Ax + B = 5x + 1$, yoki $A = 5$ва $B = -9$

Umumiy yechim

$$y_1 = (C_1 + C_2x)e^{-x} + 5x - 9.$$

$y_1' = C_2e^{-x} - (C_1 + C_2x)e^{-x} + 5$ bo'lganligi uchun, sistemaning birinchi tenglamasidan

$$y_2 = (C_1 + C_2x)e^{-x} + 6x + 14$$

ni topamiz.

**2 – Misol.** $\frac{dy_1}{dx} = 4y_1 - y_2, \quad \frac{dy_2}{dx} = y_1 + 2y_2$ tenglamalar sistemasining $y_1(0) = 0, \quad y_2(0) = 1$ kabi boshlang'ich shartlarni qanoatlantiruvchi xususiy yechimi topilsin.

**Yechish.** Birinchi tenglamadan, $\frac{d^2y_1}{dx^2} = 4\frac{dy_1}{dx} - \frac{dy_2}{dx}$ ni hosil qilamiz. U holda $\frac{d^2y_1}{dx^2} = 4(4y_1 - y_2) - (y_1 + 2y_2)$

yoki

$$\frac{d^2y_1}{dx^2} = 15y_1 - 10y_2$$

hosil bo'ladi. Birinchi tenglamadan $y_2 = 4y_1 - \frac{dy_1}{dx}$ ni yuqoridagi tenglamaga qo'yamiz:

$$\frac{d^2y_1}{dx^2} = 15y_1 - 10\left(4y_1 - \frac{dy_1}{dx}\right)$$

$$\frac{d^2y_1}{dx^2} - 10\frac{dy_1}{dx} + 25y_1 = 0$$

Bu tenglamaning umumiy yechimi $y_1 = (C_1 + C_2x)e^{5x}$

Endi $y_1$ va $y_1'$ larning qiymatlarini sistemaning birinchi tenglamasiga qo'yib, $y_2$ ni aniqlaymiz

$$y_2 = (-C_1 - C_2 - C_2 x)e^{5x}.$$

Boshlang'ich shartlarga binoan, $C_1 = 0,\ C_2 = -1$ larni topamiz. Natijada, sistemaning xususiy yechimi quyidagicha yoziladi

$$y_1 = -xe^{5x},\quad y_2 = (1+x)e^{5x}.$$

**3 – Misol.** $y_1' = y_2 + e^x,\ y_2' = y_1 + y_3,\ y_3' = y_1 + y_2$ kabi sistemaning umumiy yechimi topilsin.

**Yechish**. Birinchi tenglamani differensiallab $y_1'' = y_2' + e^x$ ni hosil qilamiz. Bu tenglamaga, ikkinchi tenglamani qo'yamiz.

$y_1'' = y_1 + y_3 + e^x$ va buni differensiallaymiz $y_1''' = y_1' + y_3' + e^x$. Bunga $y_3'$ ning qiymatini qo'yamiz $y_1''' = y_2' + y_1 + y_2 + e^x$. Birinchi tenglamadan $y_2 = y_1' - e^x$ ni yuqoridagi tenglamaga qo'yamiz $y_1''' = y_1' + y_1 + y_1' + -e^x + e^x$ yoki

$$y_1''' - 2y_1' - y_1 = 0$$

ni hosil qilamiz.

Xarakteristik tenglama $k^3 - 2k - 1 = 0$ dan, $k_1 = -1$, $k_2 = \frac{1-\sqrt{5}}{2}$, $k_3 = \frac{1+\sqrt{5}}{2}$ larni topamiz. U holda

$$y_1 = C_1 e^{-x} + C_2 e^{\frac{1-\sqrt{5}}{2}x} + C_3 e^{\frac{1+\sqrt{5}}{2}x}.$$

Uni differensiallab, $y_1'$ ning qiymatini ikkinchi tenglamaga qo'yamiz, u holda:

$$y_2 = -(C_1 + 1)e^{-x} + \frac{1-\sqrt{5}}{2}C_2 e^{\frac{1-\sqrt{5}}{2}x} + \frac{1+\sqrt{5}}{2}C_3 e^{\frac{1+\sqrt{5}}{2}x}.$$

Endi $y_2'$ ning qiymati bilan $u_1$ ni ikkinchi tenglamaga qo'yib, $u_3$ ni aniqlaymiz.

$$y_3 = e^{-x} + \frac{1-\sqrt{5}}{2}C_2 e^{\frac{1-\sqrt{5}}{2}x} + \frac{1+\sqrt{5}}{2}C_3 e^{\frac{1-\sqrt{5}}{2}x}.$$

Sistemadagi tenglamalarning soni ikki, uch va undan ortiq bo'lgan hollarda ham ko'rib o'tilgan misollardagiga o'xshash yo'ldan foydalaniladi.

**2.** Endi koeffitsientlari o'zgarmas bo'lgan birinchi tartibli chiziqli bir jinsli differensial tenglamalar sistemasining yechilish jarayonini bayon etamiz.

Oddiylik uchun uch noma'lumli uchta tenglamalar sistemasini qaraymiz.

$$\begin{cases} y_1' = a_{11}y_1 + a_{12}y_2 + a_{13}y_3, \\ y_2' = a_{21}y_1 + a_{22}y_2 + a_{23}y_3, \\ y_3' = a_{31}y_1 + a_{32}y_2 + a_{33}y_3. \end{cases} \tag{2}$$

Ushbu sistemaning xususiy yechimini quyidagi ko'rinishda axtaramiz.

$$y_1 = \alpha e^{kx}, \quad y_1 = \beta e^{kx}, \quad y_1 = \gamma e^{kx} \tag{3}$$

Bu yerdagi, α, β, γ va $k$ larni shunday aniqlash lozimki, natijada (3) funksiyalar (2) sistemaning yechimi bo'lsin.

Endi (3) ni va $y_1', y_2', y_3'$ larni (2) sistemagi qo'yib, α, β, γ larni aniqlaydigan quyidagi sistemaga ega bo'lamiz.

$$\begin{cases} (a_{11} - k)\alpha + a_{12}\beta + a_{13}\gamma = 0, \\ a_{21}\alpha + (a_{22} - k)\beta + a_{23}\gamma = 0, \\ a_{31}\alpha + a_{32}\beta + (a_{33} - k)\gamma = 0. \end{cases} \tag{4}$$

Ushbu sistema noldan farqli yechimga ega bo'lishi uchun uning asosiy determinanti nolga teng bo'lishi kerakligidan

$$\begin{vmatrix} a_{11} - k & a_{12} & a_{13} \\ a_{21} & a_{22} - k & a_{23} \\ a_{31} & a_{32} & a_{33} - k \end{vmatrix} = 0 \tag{5}$$

ni hosil qilamiz. Bu esa noma'lum $k$ ga nisbatan 3 – darajali algebraik tenglama bo'lib, uni (2) tenglamaga mos xarakteristik tenglama deb ataladi.

Quyidagi hollarni qaraymiz.

A) Xarakteristik tenglamaning ildizlari haqiqiy har hil sonlar bo'lsin. Bu holda (2) sistemaning yechimi quyidagicha yoziladi:

$$y_1 = C_1\alpha_1 e^{k_1x} + C_2\alpha_2 e^{k_2x} + C_3\alpha_3 e^{k_3x},$$
$$y_2 = C_1\beta_1 e^{k_1x} + C_2\beta_2 e^{k_2x} + C_3\beta_3 e^{k_3x},$$
$$y_3 = C_1\gamma_1 e^{k_1x} + C_2\gamma_2 e^{k_2x} + C_3\gamma_3 e^{k_3x}.$$

**1- Misol.** $\begin{cases} y_1' = -y_1 + y_2 \\ y_2' = -3y_2 \end{cases}$ ning umumiy yechimi topilsin.

__Yechish.__ Xarakterisg'chtik tenglamani tuzamız.

$\begin{vmatrix} -1-k & 1 \\ 0 & -3-k \end{vmatrix} = 0$, yoki $k^2 + 4k + 3 = 0$. Uning ildizlari $k_1 = -3$ va $k_2 = -1$.

a) $k_1 = -3$ ga mos α va β larning $\alpha_1$ va $\beta_1$ qiymatlarini aniqlaymiz. (4) ga ko'ra, $\begin{cases} 2\alpha_1 + \beta_1 = 0 \\ 0 = 0 \end{cases}$ bu yerda $\alpha_1 = 1$ deb olsak, $\beta_1 = -2$ ga teng bo'ladi. U holda xususiy yechimlar $y_1^{(1)} = e^{-3x}$ va $y_2^{(1)} = -2e^{-3x}$.

b) $k_2 = -1$ ga mos α va β larning $\alpha_2$ va $\beta_2$ qiymatlarini aniqlaymiz. (4) ga ko'ra, $\begin{cases} 0\alpha_2 + \beta_2 = 0 \\ 0\alpha_2 - 2\beta_2 = 0 \end{cases}$ dan $\alpha_2 = 1$, $\beta_2 = 0$ deb olish mumkin.

Xususiy yechimlar $y_1^{(2)} = e^{-x}$ va $y_2^{(2)} = 0$.

Endi umumiy yechimni yozamiz

$$y_1 = C_1 e^{-3x} + C_2 e^{-x}, \quad y_2 = -2C_1 e^{-3x}.$$

c) Xarakteristik tenglamaning ildizlari har xil bo'lib, ular orasida kompleks son ildizlar bor bo'lsin. Bu holda ularga mos xususiy yechimlarni Eyler formulalari orqali o'zgartiriladi.

**2-Misol.** $\begin{cases} y_1' = -7y_1 + y_2 \\ y_2' = -2y_1 - 5y_2 \end{cases}$ ning umumiy yechimi topilsin.

<u>**Yechish.**</u> Xarakteristik tenglama

$\begin{vmatrix} -7-k & 1 \\ -2 & -5-k \end{vmatrix} = 0$ yoki $k^2 + 12k + 37 = 0$ ning ildizlari $k_{1,2} = -6 \pm i$.

$k_1 = -6 + i$ ga mos $\alpha_1$ va $\beta_1$ larni topamiz.

$$\begin{cases} (-1-i)\alpha_1 + \beta_1 = 0 \\ -2\alpha_1 + (1-i)\beta_2 = 0 \end{cases} \text{ bu yerdan } \begin{cases} \alpha_1 = 1, \\ \beta_1 = 1 + i \end{cases}$$

U holda

$$y_1^{(1)} = e^{(-6+i)x} = e^{-6x}(\cos x + i\sin x)$$

$$y_2^{(1)} = (1+i)e^{(-6+i)x} = e^{-6x}[(\cos x - \sin x) + i(\cos x + \sin x)].$$

Ushbu xususiy yechimlardan, yechimlarning fundamental sistemasini tashkil etuvchi quyidagi xususiy yechimlarni topamiz.

$$\bar{y}_1^{(1)} = e^{-6x}\cos x, \qquad \bar{y}_2^{(1)} = e^{-6x}(\cos x - \sin x),$$

$$\bar{y}_1^{(1)} = e^{-6x}\sin x, \qquad \bar{y}_2^{(1)} = e^{-6x}(\cos x + \sin x).$$

Endi umumiy yechimni quyidagicha yozamiz:

$$y_1 = e^{-6x}(C_1 Cos x + C_2 \sin x),$$

$$y_2 = e^{-6x}[(C_1(\cos x - \sin x) + C_2(\cos x + \sin x)].$$

Bu yerda, $k_2 = -6 - i$ ildizdan foydalanilaganda ham, yuqoridagidek xususiy yechimlarni hosil qilinar edi. Shuning uchun ikkinchi ildizni ishlatish ortiqcha ekanligini ta'kidlab o'tamiz.

<u>**Eslatma.**</u> Biz bu yerda, xarakteristik tenglama ildizlari karrali bo'lgan holni bayon etmadik. Bu hol masalan, I.G.Petrovskiyning «Oddiy differensial tenglamalar bo'yicha ma'ruzalar» nomli kitobida va boshqa adabiyotlarda, batafsil bayon etilgan.

## MASHQLAR

Quyidagi differensial tenglamalar sistemalarining umumiy yoki (agar boshlang'ich shartlar berilgan bo'lsa) yechimlari aniqlansin.

1. $\begin{cases} y_1' = y_2 + 1, & y_1(0) = -2, \\ y_2' = y_2 + 1, & y_2(0) = 0 \end{cases}$

2. $\begin{cases} y_1' + 2y_2 = 3x, & y_1(0) = 2, \\ y_2' - 2y_1 = 4 & y_2(0) = 3 \end{cases}$

3. $\begin{cases} 4y_1' - y_2' + 3y_1 = \sin t, \\ y_1' + y_2 \cos t. \end{cases}$

4. $\begin{cases} y_1' + y_2 = 0, \\ y_2' + 4y_1 = 0 \end{cases}$

5. $\begin{cases} y_1' - y_1 + y_2 = 0, & y_1(0) = 0, \\ y_2' - y_1 - 2y_2 = 0. & y_2(0) = 1 \end{cases}$

6. $\begin{cases} y_1' - 2y_1 + y_2 = 0, \\ y_2' - 3y_1 + 2y_2 = 0. \end{cases}$

7. $\begin{cases} y_1' - y_1 - 4y_2 = 0, \\ y_2' - y_1 + 2y_2 = 0 \end{cases}$

8. $\begin{cases} y_1'' = y_2, \\ y_2'' = y_1 \end{cases}$

# 6. SONLI QATORLAR

## 6.1. Asosiy tushunchalar

**1-Ta'rif.** Faraz qilaylik, quyidagi cheksiz sonlar ketma ketligi berilgan bo'lsin:

$$a_1, a_2, a_3, \dots a_n, \dots ,$$

u holda quyidagi yig'indi

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \dots + a_n + \cdots \qquad (1)$$

*sonli qator* deb ataladi, bu yerdagi $a_1, a_2, a_3, \dots a_n, \dots$ qo'shiluvchilar qatorning hadlari deb ataladi, $a_n$ – esa qatorning umumiy hadi deyiladi.

**1 - Misol.** Agar qatorning umumiy hadi quyidagi formula bilan berilgan bo'lsa, uning birinchi to'rtta hadini yozing:

$$a_n = \frac{(-1)^{n-1}}{(3n+1)2^{n-1}}$$

**Yechish:** Agar n=1 bo'lsa, u holda:

$$a_1 = \frac{(-1)^0}{(3+1)2^0} = \frac{1}{4};$$

Agar n=2 bo'lsa, u holda:

$$a_2 = \frac{(-1)^{2-1}}{(3\cdot2+1)2^{2-1}} = -\frac{1}{7\cdot2};$$

Agar n=3 bo'lsa, u holda:

$$a_3 = \frac{(-1)^{3-1}}{(3\cdot3+1)2^{3-1}} = \frac{1}{10\cdot2^2};$$

Agar n=4 bo'lsa, u holda:

$$a_4 = \frac{(-1)^{4-1}}{(3\cdot4+1)2^{4-1}} = -\frac{1}{13\cdot2^3}.$$

Shunday qilib, berilgan qatorni quyidagicha yozish mumkin.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(3n+1)2^{n-1}} = \frac{1}{4} - \frac{1}{7\cdot2} + \frac{1}{10\cdot2^2} - \frac{1}{13\cdot2^3} + \cdots$$

**2 – Misol.** Qatorning umumiy hadini yozing:

$$\frac{2}{\sqrt[5]{7}} + \frac{4}{\sqrt[5]{14}} + \frac{8}{\sqrt[5]{21}} + \dots .$$

**Yechish:** Har bir kasrning surati ikki sonining darajasidan, mahraji esa $\sqrt[5]{7\cdot n}$ ifodadan iborat. Shuning uchun : $a_n = \frac{2^n}{\sqrt[5]{7\cdot n}}$

**3 – Misol.** Qatorning umumiy hadini yozing.

$$1 + \frac{4}{3} + \frac{7}{9} + \frac{10}{27} + \frac{13}{81} + \cdots$$

**Yechish.** Qator hadlarining hosil bo'lish qonuniyatlarini qaraymiz. Qatorning mahraji 3 sonining biror darajasi ekanligini ko'rish qiyin emas. Qatorning birinchi hadi uchun daraja nolga teng, ya'ni n = 1 va 1 – 1= 0, ikkinchi hadi uchun daraja birga teng, ya'ni n = 2 va 2 – 1= 1, beshinchi hadi uchun daraja to'rtga teng, ya'ni n = 5 va 5 – 1= 4. Bundan uchning darajasi n – 1 ga teng degan hulosaga kelamiz. O'z navbatida suratidagi son har doim $3\cdot n$ dan 2 ga kam. Demak, qatorning umumiy hadi quyidagicha bo'ladi: $a_n=\frac{3n-2}{3^{n-1}}$;

**3 – Ta'rif.** (1) qatorning dastlabki n- ta chekli hadlarining yig'indisi qatorning n- chi hususiy yig'indisi deb ataladi va quyidagicha yoziladi:

$$S_n = a_1 + a_2+a_3+ ...+ a_n \tag{2}$$

Agar

$$\lim_{n\to\infty} S_n = S$$

kabi chekli limit mavjud bo'lsa, uni (1) qatorning yig'indisi deyiladi hamda $n \to \infty$ berilgan qatorni yaqinlashuvchi, aks holda (1) qator uzoqlashuvchi deyiladi. Boshqacha aytganda, qator yaqinlashuvchi degani, qatorning cheksiz sondagi hadlari yig'indisi biror chekli S soniga teng degani, ya'ni :

$$a_1+a_2+a_3+ ...+ a_n+ ... = S$$

**4 – Misol.** Quyidagi qatorni yaqinlashishga tekshiring:

$$\sum_{n=1}^{\infty}\frac{1}{4^n} = 1+\frac{1}{4}+\frac{1}{4^2}+\frac{1}{4^3}+ ...$$

<u>**Yechish:**</u> Ma'lumki, bu cheksiz kamayuvchi geometrik progressiya. Uning hadlari yig'indisi

$$S = \frac{b_1}{1-q}$$

formula bilan hisoblanadi. Bu yerda $b_1$ – progressiyaning birinchi hadi, q esa uning maxraji. Bizning misolimizda $b_1$= 1, q=$\frac{1}{4}$. U holda:

$$S = 1+\frac{1}{4}+\frac{1}{4^2}+\frac{1}{4^3}+ ... = \frac{1}{1-\frac{1}{4}} = \frac{1}{\frac{3}{4}} = \frac{4}{3}.$$

Shunday qilib, $\frac{4}{3}$ chekli son hosil bo'ldi, demak $\sum_{n=1}^{\infty}\frac{1}{4^n}$ qator yaqinlashuvchi.

**5 – Misol.** $\sum_{n=1}^{\infty}\frac{1}{n(n+1)}$ qator berilgan. Qatorning yaqinlashuvchi ekanligini ko'rsating va uning yig'indisini toping.

<u>**Yechish.**</u> Qatorning birinchi n-ta xususiy yig'indisini yozamiz va uning shaklini almashtiramiz :

$$S_n = \frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{n(n+1)} =$$
$$= \left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\cdots++\left(\frac{1}{n}-\frac{1}{n+1}\right) = 1-\frac{1}{n+1}$$

$$S = \lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(1-\frac{1}{n+1}\right) = 1$$

Demak, berilgan qator yaqinlashuvchi va uning yig'indisi S=1.

6 – Misol. Ta'rifdan foydalanib, qatorning yaqinlashuvchi ekanligini ko'rsating va uning yig'indisini toping.

$$\frac{1}{1\cdot3}+\frac{1}{3\cdot5}+\frac{1}{5\cdot7}+\dots+\frac{1}{(2n-1)(2n+1)}+\dots$$

__Yechish.__ Qatorning umumiy hadi $a_n = \frac{1}{(2n-1)(2n+1)}$. Umumiy hadni quyidagicha yozib olamiz:

$$\frac{1}{(2n-1)(2n+1)} = \frac{A}{2n-1}+\frac{B}{2n+1} = \frac{A(2n+1)+B(2n-1)}{(2n-1)(2n+1)} = \frac{n(2A+2B)+A-B}{(2n-1)(2n+1)}.$$

A va B noma'lumlarni topish uchun aniqmas koeffitsientlar usulidan foydalanamiz: 1 = n(2A+2B) +A –B bundan:

$$\begin{cases} 2A+2B=0 \\ A-B=1 \end{cases},$$

Hosil qilingan tenglamalar sistemasini echib, A va B larni topamiz: $A=\frac{1}{2}$; $B=-\frac{1}{2}$; U holda qatorning umumiy hadi quyidagi ko'rinishga ega bo'ladi:

$$a_n = \frac{1}{(2n-1)(2n+1)} = \frac{\frac{1}{2}}{2n-1} - \frac{\frac{1}{2}}{2n+1} = \frac{1}{2}\left(\frac{1}{2n-1}-\frac{1}{2n+1}\right)$$

Umumiy haddan foydalanib qatorning birinchi n-ta xususiy yig'indisini yozamiz va uning shaklini almashtiramiz :

$$S_n = \frac{1}{1\cdot3}+\frac{1}{3\cdot5}+\frac{1}{5\cdot7}+\dots+\frac{1}{(2n-1)(2n+1)} = \frac{1}{2}\left(\left(1-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{5}\right)+\left(\frac{1}{5}-\right.\right.$$
$$\left.\left.-\frac{1}{7}\right)+\dots+\left(\frac{1}{2n-1}-\ -\frac{1}{2n+1}\right)\right) = \frac{1}{2}\left(1-\frac{1}{3}+\frac{1}{3}-\frac{1}{5}+\frac{1}{5}-\frac{1}{7}+\dots+\frac{1}{2n-1}-\frac{1}{2n+1}\right) =$$
$$= \frac{1}{2}\left(1-\frac{1}{2n+1}\right).$$

Demak, $S_n = \frac{1}{2}\left(1-\frac{1}{2n+1}\right)$. Bundan.

$$S = \lim_{n\to\infty} S_n = \lim_{n\to\infty}\frac{1}{2}\left(1+\frac{1}{n+1}\right) = \frac{1}{2}.$$

Shunday qilib, berilgan qator yaqinlashuvchi va uning yig'indisi $S=\frac{1}{2}$ ekan.

7 – Misol. Ta'rifdan foydalanib, qatorning yaqinlashuvchi ekanligini ko'rsating va uning yig'indisini toping:

$$\sum_{k=1}^{\infty}\frac{1}{9k^2+12k-5}.$$

__Yechish.__ $9k^2+12k+5 = (3k-1)(3k+5)$. U holda qatorning umumiy hadi quyidagicha bo'ladi

$$a_k = \frac{1}{(3k-1)(3k+5)}.$$

Umumiy hadni quyidagicha yozib olamiz:

$$\frac{1}{9k^2+12k-5} = \frac{1}{6}\left(\frac{1}{3k-1} - \frac{1}{3k+5}\right)$$

Undan foydalanib qatorning birinchi n-ta xususiy yig'indisining shaklini almashtiramiz :

$$S_n = \sum_{k=1}^{n} \frac{1}{9k^2+12k-5} = \frac{1}{6}\sum_{k=1}^{n} \frac{1}{6}\left(\frac{1}{3k-1} - \frac{1}{3k+5}\right) =$$

$$= \frac{1}{6}\left[\left(\frac{1}{2} - \frac{1}{8}\right) + \left(\frac{1}{5} - \frac{1}{11}\right) + \left(\frac{1}{8} - \frac{1}{14}\right) + \left(\frac{1}{11} - \frac{1}{17}\right) + \cdots\right.$$

$$\left. + \left(\frac{1}{3n-4} - \frac{1}{3n+2}\right) + \left(\frac{1}{3n-1} - \frac{1}{3n+5}\right)\right]$$

$$= \frac{1}{6}\left[\frac{1}{2} + \frac{1}{5} - \frac{1}{3n+2} + \frac{1}{3n+5}\right] =$$

$$= \frac{1}{6}\left[\frac{7}{10} - \frac{1}{3n+2} - \frac{1}{3n+5}\right].$$

Shunday qilib, quyidagini hosil qilamiz:

$$S = \lim_{n\to\infty} S_n = \frac{1}{6}\lim_{n\to\infty}\left(\frac{7}{10} - \frac{1}{3n+2} - \frac{1}{3n+5}\right) = \frac{1}{6}\cdot\frac{7}{10} = \frac{7}{60}$$

**8 – misol.** Sonli qatorning yig'indisini toping.

$$\sum_{n=1}^{\infty} \frac{3^n + 4^n}{12^n}$$

<u>**Yechish.**</u> Berilgan qatorni quyidagi ko'rinishda yozib olamiz.

$$\sum_{n=1}^{\infty} \frac{3^n+4^n}{12^n} = \sum_{n=1}^{\infty} \frac{3^n}{12^n} + \sum_{n=1}^{\infty} \frac{4^n}{12^n} =$$

$$= \sum_{n=1}^{\infty} \left(\frac{1}{4}\right)^n + \sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n .$$

Hosil bo'lgan bu ikki qator ham cheksiz kamayuvchi geometrik progressiya va ular yaqinlashuvchi. Demak berilgan qator ham yaqinlashuvchi va

$$\mathrm{S} = \mathrm{S}_1 + \mathrm{S}_2 = \frac{1}{3} + \frac{1}{2} = \frac{5}{6}.$$

## MASHQLAR

1. Berilgan umumiy had bo'yicha qatorning birinchi beshta hadini yozing:

1. $a_n = \frac{5n+1}{n^2-1}$.
2. $a_n = \frac{2^n}{n+4}$.
4. $a_n = \frac{ln(2n+1)}{n^2}$.
5. $a_n = \frac{3^n \cdot n!}{5^n}$.

3. $a_n = \frac{(-1)^n}{n^2+3}$.  6. $a_n = \sin\frac{\pi}{2} \cdot n+1$.

2. Qatorning umumiy hadini yozing:

1. $\frac{2}{ln2} + \frac{3}{ln3} + \frac{4}{ln4} + \frac{5}{ln5} +$  3. $\frac{2\cdot4}{1\cdot5} - \frac{3\cdot6}{4\cdot10} + \frac{4\cdot8}{7\cdot15} -$

2. $\frac{2}{1} + \frac{6}{1\cdot2} + \frac{12}{1\cdot2\cdot3} + \frac{20}{1\cdot2\cdot3\cdot4} +$  4. $1 - \frac{1}{2\cdot3} + \frac{1}{3\cdot9} - \frac{1}{4\cdot27} +$

3. Ta'rifdan foydalanib, qatorni yaqinlashishga tekshiring, agar qator yaqinlashuvchi bo'lsa uning yig'indisini toping:

1. $\frac{1}{1\cdot2} + \frac{1}{2\cdot3} + \ldots + \frac{1}{n(n+1)} + \ldots$
2. $\frac{1}{1\cdot3} + \frac{1}{3\cdot5} + \ldots + \frac{1}{(2n-1)(2n+1)} + \ldots$
3. $\frac{1}{1\cdot4} + \frac{1}{2\cdot5} + \ldots + \frac{1}{n(n+3)} + \ldots$
4. $\frac{1}{1\cdot7} + \frac{1}{3\cdot9} + \ldots + \frac{1}{(2n-1)(2n+5)} + \ldots$
5. $\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)}$.

## 6.2. Qator yaqinlashuvchligining zaruriy belgisi. Musbat hadli qatorlar. Musbat hadli qatorlarning yaqinlashish belgilari

Agar $\sum_{n=1}^{\infty} a_n$ qator yaqinlashuvchi bolsa, u holda uning umumiy hadi, $n \to \infty$ da nolga intiladi, ya'ni,

$$\lim_{n\to\infty} a_n = 0$$

Teskari tasdiq esa har doim ham to'g'ri emas, ya'ni, $\lim_{n\to\infty} a_n = 0$ ekanligidan berilgan qatorning yaqinlashuvchilgi kelib chiqmaydi, ammo, berilgan qator yaqinlashuvchi bo'lsa, u holda albatta $\lim_{n\to\infty} a_n = 0$ bo'ladi. Shuning uchun ham bu shart qator yaqinlashishining zaruriy sharti hisoblanadi. Agar berilgan qatorni yaqinlashishga tekshirishda $\lim_{n\to\infty} a_n = 0$ bo'lsa, uning yaqinlashishi haqidagi savol ochiq qoladi.

Agar $a_1 + a_2 + a_3 + \ldots + a_n + \ldots$ qatorning barcha hadlari musbat, ya'ni $a_n > 0$ bo'lsa, uni musbat hadli qator deyiladi. Musbat hadli qatorlarning yaqinlashish belgilarini keltiramiz. Ikkita musbat hadli qatorlarni qaraylik

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$
$$\sum_{n=1}^{\infty} b_n = b_1 + b_2 + b_3 + \cdots + b_n + \cdots$$

**Taqqoslashning birinchi belgisi.**

Agar biror $n > N$ nomerdan boshlab $a_n \le b_n$ tengsizlik bajarilsa, u holda

$\sum_{n=1}^{\infty} b_n$ qatorning yaqinlashishidan $\sum_{n=1}^{\infty} a_n$ qatorning ham yaqinlashishi kelib chiqadi.

Agar biror $n > N$ nomerdan boshlab $a_n \ge b_n$ tengsizlik bajarilsa, u holda

$\sum_{n=1}^{\infty} b_n$ qatorning uzoqlashishidan $\sum_{n=1}^{\infty} a_n$ qatorning ham uzoqlashishi kelib chiqadi.

__Izoh.__ $\sum_{n=1}^{\infty} aq^n$ geometrik progressiya, $\sum_{n=1}^{\infty} \frac{1}{n}$ garmonik qator $\sum_{n=1}^{\infty} \frac{1}{n^p}$ va umumlashgan garmonik qatorlardan taqqoslash alomatlari yordamida qatorlarni tekshirishda juda ko'p foydalaniladi.

**Taqqoslashning ikkinchi belgisi**

Chekli va noldan farqli $\lim_{n\to\infty} \frac{a_n}{b_n} = \ell$ $(\ell \neq 0;\ \ell \neq \infty)$ limit mavjud bo'lsa, u holda ikkala qator bir vaqtda yaqinlashadi yoki bir vaqtda uzoqlashadi.

**Dalamber belgisi**. Agar musbat hadli

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \dots + a_n + \dots$$

qatorda

$$\lim_{n=\infty} \frac{a_{n+1}}{a_n} = \ell$$

bo'lsa, u holda:

agar $\ell < 1$ bo'lsa, qator yaqinlashadi;

agar $\ell > 1$ bo'lsa, qator uzoqlashadi;

agar $\ell = 1$ bo'lsa, qator yaqinlashadimi yoki uzoqlashadimi degan masala hal qilinmaydi. Bunday holda qator yaqinlashishi yoki uzoqlashishini boshqa yoqinlashish belgilari yordamida tekshirish kerak bo'ladi.

**Koshiinng radikal belgisi.** Agar musbat hadli

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \dots + a_n + \dots$$

qator uchun $\lim_{n\to+\infty} \sqrt[n]{a_n} = \ell$ chekli limit mavjud bo'lsa u holda,

agar $\ell < 1$ bo'lsa, qator yaqinlashadi;

agar $\ell > 1$ bo'lsa, qator uzoqlashadi;

agar $\ell = 1$ bo'lsa, qator yaqinlashadimi yoki uzoqlashadimi degan masala hal qilinmaydi. Bunday holda qator yaqinlashishi yoki uzoqlashishini boshqa yaqinlashish belgilari yordamida tekshirish kerak bo'ladi.

**Koshining integral belgisi.** Quyidagi musbat hadli qatorni qaray

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \ldots + a_n + \ldots$$

Agar quyidagi shartlarni qanoatlantiruvchi f(x) funksiya mavjud bo'lsa

1) f(x) funksiya $[1;\infty)$ oraliqda berilgan va uzluksiz;
2) monoton kamayuvchi;
3) n = 1,2,3,... uchun $a_n = f(n)$ o'rinli bo'lsa,

u holda, $\sum_{n=1}^{\infty} a_n$ qator va $\int_1^{\infty} f(x)dx$ xosmas integral bir vaqtda yaqinlashadi yoki bir vaqtda uzoqlashadi.

**1 – Misol.** Quyidagi qatorni yaqinlashishga tekshiring $\sum_{n=1}^{\infty} \frac{n^2}{n^3+5}$

**Yechish.** Qatorning umumiy hadining limitini topamiz:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^2}{n^3+5} = \left[\frac{\infty}{\infty}\right] = \lim_{n\to\infty} \frac{\frac{1}{n}}{1+\frac{5}{n^3}} = \frac{0}{1} = 0.$$

$\lim_{n\to\infty} a_n = 0$ ekanligidan, bu qatorning yaqinlashishi yo uzoqlashishi haqida hech narsa aytolmaymiz. Keyinchalik, bu berilg qator uzoqlashuvchi ekanligi ko'rsatiladi.

**2 – Misol.** Quyidagi qatorni yaqinlashishga tekshiring:

$$\sum_{n=1}^{\infty} \frac{1}{n\cdot 2^n} = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 2^2} + \frac{1}{3\cdot 2^3} + \cdots + \frac{1}{n\cdot 2^n} + \cdots$$

**Yechish.** Qatorning umumiy hadi $a_n = \frac{1}{n\cdot 2^n}$ ga teng. Bu qator umumiy hadi $b_n = \frac{1}{2^n}$ bo'lgan qator bilan taqqoslaymiz. Ko'rin turibdiki, $a_n \le b_n$.

$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{2^n}$ qator esa maxraji $q = \frac{1}{2} < 1$ bo'lgan geometr progressiya hadlari yig'indisidan iborat va u yaqinlashuvchi qatord Taqqoslash belgisiga ko'ra berilgan qator ham yaqinlashuvchidir.

**3 – Misol.** Quyidagi qatorni yaqinlashishga tekshiring:

$$\sum_{n=1}^{\infty} \frac{1}{n+1-\cos^2 na} = \frac{1}{2-\cos^2 a} + \frac{1}{3-\cos^2 2a} + \frac{1}{4-\cos^2 3a} + \cdots + \frac{1}{n+1-\cos^2 na} + \cdots$$

**Yechish.** Qatorning umumiy hadi $a_n = \frac{1}{n+1-\cos^2 na}$ ga teng. Bu qatorni umumiy hadi $b_n = \frac{1}{n+1}$ bo'lgan

$$\sum_{n=1}^{\infty} \frac{1}{n+1} = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n+1} + \cdots$$

qator bilan taqqoslaymiz. U birinchi hadi olib tashlangan garmon qatordir, garmonik qator esa uzoqlashuvchidir. Bizning misolimiz $\frac{1}{n+1-\cos^2 na} \ge \frac{1}{n+1}$, $0 \le (\cos na)^2 \le 1$, ya'ni $a_n \ge b_n$. $\sum_{n=1}^{\infty} \frac{1}{n+}$

qatorning uzoqlashuvchiligidan, taqqoslashning birinchi belgisiga ko'ra berilgan $\sum_{n=1}^{\infty}\frac{1}{n+1-cos^2na}$ qatorning uzoqlashuvchi ekanligi kelib chiqadi.

**4 – Misol.** Quyidagi qatorni yaqinlashishga tekshiring:

$$\sum_{n=1}^{\infty}\frac{5^n}{(n+4)!}=\frac{5}{5}+\frac{5^2}{6!}+\frac{5^3}{7!}+\frac{5^4}{8!}+\cdots+\frac{5^n}{(n+4)!}+\cdots$$

**Yechish.** Qaralayotgan misolda $a_n=\frac{5^n}{(n+4)!}$. Berilgan qatorni umumiy hadi

$b_n=\frac{5^n}{n!}$ bo'qator bilan taqqoslaymiz: $a_n<b_n$, bo'lganligi uchun esa, bu

qatorni Dalamber belgisidan foydalanib yaqinlashishga tekshiramiz.

$$\lim_{n\to+\infty}\frac{b_{n+1}}{b_n}=\lim_{n\to+\infty}\frac{5^{n+1}}{(n+1)!}:\frac{5^n}{n!}=\lim_{n\to+\infty}\frac{5^{n+1}n!}{5^n(n+1)!}=$$

$=\lim\limits_{n\to+\infty}\frac{5}{n+1}=0<1$. Demak, qator yaqinlashuvchi.

Taqqoslashning birinchi belgisiga ko'ra berilgan qator ham yaqinlashuvchidir.

**5- Misol.** Quyidagi qatorni yaqinlashishga tekshiring:

$$\sum_{n=1}^{\infty}\frac{3n-4}{5n^4-3n^2+5}$$

**Yechish.** Kasrning suratida birinchi darajali ko'p had $3n-4$, maxrajida esa to'rtinchi darajali ko'phad $5n^4-3n^2+5$ turibdi. Berilgan qatorni $p=4\text{-}1=3$, bo'lgan umumlashgan garmonik qator bilan taqqoslaymiz ya'ni $\sum_{n=1}^{\infty}\frac{1}{n^3}$. Bu qator $p=3>1$, bo'lganligi uchun yaqinlashuvchidir.

Taqqoslashning ikkinchi belgisini qo'llaymiz

$$\lim_{n\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty}\left(\frac{3n-4}{5n^4-3n^2+5}:\frac{1}{n^3}\right)=\lim_{n\to\infty}\frac{3n^4-n^3}{5n^4-3n^2+5}=\frac{3}{5}\neq 0.$$

Ko'rinib turibdiki, taqqoslashning ikkinchi belgisi bajarilayapti, shuning uchun berilgan qator yaqinlashuvchidir.

**6- Misol.** Ushbu qatorni yaqinlashishga tekshiring:

$$\sum_{n=1}^{\infty}\frac{1}{2^n+1}=\frac{1}{2+1}+\frac{1}{2^2+1}+\cdots+\frac{1}{2^n+1}+\cdots$$

**Yechish.** Qaralayotgan misolda $a_n=\frac{1}{2^n+1}$. Berilgan qatorni maxraji $q=\frac{1}{2}$ bo'lgan geometrik qator bilan taqqoslaymiz, ya'ni $b_n=\frac{1}{2^n}$. U holda

$$\lim_{n\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty}\frac{\frac{1}{2^n+1}}{\frac{1}{2^n}}=\lim_{n\to\infty}\frac{2^n}{2^n+1}=\lim_{n\to\infty}\frac{1}{1+\frac{1}{2^n}}=1>0.$$

va $\sum_{n=1}^{\infty}\frac{1}{2^n}$ qator yaqinlashuvchi bo'lgani uchun $(q=\frac{1}{2}<1)$ berilgan qator ham yaqinlashadi.

Ba'zi sonli qatorlarni yaqinlashishga tekshirishda, taqqoslash belgilarini qo'llashda birinchi va ikkinchi ajoiyb limitlarning natijalaridan foydalanish ancha qulaylik yaratadi. Bularni qo'llashda $\alpha(n)$ ni cheksiz kichik miqdor deb hisoblaymiz, ya'ni $n\to\infty$ da $\alpha(n)\to 0$. Qu'yida ekvivalentlik munosobatlarini keltiramiz.

$\sin\alpha\sim\alpha,\qquad tg\alpha\sim\alpha,\qquad (1+\alpha)^k-1\sim k\alpha,$

$\arcsin\alpha\sim\alpha,\qquad \mathrm{arctg}\,\alpha\sim\alpha,\qquad \ln(1+\alpha)\sim\alpha,$

$1-\cos\alpha\sim\alpha^2.$

Endi shu munosobatlardan foydalanib bir necha misollar yechamiz.

**7 – Misol.** Ushbu qatorni yaqinlashishga tekshiring:

$$\sum_{n=1}^{\infty}\frac{1}{\sqrt[3]{n}}\cdot \ln\left(\frac{n^3+n^2+1}{n^3+2n+1}\right).$$

Yechish. $\ln\left(\frac{n^3+n^2+1}{n^3+2n+1}\right)=\ln\left(1+\frac{n^2-2n}{n^3+2n+1}\right)$ va $\frac{n^2-2n}{n^3+2n+1}\to\frac{1}{n}\to 0$ bo'lganligi uchun, $n\to+\infty$ da $\ln(1+a)\sim a$ ekvivalentlikni qo'llash mumkin. U holda qaralayotgan qatorning umumiy hadi $n\to+\infty$ da $a_n=\frac{1}{\sqrt[3]{n}}*\ln\left(\frac{n^3+n^2+1}{n^3+2n+1}\right)\sim b_n=\frac{1}{n^{\frac{4}{3}}}.(\frac{1}{n}\cdot\frac{1}{\sqrt[3]{n}}$ dan $1+\frac{1}{3}=\frac{4}{3}$ ).

Berilgan qatorni $p=1+\frac{1}{3}=\frac{4}{3}$ bo'lgan umumlashgan garmonik qator bilan taqqoslaymiz ya'ni $\sum_{n=1}^{\infty}\frac{1}{n^{\frac{4}{3}}}$. Bu qator $p=\frac{4}{3}>1$, bo'lganligi uchun yaqinlashuvchidir. Demak, taqqoslash alomatiga asosan berilgan qator ham yaqinlashuvchi.

**8 – Misol.** Quyidagi qatorni yaqinlashishga tekshiring:

$$\sum_{n=1}^{\infty} arctg\left(\frac{1}{n^2-n+1}\right)$$

Yechish. Ekvivalentlik jadvaliga asosan, berilgan qatorning umumiy hadini $n\to+\infty$ da $\lim_{n\to+\infty}\frac{1}{n^2-n+1}=0$ bo'lganligidan quyidagicha yozish mumkin

$$a_n=arctg\left(\frac{1}{n^2-n+1}\right)\sim\frac{1}{n^2-n+1}\sim b_n=\frac{1}{n^2}.$$

$\sum_{n=1}^{\infty}\frac{1}{n^2}$. Bu qator $p=2>1$, bo'lganligi uchun yaqinlashuvchidir. Demak, taqqoslash alomatiga asosan berilgan qator ham yaqinlashuvchi.

**9– Misol.** Ushbu qatorni yaqinlashishga tekshiring:

$$\sum_{n=1}^{\infty}\frac{2n-1}{3^n}=\frac{1}{3}+\frac{3}{3^2}+\frac{5}{3^3}+\frac{7}{3^4}+\dots+\frac{2n-1}{3^n}+\dots$$

**Yechish.** Dalamber belgisini qo'llaymiz;

$$a_n=\frac{2n-1}{3^n},\ a_{n+1}=\frac{2(n+1)-1}{3^{n+1}}.$$

$$\ell=\lim_{n=\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\left[\frac{2(n+1)-1}{3^{n+1}}:\frac{2n-1}{3^n}\right]=$$

$$=\lim_{n\to\infty}\frac{3^n(2n+1)}{3^{n+1}(2n-1)}=\frac{1}{3}\lim_{n\to\infty}\frac{2n+1}{2n-1}=\frac{1}{3}\lim_{n\to\infty}\frac{2+\frac{1}{n}}{2-\frac{1}{n}}=\frac{1}{3}.$$

Shunday qilib, $\ell=\frac{1}{3}<1$ va demak, berilgan qator yaqinlashuvchi.

**10 – Misol.** Ushbu qatorni yaqinlashishga tekshiring:

$$\sum_{n=1}^{\infty}\frac{2^n}{n^4}=\frac{2}{1}+\frac{4}{16}+\frac{4}{16}+\dots+\frac{2^n}{n^4}+\dots$$

**Yechish.** Dalamber belgisini qo'llaymiz; $a_n=\frac{2n}{4^n}$, $a_{n+1}=\frac{2^{n+1}}{(n+1)^n}$.

$$\ell=\lim_{n=\infty}\frac{a_{n+1}}{a_n}=\lim_{n=\infty}\left[\frac{2^{n+1}}{(n+1)^n}:\frac{2n}{4^n}\right]=$$

$$=\lim_{n=\infty}\frac{2^{n+1}n^4}{(n+1)^4 2^n}=2\lim_{n=\infty}\frac{n^4}{(n+1)^4}=2\lim_{n=\infty}\frac{1}{(1+\frac{1}{n})^4}=2.$$

$\ell=2>1$ bo'lgani uchun berilgan qator uzoqlashuvchidir.

**11 – misol.** Quyidagi qatorni yaqinlashishga tekshiring:

$$\sum_{n=1}^{\infty}\frac{n^n}{n!}$$

**Yechish.** Dalamber belgisini qo'llaymiz.

$$a_n=\frac{n^n}{n!};\quad a_{n+1}=\frac{(n+1)^{(n+1)}}{(n+1)!}.$$

$$\ell=\lim_{n=\infty}\frac{a_{n+1}}{a_n}=\lim_{n=\infty}\frac{\frac{(n+1)^{(n+1)}}{(n+1)!}}{\frac{n^n}{n!}}=\lim_{n=\infty}\frac{(n+1)^n(n+1)n!}{n^n n!(n+1)}=$$

$$=\lim_{n=\infty}\left(\frac{n+1}{n}\right)^n=\lim_{n=\infty}\left(1+\frac{1}{n}\right)^n=e.$$

(bu yerda ikkinchi ajoyib limitdan foydalandik). $\ell=e>1$, shuning uchun berilgan qator uzoqlashuvchi.

Endi $\ell=1$ bo'lgan qatorlarga doir ikkita misol ko'ramiz va ulardan biri yaqinlashuvchi, ikkinchisi esa uzoqlashuvchi ekanligini ko'rsatamiz.

**12 – Misol.** Ushbu qatorni yaqinlashishga tekshiring:

$$\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}=\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\dots+\frac{1}{\sqrt{n}}+\dots$$

**Yechish.** Dalamber belgisidan foydalanib $\ell$ ni topamiz:

$a_n = \frac{1}{\sqrt{n}}$; $\quad a_{n+1} = \frac{1}{\sqrt{n+1}}$. $\ell = \lim\limits_{n=\infty} \frac{a_{n+1}}{a_n} = \lim\limits_{n=\infty} \left(\frac{1}{\sqrt{n+1}} : \frac{1}{\sqrt{n}}\right) =$

$= \lim\limits_{n=\infty} \frac{\sqrt{n}}{\sqrt{n+1}} = 1$. Dalamber belgisi asosida qatorning yaqinlashishi yoki uzoqlashishligi haqida xulosa chiqara olmaymiz. Biroq bu uzoqlashuvchi qator ekanligini ko'rsatish oson.

Buning uchun bu qatorning

$$S_n = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}}$$

xususiy yig'indisini qaraymiz.

$\frac{1}{\sqrt{1}} > \frac{1}{\sqrt{n}}, \frac{1}{\sqrt{2}} > \frac{1}{\sqrt{n}}, \frac{1}{\sqrt{3}} > \frac{1}{\sqrt{n}}, \cdots$ bo'lgani uchun, ravshanki,

$$S_n > \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n}} + \cdots + \frac{1}{\sqrt{n}}.$$

yoki $S_n > n \cdot \frac{1}{\sqrt{n}}$, ya'ni $S_n > \sqrt{n}$. Bu yerdan $\lim\limits_{n=\infty} S_n = \infty$ ekanligi bevosita kelib chiqadi va demak, qator uzoqlashuvchi.

**13-Misol.** Ushbu qatorning yaqinlashishi yoki uzoqlashishini tekshiring:

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{n(n+1)} + \cdots$$

<u>**Yechish.**</u> Dalamber belgisidan foydalanib $\ell$ ni topamiz:

$$a_n = \frac{1}{n(n+1)}; \quad a_{n+1} = \frac{1}{(n+1)(n+2)}.$$

$$\ell = \lim_{n=\infty} \frac{a_{n+1}}{a_n} = \lim_{n=\infty} \left(\frac{1}{(n+1)(n+2)} : \frac{1}{n(n+1)}\right) = \lim_{n=\infty} \frac{n(n+1)}{(n+1)(n+2)} = 1.$$

Dalamber belgisi asosida qatorning yaqinlashishi yoki uzoqlashishi haqida xulosa chiqara olmaymiz. Biroq bu yaqinlashuvchi qator ekanligini ko'rsatish oson.

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$$

$$S_n = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{(n-1)n} + \frac{1}{(n-1)n} =$$

$$= \frac{1}{1} - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{n-1} - \frac{1}{n} + \frac{1}{n} - \frac{1}{n+1} = 1 - \frac{1}{n+1}.$$

$$S_n = 1 - \frac{1}{n+1}.$$

$$\lim_{n=\infty} S_n = \lim_{n=\infty} \left(1 - \frac{1}{n+1}\right) = 1 - \lim_{n=\infty} \frac{1}{n+1} = 1$$

chekli b o'lganligi uchun, berilgan qator yaqinlashuvchi.

**14-Misol.** Quyidagi qatorni yaqinlashishga tekshiring

$$\sum_{n=1}^{\infty} \left(\frac{n-1}{2n+1}\right)^n$$

**Yechish.** Koshining radikal belgisini qo'llaymiz $a_n = \left(\frac{n-1}{2n+1}\right)^n$ bo'lgani uchun

$$\lim_{n\to+\infty} \sqrt[n]{a_n} = \lim_{n\to+\infty} \sqrt[n]{\left(\frac{n-1}{2n+1}\right)^n} = \lim_{n\to+\infty} \frac{n-1}{2n+1} = \frac{1}{2} < 1.$$

Demak, berilgan qator yaqinlashadi.

**15- Misol.** Quyidagi qatorni yaqinlashishga tekshiring:

$$\sum_{n=1}^{\infty} \left(\frac{n-1}{n+1}\right)^{n(n-1)}$$

**Yechish.** Koshi belgisini qo'llamiz. $a_n = \left(\frac{n-1}{n+1}\right)^{n(n-1)}$ bo'lgani uchun

$$\lim_{n\to+\infty} \sqrt[n]{a_n} = \lim_{n\to+\infty} \sqrt[n]{\left(\frac{n-1}{n+1}\right)^{n(n-1)}} = \lim_{n\to+\infty} \left(\frac{n-1}{n+1}\right)^{n-1} =$$

$$= \lim_{n\to+\infty} \left(\frac{n+1-2}{n+1}\right)^{n-1} = \lim_{n\to+\infty} \left(1 + \frac{-2}{n+2}\right)^{n-1} =$$

$$= \lim_{n\to+\infty} \left(1 + \frac{-2}{n+1}\right)^{\left(\frac{n+1}{-2}\right)\left(\frac{-2}{n+1}\right)(n-1)} = e^{\lim\limits_{n\to+\infty} \frac{-2n+2}{n+1}} =$$

$= e^{-2} = \frac{1}{e^2} < 1$ (bu yerda ikkinchi ajoyib limitdan foydalandik).

$\ell = \frac{1}{e^2} < 1$. Demak, berilgan qator yaqinlashadi.

**16 - Misol.** Quyidagi qatorni yaqinlashishga tekshiring:

$$\sum_{n=1}^{\infty} \frac{1}{2^n}\left(1 + \frac{1}{n}\right)^{n^2}$$

**Yechish**. $a_n = \frac{1}{2^n}\left(1 + \frac{1}{n}\right)^{n^2}$ bo'lgani uchun Koshi belgisidan foydalanamiz:

$$\lim_{n\to+\infty} \sqrt[n]{a_n} = \lim_{n\to+\infty} \sqrt[n]{\frac{1}{2^n}\left(1 + \frac{1}{n}\right)^{n^2}} = \lim_{n\to+\infty} \frac{1}{2}\left(1 + \frac{1}{n}\right)^n = \frac{e}{2} > 1.$$

Demak, berilgan qator uzoqlashuvchi.

**17- Misol.** Quyidagi umumlashgan garmonik qator (Dirixle qatori) yaqinlashishga tekshirilsin.

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} + \cdots; \quad p \in R$$

**Yechish.** Koshining integral belgisini qo'llaymiz. $p > 0$ bo'lganda $f(x) = \frac{1}{x^p}$ funksiya Koshining integral belgisinining barcha shartlarini qanoatlantiradi.

Uchta holni qaraymiz:

1. p = 1 , bo'lganda $\sum_{n=1}^{\infty}\frac{1}{n}$ qator garmonik qator bo'ladi. f(x) = $\frac{1}{x}$ deb olib quyidagi xosmas integralni hisoblaymiz.

$$\int_1^{\infty}\frac{dx}{x}=\lim_{b\to\infty}\int_1^{b}\frac{dx}{x}=\lim_{b\to\infty}\ln x\,\Big|_1^b=\lim_{b\to\infty}(\ln b-\ln 1)=\infty.$$

Xosmas integralning uzoqlashuvchi ekanligidan, Koshining integral belgisiga asosan $\sum_{n=1}^{\infty}\frac{1}{n}$ garmonik qatorning ham uzoqlashuvchi ekanligi kelib chiqadi.

2. Agar $0<p<1$ bo'lsa, f(x) = $\frac{1}{n^p}$ deb olib quyidagi xosmas integralni hisoblaymiz.

$$\int_1^{\infty}\frac{dx}{x^p}=\lim_{b\to\infty}\int_1^{b}\frac{dx}{x^p}=\lim_{b\to\infty}\frac{x^{1-p}}{1-p}\,\Big|_1^b=\lim_{b\to\infty}\left(\frac{b^{1-p}}{1-p}-\frac{1}{1-p}\right)=\infty$$

demak, xosmas integral uzoqlashuvchi ekan. Bundan, $0<p<1$ bo'lganda umumlashgan garmonik qatorning ham uzoqlashuvchi ekanligi kelib chiqadi.

3. Agar $p>1$ bo'lsa,u holda

$$\int_1^{\infty}\frac{dx}{x^p}=\lim_{b\to\infty}\int_1^{b}\frac{dx}{x^p}=\lim_{b\to\infty}\frac{x^{1-p}}{1-p}\,\Big|_1^b=\frac{1}{1-p}\lim_{b\to\infty}\left(\frac{1}{b^{p-1}}-1\right)=\frac{1}{p-1}$$

demak, $p>1$ bo'lganda xosmas integral yaqinlashuvchi bo'lar ekan. Bundan esa Koshining integral belgisiga asosan $p>1$ bo'lganda umumlashgan garmonik qatorning ham yaqinlashuvchi ekanligi kelib chiqadi.

**18- Misol.** $\sum_{n=1}^{\infty}\frac{2x}{(x^2+1)^2}$ qatorni yaqinlashishga tekshiring.

<u>Yechish</u>. f(x) = $\frac{2x}{(x^2+1)^2}$ deb olamiz. Bu funksiya Koshining integral belgisining barcha shartlarini qanoatlantiradi.

$$\int_1^{\infty}\frac{2x\,dx}{(x^2+1)^2}=\lim_{b\to\infty}\int_1^{b}\frac{d(x^2+1)}{(x^2+1)^2}=\lim_{b\to\infty}\left(-\frac{1}{x^2+1}\right)\Big|_1^b=$$

$$=\lim_{b\to\infty}\left(\frac{-1}{b^2+1}+\frac{1}{2}\right)=\frac{1}{2}.$$

Demak, xosmas integral yaqinlashadi, shuning uchun berilgan qator ham yaqinlashadi.

**19- Misol.** $\sum_{n=2}^{\infty}\frac{1}{n\cdot \ln n}$ qatorni yaqinlashishga tekshiring.

<u>Yechish.</u> f(x) = $\frac{1}{x\cdot \ln x}$ deb olamiz. Bu funksiya Koshining integral belgisining barcha shartlarini qanoatlantiradi. Shuning uchun $\int_2^{\infty}\frac{1}{x\cdot \ln x}$ xosmas integralni hisoblaymiz.

$$\int_2^{\infty}\frac{1}{x\cdot \ln x}=\lim_{b\to\infty}\int_2^{b}\frac{1}{x\cdot \ln x}=\lim_{b\to\infty}\int_2^{b}\frac{d(\ln x)}{\ln x}=\lim_{b\to\infty}\ln\ln x\,\Big|_2^b=$$

$$=\lim_{b\to\infty}(\ln\ln b-\ln\ln 2)=\infty.$$

Demak, xosmas integral uzoqlashadi, shuning uchun berilgan qator ham uzoqlashadi.

## MASHQLAR

1. Yaqinlashishning zaruriy belgisi tekshirilsin.

1. $\sum_{n=1}^{\infty}\frac{2n+1}{5n-4}$.
2. $\sum_{n=1}^{\infty}\left(\frac{2n}{2n+1}\right)^{n}$.
3. $\sum_{n=1}^{\infty}n^2 sin\frac{1}{n^2}$.
4. $\sum_{n=1}^{\infty}cos\frac{1}{2n}$.

2. Quyidagi qatorlar taqqoslash belgisidan foydalanib, yaqinlashishga tekshirilsin

1. $\sum_{n=1}^{\infty}\frac{1}{5^n+n}$.
2. $\sum_{n=1}^{\infty}sin\frac{\pi}{2^n}$
3. $\sum_{n=1}^{\infty}\frac{n+1}{n\sqrt{n^2+1}}$.
4. $\sum_{n=1}^{\infty}\frac{\sqrt[4]{n}}{(n+1)\sqrt{n}}$.
5. $\sum_{n=1}^{\infty}\frac{5n^3}{n^5+4}$.
6. $\sum_{n=1}^{\infty}\frac{3^n}{n(3^n+1)}$.

3. Dalamber belgisidan foydalanib, quyidagi qatorlar yaqinlashishga tekshirilsin.

1. $\sum_{n=1}^{\infty}\frac{2n}{3^n(2n-1)}$
2. $\sum_{n=1}^{\infty}\frac{7^4}{n^7}$
3. $\sum_{n=1}^{\infty}\frac{n^n}{2^n n!}$
4. $\sum_{n=1}^{\infty}\frac{n^4}{5^n}$
5. $\sum_{n=1}^{\infty}\frac{n^n}{3^n n!}$
6. $\frac{2}{2}+\frac{2\cdot 3}{2\cdot 4}+\cdots+\frac{(n+1)!}{2^n n!}+\cdots$

4. Koshi belgisini qo'llab quyidagi qatorlar yaqinlashishga tekshirilsin.

1. $\sum_{n=1}^{\infty}\frac{1}{3^n}\left(\frac{n}{n+1}\right)^{n^2}$
2. $\sum_{n=1}^{\infty}\left(\frac{3n}{n+\sqrt{n}+5}\right)^{n}$
3. $\sum_{n=1}^{\infty}\left(\frac{n+2}{2n+1}\right)^{3n+4}$
4. $\sum_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^{n^2}$

1. Koshining integral belgisidan foydalanib, yaqinlashishga tekshirilsin

1. $\sum_{n=1}^{\infty}\frac{10}{3+n^2}$
2. $\sum_{n=1}^{\infty}\frac{1}{(n+1)ln^3(n+2)}$
3. $\sum_{n=1}^{\infty}\frac{1}{\sqrt{3n+1}}$
4. $\sum_{n=1}^{\infty}\frac{1}{n(ln\, n)^2}$.

## 6.3. Ishoralari o'zgaruvchan qatorlar.

### 6.3.1. Leybnits belgisi. Ishoralari almashinuvchi qatorlar

Ham musbat, ham manfiy hadlarni o'z ichiga olgan qator ishoralari o'zgaruvchan qator deb ataladi. Ishoralari o'zgaruvchan qatorga misol sifatida ushbu qatorni keltiramiz.

$$\sum_{n=1}^{\infty}(-1)^{\frac{n(n-1}{n}}\frac{1}{n^2}=\frac{1}{1^2}-\frac{1}{2^2}-\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}-\frac{1}{6^2}-\frac{1}{7^2}+$$
$$+\frac{1}{8^2}+\frac{1}{9^2}-\cdots+(-1)^{\frac{n(n-1}{n}}\frac{1}{n^2}+\cdots$$

Qatorning har bir musbat hadidan keyin manfiy had yoki har bir manfiy hadidan keyin musbat had kelsa, bunday qator ishoralari qator deyiladi. Ishoralari almashinuvchi qatorni umumiy holda quyidagicha yozish mumkin:

$$\sum_{n=1}^{\infty}(-1)^{n+1}a_n = a_1 - a_2 + a_3 - a_4+\cdots$$

(bu yerda, n = 1,2,3, …)

Ishoralari almashinuvchi qatorlarni yaqinlashishga tekshirishda Leybnits belgisidan foydalaniladi.

**Leybnits belgisi.** Agar ishoralari almashinuvchi qatorda hadlarining mutloq qiymatlaridan to'zilgan qatorning hadlari kamayuvchi bo'lsa, ya'ni,

$$a_1 > a_2 > a_3 > a_4 > \cdots > a_n > \cdots$$

va $\lim_{n\to\infty} a_n = 0$ bo'lsa, u holda bu qator yaqinlashuvchi bo'ladi, shu bilan birgalikda uning yig'indisi S, musbat va qatorning birinchi hadidan ortiq bo'lmaydi ya'ni ushbu

$$0< S < a_1$$

shartni qanoatlantiradi.

**1 -Misol.** Qatorni yaqinlashishga tekshiring

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n} = 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots+(-1)^{n-1}\frac{1}{n}+\cdots$$

<u>Yechish.</u> Berilgan qatorning hadlari mutloq qiymati bo'yicha kamayuvchi:

$$1 > \frac{1}{2} > \frac{1}{3} > \frac{1}{4} > \cdots > \frac{1}{2n-1} > \frac{1}{2n} > \cdots$$

va n→ ∞ da nolga intiladi.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{1}{n}=0$$

Leybnits belgisining barcha shartlari bajariladi. Demak, berilgan qator yaqinlashuvchi.

**2 -Misol.** Qatorni yaqinlashishga tekshiring

$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{2n-1}= 1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\dots+(-1)^{n+1}\frac{1}{2n-1}+\dots$

Yechish. Qatorning hadlari mutloq qiymati bo'yicha kamayuvchi

$$1>\frac{1}{3}>\frac{1}{5}>\frac{1}{7}>\dots>\frac{1}{2n-1}>\dots$$

Umumiy hadi n→ ∞ da nolga intiladi:

$$\lim_{n\to\infty}\frac{1}{2n-1}=0$$

Demak, berilgan qator Leybnits belgisiga asosan yaqinlashuvchidir.

### 6.3.2. Mutloq va shartli yaqinlashuvchi qatorlar

Ishoralari o'zgaruvchi bo'lgan

$$\sum_{n=1}^{\infty}(-1)^{n+1}a_n = a_1 - a_2 + a_3 - a_4+\dots+(-1)^{n+1}a_n+\dots$$

qator berilgan bo'lib, uning hadlarining mutloq qiymatlaridan tuzilgan

$$|a_1|+|a_2|+|a_3|+\dots+|a_n|+\dots$$

qator yaqinlashuvchi bo'lsa, berilgan qator mutloq yaqinlashuvchi qator deyiladi.

Agar ishoralari o'zgaruvchi qator yaqinlashuvchi bo'lib, bu qator hadlarining mutloq qiymatlaridan tuzilgan qator uzoqlashuvchi bo'lsa, u holda berilgan ishoralari o'zgaruvchi qator shartli yaqinlashuvchi qator deyiladi.

**Teorema.** Agar berilgan ishoralari o'zaruvchi

$$a_1 - a_2 + a_3 + a_4 - a_5 - a_6 + \dots$$

qatorning hadlarining mutloq qiymatlaridan tuzilgan

$$|a_1|+|a_2|+|a_3|+\dots+|a_n|+\dots$$

qator yaqinlashuvchi bo'lsa, u holda berilgan qator ham yaqinlashuvchi bo'ladi.

**3 -Misol.** Qatorni yaqinlashishga tekshiring

$$\sum_{n=1}^{\infty}(-1)^{\frac{n(n+1)}{2}}\cdot\frac{1}{2^{n-1}}= -1-\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\frac{1}{2^4}-\frac{1}{2^5}+\dots+$$
$$+(-1)^{\frac{n(n+1)}{2}}\cdot\frac{1}{2^{n-1}}+\dots$$

Yechish. Qator hadlarining mutloq qiymatlaridan tuzilgan qatorni qaraymiz:

$$1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\frac{1}{2^4}+\frac{1}{2^5}+\dots+\frac{1}{2^{n-1}}+\dots$$

Bu qator maxraji $q=\frac{1}{2}$ ,bo'lgan geometrik qator bo'lib, yaqinlashuvchidir.

Endi berilgan ishoralari o'zgaruvchi qatorga Leybnits belgisini qo'llaymiz.

1) $a_n > a_{n+1}$ ya'ni $\frac{1}{2^{n-1}} > \frac{1}{2^n}$

2) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{2^{n-1}} = 0$

Berilgan qator uchun Leybnits belgisining barcha shartlari bajariladi. Demak, berilgan ishoralari o'zgaruvchi $\sum_{n=1}^{\infty}(-1)^{\frac{n(n+1)}{2}} \cdot \frac{1}{2^{n-1}}$ qator yaqinlashuvchi. Uning mutloq qiymatlaridan tuzilgan qator ham yaqinlashuvchi. Bundan berilgan qator mutloq yaqinlashuvchi ekanligi kelib chiqadi.

**5-Misol.** Qatorni yaqinlashishga tekshiring.

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{2n+1}{n(n+1)}$$

**Yechish.** Berilgan ishorasi almashinuvchi qatorning hadlari monoton kamayuvchi va $\lim_{n\to\infty}\frac{2n+1}{n(n+1)} = 0$, u holda Leybnits alomatiga ko'ra berilgan qator yaqinlashuvchi. Berilgan qatorning mutloq qiymatlaridan tuzilgan qatorni qaraylik

$$\sum_{n=1}^{\infty}\frac{2n+1}{n(n+1)}$$

Bu qatorning umumiy hadi $x = n$ bo'lganda $f(x) = \frac{2x+1}{x(x+1)}$ funksiya orqali beriladi. Quyidagini topamiz.

$$\int_1^{\infty}\frac{2x+1}{x(x+1)}dx$$

$$= \lim_{B\to\infty}\int_1^{B}\left(\frac{1}{x}+\frac{1}{x+1}\right)dx = \lim_{B\to\infty}(ln|x| + ln|x+1|)\Big|_1^B =$$

$$= \lim_{B\to\infty}\left(\ln\left(B\cdot(B+1)\right) - ln2\right) = \infty$$

Demak, berilgan qatorning mutloq qiymatlaridan tuzilgan qator uzoqlashuvchi, shuninguchun berilgan qator shartli yaqinlashuvchidir.

**6-Misol.** Qatorni yaqinlashishga tekshiring.

$\sum_{n=1}^{\infty}(-1)^{n+1}\cdot\frac{1}{n}\cdot\frac{1}{2^n} = \frac{1}{2} - \frac{1}{2}\cdot\frac{1}{2^2} + \frac{1}{3}\cdot\frac{1}{2^3} + \cdots + (-1)^{n+1}\cdot\frac{1}{n}\cdot\frac{1}{2^n} + \cdots$

**Yechish.** Berilgan qator hadlarining mutloq qiymatlaridan tuzilgan qatorni qaraymiz.

$\sum_{n=1}^{\infty}\frac{1}{n}\cdot\frac{1}{2^n} = \frac{1}{2} + \frac{1}{2}\cdot\frac{1}{2^2} + \frac{1}{3}\cdot\frac{1}{2^3} + \cdots + \frac{1}{2}\cdot\frac{1}{2^n} + \cdots$

Bundan $a_n = \frac{1}{n} \cdot \frac{1}{2^n}$, $a_{n+1} = \frac{1}{n+1} \cdot \frac{1}{2^{n+1}}$; Dalamber belgisini qo'llaymiz

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \left| \frac{1}{n+1} \cdot \frac{1}{2^{n+1}} : \frac{1}{n+1} \cdot \frac{1}{2^n} \right| =$$

$$= \lim_{n\to\infty} \left| \frac{n \cdot 2^n}{(n+1) \cdot 2^n \cdot 2} \right| = \frac{1}{2} \lim_{n\to\infty} \left| \frac{n}{(n+1)} \right| = \frac{1}{2} \lim_{n\to\infty} \left| \frac{1}{1+\frac{1}{n}} \right| =$$

$= \frac{1}{2} < 1$. Demak, berilgan ishorasi o'zgaruvchi qator mutloq yaqinlashuvchi.

## MASHQLAR

1. Qatorni yaqinlashishga tekshiring, agar qator yaqinlashuvchi bo'lsa uni mutoq va shartli yaqinlashishga tekshiring.

1. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n+2^n}$
2. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{e^{n+1}}$.
3. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}\sqrt{n+1}}{\sqrt{n^3+2}}$
4. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}(2n-1)^3}{n \cdot 3^n}$
5. $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n\sqrt{ln}}$
6. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{7^n\sqrt{n}}$
7. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}(3^n+1)}{n \cdot 3^n}$
8. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}(n+2)}{5n+1}$
9. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(n+1)ln^2(n+1)}$
10. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}(2n)!}{n!4^n}$.

### 6.4. Funksional qatorning asosiy tushunchalari. Darajali qatorlar. Darajali qatorning yaqinlashish oralig'i va yaqinlashish radiusi.

Biror X to'plamda aniqlangan $u_1(x)$, $u_2(x)$, $u_3(x)$, $\cdots u_n(x) \cdots$ funksiyalar ketma - ketligi berilgan bo'lsin. Har bir hadi funksiyadan iborat bo'lgan quyidagi qator

$$\sum_{n=1}^{\infty} u_n(x) = u_1(x) + u_2(x) + u_3(x) + \cdots + u_n(x) + \cdots$$

o'zgaruvchi x ga nisbatan funksional qator deb ataladi.

Agar $x = x_0$ nuqtada $\sum_{n=1}^{\infty} u(x_0)$ sonli qator yaqinlashuvchi bo'lsa, $x_0$nuqta berilgan funksional qatorning yaqinlashish nuqtasi deb ataladi. Funksional qator $u_n(x)$ funksiyalarning aniqlanish sohalaridan olingan ba'zi nuqtalar uchun yaqialashuvchi, ba'zi nuqtalar uchun esa uzoqlashuvchi bo'lishi mumkin.

Funksional qatorning barcha yaqinlashuvchi nuqtalari tuplami uning yaqinlashish sohasi deb ataladi.

Har qanday erkli o'zgaruvchi $x$ ning $x = x_0 \in X$ tayin qiymatida $\sum_{n=1}^{\infty} u_n(x)$ funksional qator sonli qatorga aylanadi, shuning uchun funksional qatorni tadqiqot qilishda sonli qatorniig barcha belgilaridan foydalanish mumkin.

**1-Misol.** Ushbu funksional qatorning yaqinlashish sohasini aniqlang:

$$\frac{1}{x^2}+\frac{1}{x^4}+\frac{1}{x^6}+\cdots+\frac{1}{x^{2n}}+\cdots$$

<u>Yechish.</u> Bu qatorning hadlari maxraji $q=\frac{1}{x^2}$ bo'lgan geometrik progressiya tashkil qiladi. Bizga ma'lumki, geometrik progressiya $|q|<1$, bo'lganda yaqinlashuvchi va $|q|>1$ bo'lganda uzoqlashuvchidir. Shu sababli berilgan qator $x$ ning $\frac{1}{x^2}<1$ yoki $x^2>1$ bo'ladigan qiymatlarida yaqinlashuvchidir. SHunday qilib, qator $|x|>1$ bo'lgan barcha $x$ nuqtalar uchun yaqinlashuvchidir. Berilgan qatorning yaqinlashish sohasi ushbu ikkita cheksiz intervaldan iborat

$$-\infty<x<-1 \text{ va } 1<x<+\infty.$$

**2-Misol.** Ushbu funksional qatorning yaqinlashish sohasini va yig'indisini aniqlang:

$$1+x+x^2+x^3+\cdots+x^n+\cdots$$

<u>Yechish.</u> Bu qator $|x|<1$ shartni qanoatlantiruvchi barcha $x$ larda ya'ni $(-1, 1)$ intervaida x ning barcha qiymatlarida yaqinlashadi. $(-1, 1)$ intervalda x ning har bir qiymati uchun qatorning yig'indisi $\frac{1}{1-x}$ ga teng(maxraji $x$ bo'lgan cheksiz kamayuvchi geometrik progressiyaning yig'indisi). SHunday qilib, berilgan qator $(-1,1)$ intervalda

$$\frac{1}{1-x}=1+x+x^2+x^3+\cdots+x^n+\cdots$$

qatorning yig'indisi bo'lgan

$$S(x)=\frac{1}{1-x}$$

funksiyani aniqlaydi.

$$\sum_{n=1}^{\infty} a_n(x-x_0)^n=a_0+a_1(x-x_0)+a_2(x-x_0)^2+ \\ +a_3(x-x_0)^3+\cdots+a_n(x-x_0)^n+\cdots \quad (1)$$

ko'rinishdagi qator darajali qator deyiladi. Darajali qatorning har bir hadi darajali funksiya bo'ladi. $a_n$ (n= 0,1,2...) sonlar darajali qatorning koeffitsientlari deyiladi.

$x_0=0$ bo'lganda (1) darajali qatorning ko'rinishi quyidagicha bo'ladi:

$$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| = \lim_{n\to\infty} \left| \frac{5^{n+2}(n+3)}{5^n(n+2)} \right| = \frac{1}{3} \lim_{n\to\infty} \left| \frac{5^n 5(n+3)}{5^n(n+2)} \right| =$$
$$= 5 \lim_{n\to\infty} \frac{n+3}{n+2} = 5.$$

Agar $|x-5| < 5$ bo'lsa qator yaqinlashadi. Bundan $-5+2 < x < 5+2 \Rightarrow -3 < x < 7$. Demak, darajali qatorning yaqinlashish oralig'i ( -3; 7). Shunday qilib, $x\in(-3,7)$ bo'lganda qator mutloq yaqinlashadi, Endi $x=-3$ va $x=7$ bo'lganda ya'ni oraliqning chekka nuqtalarida qatorni yaqinlashishga tekshiramiz:

1) $x=-3$ bo'lganda, quyidagi qatorni hosil qilamiz:

$$\sum_{n=0}^{\infty} \frac{(-3-2)^{n+1}}{5^n(n+2)} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}5^{n+1}}{5^n(n+2)} = \sum_{n=0}^{\infty} \frac{5(-1)^{n+1}}{(n+2)}.$$

Bu ishorasi almashinuvchi qatordir. Uni Leybnits belgisi yordamida yaqinlashishga tekshiramiz.

1) $a_n > a_{n+1}$, ya'ni $\frac{5}{n+2} > \frac{5}{n+3}$;

2) ) $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \frac{5}{(n+2)} = 0.$

Shunday qilib, ishoralari almashinuvchi $\sum_{n=0}^{\infty} \frac{5(-1)^{n+1}}{(n+2)}$ qator uchun Leybnits belgisining ikkala sharti ham bajariladi, demak, berilgan qator yaqinlashuvchi. Hadlarining mutloq qiymatlaridan tuzi'gan qator uzoqlashuvchi. Demak yaqinlashish sohasi [-3; 7)/

**6-Misol**. Qatorning yaqinlashish sohasini toping. $\sum_{n=0}^{\infty} \frac{x^n}{n!}$

**Yechish.** Qatorning yaqinlashish radiusini topamiz,

$$R = \lim_{n\to\infty} \left(\frac{1}{n!} : \frac{1}{(n+1)!}\right) = \lim_{n\to\infty} (n+1) = \infty$$

Berilgan qator barcha sonlar to'g'ri chizig'ida yaqinlashuvchi.

## MASHQLAR

1. Quyidagi funksional qatorlarning yaqinlashish sohalari topilsin

1. $\sum_{n=2}^{\infty} \frac{2^n}{n(x+1)^n}$.
2. $\sum_{n=0}^{\infty} 5^n x^n$

2. Darajali qatorlarning yaqinlashish sohalari topilsin:

1. $\sum_{n=1}^{\infty} \frac{x^n}{5^n n!}$.
2. $\sum_{n=1}^{\infty} \frac{(x+2)^n}{n\cdot 3^{n-1}}$.
3. $\sum_{n=1}^{\infty} \frac{5^n x^n}{\sqrt{n}}$.
4. $\sum_{n=1}^{\infty} \frac{2^n x^n}{\sqrt[3]{n}}$
5. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{(2n+1)\sqrt{2n+1}}$.
6. $\sum_{n=1}^{\infty} \frac{(2n)!(x+7)^n}{3^{n-1}}$

## 6.5. Teylor va Makloren qatorlari

Agar f(x) funksiya biror oraliqda darajali qatorning yig'indisini ifodalasa, u holda shu oraliqda u darajali qatorga yoyiladi deyiladi, yoki darajali qator berilgan oraliqda f(x) funksiyaga yaqinlashadi deyiladi.

f(x) funksiya $x = x_0$ nuqtada atrofida (n+1) tartibli hosilani ham o'z ichiga olgan hosilalarga ega bo'lsin. U holda quyidagi Teylor formulasi o'rinlidir:

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots$$
$$\cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + r_n(x), \quad (1)$$

bu yerda $r_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}$ Teylor formulasining qoldiq hadi (yoki qatorning qoldig'i ).

Agar $x_0 = 0$ deb olsak, u holda Makloren qatori deb ataluvchi quyidagi qatorni hosil qilamiz. $f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots +$

$$+ \frac{f^{(n)}(0)}{n!}x^n + \cdots = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n \quad (2)$$

**Ba'zi funksiyalarning Makloren qatoriga yoyilmalari.**

1) $\sin x = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^{2n-1}}{(2n-1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + (-1)^{n-1}\frac{x^{2n-1}}{(2n-1)!} + \cdots (-\infty; +\infty)$

2) $\cos x = \sum_{n=1}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + \frac{(-1)^n x^{2n}}{(2n)!} + \cdots \ (-\infty < x < \infty)$

3) $e^x = \sum_{n=1}^{\infty}\frac{x^{n-1}}{(n-1)!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^{n-1}}{(n-1)!} + \cdots, \quad -\infty < x < +\infty$

4) $\operatorname{arctg} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots + \frac{(-1)^{n-1}x^{2n-1}}{(2n-1)} + \cdots \ -1 < x \le 1$

5) $\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + \frac{(-1)^{n+1}x^n}{n} + \cdots, \ 1 < x \le 1.$

6) $(1 + x)^m = \sum_{n=1}^{\infty}\frac{m(m-1)\ldots(m-n=1)}{n!}x^n =$

$= 1 + \frac{m}{1!}x + \frac{m(m-1)}{2!}x^2 + \frac{m(m-1)(m-2)}{3!}x^3 + \cdots + + \frac{m(m-1)\ldots(m-n+1)}{n!}x^n + \cdots, \quad 1 < x < 1$( Binomial qator)

7) $\frac{1}{1+x}=\sum_{n=1}^{\infty}(-1)^n x^n = 1-x+x^2-x^3+\cdots+(-1)^n x^n+\cdots, \quad -1<x<1$

8) $\arcsin x=\sum_{n=1}^{\infty}\frac{1\cdot3\cdot5\cdot\ldots\cdot(2n-1)}{2\cdot4\cdot6\cdot\ldots\cdot2n}\frac{x^{2n+1}}{2n+1}=x+\frac{1}{2}\frac{x^3}{3}+\frac{1}{2}\frac{3}{4}\frac{x^5}{5}+$
$+\frac{1}{2}\frac{3}{4}\frac{5}{6}\frac{x^7}{7}+\cdots++\frac{1\cdot3\cdot5\cdot\ldots\cdot(2n-1)}{2\cdot4\cdot6\cdot\ldots\cdot2n}\frac{x^{2n+1}}{2n+1}+\cdots, \quad 1<x<1$

9) $\ln\frac{1+x}{1-x}=2\left(x+\frac{x^3}{3}+\frac{x^5}{5}+\cdots++\frac{x^{2n-1}}{2n-1}+\cdots\right), \quad -1<x<1$

**1- Misol.** f(x) $=e^{-2x}$ funksiya Makloren qatoriga yoyilsin:

**Yechish.** Quyidagi

$$e^x=\sum_{n=1}^{\infty}\frac{x^{n-1}}{(n-1)!}= 1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots+\frac{x^{n-1}}{(n-1)!}+\cdots,$$

formulada x ni ( -2x) ga almashtirib izlanayotgan qatorni hosil qilamiz:

$$e^{-2x}=1+\frac{(-2x)}{1!}+\frac{(-2x)^2}{2!}+\frac{(-2x)^3}{3!}+\cdots+\frac{(-2x)^{n-1}}{(n-1)!}+\cdots, \text{yoki}$$

$$e^{-2x}=1-\frac{2x}{1!}+\frac{2^2x^2}{2!}-\frac{2^3x^3}{3!}+\cdots+(-1)^{n-1}\frac{2^{n-1}x^{n-1}}{(n-1)!}+\cdots,$$

**2- Misol.** f(x) =sin3x funksiya Makloren qatoriga yoyilsin:

**Yechish.** sinx funksiyaning Makloren qatoriga yoyilmasidan foydalanamiz:

$$\sin x = x-\frac{x^3}{3!}+\frac{x^5}{5!}+\cdots+(-1)^{n-1}\frac{x^{2n-1}}{(2n-1)!}+\cdots$$

Bu yoyilmada x ni 3x ga almashtirib, izlanayotgan qatorni hosil qilamiz.

$$\sin x = 3x-\frac{3^3x^3}{3!}+\frac{3^5x^5}{5!}+\cdots+(-1)^{n-1}\frac{3^{2n-1}x^{2n-1}}{(2n-1)!}+\cdots$$

**3- Misol.** f(x) $=\sin^2 x$ funksiya Makloren qatoriga yoyilsin:

**Yechish.** $f(x)=\sin^2 x$ funksiyani darajali qatorga yoyish uchun avval uning darajasini pasaytiramiz ya'ni $\sin^2 x=\frac{1-\cos 2x}{2}$. (5) formuladan foydalanib cos2x ni qatorga yoyiamiz: $\cos x = 1-\frac{x^2}{2}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots+\frac{(-1)^n x^{2n}}{(2n)!}+\cdots,$

formulada x ni 2x ga almashtirib quyidagini hosil qilamiz:

$$\cos 2x = 1-\frac{(2x)^2}{2}+\frac{(2x)^4}{4!}-\frac{(2x)^6}{6!}+\cdots+\frac{(-1)^n(2x)^{2n}}{(2n)!}+\cdots,$$

bundan

$$1-\cos 2x=\frac{(2x)^2}{2}-\frac{(2x)^4}{4!}+\frac{(2x)^6}{6!}+\cdots+\frac{(-1)^n(2x)^{2n}}{(2n)!}+\cdots,$$

bu ifodani 2 ga bo'lsak, izlanayotgan yoyilmani hosil qilamiz:

$$\sin^2 x = \frac{1-\cos 2x}{2} = \frac{2x^2}{2!} - \frac{2^3x^4}{4!} + \frac{2^5x^6}{6!} + \cdots + (-1)^{n-1}\frac{2^{2n-1}x^{2n}}{(2n)!} + \cdots$$

**4- Misol.** $f(x) = \frac{1}{x^2-2x-3}$ funksiya Makloren qatoriga yoyilsin:

__Yechish.__ Berilgan funksiyaning maxrajini ko'paytuvchilarga ajratamiz

$$(x-3)(x+1).$$

Shunday qilib, berilgan funksiyani quyidagicha yozib olamiz

$$f(x) = \frac{1}{x^2-2x-3} = \frac{1}{(x-3)(x+1)} = \frac{1}{4}\left(\frac{1}{x-3} - \frac{1}{x+1}\right) = -\frac{1}{12}\cdot\frac{1}{1-\frac{x}{3}} - \frac{1}{4}\cdot\frac{1}{x+1}.$$

Quyidagi yoyilmalardan foydalanamiz:

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots + (-1)^n x^n + \cdots,$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots,$$

oxirgi yoyilmada x ni $\frac{x}{3}$ ga almashtirib quyidagi yoyilmani hosil qilamiz.

$$\frac{1}{1-\frac{x}{3}} = 1 + \frac{x}{3} + \left(\frac{x}{3}\right)^2 + \left(\frac{x}{3}\right)^3 + \cdots + \left(\frac{x}{3}\right)^n + \cdots,\quad \frac{x}{3} \in (-1;1)$$

Shunday qilib, izlanayotgan qator quyidagicha bo'ladi:

$$\frac{1}{x^2-2x-3} = -\frac{1}{12}\cdot\frac{1}{1-\frac{x}{3}} - \frac{1}{4}\cdot\frac{1}{x+1} = -\frac{1}{12}\left[1 + \frac{x}{3} + \left(\frac{x}{3}\right)^2 + \cdots + + \left(\frac{x}{3}\right)^n + \cdots\right] - \frac{1}{4}[1 + x + x^2 + \cdots + x^n + \cdots\ ] = -\frac{1}{3} + \frac{2}{9}x - \frac{7}{27}x + \cdots + \left[\frac{(-1)^{n-1}}{4} - \frac{1}{12}\cdot\left(\frac{1}{3}\right)^n\right]x^n + \cdots$$

**5- Misol.** $y = x^4 - 3x^2 + 2x + 2$ funksiyani (x-1) ning darajalari bo'yicha qatorga yoying.

__Yechish.__ $x_0 = 1$ uchun Teylor formulasidan foydalanamiz. Funksiyalarning hosilalari ni va ularning $x_0 = 1$ nuqtadagi qiymatlarini topamiz:

$y(1) = 2;\qquad y'(1) = (4x^3 - 6x + 2)\big|_{x=1} = 0;$

$y''(1) = (12x^2 - 6)\big|_{x=1} = 6;\quad y'''(1) = 24x\big|_{x=1} = 24;$

$y^{IV}(1) = 24;\quad y^{V}(1) = 0$ va hokazo.

Demak,

$$y = x^4 - 3x^2 + 2x + 2 = 2 + \frac{6}{2!}(x-1)^2 + \frac{24}{3!}(x-1)^3 + \frac{24}{4!}(x-1)^4$$

yoki $\quad y = x^4 - 3x^2 + 2x + 2 = 2 + 3(x-1)^2 + 4(x-1)^3 + (x-1)^4.$

## MASHQLAR

1. Quyidagi funksiyalar Makloren qatoriga yoyilsin

1. $f(x)=\frac{1}{x+8}$ $\sum_{n=0}^{\infty}\frac{(-1)^n}{3^{n+1}}$, $(-\frac{3}{2}< x <\frac{3}{2}))$

2. $f(x)=3^x$ $1+\frac{\ln 3}{1!}x+\frac{\ln^2 3}{2!}x^2+\ldots+\frac{\ln^n 3}{n!}x^n+\ldots)$ $(-\infty<x<\infty)$

3. $\mathrm{f}(\mathrm{x})=cos^2 x$ $1-\frac{2x^2}{2!}+\frac{2^3x^4}{4!}-\frac{2^5x^6}{6!}+\frac{2^7x^8}{8!}\ldots)$. $(-\infty<x<+\infty)$

4. $f(x)=\frac{sin5x}{x}$ . $5-\frac{5^3x^2}{3!}+\frac{5^5x^4}{5!}-\frac{5^7x^6}{7!}\ldots$, $(-\infty<x<+\infty)$

2. Quyidagi funksiyalar Teylor qatoriga yoyiisin

1. $f(x)=\frac{1}{x+5}$ funksiyani $x-2$ ning darajalari bo'yicha
2. $f(x)=\sqrt{x}$ funksiyani $x-3$ ning darajalari bo'yicha
3. $f(x)=2^x$ funksiyani $x-2$ ning darajalari bo'yicha
4. $f(x)=x^3-3x^2+8x-2$ funksiyani $x-1$ning darajalari bo'yicha

### 6.6. Funksiyalarning qatorlarga yoyilmasi yordamida taqribiy hisoblashlar

Biror trigonometrikfunksiyaning x= a nuqtadagi taqribiy qyimatini hisoblash uchun shu funksiyaga mos keluvchi yoyilmada $x$ o'rniga $a$ ning qiymatini qo'yib, tuzilgan qatorning berilgan aniqlikni qanoatlantiruvchi bir necha hadlarining yig'indisi hisoblanadi.

**1-Misol.** arctg 0.9 ni $\varepsilon=0.01$ aniqlikda taqribiy hisoblang.

<u>**Yechish.**</u> arctg 0.9 ni hisoblash uchun arctgx funksiyaning yoyilmasidan foydalanamiz.

$x$ ni 0.9 bilan almashtirib quyidagi qatorni hosil qilamiz:

$$\text{arctg}\,0.9=0{,}9-\frac{0.9^3}{3}+\frac{0.9^5}{5}-\frac{0.9^7}{7}+\frac{0.9^9}{9}-\frac{0.9^{11}}{11}+\frac{0.9^{13}}{13}-\frac{0.9^{15}}{153}+$$
$$+\frac{0.9^{17}}{17}-\cdots=0{,}9-\frac{0{,}729}{3}+\frac{0.59049}{5}-\frac{0{,}4749}{7}+\frac{0{,}38742}{9}-$$
$$-\frac{0{,}313811}{11}+\frac{0{,}254187}{13}-\frac{0{,}205891}{15}+\frac{0{,}166772}{17}-\cdots=0{,}9-0{,}243+$$
$$+0{,}118-0.068+0.043-0{,}029+0{,}0196-0.014+0.0098-\cdots$$

Yoyilmaning to'qqizinchi hadi $u_9=0{,}0098<0{,}01$ bo'lganligidan berilgan aniqlikni ta'minlash uchun arctg 0.9 ning yoyilmasidagi birinchi sakkizta hadining yig'indisini olish yetarli ekan.

$\mathrm{arctg}\, 0.9 = 0,9 - 0,243 + 0,118 - 0.068 + 0.043 - 0,029 + 0,0196 - 0.014 = 0,73.$

**2-Misol.** $\cos 10^0$ ning qiymati 0,0001 aniqlikda hisoblansin.

Yechish. cosx funksiyaning yoyilmasidan foydalanamiz:

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + \frac{(-1)^n x^{2n}}{(2n)!} + \cdots, \quad -\infty < x < \infty$$

$\alpha_{rad} = \frac{\pi}{180^0}\alpha^0$ formuladan foydalanib gradusni radianga o'tkazamiz:

$$\frac{\pi}{180^0}10^0 = \frac{\pi}{18} \approx \frac{3,14}{18} \approx 0,1745.$$

cosx funksiyaning yoyilmasidagi $x$ argumentning o'rniga 0. 17453 sonini qo'yamiz va quyidagini hosil qilamiz: $\cos 10^0 = 1 - \frac{(0.17453)^2}{2} + \frac{(0.17453^4}{4!} + \cdots$

Qatorning uchinchi hadi berilgan aniqlikdan kichgina, yoki

$$\frac{(0.17453^4}{4!} < 0.0001$$

Qator ishoralari almashinuvchi bo'lgani uchun

$$|r_2| < |u_3| < 0.0001$$

Bu baholash qatorning tashlab yuborilgan barcha hadlari uchinchisidan boshlab, 0.0001 dan kichik ekanligini bildiradi.

Shunday qilib. $\cos 10^0 \approx 1 - 0,01523;$ $\cos 10^0 \approx 0,9848.$

**3-Misol.** $\sqrt{e}$ ni $\delta = 0,001$ aniqlikda hisoblanig.

Yechish. $e^x$ funksiyaning darajali qatorga yoyilmasidan foydalanamiz.

$x = \frac{1}{2}$ deb olsak, quyidagiga ega bo'lamiz:

$$\sqrt{e} = 1 + \frac{1}{2} + \frac{1}{2!2^2} + \cdots + \frac{1}{n!2^n} + \cdots$$

Bu qatorning qoldiq hadi,

$$r_n = \sum_{K=1}^{\infty} \frac{1}{(n+k)! \cdot 2^{n+k}} < \frac{1}{(n+1)! \cdot 2^n} \sum_{K=1}^{\infty} \frac{1}{2^n} = \frac{1}{(n+1)! * 2^n}$$

$(n+1)! < (n+2)! < \ldots$ bo'lganligi uchun $n=4$ da $r_n < \frac{1}{5! 24} < 0,001.$

Shunday qilib, $e^{1/2} \approx 1 + \frac{1}{2} + \frac{1}{8} + \frac{1}{48} + \frac{1}{384} \approx 1,674$

**4-Misol.** $\sqrt[5]{34}$ ni $\delta = 10^{-3}$ aniqlikda hisoblang.

Yechish. Ma'lumki, $\sqrt[5]{34} = \sqrt[5]{32+2} = \sqrt[5]{32(1 + \frac{1}{16})} =$

$$= \left(x - \frac{0.4x^3}{3} + \frac{0.08x^5}{5} - \frac{0.4^3x^7}{6\cdot7} + \frac{0.4^4x^9}{24\cdot9} - \cdots\right)\Big|_0^1 =$$
$$= 0.6 - 0.0288 + 0.0012 - 0.00004 + \cdots$$

Ishorasi almashinuvchi qator hosil qildik. Qatorning to'rtinchi hadi mutloq qymati bo'yicha berilgan aniqlikni qanoatlantiradi, ya'ni, $\frac{0.4^4x^9}{24\cdot9} \approx 0.0009 < 0.001$. Leybnits belgisiga asosan qatorning qoldig'i birinchi tashlab yuborilgan haddan katta bo'lmaydi.

Shunday qilib, berilgan integralni hisoblashda talab qilingan aniqlikni ta'minlash uchun qator yoyilmasida birnchi uchta hadining iyg'indisini olish yetarli ekan.

$$\int_0^{0.6} e^{-0.4x^2} \approx 0.6 - 0.0288 + 0.0012 = 0.5724$$

Darajali qatorlar yordamida differensial tenglamalarni integrallash mumkin. Quyidagi ko'rinishdagi chiziqli differensial tenglamani qaraylik:

$$y^{(n)}(x) + p_1(x)y^{(n-1)}(x) + p_2(x)y^{(n-2)(x)} + \cdots + p_n(x)y(x) = f(x)$$

Bu yerda $y(x)$ – noma'lum funksiya, $p_n(x), f(x)$ – ma'lum funksiyalar.

Agar barcha $p_n(x)$ koeffitsientlar va bu tenglamaning o'ng tomonidagi $f(x)$ funksya biror oraliqda yaqinlashuvchi darajali qatorga yoyilsa, u holda bu tenglamaning x= 0 nuqtaning biror kichik atrofida shu nuqtadagi boshlang'ich shartni qanoatlantiruvchi yechimi mavjud bo'ladi. Bu yechimni quyidagi darajali qator ko'rinishida tasvirlash mumkin:

$$y = c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + \cdots \qquad (*)$$

Berilgan differensial tenglamaning yechimini topish uchun noma'lum o'zgarmaslar $c_n$ larni aniqlash yetarli. Bu masalani yechishning ikki hil usuli mavjud.

1. Aniqmas koeffitsientlar usuli. Bu usul ba'zi adabiyotlarda noma'lum koeffitsientlarni taqqoslash usuli deb ham keltiriladi.
2. Ketma – ket differensiallash usuli.

**9-Misol**. Quyidagi boshlang'ich shart bilan berilgan differensial tenglamaning yechimi topilsin.

$$y'' - xy = 0, \ \ y(0) = 1, \ \ y'(0) = 0.$$

**Yechish.** Yechimni quyidagi ko'rinishda qidiramiz:

$$y(x) = c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + \cdots$$

U holda : $y' = c_1 + 2c_2x + 3c_3x^2 + 4c_4x^3 + \cdots + nc_nx^{n-1} + \cdots$

$y'' = 2c_2 + 2\cdot3\cdot c_3x + 3\cdot4c_4x^2 + 5\cdot4c_5x^3 + \cdots + n(n-1)c_nx^{n-2} + \cdots$

$y(0) = 1$, $y'(0) = 0$. Boshlang'ich shartlarni qo'llab quyidagiga ega bo'lamiz:

$$y(0) = c_0 + c_1 \cdot 0 + c_2 \cdot 0 + \dots = 1$$

$y'(0)= c_1 + 2c_2 \cdot 0 + 3c_3 \cdot 0 + 0 + \dots = 0$. Bundan $c_0 = 1$, $c_1 = 0$.

y(x), y'(x), y''(x) larning yoyilmalarini dastlabki tenglamaga qo'yamiz:

$$2c_2 + 2 \cdot 3 \cdot c_3 x + 3 \cdot 4 c_4 x^2 + 5 \cdot 4 c_5 x^3 + \dots + (n+3)(n+2)c_{n+3}x^{n+1} + \dots$$
$$= x + c_2 x^3 + c_3 x^4 + c_4 x^5 + \dots + c_n x^{n+1} + \dots$$

Chap va o'ng tomondagi x larning darajalari oldidagi koeffitsientlarni tenglashtiramiz:

$$2c_2 = 0 \Rightarrow c_2 = 0, \quad 3c_3 - 1 = 0 \Rightarrow c_3 = \frac{1}{1 \cdot 2 \cdot 3},$$
$$4c_4 - c_1 = 0 \Rightarrow c_4 = 0, \quad 5 \cdot 4c_5 - c_2 = 0 \Rightarrow c_5 = 0,$$
$$6 \cdot 5c_6 - c_3 = 0 \Rightarrow c_6 = \frac{1}{6 \cdot 5 \cdot 3!},$$

. . . . . . . . . . . . . . . . . .

$$(n+3)(n+2)c_{n+3} = c_n \Rightarrow c_{n+3} = \frac{c_n}{(n+3)(n+2)}.$$

Shunday qilib,

$$c_0 = 1, c_1 = 0, \ c_2 = 0, \ c_3 = \frac{1}{1 \cdot 2 \cdot 3}, c_4 = 0, c_5 = 0, c_6 = \frac{1}{6 \cdot 5 \cdot 3!},$$
$$c_7 = 0, \ c_8 = 0, \ c_9 = \frac{1}{9 \cdot 8} \cdot \frac{1}{6 \cdot 5 \cdot 3!} = \frac{1 \cdot 4 \cdot 7}{9!}, \ \dots$$

$3k - 3 = n$ desak, bu yerda $k = 2,3,4,5 \dots$, u holda,

$$c_k = \frac{1 \cdot 4 \cdot 7 \cdots (3k-5)}{(3k-3)!};$$

Topilgan koeffitsientlarni izlanayotgan qatorga qo'yib yechimni hosil qilamiz:

$$y = 1 + \frac{1}{6}x^3 + \frac{4}{6!}x^6 + \dots$$

**10- Misol.** Darajali qatorlar yordamida quyidagi differensial tenglama uchun Koshi masalasi yechilsin.

$$y'(x) = x^2 + y^2, \ y(0) = 1.$$

Qatorning birinchi beshta hadi hisoblansin.

<u>**Yechish.**</u> Boshlang'ich shart $x = 0$ nuqtada berilgan bo'lganligi uchun yechimni Makloren qatori ko'rinishida izlaymiz.

$$y(x) = y(0) + y'(0) + \frac{y''(0)}{2!}x^2 + \frac{y'''(0)}{3!}x^3 + \frac{y^{IV}(0)}{2!}x^2 + \dots$$

Masalaning shartiga asosan qatorninig birinchi beshta hadini yozish kerak edi. Endi yoyilmadagi hosilalarni topamiz. $y'(x) = x^2 + y^2$,

$y''(x) = 2x - 2yy'$,

$y'''(x) = 2 - 2(y')^2 - 2yy''$,

$y^{IV}(x) = -4y' \cdot y'' - 2y' \cdot y'' - 2yy''' = -6y' \cdot y'' - 2yy'''$

Berilgan boshlang'ich shartdan foydalanib topilgan hosilalarning qyimatlarini topamiz. Boshlang'ich shartga ko'ra, x = 0, y = 1.

$y'(0) = 0 - 1 = -1, \quad y''(0) = 2\cdot 0 - 2\cdot 1\cdot(-2) = 2.$

$y'''(0) = 2 - 2\cdot(-1)^2 - 2\cdot 1\cdot 2 = -4,$

$y^{IV}(0) = -6\cdot(-1)\cdot 2 - 2\cdot 1\cdot(-4) = 12 + 8 = 20.$

Topilgan hosilalarning qyimatlarini izlanayotgan qatorga qo'yib, berilgan Koshi masalasining yechimini topamiz: $y(x) = 1 - x + x^2 - \frac{2}{3}x^3 + \frac{5}{6}x^4 - \cdots$

**11- Misol**. Ketma - ket differensiallash usulini qo'llab quyidagi differnsial tenglamaning yechimini toping: $y'(x) = x^2y^2 - 1$, $y(0) = 1$,

__**Yechish.**__ Boshlang'ich shart x = 0 nuqtada berilgan bo'lganligi uchun yechimni Makloren qatori ko'rinishida izlaymiz.

$$y(x) = y(0) + y'(0) + \frac{y''(0)}{2!}x^2 + \frac{y'''(0)}{3!}x^3 + \frac{y^{IV}(0)}{4!}x^4 + \cdots$$

Yoyilmadagi y'(x), y"(x) , $y'''$,⋯ hosilalarni topamiz:

$y'(x) = x^2y^2 - 1; \quad y''(x) = 2\cdot x\cdot y^2 + 2x^2yy';$

$y'''(x) = 2\cdot y^2 + 4xyy' + 4xyy' + 2x^2(yy')' =$

$= 2\cdot y^2 + 8xyy' + 2x^2(y')^2 + 2xyy''.$

Berilgan boshlang'ich shartdan foydalanib topilgan hosilalarning qiymatlarini topamiz. Boshlang'ich shartga ko'ra, x = 0, y = 1.

$y'(0) = x^2y^2 - 1 = 0\cdot 1^2 - 1 = -1;$

$y''(0) = 2\cdot x\cdot y^2 + 2x^2yy' = 2\cdot 0\cdot 1^2 + 2\cdot 0\cdot 1\cdot(-1) = 0$

$y'''(0) = 2\cdot y^2 + 8xyy' + 2x^2(y')^2 + 2xyy'' = 2\cdot 1 + 0 + 0 + 0 = 2.$

Hosilalarning topilgan qiymatlarini izlanayotgan qatorga qo'yib yechimni hosil qilamiz. $y(x) = 1 + (-x) + \frac{1}{3}x^3 - \cdots$

## MASHQLAR

1. Quyidagi funksiyalarni Makloren qatoriga yoyilmasidan foydalanib ko'rsatilgan aniqlikda hisoblang.

1. $\cos 20^0$ ni 0.0001 aniqlikda hisoblang.
2. $\sin 1^0$ni 0.0001 aniqlikda hisoblang.

2. Quyidagi funksiyalarni $\ln\frac{1+x}{1-x}$ funksiyaning Makloren qatoriga yoyilmasidan foydalanib ko'rsatilgan aniqlikda hisoblang.

1. ln3 ni 0.0001 aniqlikda hisoblang.
2. $\lg e = \frac{1}{\ln 10}$ni 0.000001 aniqlikda hisoblang.

3. $(1+x)^n$ funksiyaning Makloren qatoriga yoyilmasidan foydalanib quyidagilarni 0.001 aniqlikda hisoblang.

1. $\sqrt[3]{30}$ 2. $\sqrt[3]{70}$

3. $\sqrt[3]{500}$ 4. $\sqrt[3]{1.105}$.

4. Quyidagi integrallarni mos funksiyalarning yoyilmalaridan foydalanib 0.001 aniqlikda hisoblang.

1. $\int_0^1 \sqrt[3]{x}\cos x dx$. 2. $\int_0^{0.8} x^{10}\, dx$.

5. Quyidagi misollarda berilgan differensial tenglamalar Teylor qatori yordamida $y$ ga nisbatan (y uchun aniq ifodasi topilsin) ikki hil usulda: noma'lum koeffitsientlar usulida va ketma – ket differensiallash usulida yechilsin.

1. $y''(x) = xy$, $y(0) = 1$, $y'(0)= 0$,
2. $y''(x) =2xy'+4y$, $y(0) = 0$, $y'(0)=1$.

### 6.7. Fure qatorlari

**Ta'rif.** Quyidagi ko'rinishdagi funksional qator

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n cosnx + +b_n sinnx)$$

trigonometrik qator deb ataladi. $a_0, a_n, b_n (n = 1,2,\cdots)$ haqiqiy sonlar trigonometrik qatorning koeffitsientlari deb ataladi.

Agar yuqoridagi qator yaqinlashuvchi bo'lsa, u holda uning yig'indisi $sinx$ va $cosx$ funksiyalar davri $2\pi$ bo'lgan davriy funksiyalar ekanligi sababli, davri $2\pi$ bo'lgan funksiyani ifodalaydi,.

**Ta'rif.** [ $-\pi, \pi$ ] oraliqda aniqlangan f(x) funksiyaning Fure qatori deb koeffitsientlari

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx. \qquad (1)$$

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(kx)dx, \qquad (2)$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(kx)dx \qquad (3)$$

formulalar bilan aniqlanuvchi

$$S(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k coskx + +b_k sinkx) \qquad (4)$$

trigonometrik qatorga aytiladi.

**Teorema. (Dirixle).** Agar f(x) funksiya $2\pi$ davrga ega bo'lsa va [ $-\pi, \pi$ ] kesmada uzluksiz yoki chekli sondagi birinchi tur uzilishga ega bo'lsa va [ $-\pi, \pi$ ] kesmani chekli sondagi kesmalarga shunday bo'lish mumkin bo'lsaki, ularning har birining ichida f(x) funksiya monoton bo'lsa, u holda f(x) funksiya uchun bu kesmada Fure qatori yaqinlashuvchi bo'ladi va shu bilan birgalikda:

1). f(x) funksiyaning barcha uzluksiz nuqtalarida uning yig'indisi f(x) ga, funksiyaning har bir $x_0$ uzilish nuqtasida esa qatorning yig'indisi quyidagiga teng

$$S(x_0)=\frac{f(x_0-0)+f(x_0+0)}{2}$$

2) $x=-\pi$ va $x=\pi$ nuqtalarda (kesmaning chetki nuqtalarida) qatorning yig'indisi quyidagiga teng bo'ladi:

$$S(-\pi)=S(\pi)=\frac{f(-\pi-0)+f(\pi+0)}{2}$$

**1-Misol.** $(-\pi,\pi)$oraliqda f(x) = 2x +3 kabi aniqlangan $2\pi$ davrli funksiyani Fure qatoriga yoying

<u>**Yechish.**</u> (1) formulaga ko'ra

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)dx=\frac{1}{\pi}\int_{-\pi}^{\pi}(2x+3)dx=\frac{1}{\pi}(x^2+3x)|_{-\pi}^{\pi}=\frac{1}{\pi}\,6\pi=6.$$

(2) formulani va bo'laklab intengrallash formulasini qo'llasak,

$$a_k=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos(kx)dx=\frac{1}{\pi}\int_{-\pi}^{\pi}(2x+3)\cos(kx)dx=$$

$$=\left|\begin{matrix}u=2x+3, du=2dx\\ dv=coskxdx, v=\frac{1}{k}sinkx\end{matrix}\right|=\frac{1}{\pi k}(2x+3)sinkx|_{-\pi}^{\pi}-$$

$$\frac{2}{\pi k}\int_{-\pi}^{\pi}sinkxdx=$$

$$=\frac{2}{\pi k^2}coskx|_{-\pi}^{\pi}=\frac{1}{\pi k^2}(cosk\pi-\cos(-k\pi))=0.$$

Aynan shu kabi (3) formulaga ko'ra,

$$b_k=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin(kx)dx=\frac{1}{\pi}\int_{-\pi}^{\pi}(2x+3)\sin(kx)dx=$$

$$=\left|\begin{matrix}u=2x+3, du=2dx\\ dv=sinkxdx, v=-\frac{1}{k}coskx\end{matrix}\right|=-$$

$$\frac{1}{\pi k}(2x+3)coskx|_{-\pi}^{\pi}+\frac{2}{\pi k}\int_{-\pi}^{\pi}coskxdx=$$

$$=-\frac{1}{\pi k}[(2\pi+3)cosk\pi-(-2\pi+3)\cos(-k\pi)]=-\frac{4\pi}{\pi k}(-1)^k=$$

$$(-1)^{k+1}\frac{4}{k}.$$

Shunday qilib, quyidagi qatorni hosil qilamiz:

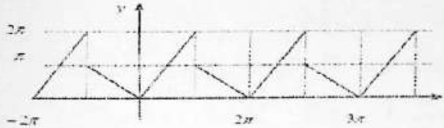
$$f(x)=2x+3=3+4\sum_{k=1}^{\infty}(-1)^{k+1}\frac{sinkx}{k}.$$

**2-Misol.** $[-\pi,\pi]$kesmada f(x) =

$$\begin{cases}2x, & 0\le x\le\pi \text{ bo'lganda}\\ -x & -\pi\le x\le 0 \text{ bo'lganda}\end{cases}$$

formula bilan berilgan va davri $2\pi$ bo'lgan f(x) funksiya Fure qatoriga yoyilsin.

<u>**Yechish.**</u> Rasmda berilgan f(x) funksiyaning grafigi tasvirlangan.




Berilgan funksiya $[-\pi,\pi]$kesmada aniqlangan, bo'lakli monoton va chekli sondagi birinchi tur uzilishga ega, demak Dirixle teoremasining shartlarini qanoatlantiradi va Fure qatoriga yoyiladi. (1) - (3) formulalar bo'yicha qatorning koeffitsientlarini topamiz.

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)dx=\frac{1}{\pi}\int_{-\pi}^{0}(-x)dx+\frac{1}{\pi}\int_{0}^{\pi}2xdx=\frac{1}{\pi}\left(-\frac{x^2}{2}|_{-\pi}^{0}+2\frac{x^2}{2}|_{0}^{\pi}\right)=\frac{3\pi}{2}.$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos(nx)dx$$

$$=\frac{1}{\pi}\int_{-\pi}^{0}(-x)\cos(nx)dx+\frac{1}{\pi}\int_{0}^{\pi}2x\cos(nx)dx;$$

Har bir integralni alohida – alohida hisoblaymiz. Integralni hisoblashda aniq integralni bo'laklab integrallash formulasidan foydalanamiz.

$$I_1=\frac{1}{\pi}\int_{-\pi}^{0}(-x)\cos(nx)dx=-\frac{1}{\pi}\int_{-\pi}^{0}x\cos(nx)dx=$$

$$=\left|\begin{matrix}u=x, & du=dx\\ dv=cosnxdx, & v=\frac{1}{n}\sin(nx)\end{matrix}\right|=-\frac{1}{\pi}\left(\frac{x}{n}\sin(nx)|_{-\pi}^{0}+\frac{1}{n}\int_{-\pi}^{0}\sin(nx)dx=\right.$$

$$=-\frac{x}{n\pi}\sin nx|_{-\pi}^{0}-\frac{1}{\pi n^2}cosnx|_{-\pi}^{0}=-\frac{1}{\pi n^2}(1-cosn\pi);$$

$$I_2=\frac{1}{\pi}\int_{0}^{\pi}2x\cos(nx)dx\left|\begin{matrix}u=2x, & du=2dx\\ dv=cosnxdx, & v=\frac{1}{n}\sin(nx)\end{matrix}\right|=$$

$$=\frac{2x}{\pi x}\sin(nx)|_{0}^{\pi}-$$

$$-\frac{1}{\pi}\frac{2}{\pi}\int_{0}^{\pi}\sin(nx)dx=\frac{2}{\pi n^2}cosnx|_{0}^{\pi}=\frac{2}{\pi n^2}(cosn\pi-1)=-\frac{2}{\pi n^2}(1-cosn\pi).$$

$$a_n=I_1+I_2=-\frac{1}{\pi n^2}(1-cosn\pi)-\frac{2}{\pi n^2}(1-cosn\pi)=-\frac{3}{\pi n^2}(1-cosn\pi)$$

Quyidagini e'tiborga olsak $cosn\pi=(-1)^n$.

U holda, $a_n=-\frac{3}{\pi n^2}(1-(-1)^n)$.

$b_n$ koeffitsientlar ham aynan shunday topiladi

$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx=$

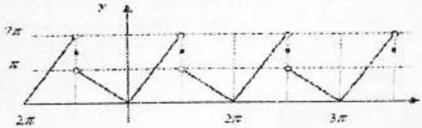
$\frac{1}{\pi}\int_{-\pi}^{0}(-x)\sin(nx)dx \ ++ \ \frac{1}{\pi}\int_{0}^{\pi} 2x\sin(nx)dx = \frac{(-1)^{n+1}}{n}.$

$f(x) = \frac{3\pi}{4} + \sum_{n=1}^{\infty}\left(-\frac{3}{\pi n^2}(1-(-1)^n)\cos nx + \frac{(-1)^{n+1}}{n}\sin(nx)\right)$

$x = \pm\pi$ nuqtalarda qator yig'indisi quyidagiga teng bo'ladi:

$$\frac{f(-\pi-0)+f(\pi+0)}{2} = \frac{\pi+2\pi}{2} = \frac{3\pi}{2}.$$

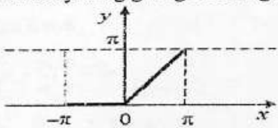
Fure qatorining yig'indisining grafigi quyidagi rasmda ko'rsatilgan.



**3-Misol .** $f(x) = \begin{cases} 0, & -\pi < x < 0, \\ x, & 0 \le x \le \pi \end{cases}$

formula bilan berilgan davri $2\pi$ bo'lgan f(x) funksiya Fure qatoriga yoyilsin.

<u>**Yechish.**</u> Quyida funksiyaning grafigi keltirilgan.



Berilgan funksiya chegaralangan bo'lakli-monoton bo'lganligi uchun uning Fure qatoriga yoyilmasi mavjud. Qatorning koeffitsientlarini topamiz.

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\int_{0}^{\pi} xdx = \frac{1}{\pi}\frac{x^2}{2}\Big|_0^{\pi} = \frac{\pi}{2}.$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos nx dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\cdot\cos nx dx + \int_{0}^{\pi} x\cdot\cos nx dx\right] =$$
$$= \frac{1}{\pi}\int_{0}^{\pi} x\cdot\cos nx dx$$

Bo'laklab integrallaymiz: $u = x$, $du = dx$, $dv = \cos nx dx$, $v = \frac{1}{n}\sin nx$ . U holda quyidagiga ega bo'lamiz:

$$a_n = \frac{1}{\pi}\left(\frac{x}{n}\sin nx\Big|_0^{\pi} - \int_0^{\pi}\frac{1}{n}\sin nx dx\right) = \frac{1}{\pi}\left[\frac{1}{n}x\sin nx + \frac{1}{n^2}\cos nx\right]\Big|_0^{\pi} =$$

$$=\frac{1}{\pi}\left[\frac{1}{n}\pi sin n\pi+\frac{1}{n^2}cos n\pi-\left(0+\frac{1}{n^2}\right)\right]=\frac{1}{\pi}\left(0+\frac{1}{n^2}cos n\pi-\frac{1}{n^2}\right)$$
$$=\frac{1}{\pi n^2}(cos n\pi-1)=\frac{1}{\pi n^2}[(-1)^n-1].$$

Bundan, $a_1=-\frac{2}{\pi}$, $a_2=0$, $a_3=-\frac{2}{3^2\pi}$, $a_4=0$, $a_5=-\frac{2}{3^2\pi}\cdots$.

$b_n$ koeffitsientni topamiz

$$b_n=\frac{1}{\pi}\int_{\pi}^{\pi} xsinnxdx=\frac{1}{\pi}\left[\int_{-\pi}^{0}0\cdot sinnxdx+\int_0^{\pi}x\cdot sinnxdx\right]=$$
$$=\frac{1}{\pi}\int_0^{\pi}x\cdot sinnxdx.$$

Bo'laklab integrallaymiz $u=x$, $du=dx$, $dv=sinnxdx$, $v=-\frac{1}{n}cosnx$. U holda quyidagiga ega bo'lamiz:

$$b_n=\frac{1}{\pi}\left[-\frac{x}{n}cosnx\Big|_0^{\pi}+\int_0^{\pi}\frac{1}{n}cosnxdx\right]=$$
$$\frac{1}{\pi}\left[\frac{1}{n}xcosnx++\frac{1}{n^2}sinnx\right]\Big|_0^{\pi}=$$
$$=\frac{1}{\pi}\left[-\frac{1}{n}\pi cosn\pi+\frac{1}{n^2}sinn\pi-(0-0)\right]=-\frac{1}{n}cosn\pi=$$
$$=\frac{1}{n}(-1)^n=\frac{1}{n}(-1)(-1)^n=\frac{1}{n}(-1)^{n+1}.$$

Bundan, $b_1=1$, $b_2=-\frac{1}{2}$ $b_3=\frac{1}{3}$ $b_4=-\frac{1}{4}$ $b_5=\frac{1}{5}\cdots$

U holda,

$$f(x)=\frac{\pi}{4}+\left(-\frac{2}{\pi}cosx+sinx\right)+\left(0-\frac{1}{2}sin2x\right)+\left(-\frac{2}{3^2\pi}cos3x+\right.$$
$$\left.+\frac{1}{3}sin3x\right)+$$
$$+\left(0-\frac{1}{4}sin4x\right)+\left(-\frac{2}{5^2\pi}cos5x+\frac{1}{5}sin5x\right)+\cdots$$

yoki,

$$f(x)=\frac{\pi}{4}+\sum_{n=1}^{\infty}\left(-\frac{2}{\pi(2n-1)^2}cos((2n-1)x)+\frac{(-1)^{n-1}}{n}sinnx\right)$$

Bu tenglik uzilish nuqtalaridan boshqa barcha nuqtalarda o'rinlidir. Har bir uzilish nuqtalarida qatorning yig'indisi chap va ung limitlarining o'rta arifmetigiga teng.

$$S(-\pi)=\frac{f(-\pi-0)+f(-\pi+0)}{2}=\frac{\pi}{2}.\quad S(\pi)=\frac{f(\pi-0)+f(\pi+0)}{2}=\frac{\pi}{2}$$

**Juft va toq funksiyalar uchun Fure qatorlari.**

1. Juft funksiya uchun $\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx.$

2. Toq funksiya uchun $\int_{-a}^{a} f(x)dx = 0.$

Agar Fure qatoriga toq funksiya yoyilayotgan bo'lsa, u holda $f(x)\cos nxdx$ toq, $f(x)\sin nxdx$ esa juft funksiya bo'ladi. Shuning uchun yuqoridagi xossalarga ko'ra toq funksiyalar uchun Fure koeffitsientlari quyidagi formulalar yordamida hisoblanadi :

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = 0 \qquad (5)$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)cosnxdx = 0 \qquad (6)$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)sinnxdx = \frac{2}{\pi}\int_{0}^{\pi} f(x)sinnxdx \qquad (7)$$

Demak, toq funksiya faqat "sinus" lar bo'yicha qatorga yoyilar ekan:

$$f(x) = \sum_{n=1}^{\infty} b_n\, sinnx$$

Xuddi shu kabi, agar Fure qatoriga juft funksiya yoyilayotgan bo'lsa

$$a_0 = \frac{2}{\pi}\int_{0}^{\pi} f(x)dx \qquad (8)$$

$$a_n = \frac{2}{\pi}\int_{0}^{\pi} f(x)cosnxdx \qquad (9)$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)sinnxdx = 0 \qquad (10)$$

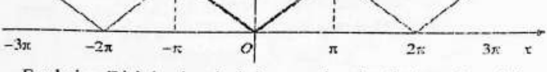
formulalarni hosil qilamiz. Demak, juft funksiya faqat "kosinus" lar bo'yicha qatorga yoyilar ekan

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\, cosnx$$

Hosil qilingan bu formulalar funksiya toq yoki juft bo'lgan hollarda qulay bo'lib, hisoblashlarni ancha kamaytiradi.

**4-Misol.** $f(x) = |x|$, $x \in [-\pi, \pi]$ funksiya uchun Fure qatorini toping.

__Yechish.__ Davriy davom ettirib funksiyaning grafigini chizamiz.




Funksiya Dirixle shartlarini qanoatlantiradi, demak uni Fure qatoriga yoyish mumkin. $[-\pi, \pi]$ oraliqda . $f(x) = |x|$ funksiya juft funksiya. Bundan kelib chiqadiki, Fure qatori bu funksiyada kosinuslardan va ular oldidagi o'zgarmas koeffitsientlardan iborat, sinuslar oldidagi koeffitsientlarning barchasi nol bo'ladi, ya'ni $b_n = 0$.

(8), (9) formulalardan foydalanib $a_0$ va $a_n$ koeffitsientlarni topamiz:

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx = \frac{2}{\pi}\int_0^{\pi} xdx = \frac{2}{\pi}\frac{x^2}{2}\Big|_0^{\pi} = \frac{\pi^2}{\pi} = \pi$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)cosnxdx = \frac{2}{\pi}\int_0^{\pi} xcosnxdx;$$

$$\int_0^{\pi} xcosnxdx = \begin{vmatrix} u = x, & du = dx \\ dv = cosnxdx, & v = \frac{1}{n}\sin nx \end{vmatrix} =$$

$$= \frac{x}{n}\sin nx\Big|_0^{\pi} - \int_0^{\pi}\frac{1}{n}\sin nxdx = \frac{1}{n^2}\ cosnx\Big|_0^{\pi} = \frac{1}{n^2}(\ cosn\pi\ -1).$$

$cosn\pi = (-1)^n$ dan

$$a_n = \frac{2}{\pi}\int_0^{\pi} xcosnxdx = \frac{2}{\pi}\frac{1}{n^2}(\ cosnx\ -1) = \frac{2}{\pi n^2}[(-1)^n - 1].$$

Agar n juft bo'lsa, u holda, $a_2 = a_4 = a_6 = \cdots = a_{2n} = \cdots = 0$, agar n toq

bo'lsa, u holda, $a_n = -\frac{4}{\pi n^2}$;

Shunday qilib, funksiyaning qatorga yoyilmasi quyidagi ko'rinishda bo'ladi.

$$f(x) = |x| = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\, cosnx = = \frac{2}{\pi} - \frac{4}{\pi}\frac{cosx}{1^2} - \frac{4}{\pi}\frac{cos3x}{3^2} - \frac{4}{\pi}\frac{cos5x}{5^2} - \frac{4}{\pi}\frac{cos7x}{7^2} - \cdots$$

$$|x| = \frac{\pi}{2} - \frac{4}{\pi}\left(\frac{cosx}{1^2} + \frac{cos3x}{3^2} + \frac{cos5x}{5^2} + \frac{cos7x}{7^2} + \cdots\right)$$

Oxirgi tenglikda x = 0 desak, quyidagiga ega bo'lamiz:

$$0 = \frac{\pi}{2} - \frac{4}{\pi}\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots\right)$$

$$\frac{\pi}{2} = \frac{4}{\pi}\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots\right) \quad \Rightarrow \quad \frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots$$

Hosil qilingan qatorni quyidagi yig'indini hisoblash uchun ham qo'llash mumkin:

$$S = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots = \sum_{n=1}^{\infty}\frac{1}{n^2}$$

$$S = \left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots\right) + \left(\frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \cdots\right) =$$

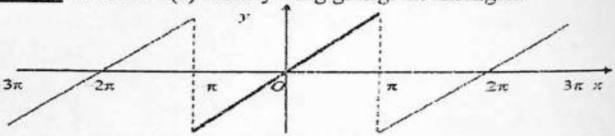
$$= \frac{\pi^2}{8} + \frac{1}{4}\left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots\right) = \frac{\pi^2}{8} + \frac{S}{4}$$

$S = \frac{\pi^2}{8} + \frac{S}{4}$ tenglikdan S ni topsak: $S - \frac{S}{4} = \frac{\pi^2}{8} \Rightarrow \frac{3S}{4} = \frac{\pi^2}{8}$

$$3S = \frac{\pi^2}{2} \Rightarrow S = \frac{\pi^2}{6} \text{ ya'ni } \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}.$$

**5 - Misol.** $f(x) = x$, $-\pi < x < \pi$ funksiyani Fure qatoriga yoying.

**Yechish.** Rasmda f(x) funksiyaning grafigi tasvirlangan.




Bu funksiya Dirixle shartlarini qaonatlantiradi, demak, uni Fure qatoriga yoyish mumkin. $[-\pi,\pi]$ oraliqda f(x) = x funksiya toq funksiya. Bundan kelib chiqadiki, Fure qatori bu funksiyada sinuslardan va ular oldidagi o'zgarmas koeffitsientlardan iborat, kosinuslar oldidagi koeffitsientlarning barchasi nol bo'ladi, ya'ni $a_n = 0$. (n= 0, 1, 2, ... )

(11) formuladan foydalanib $b_n$ koeffitsientlarni topamiz:

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)sinnxdx = \frac{2}{\pi}\int_0^{\pi} xsinnxdx$$

$$\int_0^{\pi} xsinnxdx = \left| \begin{matrix} u = x, & du = dx \\ dv = sinnxdx, & v = -\frac{1}{n}\cos nx \end{matrix} \right| =$$

$$= -\frac{x}{n}\cos nx\Big|_0^{\pi} + \int_0^{\pi}\frac{1}{n}\cos nxdx = -\frac{\pi}{n}(\cos n\pi - 1) + \frac{1}{n^2}\sin nx\Big|_0^{\pi} = -\frac{\pi}{n}\cos n\pi$$

$cosn\pi = (-1)^n$. U holda,

$$b_n = \frac{2}{\pi}\int_0^{\pi} xcosnxdx = -\frac{2}{\pi}\frac{\pi}{n}(-1)^n = -\frac{2}{n}(-1)^n = \frac{2}{n}(-1)^{n+1}$$

Shunday qilib, $a_0 = a_1 = a_2 = a_3 = a_4 = \cdots = a_n = \cdots = 0$,

$$b_1 = \frac{2}{1},\ \ b_2 = -\frac{2}{2},\ b_3 = \frac{2}{3},\ \ b_4 == -\frac{2}{4}, \cdots$$

Demak, f(x) funksiyaning Fure qatori quyidagi ko'rinishda bo'ladi:

$$f(x) = x = 2\sum_{n=1}^{\infty}(-1)^{n+1}\frac{sinnx}{n}$$

$$x = 2\left[\frac{sinx}{1} - \frac{sin2x}{2} + \frac{sin3x}{3} - \frac{sin4x}{4} + \cdots + \frac{(-1)^{n+1}}{n}sinnx + \cdots\right]$$

**Davri $2\ell$ bo'lgan funksiyalarni Fure qatoriga yoyish.**

Davri $2\ell$ (umuman olganda $\ell \neq \pi$) bo'lgan davriy funksiyalarning Fure qatoriga yoyilmasi quyidagicha bo'lib:

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k cos\frac{k\pi}{\ell}x + b_k sin\frac{k\pi}{\ell}x) \tag{11}$$

koeffitsientlari:

$$a_0 = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)dx$$

$$a_n = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)cos\frac{kn x}{\ell}dx$$

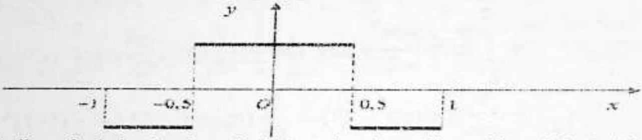
$$b_n = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\sin\frac{k\pi x}{\ell}dx$$

formulalar bilan hisoblanadi.

6 - **Misol.** Quyidagi funksiya kosinus funksiyaning karrali yoyilmari bo'yicha Fure qatoriga yoyilsin.

$$f(x) = \begin{cases} 0{,}3, & 0 < x < 0{,}5 \\ -0{,}3, & 0{,}5 < x < 1 \end{cases}$$

<u>**Yechish**</u>. Faqat kosinuslarni o'z ichiga oluvchi bu funksiyaning Fure qatoriga yoyilmasini hosil qilish uchun uni (- 1;0) oraliqda juft tarzda davom ettiramiz. Funksiyaning grafigini chizamiz.




Berilgan funksiya kosinus funksiyaning karrali yoyilmari bo'yicha Fure qatoriga yoyilayotganligi uchun u juft funksiyadir. U holda $b_n$ koeffitsientlar nolga teng bo'ladi. $a_0$ va $a_n$ koeffitsientlarni hisoblaymiz:

$$a_0 = \frac{2}{\ell}\int_0^{\ell} f(x)dx = 2\left(\int_0^{0.5} 0{,}3dx - \int_{0.5}^{1} 0{,}3dx\right) = 0.$$

$$a_n = \frac{2}{\ell}\int_0^{\ell} f(x)\cos\frac{k\pi x}{\ell}dx = 2\left(\int_0^{0.5} 0{,}3\cos\pi nxdx - \int_{0.5}^{1} 0{,}3\cos\pi nxdx\right) =$$

$$= \frac{1.2}{n\pi}\sin\frac{\pi x}{2}.$$

Agar n – juft bo'lsa u holda: $a_n = 0$.

Agar n toq bo'lsa, n = 2k-1, k = 1,2,3,... , u holda:

$$a_n = \frac{1{,}2}{(2k-1)\pi}\sin\left(k\pi - \frac{\pi}{2}\right) = (-1)^{k-1}\frac{1{,}2}{(2k-1)\pi}.$$

Topilgan koeffitsientlarni (11) formulaga qo'yib izlanayotgan Fure qatorining yoyilmasini topamiz.

$$f(x) = \frac{1{,}2}{\pi}\left(\frac{\cos\pi x}{1} + \frac{\cos 3\pi x}{3} + \frac{\cos 5\pi x}{5} + \dots + \frac{\cos(2k-1)\pi x}{2k-1} + \dots\right)$$

7- **Misol.** Davri 4 ga teng bo'lgan, davriy funksiyaning Fure qatoriga bo'lgan yoyilmasini toping. $f(x) = \begin{cases} -1, & -2 < x < 0 \\ 2, & 0 \le x \le 2 \end{cases}$

<u>**Yechish.**</u> Qatorning koeffitsientlarini topamiz. $2\ell = 4$, $\ell = 2$.

$$a_0 = \frac{1}{2}\int_{-2}^{2}(-1)dx + \int_0^{2} 2dx) = \frac{1}{2}\left(-x\Big|_{-2}^{0} + 2x\Big|_0^2\right) = \frac{1}{2}(-2+4) = 1$$

$$a_n=\frac{1}{2}\left(\int_{-2}^{0}(-1)\cos(\frac{\pi n}{2}x\Bigg)dx+\int_{0}^{2}2\cos(\frac{\pi n}{2}x)dx\right)=$$

$$=\frac{1}{2}\left(-\frac{2}{\pi n}\sin\left(\frac{\pi n}{2}x\right)\Big|_{-2}^{0}+\frac{4}{\pi n}\sin\left(\frac{\pi n}{2}x\right)\Big|_{0}^{2}\right)=0.$$

$$b_n=\frac{1}{2}(\int_{-2}^{0}(-1)sin\left(\frac{\pi n}{2}x\right)dx$$

$$+\int_{0}^{2}2sin\left(\frac{\pi n}{2}x\right)dx)=\frac{1}{2}(\frac{2}{\pi n}\cos\left(\frac{\pi n}{2}x\right)\Big|_{-2}^{0}$$

$$-\frac{4}{\pi n}(cos\pi n-1))=\frac{3}{\pi n}(cos\pi n-1)=\frac{3}{\pi n}((-1)^n-1).$$

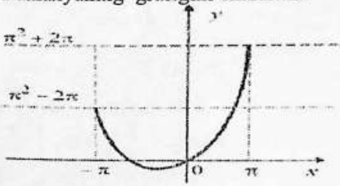
Topilgan koeffitsientlarni (11) qatorga qo'yib quyidagini hosil qilamiz.

$$f(x)=\frac{1}{2}-\frac{6}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\sin\left(\frac{(2n-1)\pi}{2}\cdot x\right).$$

**Eslatma.** $2\pi$ davrli davri funksiyalar $\int_{-\pi}^{\pi}f(x)dx=\int_{\alpha}^{\alpha+2\pi}f(x)dx$ xossaga ega bo'lganliklari uchun, Fure koeffitsientlarini aniqlashda integrallash oralig'i sifatida uzunligi $2\pi$ ga teng bo'lgan ixtiyoriy oraliqni olishimiz mumkin. Xususan ko'pchilik hollarda $(0, 2\pi)$ oraliqda integrallarni hisoblash qulay bo'ladi.

**8- Misol.** $f(x)=x^2+2x$ funksiyani a) $(-\pi;\ \pi)$ oraliqda; b) $(0;2\pi)$ oraliqda; c) $(0;2\pi)$ oraliqda sinuslar bo'yicha Fure qatoriga yoying.

**Yechish.** Funksiyaning grafigini chizamiz.




a) $(-\pi;\ \pi)$ oraliqda Fure qatoriga yoyamiz.

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)dx=\frac{1}{\pi}\int_{-\pi}^{\pi}(x^2+2x)dx=\frac{1}{\pi}(\frac{x^3}{3}+x^2)\Big|_{-\pi}^{\pi}=\frac{2}{3}\pi^2.$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(x^2+2x)cosnxdx = \begin{vmatrix} u = x^2+2x & dv = \cos(nx)\,dx \\ du = 2(x+1)dx & v = \frac{\sin(nx)}{n} \end{vmatrix} =$$

$$= \frac{1}{\pi}\Big[\frac{1}{n}\sin(nx)\cdot(x^2+2x)|_{-\pi}^{\pi} - \frac{2}{\pi}\int_{-\pi}^{\pi}(x+1)sinnxdx\Big] =$$

$$= -\frac{2}{\pi n}\int_{-\pi}^{\pi}(x+1)sinnxdx = \begin{vmatrix} u = x+1 & dv = \sin(nx)\,dx \\ du = dx & v = -\frac{\cos(nx)}{n} \end{vmatrix} =$$

$$= -\frac{2}{\pi n}[-\frac{1}{n}\text{cosnx}\cdot(\text{x}+1)\,|_{-\pi}^{\pi} - \frac{1}{\pi}\text{cosnx}|_{-\pi}^{\pi}] =$$

$$= -\frac{2}{\pi n}(-1)^n(\pi+1+\pi-1) = (-1)^n\frac{4}{n^2}.$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(x^2+2x)sinnxdx = \begin{vmatrix} u = x^2+2x & dv = \sin(nx)\,dx \\ du = 2(x+1)dx & v = -\frac{\cos(nx)}{n} \end{vmatrix}$$

=

$$= \frac{1}{\pi}\Big[-\frac{1}{n}\cos(nx)\cdot(x^2+2x)|_{-\pi}^{\pi} + \frac{2}{\pi}\int_{-\pi}^{\pi}(x+1)cosnxdx\Big] =$$

$$= -\frac{4}{n}(-1)^n + \frac{2}{\pi n}\int_{-\pi}^{\pi}(x+1)cosnxdx =$$

$$\begin{vmatrix} u = x+1 & dv = \cos(nx)\,dx \\ du = dx & v = \frac{\sin(nx)}{n} \end{vmatrix} =$$

$$= -\frac{4}{n}(-1)^n + \frac{2}{\pi n}[\frac{1}{n}\text{sin(nx)}\cdot(\text{x}+1)\,|_{-\pi}^{\pi} - \frac{1}{n}\int_{-\pi}^{\pi}sinnxdx] =$$

$$= -\frac{4}{n}(-1)^n + \frac{2}{\pi n^3}\,cosnx|_{-\pi}^{\pi} = (-1)^n\frac{4}{n}.$$

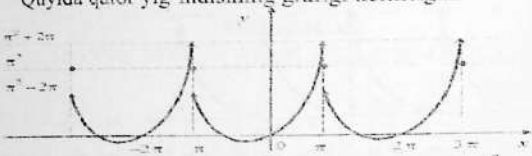
Syunday qilib,

$$x^2+2x = \frac{1}{3}\pi^2 + 4\sum_{n=1}^{\infty}(-1)^n\Big[\frac{cosnx}{n^2} - \frac{sinnx}{n}\Big], \quad x\in(-\pi;\ \pi).$$

Agar $x = \pm\pi$ desak, qatorning yig'indisi Dirixle teoremasiga asosan aniqlanib, quyidagicha bo'ladi:

$$S(\pm\pi) = \frac{f(-\pi+0)+f(\pi+0)}{2} = \frac{\pi^2-2\pi+\pi^2+2\pi}{2} = \pi^2$$

Quyida qator yig'indisining grafigi keltirilgan.




b) $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx = \frac{1}{\pi}\int_0^{2\pi}(x^2+2x)dx = \frac{1}{\pi}(\frac{x^3}{3}+x^2)\,|_0^{2\pi} =$ $\frac{4}{3}\pi^2(2\pi+3)$.

$$a_n = \frac{1}{\pi}\int_0^{2\pi}(x^2+2x)cosnxdx =$$

$$=\begin{vmatrix} u = x^2+2x & dv = \cos(nx)\,dx \\ du = 2(x+1)dx & v = \frac{\sin(nx)}{n} \end{vmatrix} =$$

$$= \frac{1}{\pi}\left[\frac{1}{n}\sin(nx)\cdot(x^2+2x)|_0^{2\pi} - \frac{2}{\pi}\int_0^{2\pi}(x+1)sinnxdx\right] =$$

$$= -\frac{2}{\pi n}\int_0^{2\pi}(x+1)sinnxdx = \begin{vmatrix} u = x+1 & dv = \sin(nx)\,dx \\ du = dx & v = -\frac{\cos(nx)}{n} \end{vmatrix} =$$

$$= -\frac{2}{\pi n}[-\frac{1}{n}\cos nx\cdot(x+1)\,|_0^{2\pi} - \frac{1}{\pi}\cos nx|_0^{2\pi}] = -\frac{2}{\pi n}2\pi = \frac{4}{n^2}.$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi}(x^2+2x)sinnxdx = \begin{vmatrix} u = x^2+2x & dv = \sin(nx)\,dx \\ du = 2(x+1)dx & v = -\frac{\cos(nx)}{n} \end{vmatrix}$$

=

$$= \frac{1}{\pi}\left[-\frac{1}{n}\cos(nx)\cdot(x^2+2x)|_0^{2\pi} + \frac{2}{\pi}\int_0^{2\pi}(x+1)cosnxdx\right] =$$

$$= -\frac{1}{\pi n}(4\pi^2+4\pi) - \frac{2}{\pi t^2}\int_0^{2\pi}(x+1)cosnxdx$$

$$=\begin{vmatrix} u = x+1 & dv = \cos(nx)\,dx \\ du = dx & v = \frac{\sin(nx)}{n} \end{vmatrix} =$$

$$= -\frac{4}{n}(\pi+1) + \frac{2}{\pi n^3}\,cosnx|_0^{2\pi} = \frac{4}{n}\,(\pi+1).$$

Shunday qilib,

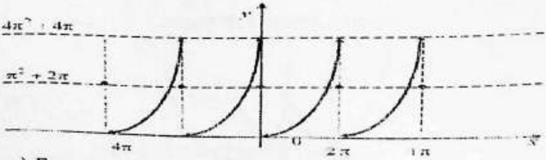
$$x^2+2x = \frac{2}{3}\pi(2\pi+3) + 4\sum_{n=1}^{\infty}(-1)^n\left[\frac{cosnx}{n^2} - (\pi+1)\frac{sinnx}{n}\right].$$

$x\in(0;\,2\pi)$.

Agar $x = 0$ yoki $x = 2\pi$ desak, qatorning yig'indisi Dirixle teoremasiga asosan aniqlanib, quyidagicha bo'ladi:

$$S(0) = S(2\pi) = \frac{f(+0)+f(2\pi+0)}{2} = \frac{0+4\pi^2+4\pi}{2} = 2\pi^2+2\pi.$$

Quyida qator yig'indisining grafigi keltirilgan.



c) Fure qatoriga sinuslar bo'yicha yoyiluvchi funksiya toq funksiya bo'lishi kerak. Shuning uchun $f(x) = x^2+2x$ funksiyani $(-\pi;\,0]$ oraliqda toq tarzda davom ettiramiz. Bunday funksiyalar uchun Fure qatori qo'yidagi ko'rinishda bo'ladi.

$$b_n = \frac{2}{\pi}\int_0^{\pi}(x^2+2x)sinnxdx =$$

$$=\begin{vmatrix} u = x^2+2x & dv = \sin(nx)\,dx \\ du = 2(x+1)dx & v = -\frac{\cos(nx)}{n} \end{vmatrix} =$$

$$=\frac{2}{\pi}\left[-\frac{1}{n}\cos(nx)\cdot(x^2+2x)|_0^{\pi} + \frac{2}{\pi}\int_0^{\pi}(x+1)cosnxdx\right] =$$

$$= -\frac{2}{\pi n}(-1)^n(\pi^2+2\pi) - \frac{4}{\pi n^2}\int_0^{\pi} sinnxdx$$

$$=\begin{vmatrix} u = x+1 & dv = \cos(nx)\,dx \\ du = dx & v = \frac{\sin(nx)}{n} \end{vmatrix} =$$

$$= -\frac{2}{\pi n}(-1)^n(\pi+2) + \frac{2}{\pi n^3}\, cosnx|_0^{2\pi} = -\frac{2}{n}(-1)^n(\pi+2) +$$

$$+\frac{4}{\pi n^3}((-1)^n\text{-}1).$$

Juft va toq nomerli koeffitsientlarni alohida – alohida hisoblaymiz.

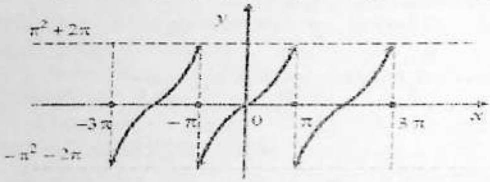
$$b_{2n} = \frac{\pi+2}{n}; \qquad b_{2n-1} = \frac{2(\pi+2)}{2n-1} - \frac{8}{\pi(2n-1)^3}.$$

Yoyilma quyidagi ko'rinishda bo'ladi.

$$(x^2+2x) = 2\sum_{n=1}^{\infty}\left(\frac{\pi+2}{2n-1} - \frac{4}{\pi(2n-1)^3}\right)\sin(2n-1)x -$$

$$-(\pi+2)\sum_{n=1}^{\infty}\frac{sin2nx}{n}.$$

Bu yerda, $x\in(0;\pi)$. Quyida qator yig'indisining grafigi keltirilgan.




## MASHQLAR

1. Quyidagi funksiyalar $(-\pi;\ \pi)$ oraliqda Fure qatoriga yoyilsin:

a) $f(x)=|x|$; b) $f(x)=\begin{cases} 0 & -\pi \le x \le 0 \\ sinx & 0 < x \le \pi \end{cases}$

c) $f(x)=\begin{cases} 0 & -\pi \le x \le 0. \\ \frac{1}{4}\pi x & 0 < x \le \pi \end{cases}$

2. Quyidagi funksiyalar $(0\ ;\ \pi)$ oraliqda kosinuslar bo'yicha Fure qatoriga yoyilsin

a) $f(x)=\begin{cases}1 & 0\le x\le 2.\\ \frac{1}{4}\pi x & 2<x\le\pi\end{cases}$ b) $f(x)=\begin{cases}\frac{\pi}{4}x & 0\le x\le\frac{\pi}{2}.\\ \frac{\pi}{4}(\pi-x) & \frac{\pi}{2}<x\le\pi\end{cases}$

3. Quyidagi funksiyalar $(0\ ;\ \pi)$ oraliqda sinuslar bo'yicha Fure qatoriga yoyilsin: a) $f(x)=x^2$; b) $f(x)=x$ ;

4. Quyidagi funksiyalar $(-\ell\ ;\ \ell)$ oraliqda Fure qatoriga yoyilsin:

a) $$f(x)=\begin{cases}1 & -\ell\le x\le 0.\\ \frac{1}{4}\pi x & 0<x\le\ell\end{cases}$$

b) $$f(x)=\begin{cases}1 & -\ell\le x\le 0.\\ \frac{1}{4}\pi x & 0<x\le\ell\end{cases}$$

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# OLIY MATEMATIKA

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