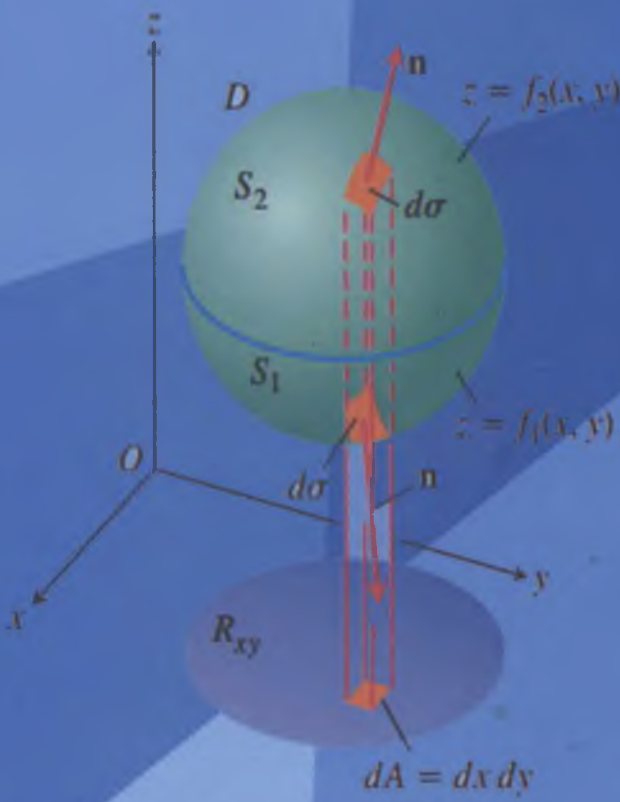
OLIY MATEMATIKA



2-QISM

O'ZBEKISTON RESPUBLIKASI OLIY VA
O'RTA MAXSUS TA'LIM VAZIRLIGI

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OLIY MATEMATIKA

2-QISM

O'zbekiston Respublikasi Oliy va o'rta maxsus ta'lim vazirligi tomonidan 300 000- Ishlab-chiqarish texnik soha, 400000-Qishloq va suv xo'jaligi, 200000-Ijtimoiy soha, iqtisod va huquq hamda barcha tabiiy va muhandis-texnik sohalarining bakalavriat ta'lim yo'nalishlari talabalari uchun darslik sifatida tavsiya etilgan

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SO‘Z BOSHI

Mazkur darslik “Oliy matematika”, 1-qismning davomi bo‘lib, oliy matematikaning R^n fazo, ko‘p o‘zgaruvchili funksiyalar, sonli va funksional qatorlar, ikki va uch karrali integrallar, birinchi va ikkinchi tur egri chiziqli integrallar, birinchi va ikkinchi tur sirt integrallari, Grin, Stoks va Ostrogradskiy formulalari, maydonlar nazariyasi elementlari, Furye qatorlari va integrallari mavzularini o‘z ichiga oladi.

Darslikning har bir boblarida mavzularning nazariy qismi, uni mustahkamlash uchun o‘z-o‘zini tekshirish savollari, amaliy mashg‘ulotlar uchun misol va masalalar, mustaqil bajarish uchun yozma ishlar materiallari berilgan.

Hozirgi vaqtda amaliyotda bir necha yaxshi rivojlangan matematik dasturlar (Mathcad, Maple, Mathematica, Mathlab va h.k.) matematik masalalarni kompyuter imkoniyatlaridan foydalanib yechishda samarali natijalar bermoqda. Shu an’anadan chetda qolmaslik uchun, qo‘llanmada ba’zi bo‘limlar bo‘yicha misol va masalalar yechishda «Maple» tizimining qo‘llanilishi va uning qulayliklari namoyish etilgan.

Mazkur kitob, asosan tabiiy, muhandis-texnik fanlar sohasidagi bakalavriat talabalari uchun mo‘ljallangan bo‘lib, undan gumanitar fanlar sohasidagi talabalar ham foydalanishlari mumkin.

Mualliflar darslikni yozishda oliy o‘quv yurtlarining muhandis-texnik, tabiiy va gumanitar yo‘nalishlardagi mutaxasislar uchun matematika fanlarining amaldagi dasturlarida tavsiya qilingan asosiy va qo‘shimcha adabiyotlardan hamda o‘zbek tilida chop yetilgan darslik va o‘quv qo‘llanmalardan keng foydalanildi.

Kitobxonlardan ushbu darslik haqidagi fikr-mulohazalarini mamnuniyat bilan qabul qilamiz.

Mualliflar

10-BOB. KO‘P O‘ZGARUVCHILI FUNKSIYALAR

10.1-§. R^m fazo va uning muhim to‘plamlari. R^m fazoda ketma-ketlik va uning limiti

10.1. Ko‘p o‘zgaruvchi miqdorlar orasidagi funksional bog‘lanish. Tabiatdagi ko‘pgina masalalarni yechish jarayonida o‘zgaruvchi miqdorlar orasida, bir o‘zgaruvchi miqdorning qiymati qolganlarining qiymatlari orqali aniqlanadigan bog‘lanishlar uchraydi.

Jismning biror xarakteristikalarini o‘rganayotganimizda jismning bir nuqtasidan boshqa nuqtasiga o‘tganda bu xarakteristikalarining o‘zgarishini hisobga olish kerak bo‘ladi. Masalan, jismning biror nuqtasidagi zichligi ρ yoki temperaturasi T, ikkinchi nuqtasiga o‘tganda, zichlikning yoki temperaturaning o‘zgarishini hisobga olish lozim bo‘ladi. Jismning har bir nuqtasi x, y, z dekart koordinatalari bilan aniqlangani uchun qaralayotgan xarakteristikalar (zichlik va temperatura) uch o‘zgaruvchi x, y, z larning qiymatlari bilan aniqlanadi. O‘zgaruvchi biror fizik jarayonni qarayotganda fizik xarakteristikalar qiymati to‘rtta o‘zgaruvchi: nuqtaning koordinatalari x, y, z va t- vaqtning qiymati bilan to‘liq aniqlanadi. Masalan, gazda tovush tebranishlarining tarqalishini o‘rganayotganda, gazning zichligi ρ va bosimi p to‘rtta o‘zgaruvchi x, y, z va t ning qiymati bilan aniqlanadi. Tabiatdagi shunga o‘xshash bog‘lanishlarni o‘rganish uchun ko‘p o‘zgaruvchili funksiya tushunchasini kiritish va ularning xususiyatlarini matematik nutai nazardan o‘rganish zaruriyati to‘g‘iladi.

10.2. m **o‘lchovli Evklid fazosi.** m ta $x_1, x_2, \ldots, x_m$ haqiqiy sonlarning barcha mumkin bo‘lgan tartiblangan $(x_1, x_2, \ldots, x_m)$ nuqtalar to‘plamiga *m-o‘lchovli koordinat fazosi* deyiladi va u

$$R^m = \{\ (x_1, x_2, \ldots, x_m) : x_k \in R, k = 1,2,\ldots,m\ \}$$

kabi belgilanadi. R^m koordinat fazosining elementini qisqacha bitta M harfi bilan belgilaymiz. $x_1, x_2, \ldots, x_m$ sonlar M nuqtaning *koordinatalari* deyiladi.

R^m koordinat fazosining ixtiyoriy ikki $M'(x'_1, x'_2, ..., x'_m)$ va $M''(x''_1, x''_2, ..., x''_m)$ nuqtalari orasidagi $\rho(M', M'')$ masofa ushbu

$$\rho(M', M'') = \sqrt{(x_1''-x_1')^2 + (x_2''-x_2')^2 + ... + (x_m''-x_m')^2} \quad (10.2.1)$$

formula bilan aniqlangan bo'lsa, u holda R^m ga *m-o'lchovli evklid fazosi* deyiladi.

(10.2.1) masofa $\rho(M', M'')$ quyidagi xossalarga ega:

1^0. $\rho(M', M'') \geq 0 \ \forall M', M'' \in R^m$; $\rho(M', M'') = 0 \Leftrightarrow M' = M''$;

2^0. $\rho(M', M'') = \rho(M'', M') \ \forall M', M'' \in R^m$;

3^0. $\rho(M', M''') \leq \rho(M', M'') + \rho(M'', M''') \ \forall M', M'', M''' \in R^m$.

Bu xossalarni isbotsiz keltiramiz.

10.2.2-eslatma. Agar R^m koordinat fazosini koordinatalari $(x_1, x_2, ..., x_m)$ bo'lgan x vektorlar to'plamidan iborat bo'lsin deb qarasak, unda $x = (x_1, x_2, ..., x_m)$, $y = (y_1, y_2, ..., y_m)$ vektorlar yig'indisi, ayirmasi

$$x \pm y = (x_1 \pm y_1, x_2 \pm y_2, ..., x_m \pm y_m), \ \forall x, y \in R^m,$$

vektorni o'zgarmas songa ko'paytmasi esa, $\lambda x = (\lambda x_1, \lambda x_2, ..., \lambda x_m)$, $\lambda \in R$ ko'rinishda aniqlangan bo'lsa, u holda R^m ni *chiziqli (vektorli) fazo* deb ataladi.

R^m fazoda nol element sifatida 0=(0, 0, …, 0) vektori olinadi.

x vektorning moduli: $|x| = \sqrt{x_1^2 + x_2^2 + ... + x_m^2}$, u holda $\rho(x, y) = |x - y|$.

10.3. R^m - evklid fazosidagi nuqtalar to'plami. $M_0 \in R^m$, $M_0(x_1^0, x_2^0, ..., x_m^0)$, r – biror musbat son berilgan bo'lsin.

1. $\{M : \rho(M, M_0) \leq r\}$ - markazi M_0 nuqtada, radiusi r bo'lgan m-o'lchovli shar deyiladi.

2. $\{M : \rho(M, M_0) < r\}$ - markazi M_0 nuqtada, radiusi r bo'lgan *ochiq shar* deyiladi.

3. $\{M : \rho(M, M_0) = r\}$ - markazi M_0 nuqtada, radiusi r bo'lgan *sfera* deyiladi.

4. $\{M : |x_1 - x_1^0| \leq d_1, |x_2 - x_2^0| \leq d_2, ..., |x_m - x_m^0| \leq d_m\}$, ($d_1, d_2, ..., d_m$ - lar musbat sonlar) – *m-o'lchovli parallelepiped* deyiladi. m=2 da bu to'plamga R^2 evklid fazosidagi to'rtburchak deyiladi.

5. $\{M: |x_1-x_1^0|<d_1, |x_2-x_2^0|<d_2, ..., |x_m-x_m^0|<d_m\}$, ($d_1,d_2,...,d_m$ - lar musbat sonlar) – *m-o'lchovli ochiq parallelepiped* deyiladi.

$\{M\}\subset R^m$, $M_0\in R^m$ bo'lsin. Markazi M_0 nuqtada, radiusi ε ga teng bo'lgan ochiq sharga, M_0 nuqtaning ε *atrofi* deyiladi.

$\{M\}\subset R^m$, $A\in\{M\}$ bo'lsin.

10.3.1-ta'rif. Agar A nuqtaning shunday $\exists U_\varepsilon(A)$ bo'lib, $\exists U_\varepsilon(A)\subset\{M\}$ bo'lsa, у холда A nuqta $\{M\}$ to'plamning *ichki nuqtasi* deyiladi (10.1-chizma).

10.1-chizma.

10. 2-chizma.

10.3.2-ta'rif. Agarda A nuqtaning ixtiyoriy ε atrofi $U_\varepsilon(A)$ da $\{M\}$ to'plamga qarashli va qarashli bo'lmagan nuqtalar mavjud bo'lsa, у холда A nuqta $\{M\}$ to'plamning *chegaraviy nuqtasi* deyiladi (10.2-chizma).Takidlaymizki, chegara nuqtalari to'plamiga qarashli va qarashli bo'lmasligi ham mumkin.

10.3.3-ta'rif. Agar $\{M\}$ to'plamning hamma nuqtalari uning ichki nuqtalari bo'lsa, u holda $\{M\}$ to'plamga *ochiq to'plam* deyiladi.

10.3.4-ta'rif. Agar $\{M\}$ to'plam, o'zining hamma chegara nuqtalarini o'zida saqlasa, u holda $\{M\}$ to'plamga *yopiq to'plam* deyiladi.

10.3.5-ta'rif. Agar A nuqtaning ixtiyoriy ε atrofida $\{M\}$ to'plamning A dan farqli hech bo'lmaganda bitta nuqtasi mavjud bo'lsa, u holda A nuqta $\{M\}$ to'plamning *limit nuqtasi* deyiladi. Takidlaymizki, to'plamning limit nuqtasi to'plamga qarashli bo'lishi ham, qarashli bo'lmasligi ham mumkin.

10.3.6--ta'rif. Agar $\{M\}$ to'plamning hamma nuqtalari biror sharning ichida joylashsa, u holda $\{M\}$ *to'plam chegaralangan* deyiladi.

$$L=\{M(x_1,x_2,...,x_m): x_1=\varphi_1(t), x_2=\varphi_2(t),...x_m=\varphi_m(t); \alpha\le t\le\beta\},$$

(bunda $\varphi_i(t), i=1,2,...m$, funksiyalar $[\alpha,\beta]$ da uzluksiz) to'plamga R^m fazodagi uzluksiz chiziq deyiladi. $A(\varphi_1(\alpha),\varphi_2(\alpha),...,\varphi_m(\alpha))$ va $B(\varphi_1(\beta),\varphi_2(\beta),...,\varphi_m(\beta))$ nuqtalar L chiziqning uchlari deyiladi.

$$\{M(x_1,x_2,...,x_m): x_1 = x_1^0 + \alpha_1 t, x_2 = x_2^0 + \alpha_2 t, ..., x_m = x_m^0 + \alpha_m t, t \in R\}$$

($x_1^0, x_2^0, ..., x_m^0; \alpha_1, \alpha_2, ..., \alpha_m$ lar sonlar) to‘plamga R^m fazodagi *to‘g‘ri chiziq* deyiladi. Ravshanki, to‘g‘ri chiziq $M_0(x_1^0, x_2^0, ..., x_m^0)$ nuqtadan o‘tadi. ($t=0$ga M_0nuqtaga mos keladi).

10.3.7-ta’rif. Agar $\{M\} \subset R^m$ to‘plamning ixtiyoriy ikki nuqtasini, shu to‘plamda butunlay yotuvchi uzluksiz chiziq bilan tutashtirish mumkin bo‘lsa, u holda $\{M\}$ to‘plamga *bog‘lamli to‘plam* deyiladi.

Ochiq bog‘lamli to‘plamga *soha* deb ham ataladi, soha bilan uning chegarasi birlashmasiga *yopiq soha* deyiladi.

10.4. R^m fazoda sonli ketma-ktlik va uning limiti. Natural sonlar to‘plami N va R^m evklid fazosi berilgan bo‘lsin. Har bir $n(n \in N)$ ga R^m ning biror muayyan $M_n = M_n(x_1^{(n)}, x_2^{(n)}, ..., x_m^{(n)})$ nuqtasi mos qo‘yilgan bo‘lsin: ya’ni

$$1 \to M_1(x_1^{(1)}, x_2^{(1)}, ..., x_m^{(1)}),$$
$$2 \to M_2(x_1^{(2)}, x_2^{(2)}, ..., x_m^{(2)}),$$
$$\dots\dots\dots\dots\dots\dots$$
$$n \to M_m(x_1^{(n)}, x_2^{(n)}, ..., x_m^{(n)}),$$
$$\dots\dots\dots\dots\dots\dots$$

Natijada, hosil bo‘lgan $M_1, M_2, \dots, M_m, \dots$ tartiblangan nuqtalaar to‘plamiga R^m da ketma-ketlik deb ataladi va u qisqacha $\{M\}(M \subset R^m)$ kabi belgilanadi.

Misollar: 1) $M_n = M_n(\frac{1}{n}, \frac{1}{n}): M_1(1,1), M_2(\frac{1}{2}, \frac{1}{2}), ..., M_n(\frac{1}{n}, \frac{1}{n}), ...$

2) $M_n = M_n(0, \frac{1}{n}): M_1(0,1), M_2(0, \frac{1}{2}), ..., M_n(0, \frac{1}{n}), ...$

3) $M_n = M_n(\frac{1}{n}, \frac{1}{n}, ..., \frac{1}{n}): M_1(1,1,...,1), M_2(\frac{1}{2}, \frac{1}{2}, ..., \frac{1}{2}), ..., M_n(\frac{1}{n}, \frac{1}{n}, ..., \frac{1}{n}), ...$

Bu ketma-ketliklarning birinchisi va ikkinchisi R^2 evklid fazosining nuqtalaridan, uchinchisi esa, R^m evklid fazosining nuqtalaridan tashkil topgan ketma-ketliklardir. R^m da

$$M_1, M_2, \dots, M_n, \dots \qquad (10.4.1)$$

nuqtalar ketma-ketligi va $A = A(a_1, a_2, ..., a_n) \in R^m$ nuqta berilgan bo‘lsin.

10.4.2-ta'rif. Agar $\forall\varepsilon>0$ son olinganda ham $\exists n_0\in N$ topilsaki, barcha $n>n_0(\varepsilon)$ lar uchun

$$\rho(M_n,A)<\varepsilon$$

tengsizlik, ya'ni $\forall\varepsilon>0,\ \exists n_0\in N,\ \forall n>n_0(\varepsilon)$: $\rho(M_n,A)<\varepsilon$ bajarilsa, A nuqta $\{M_n\}(M_n\subset R^m)$ ketma-ketlikning limiti deyiladi va $\lim\limits_{n\to\infty}M_n=A$ yoki $n\to\infty$ da $M_n\to A$ kabi belgilanadi.

Agar (10.4.1) ketma-ketlik chekli limitga ega bo'lsa, u yaqinlashuvchi ketma-ketlik deb ataladi. Limit ta'rifidagi shartni qanoatlantiruvchi A mavjud bo'lmasa, $\{M_n\}$ketma-ketlik *uzoqlashuvchi* deyiladi.

$\forall n>n_0$ da $\rho(M_r,A)<\varepsilon$ tengsizlikning bajarilishi, (10.4.1) ketma-ketlikning n_0 dan katta nomerli barcha hadlari $U_\varepsilon(A)$ atrofiga tegishli bo'lishini anglatadi. Bu hol (10.4.1) ketma-ketlik limitini quyidagicha ta'riflash imkonini beradi.

10.4.3-ta'rif. Agar $A\in R^m$ nuqtaning $\forall U_\varepsilon(A)$ atrofi olinganda ham, $\{M_n\}$ ketma-ketlikning biror hadidan keying barcha hadlari shu atrofga qarashli bo'lsa, A nuqta $\{M_n\}$ketma-ketlikning *limiti* deyiladi.

$\{M_n\}\subset R^m$ va $A\in R^m$ berilgan bo'lsin.

10.4.4-teotema. Agar $\{M_n\}$ketma-ketlik R^m fazoda $A=A(a_1,a_2,...,a_n)$ limitga ega bo'lsa, yani $\lim\limits_{n\to\infty}M_n=A$, u holda

$$\lim_{n\to\infty}x_1^{(n)}=a_1,\ \lim_{n\to\infty}x_2^{(n)}=a_2,...,\lim_{n\to\infty}x_m^{(n)}=a_m$$

bo'ladi va aksincha agar M_n nuqtaning koordinatalari $(x_1^{(n)},x_2^{(n)},...,x_m^{(n)})$ mos ravishda A nuqtaning koordinatalari $a_1,a_2,...,a_m$ ga yaqinlashsa, u holda $\{M_n\}$ ketma-ketlik $A(a_1,a_2,...,a_m)$ nuqtaga yaqinlashadi.

$\{M_n\}\subset R^m$ berilgan bo'lsin.

1^0. $\{M_n\}$ketma-ketlik yaqinlashuvchi bo'lsa, uning limiti yagona bo'ladi.

Keyingi xossalarini keltirishda avval $\{M_n\}$ ketma-ketlikning chegaralanganlik tushunchasini keltirib o'tamiz.

10.4.5-ta'rif. Agar $\{M_n\}$ ketma-ketlik hadlari qiymatlaridan tuzilgan to'plam chegaralangan bo'lsa, u holda $\{M_n\}$ ketma-ketlikga *chegaralgan* ketma-ketlik deyiladi.

10.4.6-teorema. R^m fazoda $\{M_n\}$ ketma-ketlikning chegarangan bo'lishi uchun uning koordinatalari $\{x_1^{(n)}\}$, $\{x_2^{(n)}\}$, ... , $\{x_m^{(n)}\}$ sonli ketma-ketliklarning har birining chegaralangan bo'lishi zarur va yetarli.

2^0. $\{M_n\}$ ketma-ketlik yaqinlashuvchi bo'lsa, u holda u chegaralangan bo'ladi.

3^0. Agar $\{M_n\}$ ketma-ketlik yaqinlashuvchi bo'lib, uning limiti A bo'lsa, u holda $\{kM_n\}$ $(k \in R)$ ketma-ketlik ham yaqinlashuvchi va

$$\lim_{n\to\infty} kM_n = k\lim_{n\to\infty} M_n = kA$$

bo'ladi.

4^0. Agar $\{M_n\}$ va $\{N_n\}$ ketma-ketliklar yaqinlashuvchi bo'lib, ularning limitlari mos ravishda A va B bo'lsa, u holda $\{M_n \pm N_n\}$ ketma-ketlik ham yaqinlashuvchi bo'lib, uning limiti $A \pm B$ ga teng bo'ladi, ya'ni

$$\lim_{n\to\infty}(M_n \pm N_n) = \lim_{n\to\infty} M_n \pm \lim_{n\to\infty} N_n = A \pm B.$$

$\{M_n\} \subset R^m$ berilgan bo'lsin.

10.4.7-ta'rif. Agar $\forall\varepsilon>0$ son olinganda ham $\exists n_0 \in \mathrm{N}$ topilsaki, barcha $n>n_0$ va $p>n_0$ lar uchun

$$\rho(M_n, M_p) < \varepsilon$$

tesizlik o'rinli bo'lsa, $\{M_n\}$ ketma-ketlik R^m da *fundamental ketma-ketlik* deb ataladi.

Ravshanki, agar $\{M_n\}(M_n \subset R^m)$ ketma-ketlik fundamental bo'lsa, u holda M_n ning koordinatalari $\{x_1^{(n)}\}$, $\{x_2^{(n)}\}$, ... , $\{x_m^{(n)}\}$ ketma-ketliklarning har biri ham fundamental ketma-ketlik bo'ladi va aksincha, $\{x_1^{(n)}\}$, $\{x_2^{(n)}\}$, ... , $\{x_m^{(n)}\}$ larning har biri fundamental bo'lsa, u holda $\{M_n\}$ ketma-ketlik ham fundamental bo'ladi, ya'ni quyidagi teorema o'rinli bo'ladi.

10.4.8-teorema. R^m fazoda $\{M_n\} = \{M_n(x_1^{(n)}, x_2^{(n)}, ..., x_m^{(n)})\}$ ketma-ketlikning fundamental bo'lishi uchun uning koordinatalaridan iborat $\{x_1^{(n)}\}$, $\{x_2^{(n)}\}$, ... , $\{x_m^{(n)}\}$ ketma-ketlikning har birining fundamental bo'lishi zarur va yetarli.

10.4.9-teorema (Koshi prinsipi). $\{M_n\}(\{M_n\} \subset R^m)$ ketma-ketlikning yaqinlashuvchi bo'lishi uchun uning fundamental bo'lishi zarur va yetarli.

10.5. Qismiy ketma-ketliklar. Bolsano-Veyshstrass teoremasi. $\{M_n\} \subset R^m$ berilgan bo'lsin. Bu ketma-ketlikning $n_1, n_2, ..., n_k, ...$ $(n_1 < n_2 < ... < n_k < ...;\ n_k \in \mathrm{N}, k = 1,2,...)$ nomerli hadlaridan tashkil topgan ushbu

$$M_{n_1}, M_{n_2}, \dots, M_{n_k}, \dots$$

ketma-ketlikga berilgan ketma-ketlikning *qismiy ketma-ketligi* deyiladi va u $\{M_{n_k}\}$ kabi belgilanadi.

Masalan, R^2 fazoda ushbu

$$(1,1), \left(\frac{1}{2},\frac{1}{2}\right), \left(\frac{1}{n},\frac{1}{n}\right), \dots, \left(\frac{1}{2n},\frac{1}{2n}\right), \dots,$$

$$(1,1), \left(\frac{1}{2^2},\frac{1}{2^2}\right), \left(\frac{1}{3^2},\frac{1}{3^2}\right), \dots, \left(\frac{1}{n^2},\frac{1}{n^2}\right), \dots$$

ketma-ketliklar, quyidagi

$$(1,1), \left(\frac{1}{2},\frac{1}{2}\right), \dots, \left(\frac{1}{n},\frac{1}{n}\right), \dots$$

ketma-ketlikning qismiy ketma-ketliklari bo'ladi.

10.5.1-teorema. Agar $\{M_n\}$ ketma-ketlik yaqinlashuvchi bo'lib, uning limiti $A(A \in R^m)$ bo'lsa, u holda bu ketma-ketlikning har qanday $\{M_{n_k}\}$ qismiy ketma-ketligi ham yaqinlashuvchi bo'lib, uning limiti ham A ga teng bo'ladi.

10.5.2-eslatma. Ketma-ketlikning, qismiy ketma-ketliklarining mavjud bo'lishidan berilgan ketma-ketlikning limiti mavjud bo'lishi har doim ham kelib chiqavermaydi. Masalan, (1, 1), (-1, -1), … , $((-1)^{n+1},(-1)^{n+1})$, … ketma-ketlik berilgan bo'lsin. Bu ketma-ketlik limitga ega emas, lekin uning qismiy ketma-ketliklari:

$$(1,1), (1,1), \dots; \qquad (-1,-1), (-1,-1), \dots$$

mos ravishda (1, 1) va (-1, -1) limitlarga ega.

Shunday qilib, $\{M_n\}$ ketma-ketlik limitga ega bo'lmasa ham, uning qismiy ketma-ketliklari limitga ega bo'lishi mumkin ekan.

10.5.3-teorema (Bolsano-Veyshstrass). Har qanday chegaralangan ketma-ketlikdan yaqinlashuvchi qismiy ketma-ketliklar ajratish mumkin.

10.2-§. Ko'p o'zgaruvchili funksiya va uning limiti tushunchalari

10.6. Ko'p o'zgaruvchili funksiya tushunchasi. $\{M\} \subset R^m$ to'plam berilgan bo'lsin.

10.6.1-ta'rif. $\{M\}$ to'plamning har bir M nuqtasiga biror qonun yoki qoida yordamida biror $u(u \in R)$ son mos qo'yilgan bo'lsa, $\{M\}$ to'plamga

$u=u(M)$ yoki $u=f(M)$ *funksiya aniqlangan* deyiladi. Bunda $\{M\}$ to'plam funksiyaning *aniqlanish sohasi*, $\{u\}$ to'plam esa funksiyaning *qiymatlar sohasi* deyiladi.

Misollar. 1) $u=\sqrt{9-x^2-y^2}$ funksiyaning aniqlanish sohasi $D(f)$: markazi koordinatalar boshida joylashgan radiusi 3 ga teng bo'lgan doiradan iborat, qiymatlar to'plami esa $E(f)$: $0\le u\le 3$ kesmadan iborat.

2) $u=\ln xy$ funksiyaning aniqlanish sohasi $D(f)$: $xy>0$ tengsizlikni qanoatlantiruvchi nuqtalar to'plamidan iborat, qiymatlar to'plami esa $E(f)$: $-\infty<u<+\infty$ son o'qidan iborat.

3) $u=\sqrt{1-x_1^2-x_2^2-...-x_m^2}$ funksiyaning aniqlanish sohasi $D(f)$: markazi $O(0,0,...,0)$ koordinatalar boshida joylashgan radiusi 1 ga teng bo'lgan doiradan iborat bo'lib, uning qiymatlar to'plami $E(f)$: [0; 1] kesmadan iborat.

4) $u=x_1^2+x_2^2+...+x_m^2$ funksiyaning aniqlanish sohasi $D(f)$: R^m fazodan iborat, uning qiymatlar to'plami $E(f)$: $[0;\infty]$ yarim segmentdan iborat.

10.7. Ko'p o'zgaruvchili funksiya limiti tushunchasi. $u=f(M)$ funksiya $\{M\}\subset R^m$ da aniqlangan. A nuqta $\{M\}$ to'plamning limit nuqtasi bo'lsin.

10.7.1-ta'rif (Geyne). Agar $\{M\}$ to'plamning nuqtalaridan tuzilgan va A nuqtaga intiluvchi har qanday $\{M_n\}$ $(M_n\ne A, n=1,2,...)$ ketma-ketlik olinganda ham unga mos kelgan funksiyaning qiymatlar ketma-ketligi $\{f(M_n)\}$ hamma vaqt yagona B songa (chekli yoki cheksiz) limitga intilsa, shu B $f(M)$ funksiyaning A nuqtadagi (yoki $M\to A$ dagi) *limiti* deb ataladi va u $\lim\limits_{M\to A} f(M)=B$ yoki $M\to A$ da $f(M)\to B$ yoki $\lim\limits_{\substack{x_1\to a_1\\ x_2\to a_2\\ \dots\\ x_m\to a_m}} f(x_1,x_2,...,x_m)=B$ kabi belgilanadi.

10.7.2-ta'rif (Koshi). Agar $M\to A \Leftrightarrow \begin{cases} x_1\to a_1\\ x_2\to a_2\\ \dots\dots\dots\\ x_m\to a_m\end{cases}$ $\forall\varepsilon>0$ son olinganda ham $\exists\delta=\delta(\varepsilon)>0$ topilsaki, $0<\rho(M,A)<\varepsilon$ tengsizlikni qanoatlantiruvchi barcha $M\in\{M\}$ nuqtalarda

$$|f(M)-B|<\varepsilon$$

tengsizlik bajarilsa, shu B ga $f(M)$ funksiyaning A nuqtadagi ($M\to A$ dagi) *limiti* deyiladi.

10.7.1- va 10.7.2- ta'riflar o'zaro ekvivalent ta'riflardir. Buning isboti bir o'zgaruvchili funksiyalarning limitik qiymatining ikki ta'rifi ekvivalentligining isboti kabi ko'rsatiladi.

10.7.3-ta'rif. Agar $\forall\varepsilon>0$ uchun $\exists\delta>0$ bo'lib, $0<\rho(M,A)<\delta$ tengsizlikni qanoatlantiruvchi barcha $M\in\{M\}$ nuqtalarda $|f(M)|>\varepsilon$ ($f(M)>\varepsilon\,;f(M)<-\varepsilon$) bo'lsa, $f(M)$ funksiya A nuqtadagi ($M\to A$ dagi) $\infty\ (+\infty,-\infty)$ deyiladi.

Funksiyaning limiti uning argumentlari $x_1,x_2,\ldots,x_m$ larning bir vaqtda mos ravishda $a_1,a_2,\ldots,a_m$ sonlarga intilgandagi limitdan iborat ekan. Biz bundan keyin, bu limitni *karrali limit* deb ataymiz.

Limitga ega bo'lgan m-o'zgaruvchili funksiyalar ustida arifmetik amallar, ya'ni quyiagi tasdiq o'rinli bo'ladi:

10.7.4-teorema. $f(M)$ va $g(M)$ funksiyalar bir xil $\{M\}$ to'plamda aniqlangan va A nuqtada mos ravishda B va C limitlarga ega bo'lsa, u holda $f(M)\pm g(M)$, $f(M)\,g(M)$ va $\dfrac{f(M)}{g(M)}$ funksiyalar ham A nuqtada $(C\neq0)$ mos ravishda $B\pm C$, BC va $\dfrac{B}{C}$ limitlarga ega bo'ladi.

10.7.5-eslatma. $f(M)$ va $g(M)$ funksiyalar yig'indisi, ayirmasi, ko'paytmasi va nisbatining limitga ega bo'lishidan, bu funksiyalarning har birining limitga ega bo'lishi har doim ham kelib chiqavermaydi.

$\alpha(M)$ funksiya $\{M\}\subset R^m$ to'plamda berilgan bo'lib, A $(A\in R^m)$ nuqta esa, shu to'plamning limit nuqtasi bo'lsin.

10.7.6-ta'rif. Agar $M\to A$ da $\alpha(M)$ funksiyaning limiti nol, ya'ni $\lim\limits_{M\to A}\alpha(M)=0$ bo'lsa, u holda $\alpha(M)$ funksiya $M\to A$ da *cheksiz kichik funksiya* deyiladi.

Masalan, $f(M)=(x_1-a_1)^{n_1}+(x_2-a_2)^{n_2}+\ldots+(x_m-a_m)^{n_m}$ (bunda $n_1,n_2,\ldots,n_m$ lar musbat sonlar) funksiya $A(a_1,a_2,\ldots,a_m)$ nuqtada cheksiz kichik bo'ladi.

Agar $u=f(M)$ funksiya A nuqtada B limitga ega bo'lsa, u holda $\alpha(M)=f(M)-B$ funksiya A nuqtada cheksiz kichik bo'ladi.

Demak, limitga ega bo'lgan funksiyani

$$f(M) = B + \alpha(M)$$

ko'rinishida tasvirlash mumkin ekan, bunda $\lim\limits_{M \to A} \alpha(M) = 0$. Bundan $f(M)$ funksiyaning A nuqtaning atrofida B dan cheksiz kichik funksiyaga farq qilishi, ya'ni $f(M)$ funksiya A nuqtaning atrofida B ga asimptotik yaqinlashar ekan degan xulosa kelib chiqadi.

Cheksiz kichik funksiyalar quyidagi xossalarga ega:

1^0. Agar $M \to A$ da $\alpha(M)$ va $\beta(M)$ funksiyalar cheksiz kichik funksiyalar bo'lsa, u holda ularning yig'indisi $\alpha(M) + \beta(M)$ ham A nuqtada cheksiz kichik bo'ladi.

2^0. Agar $M \to A$ da $\alpha(M)$ funksiya cheksiz kichik bo'lib, $\beta(M)$ funksiya chegaralangan bo'lsa, u holda $\alpha(M)$, $\beta(M)$ funksiya cheksiz kichik bo'ladi.

10.7.7-ta'rif. Agar $h(M)$ funksiya $\{M\}$ to'plamda aniqlangan bo'lib, $\lim\limits_{M \to A} h(M) = \infty$ bo'lsa, $h(M)$ funksiya $M \to A$ da *cheksiz katta funksiya* deyiladi.

3^0. Agar $M \to A$ da $\alpha(M)(\alpha(M) \neq 0)$ cheksiz kichik funksiya bo'lsa, u holda $\dfrac{1}{\alpha(M)}$ funksiya $M \to A$ da cheksiz katta funksiya funksiya bo'ladi.

4^0. Agar $M \to A$ da $h(M)$ cheksiz katta funksiya bo'lsa, $\dfrac{1}{h(M)}$ funksiya $M \to A$ da cheksiz kichik funksiya bo'ladi.

10.7.8-teorema (Koshi kritetiyasi). $u = f(M)$ funksiyaning limitga ega bo'lishi uchun $\forall \varepsilon > 0$ son olinganda ham $\exists \delta > 0$ mavjud bo'lib, ushbu

$$0 < \rho(M', A) < \delta, \ 0 < \rho(M'', A) < \delta$$

tengsizliklarni qanoatlantiruvchi barcha $M', M'' \in \{M\}$ nuqtalarda

$$|f(M') - f(M'')| < \varepsilon$$

tengsizlikning bajarilishi zarur va yetarli.

Bu teoremaning isboti ham bir o'zgaruvchili funksiyaning limitga ega bo'lishi haqidagi Koshi kriteriyasining isboti kabi bo'ladi.

10.8. Takroriy limitlar. Yuqorida biz $u = f(M) = f(x_1, x_2, ..., x_m)$ funksiyaning $A = A(a_1, a_2, ..., a_m)$ nuqtadagi limiti

$$\lim_{M \to A} f(M) = B \quad \left(\lim_{\substack{x_1 \to a_1 \\ x_2 \to a_2 \\ \dots \\ x_m \to a_m}} f(x_1, x_2, \dots, x_m) = B \right)$$

bilan tanishgan edik. Bundan, funksiyaning limiti, uning argumentlari $x_1, x_2, \dots, x_m$ larning birdaniga mos ravishda $a_1, a_2, \dots, a_m$ sonlarga intilgandagi limitidan iborat ekanligiga ishonch hosil qilamiz. Biz bundan keyin bu limitni *karrali limit* deb ataymiz. Faqat ko'p o'zgaruvchili funksiyalarga xos bo'lgan boshqa ko'rinishdagi limit tushunchasini ham kiritamiz: har bir argumenti bo'yicha (qolganlarini o'zgarmas deb qarab) ham limit tushunchasini kiritish mumkin. Bundan, takroriy limit deb ataluvchi limit tushunchasi kelib chiqadi.

$u = f(M)$ funksiya $\{M\} \subset R^m$ to'plamda berilgan bo'lib, $A = A(a_1, a_2, \dots, a_m)$ nuqta $\{M\}$ to'plamning limit nuqtasi bo'lsin.

Berilgan funksiyaning $x_1 \to a_1$ (qolgan barcha argumentlarini tayinlab) dagi limiti mavjud bo'lsin deb

$$\lim_{x_1 \to a_1} f(x_1, x_2, \dots, x_m)$$

ni qaraylik, bu limit $x_2, x_3, \dots, x_m$ o'zgaruvchilarga bog'liq bo'ladi:

$$\lim_{x_1 \to a_1} f(x_1, x_2, \dots, x_m) = \varphi_1(x_2, x_3, \dots, x_m).$$

Endi $\varphi_1(x_2, x_3, \dots, x_m)$ funksiyaning x_2 argumentni o'zgaruvchi (qolgan barcha argumentlarini tayinlab) uning $x_2 \to a_2$ dagi limiti mavjud deb

$$\lim_{x_2 \to a_2} \varphi_1(x_2, x_3, \dots, x_m)$$

ni qaraymiz. Bu limit $x_3, x_4, \dots, x_m$ o'zgaruvchilarga bog'liq bo'ladi:

$$\lim_{x_2 \to a_2} \varphi_1(x_2, x_3, \dots, x_m) = \varphi_2(x_3, x_4, \dots, x_m).$$

Xuddi shunday birin-ketin $x_3 \to a_3$, $x_4 \to a_4$, ... , $x_m \to a_m$ da limitga o'tib (bu limitlarni mavjud deb), natijada

$$\lim_{x_1 \to a_1} \lim_{x_2 \to a_2} \dots \lim_{x_m \to a_m} f(x_1, x_2, \dots, x_m)$$

ni hosil qilamiz. Bu limitga $f(x_1, x_2, \dots, x_m)$ funksiyaning *takroriy limiti* deb ataladi.

Ravshanki, $f(x_1,x_2,...,x_m)$ funksiyaning argumentlari $x_1,x_2,...,x_m$ lar mos ravishda $a_1,a_2,...,a_m$ sonlarga turli tartibda intilganda funksiyaning turli takroriy limitlari hosil bo'ladi.

Masalan, $f(x_1,x_2,...,x_m)$ funksiyaning $x_{i_1},x_{i_2},...,x_{i_k}$ argumentlari mos ravishda $a_{i_1},a_{i_2},...,a_{i_k}$ larga intilgandagi takroriy limitlari

$$\lim_{x_{i_1}\to a_{i_1}}\lim_{x_{i_2}\to a_{i_2}}...\lim_{x_{i_k}\to a_{i_k}} f(x_1,x_2,...,x_m)$$

ni qarash ham mumkin.

Biz quyida funksiyaning karrali limiti bilan takroriy limitlar orasidagi bog'lanish hamda ularning ma'lum shartlarda o'zaro tengligi haqidagi teoremani keltiramiz va uni isbotlaymiz. $u=f(M)$ funksiya

$$\{M\}=\{(x_1,x_2)\in R^2, \left|x_1-x_1^0\right|<d_1, \quad \left|x_2-x_2^0\right|<d_2\}$$

to'plamda berilgan bo'lsin.

10.8.1-teorema. Agar 1) $M(x_1,x_2)\to M_0(x_1^0\cdot x_2^0)$ da $f(x_1,x_2)$ funksiyaning karrali limiti mavjud:

$$\lim_{\substack{x_1\to x_1^0\\ x_2\to x_2^0}} f(x_1,x_2)=B.$$

2) Har bir tayinlangan x_1 da quyidagi $\lim\limits_{x_2\to x_2^0} f(x_1,x_2)=\varphi(x_1)$ limit, har bir tayinlangan x_2 da $\lim\limits_{x_1\to x_1^0} f(x_1,x_2)=\varphi(x_2)$ limit mavjud bulsa, u holda $\lim\limits_{x_1\to x_1^0}\lim\limits_{x_2\to x_2^0} f(x_1,x_2)$ va $\lim\limits_{x_2\to x_2^0}\lim\limits_{x_1\to x_1^0} f(x_1,x_2)$ takroriy limitlar mavjud bo'ladi va ular B ga teng bo'ladi.

10.8.2-natija. 10.8.1-teoremaning shartlari bajarilganda

$$\lim_{\substack{x_1\to x_1^0\\ x_2\to x_2^0}} f(x_1,x_2)=\lim_{x_1\to x_1^0}\lim_{x_2\to x_2^0} f(x_1,x_2)=\lim_{x_2\to x_2^0}\lim_{x_1\to x_1^0} f(x_1,x_2)$$

munosabat o'rinli bo'ladi. Xuddi yuqoridagidek $u=f(x_1,x_2,...,x_m)$ funksiyaning $x_{i_1},x_{i_2},...,x_{i_k}$ o'zgaruvchilari bo'yicha ham

$$\lim_{\substack{x_{i_1}\to x_{i_1}^0\\ x_{i_2}\to x_{i_2}^0\\ \dots\dots\\ x_{i_k}\to x_{i_k}^0}} f(x_1,x_2,...,x_m)$$

karrali hamda

$$\lim_{x_{i_1}\to x_{i_1}^0}\lim_{x_{i_2}\to x_{i_2}^0}\dots\lim_{x_{i_m}\to x_{i_m}^0} f(x_1,x_2,\dots,x_m)$$

takroriy limitlari va ular orasidagi bog'lanishlarni qarash mumkin.

10.3-§. Ko'p o'zgaruvchili funksiyalarning uzkulsizligi

10.9. Ko'p o'zgaruvchili funksiyalarning uzluksizligi. $u=f(M)=f(x_1,x_2,\dots,x_m)$ funksiya $\{M\}\subset R^m$ to'plamda aniqlangan bo'lib, $A(a_1,a_2,\dots,a_m)$ nuqta $\{M\}$ to'plamning limit nuqtasi bo'lib, u $A\in\{M\}$ bo'lsin.

10.9.1-ta'rif. Agar $\exists\lim\limits_{M\to A} f(M)=f(A)$ bo'lsa, u holda $f(M)$ funksiya A *nuqtada uzluksiz* deyiladi.

$A=\lim\limits_{M\to A} M$ ekanligini e'tiborga olsak funksiyaning uzluksizlik shartini ushbu

$$\lim_{M\to A} f(M)=f(\lim_{M\to A} M)$$

ko'rinishida ham yozish mumkin.

10.9.2-ta'rif. Agar $\forall\varepsilon>0$ son uchun $\exists\delta=\delta(\varepsilon)$ son topilsaki, ushbu $\rho(M,A)<\varepsilon$ tengsizlikni qanoatlantiruvchi barcha $M\in\{M\}$ nuqtalarda

$$|f(M)-f(A)|<\varepsilon$$

tengsizlik bajarilsa, $f(M)$ funksiya A *nuqtada uzluksiz* deyiladi.

10.9.1-, 10.9.2-ta'riflar o'zaro ekvivalent ta'riflardir.

10.9.3-ta'rif. Agar $f(M)$ funksiya $\{M\}$ to'plamning har bir nuqtasida uzluksiz bo'lsa, u holda u $\{M\}$ *to'plamda uzluksiz* deyiladi.

Ushbu

$$\Delta u=f(M)-f(A) \tag{10.9.4}$$

ayirmaga $f(M)$ funksiya A nuqtadagi *ortirmasi (to'liq)* deyiladi, bunda $\forall M\in\{M\}$.

Agar $x_1-a_1=\Delta x_1$, $x_2-a_2=\Delta x_2$, … , $x_m-a_m=\Delta x_m$ deb belgilasak, u holda (10.9.4) ni quyidagi

$$\Delta u=f(a_1+\Delta x_1,a_2+\Delta x_2,\dots,a_m+\Delta x_m)-f(a_1,a_2,\dots,a_m)$$

ko'rinishida ham yozish mumkin.

Ravshanki, $\Delta u = f(M) - f(A)$ funksiyaning A nuqtadagi uzluksizlik sharti $\lim\limits_{M \to A}(f(M) - f(A)) = 0$ yoki $\lim\limits_{\substack{\Delta x_1 \to 0 \\ \Delta x_2 \to 0 \\ \dots\dots \\ \Delta x_m \to 0}} \Delta u = 0$ shartga ekvivalent.

$u = f(x_1, x_2, \dots, x_m)$ funksiyaning $x_k (k = \overline{1,m})$ argumentini o'zgaruvchi deb, qolgan argumentlarini o'zgarmas deb x_k argumentiga ixtiyoriy Δx_k ortirma beramiz. U holda $u = f(x_1, x_2, \dots, x_m)$ funksiya ham ortirma oladi, ya'ni

$$\Delta_{x_k} u = f(x_1, x_2, \dots, x_{k-1}, x_k + \Delta x_k, x_{k+1}, \dots, x_m) - f(x_1, x_2, \dots, x_m)$$

bu ayirmani funksiyaning A nuqtadagi xususiy ortirmasi deyiladi.

10.9.5-ta'rif. Agar

$$\lim_{\Delta x_k \to 0} \Delta_{x_k} u = 0$$

bo'lsa, u holda $u = f(x_1, x_2, \dots, x_m)$ funksiya A nuqtada x_k *arumenti bo'yicha uzluksiz* deyiladi.

$u = f(x_1, x_2, \dots, x_m)$ funksiyaning x_k arumentidan boshqa qolgan barcha argumentlari o'zgarmas deb olinganda funksiya bir argumentli funksiyaga aylanadi. Shu sababli, funksiyaning x_k arumenti bo'yicha uzluksizligi, bir argumentli funksiyaning uzluksizligini anglatadi.

Ravshanki, $u = f(x_1, x_2, \dots, x_m)$ funksiyaning berilgan A nuqtadagi uzluksizligidan, funksiyaning shu nuqtadagi har bir argumentlari bo'yicha uzluksizligi kelib chiqadi, ya'ni quyidagi tasdiq o'rinli bo'ladi:

Agar $u = f(x_1, x_2, \dots, x_m)$ funksiya A nuqtaning biror atrofida aniqlangan va A nuqtada uzluksiz bo'lsa, u holda u bu nuqtada har bir $x_1, x_2, \dots, x_m$ argumentlari bo'yicha ham uzluksiz bo'ladi.

Bu tasdiqning teskarisi o'rinli emas. Masalan, ushbu

$$f(x_1, x_2) = \begin{cases} \dfrac{2x_1 x_2}{x_1^2 + x_2^2}, x_1^2 + x_2^2 \neq 0, \\ 0, x_1^2 + x_2^2 = 0 \end{cases}$$

funksiyani qaraylik. $x_2 \neq 0$ va x_0 - berilgan son bo'lsin. U holda

$$\lim_{x_1 \to x_0} f(x_1, x_2) = \lim_{x_1 \to x_0} \frac{2x_1 x_2}{x_1^2 + x_2^2} = \frac{2x_0 x_2}{x_0^2 + x_2^2} = f(x_0, x_2).$$

Agar $x_2 = 0$ bo'lsa, u holda $\forall x_0 \neq 0$ da $\lim\limits_{x_1 \to x_0} f(x_1, 0) = f(x_0, 0) = 0$, agar $x_2 = 0$ va $x_1 = 0$ bo'lsa, $\lim\limits_{x_1 \to 0} f(x_1, 0) = f(0,0) = 0$.

Demak, har bir berilgan x_2 larda $f(x_1, x_2)$ funksiya x_1 bo'yicha uzluksiz ekan. Xuddi shunday har bir berilgan x_1 larda x_2 bo'yicha uzluksizligi ko'rsatiladi. Lekin bu funksiya (0, 0) nuqtada hamma argumentlari bo'yicha birdaniga uzluksiz emas.

Haqiqatdan ham, ikkita $\left\{\frac{1}{n}, \frac{1}{n}\right\}$ va $\left\{\frac{2}{n}, \frac{1}{n}\right\}$, $n \to \infty$ da (0, 0) nuqtaga intilvchi ketma-ketlik olinganda, unga mos kelgan $\left\{f\left(\frac{1}{n}, \frac{1}{n}\right)\right\}$, $\left\{f\left(\frac{2}{n}, \frac{1}{n}\right)\right\}$ ketma-ketliklar $n \to \infty$ da har xil limitga ega:

$$f\left(\frac{1}{n}, \frac{1}{n}\right) = \frac{\frac{2}{n^2}}{\frac{1}{n^2} + \frac{1}{n^2}} \to 1, \quad f\left(\frac{2}{n}, \frac{1}{n}\right) = \frac{\frac{4}{n^2}}{\frac{4}{n^2} + \frac{1}{n^2}} \to \frac{4}{5}.$$

10.10. Uzluksiz funksiyalar ustida arifmetik amallar. $u = f(M)$ va $v = g(M)$ funksiyalar $\{M\} \subset R^m$ aniqlangan. $A \in R^m$ nuqta esa, $\{M\}$ to'plamning limit nuqtasi bo'lib, $A \in \{M\}$.

10.10.1-teorema. Agar $f(M)$ va $g(M)$ funksiyalar $A \in \{M\}$ nuqtada uzluksiz bo'lsa, $f(M) \pm g(M)$, $f(M)\, g(M)$, $\frac{f(M)}{g(M)} (g(A) \neq 0)$ funksiyalar ham shu nuqtada uzluksiz bo'ladi.

10.11. Murakkab funksiyaning uzluksizligi. $u = f(M) = f(x_1, x_2, .., x_m)$ funksiya $\{M\} \subset R^m$ to'plamda berilgan bo'lib, $x_1, x_2, ..., x_m$ o'zgaruvchilarning har biri $\{N\} \subset R^k$ to'plamda beilgan funksiyalar bo'lsin:

$$\begin{cases} x_1 = \varphi_1(t_1, t_2, ..., t_m), \\ x_2 = \varphi_2(t_1, t_2, ..., t_m), \\ \dots\dots\dots\dots\dots\dots \\ x_m = \varphi_m(t_1, t_2, ..., t_m). \end{cases} \qquad (10.11.1)$$

U holda har bir $N(t_1, t_2, ..., t_k) \in \{N\}$ nuqtaga (10.11.1) formula yordamida $M(x_1, x_2, ..., x_m) \in \{M\}$ nuqta mos qo'yiladi. Bu holda, $\{N\} \subset R^k$ to'plamda

$$u=f(\varphi_1(t_1,t_2,...,t_k),...,\varphi_m(t_1,t_2,...,t_k))=\Phi(t_1,t_2,...,t_k)$$

murakkab funksiya aniqlangan bo'ladi.

10.11.2-teorema. Agar $x_i=\varphi_i(t_1,t_2,...,t_k)$ $(i=\overline{1,m})$ funksiyaning har biri $A(a_1,a_2,...,a_k)$ nuqtada uzluksiz, $u=f(x_1,x_2,...,x_m)$ funksiya esa, $B(b_1,b_2,...,b_m)(b_i=\psi(a_1,a_2,...,a_k),(i=\overline{1,m}))$ nuqtada uzluksiz bo'lsa, u holda $\Phi(t_1,t_2,...,t_k)$ murakkab funksiya $A(a_1,a_2,...,a_k)$nuqtada uzluksiz bo'ladi.

10.12. Ko'p o'zgaruvchili uzluksiz funksiyalarning local xossalari. Ko'p o'zgaruvchili uzluksiz funksiyalar ham xuddi bir o'zgaruvchili uzluksiz funksiyalar kabi xossalarga ega. Bu xossalarning isboti bir o'zgaruvchili uzluksiz funksiyalarning isboti kabi bo'lganligi sababli, ularni isbotsiz keltiramiz.

$u=f(M)$ funksiya $\{M\}\subset R^m$ to'plamda berilgan bo'lsin. Bu to'plamdan $M_0(x_1^0,x_2^0,...,x_m^0)\in\{M\}$ nuqtani olib, uning yetarli kichik atrofini qaraymiz.

1⁰. Agar $f(M)$ funksiya M_0 nuqtada uzluksiz bo'lsa, u holda u M_0 nuqtaning yetarli kichik atrofida chegaralangan bo'ladi.

2⁰. Agar $f(M)$ funksiya M_0 nuqtada uzluksiz bo'lib, $f(M_0)>0$ $(f(M_0)<0)$ bo'lsa, M_0 nuqtaning yetarli kichik atrofidagi $\forall M$ nuqtalarda ham $f(M)>0$ $(f(M_0)<0)$ bo'ladi.

3⁰. Agar $f(M)$ funksiya M_0 nuqtada uzluksiz bo'lsa, u holda M_0 nuqtaning yetarli kichik atrofidagi ixtiyoriy $M_1\in\{M\}, M_2\in\{M\}$ nuqtalar uchun

$$|f(M_1)-f(M_2)|<\varepsilon$$

(ε - ixtiyoriy istalgancha kichik son) tengsizlik o'rinli bo'ladi.

10.13. Ko'p o'zgaruvchili funksiyalarning global xossalari

10.13.1-ta'rif. Agar shunday c va C sonlar mavjud bo'lib, $\forall M\in\{M\}$ uchun $c\leq f(M)\leq C$ tengsizlik o'rinli bo'lsa, $u=f(M)$ funksiya shu to'plamda chegaralangan deyiladi.

1⁰ (Bolsano – Koshining birinchi teoremasi). $u=f(M)$ funksiya $\{M\}\subset R^m$ bog'lamli to'plamda berilgan va uzluksiz bo'lsin. Agar berilgan funksiya $\{M\}$ to'plamning ikkita $A(a_1,a_2,...,a_m)$ va $B(b_1,b_2,...,b_m)$nuqtalarida har xil ishorali qiymatlarga ega bo'lsa, u holda shunday $C(c_1,c_2,...,c_m)\in\{M\}$ nuqta topiladiki, shu nuqtada funksiya nolga aylanadi, ya'ni $f(C)=f(c_1,c_2,...,c_m)=0$.

2^0 ***(Bolsano – Koshining ikkinchi teoremasi).*** $u=f(M)$ funksiya $\{M\}\subset R^m$ bog'lamli to'plamda berilgan va uzluksiz bo'lib, $\{M\}$ to'plamning ikkita $A(a_1,a_2,...,a_m)$ va $B(b_1,b_2,...,b_m)$ nuqtalarida $f(A)\neq f(B)$ bo'lsa, u holda $f(A)$ va $f(B)$ qiymatlar orasida har qanday C son olinsa ham, $\{M\}$ to'plamda shunday $c(c_1,c_2,...,c_m)$ nuqta topiladiki,

$$f(c)=f(c_1,c_2,...,c_m)=C$$

bo'ladi.

3^0 ***(Veyishtrassning birinchi teoremasi).*** Agar $f(M)$ funksiya $\{M\}$ $(\{M\}\subset R^m)$–chegaralangan yopiq to'plamda berilgan va uzluksiz bo'lsa, bu funksiya shu $\{M\}$ to'plamda chegaralangan bo'ladi.

4^0 ***(Veyishtrassning ikkinchi teoremasi).*** Agar $f(M)$ funksiya chegaralangan yopiq $\{M\}\subset R^m$ to'plamda berilgan va uzluksiz bo'lsa, u shu to'plamda o'zining aniq yuqori hamda aniq kuyi chegarasiga erishadi.

10.4-§. Ko'p o'zgaruvchili funksiyalarning hosila va differensiali

10.14. Ko'p o'zgaruvchili funksiyalarning xususiy hosilalari. $u=f(M)=f(x_1,x_2,...,x_m)$ funksiya $\{M\}\subset R^m$ to'plamda berilgan bo'lib, $M(x_1,x_2,...,x_m)\{M\}$ to'plamning ichki nuqtasi bo'lsin. Bu $\{M\}\subset R^m$ to'plamning $M(x_1,x_2,...,x_m)$ nuqtasini olib, uning x_k koordinatasiga $\Delta x_k\neq 0$ orttirma (qolganlarini o'zgarmas deb) belgilaylikki $M_1(x_1,x_2,...,x_{k-1},x_k+\Delta x_k,x_{k+1},...,x_m)\in\{M\}$ bo'lsin. Natijada $u=f(x_1,x_2,...,x_n)$ funksiya ham $M(x_1,x_2,...,x_m)$ nuqtada x_k argumenti bo'yicha xususiy $\Delta_{x_k}u$ orttirma oladi:

$$\Delta_{x_k}u=f(x_1,x_2,...,x_{k-1},x_k+\Delta x_k,x_{k+1},...,x_m)-f(x_1,x_2,...,x_m). \quad (10.14.1)$$

Uning xususiy $\Delta_{x_k}u$ orttirmasi (10.14.1) formula orqali ifoda qilinadi. (10.14.1) ning ikkala tomonini Δx_k ga bo'lamiz, natijada

$$\frac{\Delta_{x_k}u}{\Delta_{x_k}k}=\frac{f(x_1,x_2,...,x_{k-1},x_k+\Delta x_k,x_{k+1},...,x_m)-f(x_1,x_2,...,x_m)}{\Delta x_k} \quad (10.14.2)$$

nisbatga ega bo'lamiz.

10.14.3-ta'rif. Agar (10.14.2) nisbatning $\Delta x_k \to 0$ dagi limiti mavjud va chekli bo'lsa, bu limitga $u = f(x_1, x_2, ..., x_m)$ funksiyaning $M(x_1, x_2, ..., x_m)$ nuqtadagi x_k argumenti bo'yicha *xususiy hosilasi* deyiladi va u

$$\frac{\partial f(x_1, ..., x_m)}{\partial x_k}, \ \frac{\partial f}{\partial x_k}, \ u'_{x_k}, \ f'_{x_k}(x_1, x_2, ..., x_m), \ f'_{x_k}$$

kabi belgilanadi.

Shunday qilib, ta'rifga ko'ra,

$$\frac{\partial u}{\partial x_k} = \lim_{\Delta x_k \to 0} \frac{\Delta_{x_k} u}{\Delta x_k} \qquad (10.14.4)$$

bo'ladi.

10.14.5-misol. 10.14.3-ta'rifdan foydalanib, ushbu

$$f(x, y) = 3^{2x+y}$$

funksiyaning $O(0,0)$ nuqtadagi f'_x, f'_y xususiy hosilalarini hisoblang.

Yechilishi. 10.14.3-ta'rifga ko'ra, f'_x, f'_y xususiy hosilalarini topamiz:

$$\frac{\partial f(0,0)}{\partial x} = \lim_{\Delta x \to 0} \frac{f(0+\Delta x, 0) - f(0,0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{3^{2\Delta x} - 1}{\Delta x} = 2\ln 3 = \ln 9,$$

$$\frac{\partial f(0,0)}{\partial y} = \lim_{\Delta y \to 0} \frac{f(0, 0+\Delta y) - f(0,0)}{\Delta y} = \lim_{\Delta y \to 0} \frac{3^{\Delta y} - 1}{\Delta y} = \ln 3.$$

Demak, $f'_x(0,0) = \ln 9$, $f'_y(0,0) = \ln 3$.

10.14.6-eslatma. $u = f(x_1, x_2, ..., x_m)$ funksiyaning x_k argumenti bo'yicha xususiy hosilasi, bir argumentli funksiya hosilasi singari bo'ladi. Shuning uchun xususiy hosilasi hisoblanganda, oddiy hosilani hisoblash qoidasiga rioya qilinadi.

10.14.7-eslatma. Berilgan nuqtada funksiyaning hamma xususiy hosilalarining mavjudligidan uning shu nuqtada uzluksizligi har doim ham kelib chiqavermaydi. Masalan, ushbu

$$u = \begin{cases} \dfrac{xy}{x^2 + y^2}, x^2 + y^2 \neq 0 bo'lg\,a\,nda, \\ 0, \quad x^2 + y^2 = 0 bo'lg\,a\,nda. \end{cases}$$

Ma'lumki, bu funksiya $M_0(0,0)$ nuqtada uzluksiz emas, lekin bu funksiya ko'rsatilgan nuqtada x va y argumentlari bo'yicha xususiy hosilalari mavjud:

$$\left.\frac{\partial u}{\partial x}\right|_{(0,0)} = 0; \quad \left.\frac{\partial u}{\partial y}\right|_{(0,0)} = 0,$$

chunki $f(x,0)=0$, $f(0,y)=0$.

10.14.8-eslatma. Agar M nuqta funksiya aniqlanish sohasi chegarasida bo'lsa, u holda bu nuqtada funksiya xususiy hosilasi tarifi o'rinli bo'lmaydi.

10.15. Ko'p o'zgaruvchili funksiyaning differensiallanuvchiligi. $u=f(M)=f(x_1,x_2,...,x_m)$ funksiya $\{M\}\subset R^m$ to'plamda aniqlangan bo'lib, uning $\{M\}$to'plamning ichki $M(x_1,x_2,...,x_m)$ nuqtasidagi Δu to'liq orttirmasini qaraylik:

$$\Delta u = f(x_1+\Delta x_1, x_2+\Delta x_2,...,x_m+\Delta x_m) - f(x_1,x_2,...,x_m).$$

10.15.1-ta'rif. Agar $u=f(M)$funksiyaning M nuqtadagi Δu to'liq orttirmasi ushbu

$$\Delta u = A_1\Delta x_1 + A_2\Delta x_2 + ... + A_m\Delta x_m + \alpha_1\Delta x_1 + \alpha_2\Delta x_2 + ... + \alpha_m\Delta x_m \qquad (10.15.2)$$

ko'rinishda tasvirlansa (bunda $A_1,A_2,...,A_m$ lar mos ravishda $\Delta x_1,\Delta x_2,...,\Delta x_m$ larga bog'liq bo'lmagan sonlar, $\alpha_1,\alpha_2,...,\alpha_m$ lar esa, $\Delta x_1\to 0,\Delta x_2\to,...,\Delta x_m\to 0$ da cheksiz kichik ifodalar bo'lib, ular$\Delta x_1=\Delta x_2=...=\Delta x_m=0$ da nolga teng deb olinadi), u holda u $M(x_1,x_2,...,x_m)$ nuqtada *differensialanuvchi* deyiladi.

(10.15.2) ni boshqacha ham yozish mumkin. Funksiyaning $M(x_1,x_2,...,x_m)$ nuqtadagi differensiallanuvchilik sharti (10.15.2) ni quyidagicha

$$\Delta u = A_1\Delta x_1 + A_2\Delta x_2 + ... + A_m\Delta x_m + o(\rho) \qquad (10.15.3)$$

ko'rinishida ham ifodalash mumkinligini ko'rsatamiz, bundaρ, $M(x_1,x_2,...,x_m)$ va $M_1(x_1+\Delta x_1,x_2+\Delta x_2,...,x_m+\Delta x_m)$ nuqtalar orasidagi masofa:

$$\rho=\sqrt{(\Delta x_1)^2+(\Delta x_2)^2+...+(\Delta x_m)^2},$$

bunda $\Delta x_1\to 0,\Delta x_2\to,...,\Delta x_m\to 0$ da $\rho\to 0$. Endi $\Delta x_1\to 0,\Delta x_2\to,...,\Delta x_m\to 0$ da (10.15.2) dagi $\alpha_1\Delta x_1+\alpha_2\Delta x_2+...+\alpha_m\Delta x_m$ miqdorning ρ ga nisbatan yuqori tartibli cheksiz kichik miqdor ekanligini ko'rsatamiz. $\rho\neq 0$ bo'lganda $\frac{|\Delta x_i|}{\rho}\leq 1$ $(i=\overline{1,m})$ tengsizlik o'rinli, shuning uchun

$$|\alpha_1\Delta x_1+\alpha_2\Delta x_2+...+\alpha_m\Delta x_m|\le\rho\left\{|\alpha_1|\frac{|\Delta x_1|}{\rho}+|\alpha_2|\frac{|\Delta x_2|}{\rho}+...+|\alpha_m|\frac{|\Delta x_m|}{\rho}\right\}\le$$
$$\le\{|\alpha_1|+|\alpha_2|+...+|\alpha_m|\}=o(\rho).$$

Shunday qilib differensiallanuvchi funksiyaning (10.15.2) differensiallanuvchilik shartini (10.15.3) ko'rinishida ham ifodalash mumkin ekan. (10.15.2) shart bilan (10.15.3) o'zaro ekvivalent.

10.15.4-teorema. Agar $u=f(M)$ funksiya M nuqtada $(M\in\{M\}\subset R^m)$ diferensiallanuvchi bo'lsa, u holda uning barcha argumentlari bo'yicha M nuqtadagi xususiy hosilalari mavjud va $\frac{\partial u(M)}{\partial x_i}=A_i\,(i=\overline{1,m})$ bo'ladi, bunda $A_i\,(i=\overline{1,m})$ (10.15.2) yoki (10.15.3) shartdan aniqlanadi.

Isboti. Shartga ko'ra $u=f(M)$ funksiya M nuqtada differansiallanuvchi bo'lgani uchun (10.15.2) shartdan

$$\Delta x_i\ne 0,\ \Delta x_1=\Delta x_2=...=\Delta x_{i-1}=\Delta x_{i+1}=...=\Delta x_m=0$$

deb olsak, u holda (10.15.2) dan $\Delta_{x_i}u=A_i\Delta x_i+\alpha_i\Delta x_i$, bundan $\frac{\Delta_{x_i}u}{\Delta x_i}=A_i+\alpha_i(\Delta x_i)$, $\Delta x_i\to 0$ da $\alpha_i(\Delta x_i)\to 0$ bo'lgani uchun

$$\lim_{\Delta x_i\to 0}\frac{\Delta_{x_i}u}{\Delta x_i}=A_i,\ \frac{\partial u}{\partial x_i}=A_i\,(i=\overline{1,m}).\blacksquare$$

10.15.5-natija. $u=f(M)$ funksiyaning M nuqtadagi (10.15.3) differensiallanuvchanlik shartini

$$\Delta u=\frac{\partial u}{\partial x_1}\Delta x_1+\frac{\partial u}{\partial x_2}\Delta x_2+...+\frac{\partial u}{\partial x_m}\Delta x_m+o(\rho)$$

ko'rinishda yozish mumkin.

10.5.6-teorema. Agar $u=f(x_1,x_2,...,x_m)$ funksiya M nuqtada differensiallanuvchi bo'lsa, u shu nuqtada uzluksiz bo'ladi.

Isboti. $u=f(x_1,x_2,...,x_m)$ funksiya M nuqtada differensiallanuvchi bo'lsin. U holda, 10.15.1-ta'rifdagi (10.15.2) shart o'rinli bo'ladi. (10.15.3) dan

$$\lim_{\substack{\Delta x_1 \to 0 \\ \Delta x_2 \to 0 \\ \dots\dots \\ \Delta x_m \to 0}} \Delta u = 0,$$

bundan $u = f(M)$funksiyaning M nuqtada uzluksizligi kelib chiqadi.■

10.5.7-teorema.Agar $u = f(x_1, x_2, ..., x_m)$ funksiya $\{M\} \subset R^m$da aniqlangan bo'lib, funksiya $M_0(x_1^0, x_2^0, ..., x_m^0) \in \{M\}$ nuqtaning biror atrofida hamma argumentlari bo'yicha xususiy hosilalarga ega bo'lib, bu xususiy hosilalar M_0 nuqtada uzluksiz bo'lsa, u shu nuqtada differensiallanuvchi bo'ladi.

Shunday qilib, 10.15-banddagi teoremalardan quidagi xulosalarni chiqarish mumkin:

1) Bir o'zgaruvchili funksiyalarda ham, ko'p o'zgaruvchili funksiyalarda ham funksiyaning biror nuqtada differensiallanuvchi bo'lishidan uning chu nuqtada uzluksiz bo'lishi kelib chiqadi.

2) Ma'lumki, bir o'zgaruvchi funksiyalarda funksiyaning biror nuqtada differensiallanuvchi bo'lishidan uning shu nuqtada chekli hosilaga ega bo'lishi kelib chiqadi va aksincha funksiyaning biror nuqtada chekli hosilaga ega bo'lishidan uning shu nuqtada differensiallanuvchi bo'lishi kelib chiqadi.

3) Ko'p o'zgaruvchili funksiyalarda esa, funksiyaning biror nuqtada differensiallanuvchi bo'lishidan shu nuqtada barcha chekli xususiy hosilalarga ega bo'lishi kelib chiqadi, lekin funksiyaning biror nuqtada barcha chekli xususiy hosilalarga ega bo'lishidan uning shu nuqtada differensiallanuvchi bo'lishi har doim ham kelib chiqavermaydi

10.16. Ko'p o'zgaruvchili funksiyaning differensiali tushunchasi. $u = f(M) = f(x_1, x_2, ..., x_m)$ funksiya ochiq $\{M\} \subset R^m$ to'plamda aniqlangan bo'lib, u $M \in \{M\}$ nuqtada differensiallanuvchi bo'lsin. 10.15.1-ta'rifga ko'ra uning to'liq orttirmasi Δu uchun (10.15.3) formula o'rinli bo'ladi. Unda $A_i = \dfrac{\partial u(M)}{\partial x_i}\ (i = \overline{1, m})$. (10.15.3) tenglikning o'ng tomoni ikki qismdan iborat: 1) $\Delta x_1, \Delta x_2, ..., \Delta x_m$ larga nisbatan chiziqli $A_1\Delta x_1 + A_2\Delta x_2 + ... + A_m\Delta x_m$ ifodadan; 2) $\Delta x_1 \to 0, \Delta x_2 \to 0, .., \Delta x_m \to 0$ da $\alpha_1 \to 0, \alpha_2 \to 0, ..., \alpha_m \to 0$, yoki $\rho \to 0$da ρ ga nisbatan yuqori tartibli cheksiz kichik ifodalar $o(\rho)$ dan iborat.

10.16.1-ta'rif. $u = f(M)$ funksiya orttirmasi Δu ning $\Delta x_1, \Delta x_2, ..., \Delta x_m$ larga nisbatan chiziqli bosh qismi, $u = f(M)$ funksiyaning M nuqtadagi *differensiali* (*to'liq differensiali*) deb ataladi va du, df yoki $df(M)$ kabi belgilanadi.

Demak,

$$du = df = df(x_1, x_2, ..., x_m) = A_1\Delta x_1 + A_2\Delta x_2 + ... + A_m\Delta x_m.$$

Agar (10.15.2) yoki (10.15.3) da $A_i = \frac{\partial u(M)}{\partial x_i}$ $(i = \overline{1,m})$ larning hammasi $A_i = 0$ $(i = \overline{1,m})$ bo'lsa, u holda funksiyaning M nuqtadagi differensiali $df(M) = 0$ deb olinadi.

x_i – erkli o'zgaruvchining differensiali dx_i deb, $x_1, x_2, ..., x_m$ larga bog'liq bo'lmagan ixtiyoriy son tushuniladi, ya'ni bundan keyin, uni erkli o'zgaruvchi x_i ning orttormasi Δx_i deb olishga kelishib olamiz: $dx_i = \Delta x_i$ $(i = \overline{1,m})$. Bu kelishuvga asosan, $A_i = \frac{\partial u(M)}{\partial x_i}$ $(i = \overline{1,m})$ ekanligini e'tiborga olsak, funksiyaning M nuqtadagi differensialini

$$du = \frac{\partial u}{\partial x_1}dx_1 + \frac{\partial u}{\partial x_2}dx_2 + ... + \frac{\partial u}{\partial x_m}dx_m \qquad (10.16.2)$$

ko'rinishda yozish mumkin bo'ladi.

10.17. Ko'p o'zgaruvchili murakkab funksiyaning differensial-lanuvchiligi. $u = f(M) = f(x_1, x_2, ..., x_m)$ funksiya $\{M\} \subset R^m$ to'plamda berilgan bo'lib, $x_1, x_2, ..., x_m$ o'zgaruvchilarning har biri o'z navbatida $t_1, t_2, ..., t_k$ o'zgaruvchilarning funksiyasi sifatida $\{N\} \subset R^k$ to'plamida berilgan funksiyalar bo'lsin:

$$\begin{cases} x_1 = \varphi_1(t_1, t_2, ..., t_k), \\ x_2 = \varphi_2(t_1, t_2, ..., t_k), \\ .... \\ x_m = \varphi_m(t_1, t_2, ..., t_k), \end{cases} \qquad (10.17.1)$$

bunda $(t_1, t_2, ..., t_m) \in \{N\}$ bo'lganda unga mos kelgan $(x_1, x_2, ..., x_m) \in \{M\}$ bulsin. Natijada, ushbu

$$u = f(\varphi_1(t_1, t_2, ..., t_k), \varphi_2(t_1, t_2, ..., t_k), ..., \varphi_m(t_1, t_2, ..., t_k))$$

murakkab funksiyaga ega bo'lamiz.

10.17.2-teorema. Agar (10.17.1) fnksiyalarning har biri $N_0(t_1^0, t_2^0, ..., t_k^0) \in \{N\}$ nuqtada differensiallanuvchi bo'lib, $y = f(x_1, x_2, ..., x_m)$ funksiya esa unga mos $M_0(x_1^0, x_2^0, ..., x_k^0) \in \{M\}$

$$\left(x_1^0 = \varphi_1(t_1^0, t_2^0, ..., t_k^0), x_2^0 = \varphi_2(t_1^0, t_2^0, ..., t_k^0), ..., x_m^0 = \varphi_m(t_1^0, t_2^0, ..., t_k^0)\right)$$

nuqtada differensiallanuvchi bo'lsa, u holda murakkab funksiya $f(\varphi_1(t_1,t_2,\dots,t_k),\dots,\varphi_m(t_1,t_2,\dots,t_k))$ ham $N_0(t_1^0,t_2^0,\dots,t_k^0)$ nuqtada differensial-lanuvchi bo'ladi va uning xususiy hosilalari ushbu

$$\begin{aligned}
&\frac{\partial u}{\partial t_1}=\frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t_1}+\frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial t_1}+\dots+\frac{\partial u}{\partial x_m}\frac{\partial x_m}{\partial t_1},\\
&\frac{\partial u}{\partial t_2}=\frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t_2}+\frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial t_2}+\dots+\frac{\partial u}{\partial x_m}\frac{\partial x_m}{\partial t_2},\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
&\frac{\partial u}{\partial t_1}=\frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t_k}+\frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial t_k}+\dots+\frac{\partial u}{\partial x_m}\frac{\partial x_m}{\partial t_k}.
\end{aligned}\tag{10.17.3}$$

formula orqali topiladi.

10.17.8-eslatma. Xususiy holda, (10.17.1) dagi funksiyalarning har biri faqat bitta t ga bog'liq bo'lsa, u holda biz faqat t ga bog'liq bo'lgan $u=f(x_1,x_2,\dots,x_m)$, $x_i=\varphi_i(t)$ murakkab funksiyaga ega bo'lamiz. Bu murakkab funksiyaning hosilasi

$$\frac{du}{dt}=\frac{du}{dx_1}\cdot\frac{dx_1}{dt}+\frac{du}{dx_2}\cdot\frac{dx_2}{dt}+\dots+\frac{du}{dx_m}\cdot\frac{dx_m}{dt}\tag{10.17.9}$$

formula bilan topiladi.

10.18. Bir jinsli funksiya tushunchasi. $u=f(x_1,x_2,\dots,x_m)$ funksiya ochiq $\{M\}\subset R^m$ to'plamda aniqlangan bo'lib, $M(x_1,x_2,\dots,x_m)$ nuqta bilan $M_1(tx_1,tx_2,\dots,tx_m)\,(-\infty<t<\infty)$ nuqta ham $\{M\}$to'plamga tegishli bo'lsin.

10.18.1-ta'rif. Agar $\{M\}$ to'plamning har bir nuqtasida va har bir t lar uchun

$$f(tx_1,tx_2,\dots,tx_m)=t^p f(x_1,x_2,\dots,x_m)\tag{10.18.2}$$

tenglik bajarilsa, $\{M\}$ to'plamda $u=f(x_1,x_2,\dots,x_m)$ funksiya p-darajali *bir jinsli funksiya* deyiladi

Misollar. 1) $f(x,y)=x\dfrac{\sqrt{x^4+y^4}}{x-y}\ln\dfrac{x}{y}$ funksiya ikkinchi darajali bir jinsli funksiya bo'ladi. Haqiqatan ham, ya'ni

$$f(tx,ty)=tx\frac{\sqrt{(tx)^4+(ty)^4}}{tx-ty}\ln\frac{tx}{ty}=t^2f(x,y).$$

2) $f(x,y)=e^{\sin xy}$ funksiya uchun (10.18.2) shart bajarilmaydi.

Demak bu funksiya bir jinsi funksiya emas.

Agar $u=f(x_1,x_2,...,x_m)$ funksiya nolinchi tartibli birjinsli funksiya bo'lsa, ya'ni $f(tx_1,tx_2,...,tx_m)=f(x_1,x_2,...,x_m)$, u holda, $t=\frac{1}{x_1}$ deb olsak,

$$f(x_1,x_2,...,x_m)=f\left(1,\frac{x_2}{x_1},...,\frac{x_m}{x_1}\right)$$

ega bo'lamiz. Agar $n-1$ argumentli ushbu

$$\varphi(x_1,x_2,...,x_{n-1})=f(1,u_1,u_2,...,u_{n-1})$$

funksiyani kiritsak, u holda $f(x_1,x_2,...,x_n)=\varphi\left(\frac{x_2}{x_1},\frac{x_3}{x_1},...,\frac{x_n}{x_1}\right)$ bo'ladi.

Demak, har qanday nolinchi tartibli bir jinsli funksiya hamma argumentlarning ularning birortasiga nisbati shaklida tasvirlanar ekan.

Agar $u=f(x_1,x_2,...,x_m)$ funksiya m-tartibli bir jinsli funksiya bo'lsa, u holda, $\frac{f(x_1,x_2,...,x_m)}{x_1^m}$ funksiya nolinchi tartibli bir jinsli funksiy bo'ladi, ya'ni

$$\frac{f(x_1,x_2,...,x_m)}{x_1^m}=\varphi\left(\frac{x_2}{x_1},\frac{x_3}{x_1},...,\frac{x_m}{x_1}\right).$$

Shunday qilib, m-tartibli bir jinsli funksiyaning umumiy ko'rinishi

$$f(x_1,x_2,\ldots,x_m)=x_1^m\varphi\left(\frac{x_2}{x_1},\frac{x_3}{x_1},\ldots,\frac{x_m}{x_1}\right)$$

shaklida bo'lar ekan.

10.18.3-teorema (Bir jinsli funksiyalar haqida Eyler teoremasi). Agar $u=f(x_1,x_2,...,x_m)$ funksiya ochiq $\{M\}$ da differensiallanuvchi p-darajali bir jinsli funksiya bo'lsa, u holda $\{M\}$ to'plamning har bir $M(x_1,x_2,...,x_m)$ nuqtasida

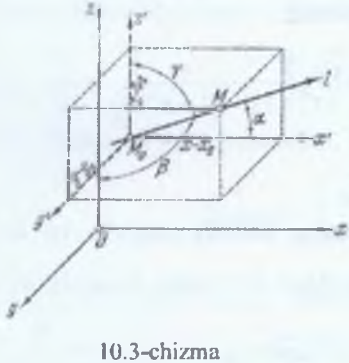
$$\frac{\partial u}{\partial x_1}x_1+\frac{\partial u}{\partial x_2}x_2+...+\frac{\partial u}{\partial x_m}x_m=pu \qquad (10.18.4)$$

tenglik o'rinli bo'ladi.

10.19. Yo'nalish bo'yicha hosila. Gradiyen. Ma'dumki, bir o'zgaruvchili $y=f(x)$ funksiyaning $(x\in R, y\in R)$ $\frac{df}{dx}$ hosilasi berilgan funksiyaning *o'zgarish tezligini* bildiradi. Ko'p o'zgaruvchili $u=f(x_1,x_2,...,x_m)=f(M)$ funksiyaning $((x_1,x_2,...,x_m)\in R^m, u\in R)$ Xususiy hosilalari ham bir o'zgaruvchili funksiyaning hosilasi kabi ekanligini e'tiborga olib, bu $\frac{\partial f}{\partial x_1},\frac{\partial f}{\partial x_2},...,\frac{\partial f}{\partial x_m}$ funksiyalar ham $u=f(x_1,x_2,...,x_m)$ funksiyaning mos ravishda $Ox_1,Ox_2,...,Ox_m$ o'qlar bo'yicha R^m da *o'zgarish tezligini* ifodalaydi deb qarash mumkin. Endi ko'p o'zgaruvchili funksiyaning ixtiyoriy yo'nalish bo'yicha o'zgarish tezligini ifodalovchi tushuncha bilan tanishamiz. Soddalik uchun uch o'zgaruvchili funksiyani qaraymiz. $u=f(x,y,z)=f(M)$ funksiya ochiq $\{M\}\subset R^3$ to'plamda berilgan bo'lsin. Birlik vektori $\vec{r}(\cos\alpha,\cos\beta,\cos\gamma)$ bilan aniqlanadigan biror ℓ yo'nalishni qaraylik. $\{M\}$ to'plamda ixtiyoriy $M_0(x_0,y_0,z_0)$ nuqtani olib, bu nuqta orqali o'tuvchi, yo'nalishi $\vec{r}(\cos\alpha,\cos\beta,\cos\gamma)$ vektor yo'nalishga mos kelgan $\vec{\ell}$ yo'nalishni qaraylik (10.3-chizma). $M_0(x_0,y_0,z_0)$ nuqtaga yaqin $\vec{\ell}$ yo'nalishda yotgan ixtiyoriy o'zgaruvchi $M(x,y,z)$ nuqtani olamiz $(M\in\{M\})$. M_0M kesma $\{M\}\subset R^3$ to'plamga tegishli.

Ma'lumki,

$$\frac{x-x_0}{\rho(M_0,M)}=\cos\alpha,\quad \frac{y-y_0}{\rho(M_0,M)}=\cos\beta,\quad \frac{z-z_0}{\rho(M_0,M)}=\cos\gamma \qquad (10.19.1)$$



10.3-chizma

10.19.2-ta'rif. Agar $M(x,y,z)$ nuqta $\vec{\ell}$ yo'nalgan to'g'ri chiziq bo'ylab $M_0(x_0,y_0,z_0)$ nuqtaga intilganda $(M\to M_0)$ ushbu nisbatning

$$\frac{f(M)-f(M_0)}{\rho(M_0,M)}=\frac{f(x,y,z)-f(x_0,y_0,z_0)}{\rho((x_0,y_0,z_0),(x,y,z))}$$

limiti mavjud bo'lsa, bu limitga $f(x,y,z)=f(M)$ funksiyaning $M_0(x_0,y_0,z_0)$ nuqtadagi $\vec{\ell}$ *yo'nalish bo'yicha hosilasi* deb ataladi va u

$$\frac{\partial f(M_0)}{\partial \ell} \quad \text{yoki} \quad \frac{\partial f(x_0,y_0,z_0)}{\partial \ell}$$

kabi belgilanadi.

Demak, ta'rifga ko'ra,

$$\frac{\partial f}{\partial \ell}=\lim_{M\to M_0}\frac{f(M)-f(M_0)}{\rho(M_0,M)}$$

bo'ladi.

10.19.3-teorema. Agar $u=f(x,y,z)=f(M)$ funksiya ochiq $\{M\}\subset R^3$ to'plamda berilgan bo'lib, u $M_0(x_0,y_0,z_0)\in\{M\}$ nuqtada differensiallanuvchi bo'lsa, u holda funksiya shu nuqtada har qanday $\vec{\ell}$ yo'nalish bo'yicha hosilaga ega bo'ladi va bu hosila ushbu

$$\frac{\partial e(M_0)}{\partial \ell}=\frac{\partial f(x_0,y_0,z_0)}{\partial \ell}=$$
$$=\frac{\partial f(x_0,y_0,z_0)}{\partial x}\cos\alpha+\frac{\partial f(x_0,y_0,z_0)}{\partial y}\cos\beta+\frac{\partial f(x_0,y_0,z_0)}{\partial z}\cos\gamma \qquad (10.19.4)$$

formula orqali topiladi.

Misollar: 1)Ushbu $u=f(x,y)=arctg\frac{x}{y}$ funksiyani qaraylik. $\vec{\ell}$ birinchi kvadratning $M_0(1,1)$ nuqtasidan o'tuvchi va (0,0) nuqtadan $M_0(1,1)$ nuqtaga qarab yo'nalgan bissektrissadan iborat bo'lsin. Berilgan funksiyaning $M_0(1,1)$ nuqtadagi $\vec{\ell}$ yo'nalish bo'yicha hosilasi topilsin. Berilgan funksiya $M_0(1,1)$ nuqtada differensiallanuvchi bo'lgani uchun uning yo'nalish bo'yicha holsilasini (10.19.4) formula bo'yicha topamiz:

$$\frac{\partial f(1,1)}{\partial \ell}=\frac{\partial f(1,1)}{\partial x}\cos\frac{\pi}{4}+\frac{\partial f(1,1)}{\partial y}\cos\frac{\pi}{4}=\left(\frac{y}{x^2+y_2}-\frac{x}{x^2+y^2}\right)_{\substack{x=1\\y=1}}=\frac{\sqrt{2}}{2}=0\,.$$

Demak, $\frac{\partial f(1,1)}{\partial \ell}=0$.

2) Ushbu $f(x,y)=\sqrt{x^2+y^2}$ funksiyaning M(0,0) nuqtada ixtiyoriy $\vec{\ell}$ yo'nalish bo'yicha hosilasi $\frac{\partial f(0,0)}{\partial \ell}=1$ ekanligini ko'rsating. Haqiqatan ham,

$$\frac{f(M)-f(M_0)}{\rho(M_0,M)}=\frac{\sqrt{x^2+y^2}}{\rho}=\frac{\rho}{\rho}=1,$$

bundan

$$\lim_{\rho\to 0}\frac{f(M)-f(M_0)}{\rho(M_0,M)}=1.$$

3) Ushbu $f(x,y)=x+|y|$ funksiyaning $M_0(0,0)$ nuqtada Ox koordinata o‘qi bo‘yicha hosilasi 1 ga teng bo‘lib, Oy koordinata o‘qi bo‘yicha hosilasi mavjud emas.

10.19.5-eslatma. Funksiya biror nuqtada differensiallanuvchi bo‘lmasa ham, u shu nuqtada biror yo‘nalish bo‘yicha va hatto har qanday yo‘nalish bo‘yicha hosilaga ega bo‘lishi mumkin. Masalan, ushbu $f(x,y)=\sqrt{x^2+y^2}$ funksiya $M_0(0,0)$ nuqtada differensiallanuvchi emas, lekin biz yuqorida ko‘rdikki, bu funksiya $M_0(0,0)$ nuqtada ixtiyoriy yo‘nalish bo‘yicha hosilaga ega.

10.19.6-ta’rif. Komponentlari (koordinatalari) $\frac{\partial u}{\partial x},\frac{\partial u}{\partial y},\frac{\partial u}{\partial z}$ bo‘lgan vektorga, $u=f(x,y,z)=f(M)$ funksiyaning $M_0(x_0,y_0,z_0)$ nuqtadagi *gradiyenti* deb ataladi va u $grad u=\frac{\partial u}{\partial x}\vec{i}+\frac{\partial u}{\partial y}\vec{j}+\frac{\partial u}{\partial z}\vec{k}$ kabi belgilanadi, bunda $\frac{\partial u}{\partial x},\frac{\partial u}{\partial y},\frac{\partial u}{\partial z}$ xususiy hosilalar $M_0(x_0,y_0,z_0)$ nuqtada hisoblangan. $\vec{r}(\cos\alpha,\cos\beta,\cos\gamma)$ ni e’tiborga olsak, u holda (10.19.4) ni ushbu

$$\frac{\partial u}{\partial \ell}=\left(\vec{r},\, grad u\right) \tag{10.19.7}$$

ko‘rinishda yozish mumkin. (10.19.7) formulani

$$\frac{\partial u}{\partial \ell}=|\vec{r}|\cdot|grad u|\cdot\cos\varphi$$

ko‘rinishda ham yozish mumkin, bunda

$$|grad u|=\sqrt{\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2},$$

φ, $\vec{r}$ bilan $grad u$ ning orasidagi burchak. $|\vec{r}|=1$ bo‘lgani uchun $\frac{\partial u}{\partial \ell}=|grad u|\cdot\cos\varphi$. Bu formuladan ko‘rinadiki, yo‘nalish bo‘yicha hosila o‘zining maksimum $\left(\frac{\partial u}{\partial \ell}\right)_{\max}$ qiymatiga $\cos\varphi=1$ bo‘lganda erishadi, ya’ni $\vec{r}$ vektorning yo‘nalishi $gradu$ ning yo‘nalishga mos tushsa, $\left(\frac{\partial u}{\partial \ell}\right)_{\max}=|gradu|$.

Agar $\vec{r}$ ning yo‘nalishi $gradu$ ga perpendikulyar yo‘nalgan bo‘lsa, $\frac{\partial u}{\partial \ell}=0$ bo‘ladi, chunki $\varphi=\frac{\pi}{2}$, $\cos\frac{\pi}{2}=0$.

10.20. Birinchi tartibli differensial formasining invariantligi. Biz 10.16-bandda $u=f(x_1,x_2,...,x_m)$ funksiyaning argumentlari $x_1,x_2,...,x_m$ lar erkli o‘zgaruvchi bo‘lganda uning differensiali (10.16.2) formula bilan ifoda qilinishini ko‘rgan edik. Endi biz bu bandda argument $x_1,x_2,...,x_m$ lar erksiz bo‘lib, ularning o‘zlari ham $t_1,t_2,...,t_m$ larning differensiallanuvchi funksiyalari bo‘lgan holda ham (10.16.2) formula o‘rinli ekanligini ko‘rsatamiz. Odatda birinchi tartibli differensialning bu xossani, *birinchi tartibli differensial formasi invarianlig xossasi* deb ataladi.

$u=f(M)=f(x_1,x_2,...,x_m)$ funksiya $\{M\}\subset R^m$ to‘plamda berilgan bo‘lsin. Biz yuqorida ko‘rgan edikki, agar $x_1,...,x_m$ argumentlar erkli o‘zgaruvchilar bo‘lganda uning differensiali (to‘liq differensiali)

$$du=\frac{\partial u}{\partial x_1}dx_1+\frac{\partial u}{\partial x_2}dx_2+...+\frac{\partial u}{\partial x_m}dx_m \qquad (10.20.1)$$

ko‘rinishda tasvirlanadi. Endi $x_1,x_2,...,x_m$ argumentlarni erksiz, ya’ni biror $N_0(t_1^0,t_2^0,...,t_k^0)\in\{N\}$ nuqtada, $u=f(x_1,x_2,...,x_m)$ funksiya esa, $M_0(x_1^0,x_2^0,...,x_m^0)$ nuqtada differensiallanuvchi bo‘lsin deb faraz qilaylik, bunda $x_i=\varphi_i(t_1,t_2,...,t_k)$. Bu holda $u=f(x_1,x_2,...,x_m)$ funksiyani $t_1,t_2,...,t_k$ larning murakkab funksiyasi deb qaraymiz. **10.17.2**-teoremaga asosan, bu murakkab funksiya N_0 nuqtada differensiallanuvchi bo‘ladi va uning differensiali ushbu

$$du=\frac{\partial u}{\partial t_1}dt_1+\frac{\partial u}{dt_2}dt_2+...+\frac{\partial u}{\partial t_m}dt_m \qquad (10.20.2)$$

ko‘rinishda tasvirlanadi, bunda $\frac{\partial u}{\partial t_i}$ (10.17.3) formula orqali topiladi. $\frac{\partial u}{\partial t_i}$ larni (10.17.3) dan (10.20.2) ga olib kelib qo‘ysak va $\frac{\partial u}{\partial x_i}$ larning koyeffisentlarini jamlash natijasida

$$du = \frac{\partial u}{\partial x_1}\left(\frac{\partial x_1}{\partial t_1}dt_1 + \frac{\partial x_1}{\partial t_2}dt_2 + ... + \frac{\partial x_1}{\partial t_k}dt\right) + ... +$$
$$+ \frac{\partial u}{\partial x_m}\left(\frac{\partial x_m}{\partial t_1}dt_1 + \frac{\partial x_m}{\partial t_2}dt_2 + ... + \frac{\partial x_m}{\partial t_k}dt_k\right)$$

ni hosil qilamiz.. Ma'lumki $\frac{\partial u}{\partial x_i}$ $(i = 1,2,...,m)$ ning koyeffisenti $x_i = \varphi_i(t_1,t_2,...,t_k)$ funksiyaning dx_i differensialini ifodalaydi.

Demak, $x_1, x_2, ..., x_m$ lar erksiz uzgaruvchilar bo'lganda ham $u = f(x_1, x_2, ..., x_m)$ funksiyaning differensiali (10.20.1) ko'rinishda bo'lar ekan, ya'ni ko'p o'zgaruvchili murakkab funksiya birinchi tartibli differensiali formasi invariantligi (ko'rinishi) saqlanar ekan. Birinchi tartibli differensial formasining invariantligi quyidagi qoidalarning o'rinli ekanini ko'rsatishga imkon beradi u va v ko'p o'zgaruvchi funksiyalar differensiallanuvchi bo'lsa, u holda

$$d(Cu) = Cdu\,(C = const),$$
$$d(u \pm v) = du \pm dv,$$
$$d(uv) = udv + vdu,$$
$$d\left(\frac{u}{v}\right) = \frac{vdu - udv}{v^2}$$

formulalar o'rinli bo'ladi. Bu formulalarning uchinchisining o'rinli ekanini (qolganlari xam shu usulda ko'rsatiladi) ko'rsatamiz. $W = u \cdot v$ funksiyani qaraylik, bu funksiyani u va v o'zgaruvchilarning funksiyasi deb qaralsa, u holda bu funksiyaning differensiali

$$dw = \frac{\partial w}{\partial u}du + \frac{\partial w}{\partial v}dv$$

bo'ladi.

$$\frac{\partial w}{\partial u} = v, \quad \frac{\partial w}{\partial v} = u$$

bo'lgani uchun $du = udv + vdu$. Birinchi tartibli differensial formasi invariantligining saklanish xossasiga asosan, $udv + vdu$ ifoda, u va v lar differensiallanuvchi funksiyalar bulganda $u\,v$ funksiyaning differensialini ifodalaydi.

10.5-§. Ko'p o'zgaruvchili funksiyaning yuqori tartibli xususiy hosilalari va differensiallari

10.21. Yuqori tartibli xususiy hosilalar. $u=f(x_1,x_2,...,x_m)=f(M)$ funksiya ochiq $\{M\}\subset R^m$ to'plamda berilgan bo'lib, uning har bir $M(x_1,x_2,...,x_m)$ nuqtasida $\frac{\partial f}{\partial x_1},\frac{\partial f}{\partial x_2},...,\frac{\partial f}{\partial x_m}$ xususiy hosilalarga ega bo'lsin. Bu xususiy hosilalari o'z navbatida $x_1,x_2,...,x_m$ o'zgaruvchilarning funksiyasi sifatida $\{M\}$ to'plamda aniqlangan bo'lsin.

Endi $\frac{\partial f}{\partial x_1},\frac{\partial f}{\partial x_2},...,\frac{\partial f}{\partial x_m}$ funksiyalar ham biror $M\in\{M\}$ nuqtada x_k argument bo'yicha xususiy hosilaga ega bo'lishi mumkin. Bu x_k argument bo'yicha olingan hosila berilgan $u=f(x_1,x_2,...,x_m)$ funksiyaning *ikkinchi tartibli xususiy hosilasi* deb ataladi va u $\frac{\partial^2 u}{\partial x_k\partial x_i}$, $f^{(2)}_{x_i x_k}$, $u^{(2)}_{x_i x_k}$ $(i=1,2,...,m;\ k=1,2,...,m)$ kabi belgilanadi, bunda $i\neq k$ bo'lsa, u holda $\frac{\partial^2 u}{\partial x_k\partial x_i}$ xususiy hosilaga *aralash xususiy hosila* deyiladi, $k=i$ bo'lganda $\frac{\partial^2 u}{\partial x_k\partial x_k}$ deb yozish o'rniga $\frac{\partial^2 u}{\partial x_k^2}=f''_{x_k^2}$ deb yoziladi.

Xuddi shunday, $f(x_1,x_2,...,x_m)$ funksiyaning uchinchi, to'rtinchi, va xokazo tartibli xususiy hosilalari ta'riflanadi. $f(x_1,x_2,...,x_m)$ funksiya $x_{i_1},x_{i_2},...,x_{i_{n-1}}$ argumentlari bo'yicha $(n-1)$ tartibli xususiy hosilalarga ega bo'lsin. Bu $(n-1)$ tartibli xususiy hosilalar ham $M(x_1,x_2,....,x_m)\in\{M\}$ nuqtada x_{i_n} argumenti bo'yicha hosilaga ega bo'lsin. Bu hosila $u=f(x_1,x_2,...,x_m)$ funksiyaning $x_{i_1},x_{i_2},...,x_{i_{n-1}}$ argumentlar bo'yicha $M\in\{M\}$ nuqtadagi *n-tartibli xususiy hosilasi* deyiladi. Shunday qilib $x_{i_1},x_{i_2},...,x_{i_{n-1}},x_{i_n}$ argumentlar bo'yicha n-tartibli xususiy hosilani

$$\frac{\partial^n u}{\partial x_{i_n}\partial x_{i_{n-1}},...,\partial x_{i_2}\partial x_{i_1}}=\frac{\partial}{\partial x_{i_n}}\left(\frac{\partial^{n-1}u}{\partial x_{i_{n-1}}....\partial x_{i_2}\partial x_{i_1}}\right)$$

deb yozish mumkin.

Agar $i_1, i_2, \ldots, i_n$ indekslarning hammasi birdaniga bir-biriga teng bo'lmasa, u holda $\dfrac{\partial^n u}{\partial x_{i_n} \ldots \partial x_{i_2} \partial x_{i_1}}$ xususiy hosila n-tartibli *aralish xususiy hosila* deyiladi.

10.21.1-misol. Ushbu

$$u = arctg \frac{x}{y}$$

funksiyaning 2-tartibli xususiy hosilalarini toping.

Yechilishi. Berilgan funksiyaning 1- va 2-tartibli xususiy hosilalarini topamiz:

$$\frac{\partial u}{\partial x} = \frac{y}{x^2 + y^2}, \quad \frac{\partial u}{\partial y} = -\frac{x}{x^2 + y^2},$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{x^2 + y^2}{\left(x^2 + y^2\right)^2}, \quad \frac{\partial^2 u}{\partial x^2} = -\frac{2xy}{\left(x^2 + y^2\right)^2},$$

$$\frac{\partial u}{\partial y \partial x} = \frac{x^2 - y^2}{\left(x^2 + y^2\right)^2}, \quad \frac{\partial^2 u}{\partial y^2} = -\frac{2x}{\left(x^2 + y^2\right)^2}.$$

Bu **10.21.1**-misolda $\dfrac{\partial^2 u}{\partial y \partial x}$ va $\dfrac{\partial^2 u}{\partial x \partial y}$ aralish xususiy hosilalar bir-biriga teng bo'ladi. Umumiy holda bu aralash xususiy hosilalar bir-biriga teng bo'lmasligi ham mumkin.

10.21.2-misol. Ushbu

$$f(x, y) = \begin{cases} xy \dfrac{x^2 - y^2}{x^2 + y^2}, & x^2 + y^2 \neq 0, \\ 0, & x^2 + y^2 = 0 \end{cases}$$

funksiyaning $O(0;0)$ nuqtada $f''_{yx}(x, y)$, $f''_{xy}(x, y)$ hosilalarga ega va ular bir – biriga teng emasligini isbot qiling.

Yechilishi. Haqiqatan ham,

$$f'_x(x, y) = y \frac{x^2 - y^2}{x^2 + y^2} + \frac{4x^2 y^3}{\left(x^2 + y^2\right)^2}; \; f'_y(x, y) = x \frac{x^2 y^2}{x^2 + y^2} - \frac{4x^3 y^2}{\left(x^2 + y^2\right)^2}.$$

Agar $x = y = 0$ bo'lsa, u holda $f'_x(0,0)$ va $f'_y(0,0)$ hosilalarni bevosita hosilaning ta'rifi bo'yicha topamiz:

$$f'_x(0,0)=\lim_{\Delta x\to} f\frac{(\Delta x,0)-f(0,0)}{\Delta x}=\lim_{\Delta x\to 0}\frac{0}{\Delta x}=0,$$

$$f'_y(0,0)\lim_{\Delta y\to 0}\frac{f(0,\Delta y)-f(0,0)}{\Delta y}=\lim_{\Delta y\to 0}\frac{0}{\Delta y}=0$$

endi, bu qiymatlardan foydalanib aralash hosilalarni topamiz:

$$f''_{xy}(0;0)=\lim_{\Delta y\to 0}\frac{f'_x(0,\Delta y)-f'_x(0,0)}{\Delta y}=\lim_{\Delta y\to 0}\frac{-\Delta y^3}{\Delta y^3}=-1;$$

$$f''_{yx}(0;0)=\lim_{\Delta x\to 0}\frac{f'_y(\Delta x,0)-f'_y(0,0)}{\Delta x}=\lim_{\Delta x\to 0}\frac{\Delta x^3}{\Delta x^3}=1.$$

Bundan $f''_{xy}(0;0)\neq f''_{yx}(0;0)$ ekanligini topamiz.

Yuqoridagi keltirilgan misollardan ko'rinadiki, $\frac{\partial^2 f}{\partial x\partial y}$ va $\frac{\partial^2 f}{\partial y\partial x}$ aralash hosilalar teng bo'lishi ham teng bo'lmasligi ham mumkin ekan.

Endi shunday savol tug'iladi, qanday shartlarda aralash hosilalar differensiallashni qanday tartibda olib borishga bog'liq bo'lmaydi. Bu savolga quyidagi teorema javob beradi.

10.21.3-teorema. $u=f(x,y)$ funksiya $\{M\}\subset R^2$ ochiq to'plamda aniqlangan bo'lib, shu to'plamda $f'_x, f'_y, f''_{xy}, f''_{yx}$ hosilalarga ega bo'lsin. Agar aralash xususiy hosilalar $M_0(x_0,y_0)\in\{M\}$ nuqtada uzluksiz bo'lsa, u holda shu nuqtada

$$f''_{xy}(x_0,y_0)=f''_{yx}(x_0,y_0) \tag{10.21.4}$$

bo'ladi.

Aralash xosilalar uchun quyidagi umumiy teorema o'rinli.

10.21.5-teorema. $u=f(x_1,x_2,...,x_m)$ funksiya $\{M\}\subset R^m$ ochiq to'plamda aniqlangan bo'lib, bu to'plamda mumkin bo'lgan hamma $(n-1)$ tartibgacha xususiy hosilalarga va n-tartibli aralash xususiy hosilaga ega bo'lib, bu hosilalar $\{M\}$ to'plamda uzluksiz bo'lsa, u xolda ixtiyoriy n-tartibli aralash hosila, hosilani topish tartibiga bog'liq bo'lmaydi.

10.22. Yuqori tartibli differensiyallar. $u=f(x_1,x_2,....,x_m)=f(M)$ funksiya ochiq $\{M\}(\{M\}\subset R^m)$ to'plamda f'_{x_1}, $f'_{x_2},....,f'_{x_m}$ xususiy hosilalarga ega bo'lsin.

Agar bu xususiy hosilalar $M_0\in\{M\}$ nuqtada differensiallanuvchi bo'lsa, $f(M)$ funksiya shu nuqtada *ikki marta differensiallanuvchi funksiya* deyiladi.

Agar $u=f(x_1,x_2,...,x_m)$ funksiya $\{M\}$ to'plamda barcha $n-1$ tartibli xususiy hosilalarga ega bo'lib, bu xususiy hosilalar $M_0\in\{M\}$ nuqtada differensiallanuvchi bo'lsa, $f(M)$ funksiya M_0 nuqtada *$n-$ marta differensiallanuvchi funksiya* deyiladi.

10.22.1-teorema. Agar ochiq $\{M\}$ to'plamda $f(M)$ funksiyaning barcha $n-$tartibli xususiy hosilalari mavjud va $M_0\in\{M\}$ nuqtada uzluksiz bo'lsa, $f(M)$ funksiya M_0 nuqtada $n-$ marta differensiallanuvchi bo'ladi.

$u=f(x_1,x_2,...,x_m)$ funksiya ochiq $\{M\}$ $(\{M\}\subset R^m)$ to'plamda aniqlangan bo'lib, u $M(x_1,x_2,...,x_m)\in\{M\}$ nuqtada differensiallanuvchi bo'lsin. U holda bu funksiyaning M nuqtadagi differensiali

$$du=\frac{\partial f}{\partial x_1}dx_1+\frac{\partial f}{\partial x_2}dx_2+....+\frac{\partial f}{\partial x_m}dx_m \qquad (10.22.2)$$

ko'rinishda bo'lishini yuqoridagi 10.16-bandda ko'rgan edik, bunda $dx_1,dx_2,...,dx_m$ lar $x_1,x_2,....,x_m-$ o'zgaruvchilarning ixtiyoriy ortirmalaridir. Faraz qilaylshik $f(M)$ funksiya $M\in\{M\}$ nuqtada ikki marta differensiallanuvchi bo'lsin.

10.22.3-ta'rif. $u=f(M)$ funksiyaning M nuqtadagi differensiali $df(M)$ ning differensialiga berilgan $f(M)$ funksiyaning ikkinchi tartibli differensiali deyiladi va u d^2u yoki d^2f kabi belgilanadi. (10.22.2) ni e'tiborga olgan holda, differensiallash qoidalaridan foydalanib quyidagini topamiz:

$$d^2f=d(df)=d\left(\frac{\partial f}{\partial x_1}dx_1+\frac{\partial f}{\partial x_2}dx_2+...+\frac{\partial f}{\partial x_m}dx_m\right)=$$

$$=dx_1d\left(\frac{\partial f}{\partial x_1}\right)+dx_2d\left(\frac{\partial f}{\partial x_2}\right)+.....+dx_md\left(\frac{\partial f}{\partial x_m}\right)=$$

$$=\left(\frac{\partial^2 f}{\partial x_1^2}dx_1+\frac{\partial^2 f}{\partial x_1\partial x_2}dx_2+...+\frac{\partial^2 f}{\partial x_1\partial x_m}\right)dx_1+\left(\frac{\partial^2 f}{\partial x_2\partial x_1}dx_1+\frac{\partial^2 f}{\partial x_2^2}dx_2+....+\frac{\partial^2 f}{\partial x_2\partial x_m}\right)dx_2+$$

$$+....+\left(\frac{\partial^2 f}{\partial x_m\partial x_1}dx_1+\frac{\partial^2 f}{\partial x_m\partial x_2}dx_2+......+\frac{\partial^2 f}{\partial x_m^2}dx_m\right)dx_m=$$

$$=\frac{\partial^2 f}{\partial x_1^2}dx_1^2+\frac{\partial^2 f}{\partial x_2^2}dx_2^2+...+\frac{\partial^2 f}{\partial x_m^2}dx_m^2+2\frac{\partial^2 f}{\partial x_1\partial x_2}dx_1dx_2+2\frac{\partial^2 f}{\partial x_1\partial x_3}dx_1dx_3+....+$$

$$+2\frac{\partial^2 f}{\partial x_1\partial x_m}dx_1dx_m+\frac{2\partial^2 f}{\partial x_2\partial x_3}dx_2dx_3+2\frac{\partial^2 f}{\partial x_2\partial x_4}dx_2dx_4+....+2\frac{\partial^2 f}{\partial x_2\partial x_m}dx_2dx_m+$$

$$+...+2\frac{\partial^2 f}{\partial x_{m-1}\partial xm}dx_{m-1}dx_m. \qquad (10.22.4)$$

$f(x_1,x_2,....,x_m)$ funksiyaning $M(x_1,x_2,....,x_m)$ nuqtadagi uchinchi, to'rtinchi va hokazo differensiallari ham xuddi yuqoridagidek ta'riflandi. Umumiy holda $f(M)$ funksiyaning M nuqtadagi $(n-1)$ tartibli defferensiali $d^{n-1}f(M)$ ning defferensialligiga berilgan $f(M)$ funksiyaning shu nuqtadagi $n-$ *tartibli defferensiali* deyiladi va u $d^n f$ kabi belgilanadi.

Shunday qilib, $d^n f = d(d^{n-1}f)$. Biz yuqorida $f(M)$ funksiyaning ikkinchi tartibli defferensialining (10.22.4) ko'rnishda tasvirlashini ko'rdik. Funksiyaning keyingi tartibli defferensiallarining uning xususiy hosilalari orqali ifodasi borgan sari murakkablashib boradi. Shuning uchun yuqori tartibli defferensiallarini, simvolik ravishda, soddaroq qulayroq formada ifoda qilamiz:

$$df = \frac{\partial u}{\partial x_1}dx_1 + \frac{\partial u}{\partial x_2}dx_2 + ... + \frac{\partial f}{\partial x_m}dx_m$$

ni simvolik ravishda (f ni formal ravishda qavs taшqarisiga chiqarib) quyidagicha

$$df = \left(\frac{\partial}{\partial x_1}dx_1 + \frac{\partial}{\partial x_2} + ... + \frac{\partial}{\partial x_m}dx_m\right)f$$

yozamiz. U holda funksiyaning ikkinchi tartibli defferensialini

$$d^2 f = \left(\frac{\partial}{\partial x_1}dx_1 + \frac{\partial}{\partial x_2} + ... + \frac{\partial}{\partial x_m}dx_m\right)^2 f$$

ko'rinishda yozish mumkin. Bunda qavs ichidagi yig'indilar kvadratga ko'tarilib, so'ng $f$ga ko'paytiriladi. Bunda daraja ko'rsatgichlar xususiy hosilalar tartibini bildiradi.

Induksiya usuli bo'yicha funksiyaning $n-$darajali defferensialini simvolik ko'rinishda

$$d^n f = \left(\frac{\partial}{\partial x_1}dx_1 + \frac{\partial}{\partial x_2} + ... + \frac{\partial}{\partial x_m}dx_m\right)^n f$$

deb yozish mumkin.

Xususiy hollarda, ikki o'zgaruvchili funksiyalar uchun ushbu formulalar urinli:

$$d^2z = d(dz) = \frac{\partial^2 f(x,y)}{\partial x^2}dx^2 + 2\frac{\partial^2 f(x,y)}{\partial x\partial y}dxdy + \frac{\partial^2 f(x,y)}{\partial y^2}dy^2,$$

$$d^3z = d\left(d^2z\right) = \frac{\partial^3 f(x,y)}{\partial x^3}dx^3 + 3\frac{\partial^3 f(x,y)}{\partial x^2\partial y}dx^2dy +$$

$$+3\frac{\partial^3 f(x,y)}{\partial x\partial y^2}dxdy^2 + \frac{\partial^3 f(x,y)}{\partial y^3}dy^3 = \left(dx\frac{\partial}{\partial x} + dy\frac{\partial}{\partial y}\right)^3 f,$$

$$d^nz = \left(dx\frac{\partial}{\partial x} + dy\frac{\partial}{\partial y}\right)^n f.$$

10.23. Ko'p o'zgaruvchili funksiyaning Teylor formulasi. $u = f(x_1, x_2, ..., x_m) = f(M)$ funksiya ochiq bog'lamli $\{M\} \subset R^m$ to'plamda aniqlangan bo'lsin. Bu funksiyaning k-tartibli differensiyalining $M_0 \in \{M\}$ nuqtadagi qiymatini $d^k u\big|_{M_0}$ deb belgilaymiz.

10.23.1-teorema. $u = f(M)$ funksiya $M_0(x_1^0, x_2^0, ..., x_m^0)$ nuqtaning biror $U_\delta(M_0) \subset \{M\}$ $\delta > 0$ atrofida aniqlangan va $n+1$ marta differensiyalanuvchi bo'lsin. U holda uning M_0 nuqtadagi to'liq orttirmasi $\Delta u = f(M) - f(M_0)$ uchun ushbu

$$\Delta u = du\Big|_{M_0} + \frac{1}{2!}d^2u\Big|_{M_0} + ... + \frac{1}{n!}d^nu\Big|_{M_0} + \frac{1}{(n+1)!}d^{n+1}u\Big|_N \qquad (10.23.2)$$

tenglik o'rinli bo'ladi, bunda $N - U_\delta(M_0)$ atrofdagi $M(x_1, x_2, ..., x_m)$ nuqtaga bog'liq bo'lgan biror nuqta, $d^k u\big|_{M_0}$ va $d^{n+1}u\big|_N$ ifodalarda qatnashuvchi dx_i lar $\Delta x_i = x_i - x_i^0$ ga teng.

(10.23.2) formulaga $u = f(x_1, x_2, ..., x_m)$ funksiyaning M_0 nuqtadagi *Teylor formulasi* deyiladi.

10.6-§. Ko'p o'zgaruvchi funksiyaning ekstremumlari

10.24. Funksiya ekstremumining zaruriy sharti. $u = f(x_1, x_2, ..., x_m) = f(M)$ funksiya ochiq $\{M\} \subset R^m$ to'plamda aniqlangan bo'lib, $M_0(x_1^0, x_2^0, ..., x_m^0) \in \{M\}$ bo'lsin.

10.24.1-ta'rif. Agar M_0 nuqtaning shunday $U_\delta(M_0)\subset\{M\}(\delta>0)$ atrofi mavjud bo'lib, $\forall M\in U_\delta(M_0)$ uchun $f(M)\le f(M_0)$ $(f(M)\ge f(M_0))$ bo'lsa, $f(M)$ funksiya M_0 *nuqtada lokal maksimumga (minimumga)* ega deyiladi.

10.24.2-ta'rif. Agar M_0 nuqtaning shunday $U_\delta(M_0)\subset\{M\}(\delta>0)$ atrofi mavjud bo'lsaki, $\forall M\in U_\delta(M_0)\setminus\{M_0\}$ uchun $f(M)<f(M_0)$ $(f(M)<f(M_0))$ bo'lsa, $f(M)$ funksiya M_0 nuqtada *qat'iy maksimumga (qat'iy minimumga)* ega deyiladi.

Funksiyaning lokal maksimumi, lokal minimumi umumiy nom bilan funksiyaning lokal ekstremumi deyiladi. Bunda M_0 nuqta $f(M)$ funksiyaning lokal *ekstremum nuqtasi* deyiladi, $f(M_0)$ ga esa, funksiyaning lokal *ekstremum qiymati* deyiladi.

Funksiyaning maksimum(minimum) qiymatlari

$$f(M_0)=\max_{M\in U_\delta(M_0)}\{f(M)\}\quad f(M_0)=\min_{M\in U_\delta(M_0)}\{f(M)\}$$

kabi belgilanadi.

Ma'lumki, $\Delta u=f(M)-f(M_0)$ ayirma funksiyaning M_0 nuqtadagi to'liq orttirmasi deyilar edi.

$f(M)$ funksiyaning M_0 nuqtada lokal maksimum(minimum) ga erishsa, u holda $\forall M\in U_\delta(M)$ da $\Delta f(M_0)\le 0(\Delta f(M_0)\ge 0)$ bo'ladi.

10.24.3-teorema (funksiya ekstremumining zaruriy sharti). Agar $f(M)$ funksiya M_0 nuqtada lokal exstremumga erishib, bu nuqtada barcha $f'_{x_1},f'_{x_2},...,f'_{x_m}$ xususiy hosilalarga ega bo'lsa, u holda

$$f'_{x_1}(M_0)=0,\ f'_{x_2}(M_0)=0,...,f'_{x_m}(M_0)=0$$

bo'ladi.

10.24.5-natija. Agar $u=f(M)$ funksiya M_0 nuqtada ekstrimumga ega bo'lib, u shu nuqtada differensiyallanuvchi bo'lsa, u holda $du(M_0)=0$ bo'ladi.

10.24.6-eslatma. (10.24.4) shart, funksiyaning M_0 nuqtada ekstrimumga ega bo'lishi uchun yetarli shart bo'la olmaydi. Masalan: $u=f(x,y)=x^2y$ funksiya xususiy hosilalarga ega: $\frac{\partial u}{\partial x}=2xy$, $\frac{\partial u}{\partial y}=x^2$. Bu xususiy hosilalar $M_0(0,0)$ nuqtada nolga aylanadi:

$$\left.\frac{\partial u}{\partial x}\right|_{M_0}=0,\ \left.\frac{\partial u}{\partial y}\right|_{M_0}=0.$$

Lekin, $M_0(0,0)$ nuqta berilgan funksiyaning ekstrimum nuqtasi bo'la olmaydi. Chunki, $M_0(0,0)$ nuqtaning ixtiyoriy $U_\delta(M_0)$ atrofida $\Delta u = x^2y - 0 = x^2y$ ifoda faqat manfiy yoki faqat musbat qiymatlarni qabul qiladi.

$u = f(x_1, x_2, ..., x_m) = f(M)$ funksiya xususiy hosilalarni nolga aylantiradigan nuqtalar uning *statsionar nuqtalari* (*ekstrimumga shubhali nuqtalari*) deyiladi. Ekstremumga shubhali nuqtalarni, ya'ni statsionar nuqtalarni toppish uchun quyidagi

$$\begin{cases} \dfrac{\partial f(M)}{\partial x_1} = 0, \\ \dfrac{\partial f(M)}{\partial x_2} = 0, \\ \dots\dots\dots\dots\dots, \\ \dfrac{\partial f(M)}{\partial x_m} = 0 \end{cases}$$

tenglamalar sistemasini yechish zarur.

10.25. Funksiyaning ekstremumga ega bolishning etarli sharti. $u = f(x_1, x_2, ..., x_m) = f(M)$ funksiyaning ekstremumga ega bo'lishining yetarli shartini keltirishdan, avvalo, algebra kursida batafsil o'rganiladigan kvadratik forma haqida qisqacha tushuncha beramiz:

$$Q(x_1, x_2, ..., x_m) = a_{11}x_1^2 + a_{12}x_1x_2 + ... + a_{1m}x_1x_m + a_{21}x_2x_1 + a_{22}x_2^2 + ... + a_{mm}x_m^2$$

yoki

$$Q(x_1, x_2, ..., x_m) = \sum_{i=1}^{m} \sum_{j=1}^{m} a_{ij}x_ix_j \qquad (10.25.1)$$

funksiyaga $x_1, x_2, ..., x_n$ o'zgaruvchilarning *kvadratik formasi* deyiladi, bunda $a_{ij} = a_{ji}$, $(i, j = \overline{1, m})$ sonlar kvadratik formaning *koeffitsiyentlari* deyiladi. Agar $x_1 = x_2 = ... = x_n = 0$ bo'lsa, har qanday kvadratik forma uchun $Q(0,0,...,0) = 0$ bo'ladi.

Kvadratik formaning koeffitsiyentlari a_{ij} lardan tuzulgan ushbu

$$A = \begin{pmatrix} a_{11} & a_{12} & ... & a_{1m} \\ a_{21} & a_{22} & ... & a_{2m} \\ ... & ... & ... & ... \\ a_{m1} & a_{m2} & ... & a_{mm} \end{pmatrix}$$

simmetrik matritsaga kvadratik formaning *matritsasi* deyiladi.

Ushbu

$$\delta_1 = a_{11}\ ,\ \delta_2 = \begin{vmatrix} a_{11}a_{12} \\ a_{21}a_{22} \end{vmatrix},\ \delta_3 = \begin{vmatrix} a_{11}a_{12}a_{13} \\ a_{21}a_{22}a_{23} \\ a_{31}a_{32}a_{33} \end{vmatrix},\ \dots,\ \delta_m = \begin{vmatrix} a_{11}a_{12}\dots\dots\dots\dots a_{1m} \\ a_{11}a_{22}\dots\dots\dots\dots a_{2m} \\ \dots\dots\dots\dots\dots\dots\dots \\ a_{m1}a_{m2}\dots\dots\dots\dots a_{mm} \end{vmatrix}$$

detereminantlarga A matritsaning *bosh minorlari* deyiladi.

Agarda birdaniga hammasi nolga aylanmaydigan $x_1, x_2, ..., x_m$ o‘zgaruvchilarning harqanday qiymatida $Q(x_1, x_2, ..., x_m)$ kvadratik forma qat’iy musbat (qat’iy manfiy) qiymat, ya’ni $Q(x_1, x_2, ..., x_m) > 0$ ($Q(x_1, x_2, ..., x_m) < 0$) qabul qilsa, u holda *kvadratik forma musbat aniqlangan (manfiy aniqlangan)* deyiladi. Masalan, $Q(x_1, x_2) = x_1^2 + 2x_2^2$ kvadratik forma musbat aniqlangan, chunki $(0,0)$ nuqtadan tashqari qolgan hamma (x_1, x_2) nuqtalarda $Q(x_1, x_2) > 0$ bo‘ladi.

Agar kvadratik forma faqat musbat yoki faqat manfiy aniqlangan bo‘lsa, u holda u *ishora saqlaydi* deyiladi.

Agar kvadratik forma faqat barcha nuqtalarda $Q(x_1, x_2, ..., x_m) \geq 0$ (faqat $Q(x_1, x_2, ..., x_m) \leq 0$) qiymatni qabul qilsa, ular orasida shunday $(x_1, x_2, ..., x_m)$ nuqtalar ham borki $x_1 = x_2 = ... = x_m = 0$ dan tashqari $Q(x_1, x_2, ..., x_m) = 0$. Bu holda kvadratik forma *yarim musbat (yarim manfiy) aniqlangan* deyiladi. Masalan, $Q(x_1, x_2) = x_1^2 + 2x_1x_2 + x_2^2$ kvadratik forma yarim musbat aniqlangan, chunki hamma (x_1, x_2) nuqtalarda $Q(x_1, x_2) = (x_1 + x_2)^2 \geq 0$, kvadratik forma nafaqat $(0,0)$ nuqtada $Q(x_1, x_2) = 0$ bo‘ladi, $(1,-1)$ nuqtada ham $Q(1,-1) = 0$ bo‘ladi.

Agar ba’zi $(x_1, x_2, ..., x_m)$ nuqtalar uchun $Q(x_1, x_2, ..., x_m) > 0$ bo‘lsa, ba’zi $(x_1, x_2, ..., x_m)$ nuqtalar uchun $Q(x_1, x_2, ..., x_m) < 0$ bo‘lsa, kvadratik forma *noaniq* deyiladi. Masalan, $Q(x_1, x_2) = x_1^2 - x_1x_2 - x_2^2$ kvadratik forma noaniq, chunki u $Q(1,0) = 1 > 0$, $Q(0,1) = -1 < 0$.

Ekstremumning yetarli shartini aniqlashda kvadratik formaning musbat yoki manfiy aniqlanganligini aniqlash muhim ro‘l o‘ynaydi. Kvadratik formaning musbat va manfiy aniqlanganligini algebra kursidan ma’lum bo‘lgan Sil’vestr alomatidan foydalanib topish mumkin. Bu alomatni ham qisqacha isbotsiz keltiramiz.

10.25.2. Sil'vestr alomati. Ushbu $Q(x_1,x_2,...,x_m)=\sum_{i=1}^{m}\sum_{j=1}^{m}a_{ij}x_ix_j$ kvadratik formaning musbat aniqlangan bo'lishi uchun, ushbu $\delta_1>0,\delta_2>0,...,\delta_m>0$ tengsizliklarning bajarilishi zarur va yetarli;

$Q(x_1,x_2,...,x_m)$ kavadratik formaning manfiy aniqlangan bo'lishi uchun, ushbu $\delta_1<0,\delta_2>0,\delta_3<0...,(-1)^m\delta_m>0$ tengsizliklarning bajarilishi zarur va yetarli.

Biz yuqorida 10.24-bandda $f(M)$ funksiyaning M_0 nuqtada ekstremumga ega bo'lishining zaruriy shartini ko'rsatdik. Endi 10.25-bandda funksiyaning ekstremumga erishishning yetarli shartini ko'rsatamiz.

$f(M)$ funksiya $M_0\in\{M\}\subset R^m$ nuqtaning biror $U_\delta(M_0)(\delta>0)$ atrofida aniqlangan bo'lib, M_0 nuqta $f(M)$ funksiyaning statsionar nuqtasi bo'lsin.

$u=f(x_1,x_2,...,x_m)=f(M)$ funksiyaning $x_1,x_2,...,x_n$ lar erkli o'zgaruvchilar bo'lganda, $M_0(x_1^0,x_2^0,...,x_m^0)$ nuqtadagi ikkinchi tartibini differensiyalini ushbu

$$d^2u\big|_{M_0}=\sum_{i=1}^{m}\sum_{j=1}^{m}\frac{\partial^2 f(M_0)}{\partial x_i\partial x_j}dx_idx_j \tag{10.25.3}$$

ko'rinishda yozish mumkin. Bu (10.25.3) ifoda, $u=f(x_1,x_2,...,x_m)$ funksiyaning M_0 nuqtadagi ikkinchi tartibli differensiyali bo'lib, $dx_1,dx_2,...,dx_m$ o'zgaruvchilarga nisbatan esa, kvadratik forma ekanini ifodalaydi. Kvadratik formaning koeffitsiyentlarini

$$a_{ij}=\frac{\partial^2 f(M_0)}{\partial x_i\partial x_j}\left(i,j=\overline{1,m}\right) \tag{10.25.4}$$

deb belgilaymiz.

10.25.5-teorema (ekstremumning yetarli sharti). $u=f(x_1,x_2,...,x_m)=f(M)$ funksiya $M_0(x_1^0,x_2^0,...,x_m^0)$ nuqtaning biror $U_\delta(M_0)(\delta>0)$atrofida differensiallanuvchi va M_0 nuqtada ikki marta differensiallanuvchi hamda M_0 nuqta statsionar nuqta, ya'ni $du|_{M_0}=0$ bo'lsin. U holda, agar $dx_1,dx_2,...,dx_m$ o'zgaruvchilarga nisbatan kvadratik forma ikkinchi tartibli $d^2u|_{M_0}$ musbat (manfiy) aniqlangan bo'lsa, $u=f(M)$ funksiya M_0 nuqtada minimum (maksimum) ega bo'ladi.

Agar $d^2u|_{M_0}$ ifoda noaniq aniqlangan kvadratik forma bo'lsa, $u = f(M)$ funksiya M_0 nuqtada ekstremumga ega bo'lmaydi.

10.25.6-eslatma. Agar M_0 statsionar nuqtda ikki marta differentsiallanuvchi $u = f(M)$ funksiyaning ikkinchi tartibli differentsiali yarim musbat (yarim manfiy) aniqlangan bo'lsa, u holda funksiyaning M_0 nuqtada ekstremumga ega bo'lishi yoki ekstremumga ega bo'lmasligi to'g'risida hech narsa aytib bo'lmaydi. Masalan, $u_1 = x^3 + y^3$ va $u_2 = x^4 + y^4$ ikkala funksiyalarning ham ikkinchi tartibli differentsialari $M_0(0,0)$-statsionar nuqtalarda aynan nolga teng (ya'ni ular yarim musbat (yarim manfiy) aniqlangan), lekin ularning ikkinchisi ekstremumga ega.

Bu teoremani xususiy holda, funksiya ikki o'zgaruvchiga bo'gliq bo'lgan holda qaraylik.

$u = f(M)$ funksiya $M_0(x_0, y_0)$ nuqtaning biror $U_\delta(M_0)(\delta > 0)$ atrofida differensiallanuvchi va M_0 nuqtada ikki marta differensiallanuvchi (M_0-statsionar nuqta, ya'ni $du|_{M_0} = 0$) bo'lsin. Quyidagicha belgilashlarni kiritamiz: $a_{11} = \frac{\partial^2 u}{\partial x_2}(M_0)$, $a_{12} = \frac{\partial^2 u}{\partial x \partial y}(M_0)$, $a_{22} = \frac{\partial^2 u}{\partial y^2}(M_0)$. U holda, **10.25.5.** - teorema va **10.25.2**-Silvestr alomatiga asosan,

1) agar $D = a_{11}a_{22} - a_{12}^2 > 0$ va $a_{11} > 0$ bo'lsa, $f(M)$ funksiya M_0 nuqtada minimumga erishadi:

2) agar $D > 0$ va $a_{11} < 0$ bo'lsa, $f(M)$ funksiya M_0 nuqtada maksimumga erishadi.

3) $D < 0$ bo'lsa, $f(M)$ funksiya M_0 nuqtada ekstremumga erishmaydi.

4) $D = 0$ bo'lsa, $f(M)$ funksiya M_0 nuqtada ekstremumga erishishi ham mumkin, erishmasligi ham mumkin. Bu hol qo'shimcha tekshirish yordamida aniqlanadi.

10-bob bo'yicha nazariy materiallarni mustahkamlash uchun topshiriqlar

10.1. Ko'p o'zgaruvchili funksiya tushunchasi ([3], 2-q., 33-35 betlar; [5], 1-t., 352-354 betlar; [10], 1-q., 442-460 betlar; [12], 2-q., 31-34 betlar, [30], 14- bo'lim).

10.2. Evklid tekisligi va fazosi haqida tushuncha ([3], 2-q., 16-23 betlar; [5], 1-t., 340-360 betlar; [10], 1-q., 442-460 betlar; [12], 2-q.,2-12 betlar; [30], 14- bo'lim).

10.3. R^m fazoda ketma – ketlik va uning limiti ([3], 2-q., 23-30 betlar; [5], 1-t., 340-360 betlar; [10], 1-q., 442-460 betlar; [12], 2-q.,17-25 betlar; [30], 14- bo'lim).

10.4. R^m fazoda yaqinlashuvchi ketma – ketlik ([3], 2-q., 29-30 betlar; [5], 1-t., 340-360 betlar, [10], 1-q., 442-460 betlar; [12], 2-q.,18-25 betlar).

10.5. R^m fazoda ketma – ketlik yaqinlashuvchi bo'lishining Koshi kriteriysi ([3], 2-q., 30-35 betlar; [5], 1-t., 340-360 betlar; [10], 1-q., 442-460 betlar; [12], 2-q., 25-28 betlar) .

10.6. Ko'p o'zgaruvchili funksiyaning limiti ([3], 2-q., 33-35 betlar; [5], 1-t., 340-360 betlar; [10], 1-q., 442-460 betlar; [12], 2-q., 34-36 betlar, [30], 14- bo'lim).

10.7. Karrali va takroriy limitlarning tengligi haqidagi teorema ([3], 2-q., 35-45 betlar; [5], 1-t., 340-360 betlar, [10], 1-q., 442-460 betlar; [12], 2-q.,42-44 betlar, [30], 14- bo'lim).

10.8. Ko'p o'zgaruvchili funksiyaning uzluksizligi ([3], 2-q., 46-49 betlar; [5], 1-t., 362-373 betlar; [10], 1-q., 460-469 betlar; [12], 2-q.,46-51 betlar, [30], 14- bo'lim).

10.9. Ko'p o'zgaruvchili uzluksiz funksiyaning asosiy xossalari ([3], 2-q., 49-52 betlar; [5], 1-t., 362-373 betlar, [10], 1-q., 460-469 betlar; [12], 2-q.,54-62 betlar).

10.10. Ko'p o'zgaruvchili funksiyaning xususiy hosilasi va differensiali ([3], 2-q., 59-72 betlar; [5], 1-t., 375-400 betlar, [12], 2-q.,62-75 betlar; [30], 14- bo'lim).

10.11. Yuqori tartibli xususiy hosilalar va differensiallar ([3], 2-q., 88-93 betlar; [5], 1-t., 375-400 betlar; [12], 2-q.,87-93 betlar; [30], 14- bo'lim).

10.12. Yo'nalish bo'yicha hosila va skalyar maydon gradiyenti ([3], 2-q., 74-77 betlar; [5], 1-t., 375-400 betlar; [30], 14- bo'lim).

10.13. Ko'p o'zgaruvchili funksiyalar uchun Teylor formulasi ([3], 2-q., 93-95 betlar; [5], 1-t., 402-414 betlar; [12], 2-q.,96-98 betlar)

Agar $d^2u|_{M_0}$ ifoda noaniq aniqlangan kvadratik forma bo'lsa, $u=f(M)$ funksiya M_0 nuqtada ekstremumga ega bo'lmaydi.

10.25.6-eslatma. Agar M_0 statsionar nuqtda ikki marta differentsiallanuvchi $u=f(M)$ funksiyaning ikkinchi tartibli differentsiali yarim musbat (yarim manfiy) aniqlangan bo'lsa, u holda funksiyaning M_0 nuqtada ekstremumga ega bo'lishi yoki ekstremumga ega bo'lmasligi to'g'risida hech narsa aytib bo'lmaydi. Masalan, $u_1 = x^3 + y^3$ va $u_2 = x^4 + y^4$ ikkala funksiyalarning ham ikkinchi tartibli differentsialari $M_0(0{,}0)$-statsionar nuqtalarda aynan nolga teng (ya'ni ular yarim musbat (yarim manfiy) aniqlangan), lekin ularning ikkinchisi ekstremumga ega.

Bu teoremani xususiy holda, funksiya ikki o'zgaruvchiga bo'gliq bo'lgan holda qaraylik.

$u=f(M)$ funksiya $M_0(x_0,y_0)$ nuqtaning biror $U_\delta(M_0)(\delta>0)$ atrofida differensiallanuvchi va M_0 nuqtada ikki marta differensiallanuvchi (M_0-statsionar nuqta, ya'ni $du|_{M_0}=0$) bo'lsin. Quyidagicha belgilashlarni kiritamiz: $a_{11} = \frac{\partial^2 u}{\partial x_2}(M_0)$, $a_{12} = \frac{\partial^2 u}{\partial x \partial y}(M_0)$, $a_{22} = \frac{\partial^2 u}{\partial y^2}(M_0)$. U holda, **10.25.5.** -teorema va **10.25.2**-Silvestr alomatiga asosan,

1) agar $D = a_{11}a_{22} - a_{12}^2 > 0$ va $a_{11} > 0$ bo'lsa, $f(M)$ funksiya M_0 nuqtada minimumga erishadi:

2) agar $D>0$ va $a_{11}<0$ bo'lsa, $f(M)$ funksiya M_0 nuqtada maksimumga erishadi.

3) $D<0$ bo'lsa, $f(M)$ funksiya M_0 nuqtada ekstremumga erishmaydi.

4) $D=0$ bo'lsa, $f(M)$ funksiya M_0 nuqtada ekstremumga erishishi ham mumkin, erishmasligi ham mumkin. Bu hol qo'shimcha tekshirish yordamida aniqlanadi.

10-bob bo'yicha nazariy materiallarni mustahkamlash uchun topshiriqlar

10.1. Ko'p o'zgaruvchili funksiya tushunchasi ([3], 2-q., 33-35 betlar; [5], 1-t., 352-354 betlar; [10], 1-q., 442-460 betlar; [12], 2-q., 31-34 betlar, [30], 14- bo'lim).

10.2. Evklid tekisligi va fazosi haqida tushuncha ([3], 2-q., 16-23 betlar; [5], 1-t., 340-360 betlar; [10], 1-q., 442-460 betlar; [12], 2-q.,2-12 betlar; [30], 14- bo'lim).

10.3. R^m fazoda ketma – ketlik va uning limiti ([3], 2-q., 23-30 betlar; [5], 1-t., 340-360 betlar; [10], 1-q., 442-460 betlar; [12], 2-q.,17-25 betlar; [30], 14- bo'lim).

10.4. R^m fazoda yaqinlashuvchi ketma – ketlik ([3], 2-q., 29-30 betlar; [5], 1-t., 340-360 betlar, [10], 1-q., 442-460 betlar; [12], 2-q.,18-25 betlar).

10.5. R^m fazoda ketma – ketlik yaqinlashuvchi bo'lishining Koshi kriteriysi ([3], 2-q., 30-35 betlar; [5], 1-t., 340-360 betlar; [10], 1-q., 442-460 betlar; [12], 2-q., 25-28 betlar) .

10.6. Ko'p o'zgaruvchili funksiyaning limiti ([3], 2-q., 33-35 betlar; [5], 1-t., 340-360 betlar; [10], 1-q., 442-460 betlar; [12], 2-q., 34-36 betlar, [30], 14- bo'lim).

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3) $D<0$ bo'lsa, $f(M)$ funksiya M_0 nuqtada ekstremumga erishmaydi.

4) $D=0$ bo'lsa, $f(M)$ funksiya M_0 nuqtada ekstremumga erishishi ham mumkin, erishmasligi ham mumkin. Bu hol qo'shimcha tekshirish yordamida aniqlanadi.

10-bob bo'yicha nazariy materiallarni mustahkamlash uchun topshiriqlar

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10.6. Ko'p o'zgaruvchili funksiyaning limiti ([3], 2-q., 33-35 betlar; [5], 1-t., 340-360 betlar; [10], 1-q., 442-460 betlar; [12], 2-q., 34-36 betlar, [30], 14- bo'lim).

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10.8. Ko'p o'zgaruvchili funksiyaning uzluksizligi ([3], 2-q., 46-49 betlar; [5], 1-t., 362-373 betlar; [10], 1-q., 460-469 betlar; [12], 2-q.,46-51 betlar, [30], 14- bo'lim).

10.9. Ko'p o'zgaruvchili uzluksiz funksiyaning asosiy xossalari ([3], 2-q., 49-52 betlar; [5], 1-t., 362-373 betlar, [10], 1-q., 460-469 betlar; [12], 2-q.,54-62 betlar).

10.10. Ko'p o'zgaruvchili funksiyaning xususiy hosilasi va differensiali ([3], 2-q., 59-72 betlar; [5], 1-t., 375-400 betlar, [12], 2-q.,62-75 betlar; [30], 14- bo'lim).

10.11. Yuqori tartibli xususiy hosilalar va differensiallar ([3], 2-q., 88-93 betlar; [5], 1-t., 375-400 betlar; [12], 2-q.,87-93 betlar; [30], 14- bo'lim).

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10.13. Ko'p o'zgaruvchili funksiyalar uchun Teylor formulasi ([3], 2-q., 93-95 betlar; [5], 1-t., 402-414 betlar; [12], 2-q.,96-98 betlar)

10.14. Ko'p o'zgaruvchili funksiyalarning ekstremum qiymatlari ([3], 2-q., 99-100 betlar; [5], 1-t., 417-431 betlar; [12], 2-q.,99-100 betlar; [30], 14-bo'lim).

10.15. Ekstremumning zaruriy va yetarli shartlari ([3], 2-q., 100-108 betlar; [5], 1-t., 417-431 betlar; [12], 2-q.,100-110 betlar; [30], 14- bo'lim).

10.16. Oshkormas funksiyaning hosilasi ([3], 2-q., 109-108 betlar; [5], 1-t., 441-463 betlar; [30], 14- bo'lim).

10.1-amaliy mashg'ulot.
Ko'p argumentli funksiyalar limitlarini hisoblash. Ko'p o'zgaruvchili funksiyaning uzluksizligi

1-misol. Ushbu $f(x,y)=\dfrac{2x^2y}{x^4+y^2}$ funksiya, (x,y) nuqta $(0,0)$ nuqtaga intilganda, limitga ega emasligini ko'rsating.

Yechilishi. Ravshanki, $y=kx^2$, $x\neq 0$ chiziq bo'ylab o'zgarmas qiymat qabul qilinadi, ya'ni

$$f(x,y)\big|_{y=kx^2}=\frac{2x^2y}{x^4+y^2}\bigg|_{y=kx^2}=\frac{2x^2kx^2}{x^4+\left(kx^2\right)^2}=\frac{2k}{1+k^2}.$$

Demak,

$$\lim_{(x,y)\to(0,0)}f(x,y)=\lim_{(x,y)\to(0,0)}f(x,y)\big|_{y=kx^2}=\frac{2k}{1+k^2}.$$

Bu limit $y=kx^2$ chiziq bo'ylab intilish yo'liga bog'liq ravishda o'zgaradi. Agar (x,y) nuqta $(0,0)$ nuqtaga $y=x^2$ parabola bo'ylab intilsa, ya'ni $k=1$ bo'lsa, limit 1 ga teng bo'ladi. Agar (x,y) nuqta $(0,0)$ nuqtaga Ox o'q bo'ylab intilsa, ya'ni $k=0$ bo'lsa, limit 0 ga teng bo'ladi. Bu esa, yuqoridagi ikki yo'l qoidasiga binoan, $f(x,y)$ funksiya, (x,y) nuqta $(0,0)$ nuqtaga intilganda, limitga ega emasligini anglatadi.

2-misol. Quyidagi $f(x,y)=\begin{cases}\dfrac{x^4+y^4}{x^2+y^2}, & x^2+y^2\neq 0,\\ 0, & x^2+y^2=0\end{cases}$ funksiyani $O(0,0)$, $A(1,-1)$ nuqtalarda har bir o'zgaruvchi bo'yicha xususiy va ikkala o'zgaruvchi bo'yicha birgalikda uzluksizlikka tekshiring.

Yechilishi. Funksiyaning $O(0,0)$, $A(1,-1)$ nuqtalarda har bir o'zgaruvchi bo'yicha xususiy uzluksizligini ko'rsatamiz: $y\neq 0$ va $x\to x_0\neq 0$ bo'lsa,

$$\lim_{x\to x_0} f(x,y)=\lim_{x\to x_0}\frac{x^4+y^4}{x^2+y^2}=\frac{x_0^4+y^4}{x_0^2+y^2}=f(x_0,y),$$

$y=0$ va $x\to x_0\neq 0$ bo'lsa,

$$\lim_{x\to x_0} f(x,0)=\lim_{x\to x_0}\frac{x^4+0}{x^2+0}=\lim_{x\to x_0}x^2=x_0^2=f(x_0,0),$$

$y=0$ va $x\to 0$ bo'lsa,

$$\lim_{x\to 0} f(x,0)=\lim_{x\to 0}\frac{x^4+0}{x^2+0}=\lim_{x\to 0}x^2=0=f(0,0),$$

$x\neq 0$ va $y\to y_0\neq 0$ bo'lsa,

$$\lim_{y\to y_0} f(x,y)=\lim_{y\to y_0}\frac{x^4+y^4}{x^2+y^2}=\frac{x^4+y_0^4}{x^2+y_0^2}=f(x,y_0),$$

$x=0$ va $y\to y_0\neq 0$ bo'lsa,

$$\lim_{y\to y_0} f(0,y)=\lim_{y\to y_0}\frac{0+y^4}{0+y^2}=y_0^2=f(0,y_0),$$

$x=0$ va $y\to 0$ bo'lsa,

$$\lim_{y\to 0} f(0,y)=\lim_{x\to 0}\frac{0+y^4}{0+y^2}=\lim_{y\to 0}y^2=0=f(0,0).$$

Ravshanki, yuqoridagidek, funksiya $A(1,-1)$ nuqtada har bir o'zgaruvchi bo'yicha xususiy uzluksiz ekanligini ko'rsatish qiyin emas.

Berilgan funksiyaning $O(0,0)$, $A(1,-1)$ nuqtalarda ikkala o'zgaruvchi bo'yicha ham bir yo'la uzluksiz ekanligini ko'rsatamiz. Agar o'zgaruvchilar, $x=\rho\cos\varphi,\quad y=\rho\sin\varphi$ deyilsa,

$$\lim_{\substack{x\to 0\\ y\to 0}} f(x,y)=\lim_{\rho\to 0} f(\rho\cos\varphi,\ \rho\sin\varphi)=\lim_{\rho\to 0}\rho^2\left(\cos^4\varphi+\sin^4\varphi\right)=0,$$

$$\lim_{\substack{x\to 1\\ y\to -1}} f(x,y)=\lim_{\substack{x\to 1\\ y\to -1}}\frac{x^4+y^4}{x^2+y^2}=\frac{1^4+(-1)^4}{1^2+(-1)^2}=1=f(1;-1)$$

bo'ladi. Bundan, berilgan funksiyaning $O(0,0)$, $A(1,-1)$ nuqtalarda ikkala o'zgaruvchi bo'yicha ham bir yo'la uzluksiz ekanligi kelib chiqadi.

3-misol. Ushbu

$$f(x_1,x_2)=\begin{cases}\dfrac{x_1x_2}{\sqrt{x_1^2+x_2^2}}, x_1^2+x_2^2>0\ bo'lsa,\\[2mm] 0,\ x_1^2+x_2^2=0\ \ bo'lsa\end{cases}$$

funksiyaning $M(x_1,x_2)\to A(0,0)$ ga karrali va takroriy limitlarini hisoblang.

Yechilishi. a) Berilgan funksiyaning $x_1 \to 0, x_2 \to 0$ karrali limitini hisoblaymiz. Buni koshi ta'rifi bo'yicha hisoblaymiz: $\forall\varepsilon > 0, \delta = 2\varepsilon$ deb olsak, unda $0 < \rho(M,O) < \delta$ ni qanoatlantiruvchi $\forall M(x_1,x_2)$ va

$$|f(x_1,x_2) - f(0,0)| = \frac{|x_1x_2|}{\sqrt{x_1^2+x_2^2}} = \frac{|x_1||x_2|}{\sqrt{x_1^2+x_2^2}} \le \frac{1}{2}\sqrt{x_1^2+x_2^2} = \frac{1}{2}\rho(M,O) < \frac{1}{2}\delta = \varepsilon$$

Tengsizlik o'rinli bo'ladi. Bundan esa,

$$\lim_{M\to O(0,0)} f(M) = \lim_{\substack{x_1\to 0\\ x_2\to 0}} \frac{x_1x_2}{\sqrt{x_1^2+x_2^2}} = 0.$$

b) Endi berilgan funksiyaning takroriy limitlari mavjudligi va ular 0 ga teng ekanligini ko'rsatamiz:

$$\lim_{x_1\to 0} f(x_1,x_2) = \lim_{x_1\to 0} \frac{x_1x_2}{\sqrt{x_1^2+x_2^2}} = 0 \quad \lim_{x_2\to 0}\lim_{x_1\to 0} f(x_1,x_2) = 0,$$

$$\lim_{x_2\to 0} f(x_1,x_2) = \lim_{x_2\to 0} \frac{x_1x_2}{\sqrt{x_1^2+x_2^2}} = 0, \quad \lim_{x_1\to 0}\lim_{x_2\to 0} f(x_1,x_2) = 0.$$

Shunday qilib, berilgan funksiyaning takroriy limitlari mavjud va ular bir-biriga teng. Ular esa karrali limitga teng.

4-misol. Ushbu $f(M) = \dfrac{x_1x_2}{x_1^2+x_2^2}$ funksiyaning $M(x_1,x_2) \to M_0(0,0)$ dagi limitining mavjud emasligini ko'rsating.

Yechilishi. Ikkita $M_n(\frac{1}{n},\frac{1}{n}) \to M_0(0,0)$, $M_n(\frac{2}{n},\frac{1}{n}) \to M_0(0,0)$ ketma-ketliklarni olib, ularga mos kelgan fuksiyaning qiymatlaridan tuzilgan ketma-ketlikni qaraymiz;

$$f(M_n) = f(\frac{1}{n},\frac{1}{n}) = \frac{1}{2}, \; f(M_n) = f(\frac{2}{n},\frac{1}{n}) = \frac{2}{5}.$$

Bundan, $M(x_1,x_2) \to M_0(0,0)$ da berilgan funksiyaning limitga ega emasligi kelib chiqadi.

5-misol. Ushbu

$$f(x_1,x_2) = \begin{cases} x_1 + x_2\sin\dfrac{1}{x_1}, & x_1 \ne 0 \quad bo'lganda, \\ 0, & x_1 = 0 \quad bo'lganda \end{cases}$$

funksiyaning bitta takroriy limiti mavjud, ikkinchisi mavjud emas, karrali limiti esa mavjud ekanligini ko‘rsating.

Yechilishi. Haqiqatan ham,

$$\lim_{x_2\to 0} f(x_1,x_2)=x_1, \quad \lim_{x_1\to 0}\lim_{x_2\to 0} f(x_1,x_2)=0$$

bo‘lib,

$$\lim_{x_2\to 0}\lim_{x_1\to 0} f(x_1,x_2)$$

mavjud emas, lekin

$$|f(x_1,x_2)-0|=\left|x_1+x_2\sin\frac{1}{x_1}\right|\le|x_1|+|x_2| \quad (x_1\ne 0)$$

tengsizlikdan ko‘rinadiki $M(x_1,x_2)\to A(0,0)$ da $f(x_1,x_2)$ funksiyaning karrali limiti mavjud va u

$$\lim_{\substack{x_1\to 0\\ x_2\to 0}} f(x_1,x_2)=0$$

ga teng.

Yuqoridagi keltirilgan misollardan ko‘rinadiki, funkisiyaning biror nuqtada karrali limitining mavjudligidan, uning shu nuqtada takroriy limitlarining mavjudligi va aksincha funksiyaning biror nuqtada takroriy limitlarining mavjudligidan, uning shu nuqtada karrali limitlarining mavjudligi har doim kelib chiqavermas ekan. Bundan tashqari takroriy limitlar har doim bir-biriga teng bo‘lavermas ekan.

Mustaqil yechish uchun misollar

Quyidagi limitlarni hisoblang:

1. $\lim\limits_{\substack{x\to\pi\\ y\to\ln 2}} e^y\cos x.$

2. $\lim\limits_{\substack{x\to 1\\ y\to 1}} \dfrac{x-y}{x^2-y^2}.$

3. $\lim\limits_{\substack{x\to 0\\ y\to 1\\ z\to e}} \ln|x+y+z|.$

4. $\lim\limits_{\substack{x\to 0\\ y\to a}} \dfrac{\sin xy}{x}.$

5. $\lim\limits_{\substack{x\to 0\\ y\to 0}} \dfrac{tg\,2xy}{x^2y}.$

6. $\lim\limits_{\substack{x\to 0\\ y\to 3}} \left(1+xy^2\right)^{\frac{y}{x^2y+xy^2}}.$

7. $\lim\limits_{\substack{x\to\infty\\ y\to\infty}} \dfrac{ax+by}{x^2+xy+y^2}.$

8. $\lim\limits_{\substack{x\to\infty\\ y\to\infty}} \dfrac{x^2+y^2}{x^4+y^4}.$

9. $\lim\limits_{\substack{x\to+\infty \\ y\to+\infty}} \dfrac{x^2+y^2}{|x^3|+|y^3|}$. 10. $\lim\limits_{\substack{x\to+\infty \\ y\to+\infty}} (x+y)^{-(x^2+y^2)}$

11. $\lim\limits_{\substack{x\to0 \\ y\to0}} (x^2+y^2)^{x^2y^2}$.

12. Ushbu $\lim\limits_{\substack{x\to0 \\ y\to0}} f(x,y)$ni hisoblang, bunda

$$f(x,y)=\begin{cases} \dfrac{x^2y}{\sqrt{1+x^2y}-1}, & x^2y\neq0 \text{ } bo\text{'}lg\ anda, \\ 2, & x^2y=0 \text{ } bo\text{'}lg\ anda. \end{cases}$$

Quyidagi karrali limitlarni hisoblang.

13. $\lim\limits_{\substack{x\to0 \\ y\to0}} \dfrac{\sqrt{1+x^2y^2}-1}{x^2+y^2}$. 14. $\lim\limits_{\substack{x\to0 \\ y\to0}} \dfrac{\sin(x^4y^2)}{(x^2+y^2)^2}$.

15. $\lim\limits_{\substack{x\to0 \\ y\to0}} \dfrac{(x^2+y^2)}{1-\cos(x^2+y^2)}$. 16. $\lim\limits_{\substack{x\to0 \\ y\to0}} \dfrac{e^{\frac{1}{x^2+y^2}}}{x^4+y^4}$.

17. $\lim\limits_{\substack{x\to0 \\ y\to0}} (1+x^2y^2)^{\frac{1}{x^2+y^2}}$. 18. $\lim\limits_{\substack{x\to+\infty \\ y\to+\infty}} \dfrac{(x+y)^2}{(x^2+y^4)^2}$.

19. $\lim\limits_{\substack{x\to+\infty \\ y\to+\infty}} (x^2+y^2)\sin\dfrac{1}{x^2+y^2}$.

20. Ushbu $\lim\limits_{\substack{x\to1 \\ y\to1}} f(x,y)$ limitni hisoblang, bunda

$$f(x,y)=\begin{cases} \dfrac{x^2+2xy-3y^2}{x^3-y^3}, & x\neq y \text{ } bo\text{'}lg\ anda, \\ 4/3, & x=y \text{ } bo\text{'}lg\ anda. \end{cases}$$

Quyidagi karrali limitlarning mavjud emasligini isbotlang.

21. $\lim\limits_{\substack{x\to0 \\ y\to0}} \dfrac{x}{x+y}$. 22. $\lim\limits_{\substack{x\to0 \\ y\to0}} \dfrac{x-y}{x+y}$.

23. $\lim\limits_{\substack{x\to0 \\ y\to0}} \dfrac{x^2-y^2}{x^2+y^2}$. 24. $\lim\limits_{\substack{x\to1 \\ y\to0}} \dfrac{\ln(x+y)}{y}$

25. Ushbu $f(x,y)=\begin{cases} \dfrac{x^3+y^3}{x^2+y^2}, & (x,y)\neq(0,0) \text{ } bo\text{'}lg\ anda, \\ 0, & (x,y)=(0,0) \text{ } bo\text{'}lg\ anda. \end{cases}$

funksiyaning $(0,0)$ nuqtada karrali limiti mavjud emasligini ko‘rsating.

26. Ushbu

$$f(x,y)=\begin{cases}\left(1+\dfrac{1}{x+y}\right)^{x+y}, & x+y\neq 0 \quad bo'lganda,\\ 1, & x+y=0 \quad bo'lganda\end{cases}$$

funksiyaning $x\to\infty,\ y\to\infty$ dagi karrali limiti mavjud emasligini isbotlang.

Quyidagi $\lim\limits_{x\to x^0}\lim\limits_{y\to y^0} f(x,y)$ va $\lim\limits_{y\to y^0}\lim\limits_{x\to x^0} f(x,y)$ takroriy limitlarni hisoblang.

27. $f(x,y)=\dfrac{x^2+xy+y^2}{x^2-xy+y^2},\quad x^0=0,\ y^0=0.$

28. $f(x,y)=\dfrac{\sin(x+y)}{2x+3y},\quad x^0=0,\ y^0=0.$

29. $f(x,y)=\dfrac{\cos x-\cos y}{x^2+y^2},\quad x^0=0,\ y^0=0.$

30. $f(x,y)=\dfrac{x^2+y^2}{x^2+y^4},\quad x^0=\infty,\ y^0=\infty.$

31. $f(x,y)=\dfrac{x^y}{1+x^y},\quad x^0=\infty,\ y^0=0.$

32. $f(x,y)=\sin\dfrac{\pi x}{2x+y},\quad x^0=\infty,\ y^0=\infty.$

33. $f(x,y)=\dfrac{1}{xy}tg\dfrac{xy}{1+xy},\quad x^0=0,\ y^0=\infty.$

34. $f(x,y)=\log_x(x+y),\quad x^0=1,\ y^0=0.$

35. $f(x,y)=\dfrac{\sin 3x+tg 2y}{6x+3y}tg\dfrac{xy}{1+xy},\quad x^0=0,\ y^0=0.$

Quyidagi berilgan funksiyalarning (x^0,y^0) nuqtada karrali va takroriy limitlari mavjudmi?

36. $f(x,y)=\dfrac{x^2-y^2}{x^2+y^2},\quad x^0=0,\ y^0=0.$

37. $f(x,y)=\log_x(x+y),\quad x^0=1,\ y^0=0.$

38. $f(x,y)=\dfrac{\sin x+\sin y}{x+y},\quad x^0=0,\ y^0=0.$

Quyidagi funksiyalarning uzilish nuqtalarini toping.

39. $u=\dfrac{1}{\sqrt{x^2+y^2}}.$

40. $u=\dfrac{1}{x^2+y^2}.$

41. $u=\ln(9-x^2-y^2)$

42. $u=\dfrac{xy}{x+y}.$

Mustaqil yechish uchun misollarning javoblari

1. -2. **2.** $0{,}5$. **3.** 1. **4.** a. **5.** 2. **6.** e^{1}. **7.** 0. **8.** 0. **9.** 0. **10.** 0. **11.** 0. **12.** 2. **13.** 0. **14.** 0. **15.** 0. **16.** 0. **17.** 1. **18.** 0. **19.** 1. **20.** $\frac{4}{3}$. **27.** 1, 1. **28.** $\frac{\text{[illegible]}}{2}$, $\frac{1}{3}$. **29.** $\frac{1}{2}$ va $-\frac{1}{2}$. **30.** 0 va 1. **31.** $\frac{1}{2}$ va 1. **32.** 0 va 1. **33.** 0 va 1. **34.** 1 va ∞. **35.** $\frac{1}{2}$ va $-\frac{2}{3}$. **36.** Karrali limit mavjud emas, $\lim\limits_{x\to 0}\lim\limits_{y\to 0} f(x,y)=-1$ va $\lim\limits_{y\to 0}\lim\limits_{x\to 0} f(x,y)=1$. **37.** Karrali limit mavjud emas, $\lim\limits_{x\to 1}\lim\limits_{y\to 1} f(x,y)=1$ va $\lim\limits_{y\to 0}\lim\limits_{x\to 1} f(x,y)=\infty$. **38.** $\lim\limits_{\substack{x\to 0\\ y\to 0}} f(x,y)=\lim\limits_{x\to 0}\lim\limits_{y\to 0} f(x,y) =\lim\limits_{y\to 0}\lim\limits_{x\to 0} f(x,y)=1$. **39.** $O(0;0)$. **40.** $0(0;0)$. **41.** $x^2+y^2=9-$ aylananing hamma nuqtalari. **42.** $x+y=0$ chiziqning hamma nuqtalari.

10.2-amaliy mashg'ulot.
Ko'p o'zgaruvchili funksiyaning xususiy hosilalari va differensiallari

1-misol. Ushbu

$$z=x^2-xy+y^2+1$$

funksiyaning z'_x, z'_y xususiy hosilalarini ta'rif bo'yicha toping.

Yechilishi. Berilgan funksiyaning x bo'yicha xususiy orttirmasini tuzamiz:

$$\Delta_x z=f(x+\Delta x,y)-f(x,y)=(x+\Delta x)^2-(x+\Delta x)y+y^2+1-x^2+xy-y^2-1=$$
$$=2x\Delta x+\Delta x^2-y\Delta x.$$

$\Delta x\to 0$ da $\frac{\Delta_x z}{\Delta x}$ nisbatning limitini hisoblaymiz:

$$\frac{\partial z}{\partial x}=\lim_{\Delta x\to 0}\frac{\Delta_x z}{\Delta x}=\lim_{\Delta x\to 0}\frac{2x\Delta x+\Delta x^2-y\Delta x}{\Delta x}=2x-y.$$

Demak, $\frac{\partial z}{\partial x}=2x-y$.

Xuddi z'_x xususiy hosilaga o'xshash holda z'_y xususiy hosilani topamiz: $\frac{\partial z}{\partial y}=2y-x$.

2-misol. $f(x,y)=arctg(xy)$ funksiyaning $f'_x(x,y)$ va $f'_y(x,y)$ xususiy hosilalarini toping.

Yechilishi. Ushbu $(\mathrm{arctg} u)' = \frac{u'}{1+u^2}$ formulaga asosan, $f'_x(x,y)$ va $f'_y(x,y)$ xususiy hosilalarni topamiz:

$$f'_x(x,y) = \left(arctg(xy)\right)'_x = \frac{y}{1+(xy)^2},\ f'_y(x,y) = \left(arctg(xy)\right)'_y = \frac{x}{1+(xy)^2}.$$

Mustaqil yechish uchun misollar

Quyidagi funksiyalarning xususiy hosilalarini toping.

1. $u = x^2 + y^2 + 3x^2y^3.$ **2.** $u = \frac{x(x-y)}{y^2}$

3. $u = xyz + \frac{x}{yz}$ **4.** $u = \sin(xy + yz).$

5. $u = tg(x+y) \cdot e^{x/y}.$ **6.** $u = \sin\frac{x}{y} \cdot \cos\frac{y}{x}.$

7. $u = e^x(\cos y + x\sin y).$ **8.** $u = x^y.$

9. $u = \left(\frac{y}{x}\right)^x.$ **10.** $u = \ln\frac{\sqrt{x^2+y^2}-x}{\sqrt{x^2+y^2}+x}.$

Quyidagi funksiyalarning berilgan nuqtadagi xususiy hosilalarini toping.

11. $u = \frac{x}{y^2},\ (1;1)$. **12.** $u = \ln\left(1+\frac{x}{y}\right),\ (1;2)$.

13. $u = xye^{\sin xy},\ (1;1)$. **14.** $u = (2x+y)^{2x+y},\ (1;-1)$.

15. Ushbu $u = \sqrt[3]{xy}$ funksiyaning $O(0;0)$ nuqtadagi xususiy hosilalarini toping. Bu funksiya $O(0;0)$ nuqtada differensiallanuvchi bo'ladimi?

Quyidagi berilgan $u(x,y)$ funksiyalar $O(0;0)$ nuqtada xususiy hosilalarga egami; $O(0;0)$ nuqtada differensiallanuvchi bo'ladimi?

16. $u = \sqrt{x^2+y^4}.$ **17.** $u = \sqrt{x^4+y^4}$.

18. $u = \sqrt[3]{xy}$ **19.** $u = \sqrt[3]{x^2y^2}$.

20. $u = \begin{cases} e^{-\frac{1}{x^2+y^2}}, & x^2+y^2 \neq 0\ bo'lganda, \\ 0, & x^2+y^2 = 0\ bo'lganda \end{cases}$

21. $u = \begin{cases} \frac{x^4+y^4}{x^2+y^2}, & x^2+y^2 \neq 0\ bo'lganda, \\ 0, & x^2+y^2 = 0\ bo'lganda \end{cases}$

Quyidagi berilgan funksiyalarning differensialini toping.

22. $u=2x^4-3x^2y^2+x^3y$.

23. $u=(y^3+2x^2+3)^x$.

24. $u=\frac{y}{x}+\frac{x}{y}$.

25. $u=-\frac{x}{\sqrt{x^2+y^2}}$.

26. $u=a^{-\frac{y}{x}}$.

27. $u=\ln\left(x+\sqrt{x^2+y^2}\right)$.

28. $u=\ln\sin\frac{x+1}{\sqrt{y}}$.

29. $u=arctg\frac{x+y}{x-y}$.

Quyidagi funksiyalarning berilgan nuqtalardagi differensialini toping.

30. $u=\frac{x^2-y^2}{x^2+y^2}$. *a)* $(1;1)$; *b)* $(0;1)$. 31. $u=\sqrt{xy+\frac{x}{y}}$, $(2;1)$.

32. $u=\cos(xy+xz)$, $M(x,y,z)$ va $N\left(1;\frac{\pi}{6};\frac{\pi}{6}\right)$ nuqtalarda.

33. $u=e^{xy}$. $M(x,y)$ va $O(0;0)$ nuqtada.

34. $u=x^y$, $M(x,y)$ va $M_0(2;3)$nuqtalarda.

35. $u=x\ln(xy)$, $M(x,y)$ va $M_0(-1;-1)$ nuqtalarda.

36. $u=\frac{x}{x^2+y^2+z^2}$, $M(1,0,1)$. 37. $u=arctg\frac{xy}{z^2}$, $M(3,2,1)$.

Quyidagi berilgan $f(u)$, $f(u,v)$, $f(u,v,w)$ funksiyalarni differensiallanuvchi va ularning f_u, f_v, f_w hosilalari aniq deb faraz qilib, quyidagi φ funksiyaning differensialini toping.

38. $\varphi=f(u)$, $u=xy+\frac{y^3}{x}$.

39. $\varphi=f(u,v,w)$, $u=x^2+y^2+z^2$, $v=x+y+z$, $w=xyz$.

40. Agar $W=\sin(xy+\pi)$, $x=e^t$ va $y=\ln(t+1)$ bo'lsa, $t=0$ da $\frac{dw}{dt}$ hosilani hisoblang.

Quyidagi misollarda: *a)*zanjir qoidasidan foydalanib; *b)* bevosita t bo'yicha differensialab, $\frac{dW}{dt}$ ni t ning funksiyasi sifatida ifodalang, so'ngra $\frac{dW}{dt}$ ning berilgan $t=t_0$ nuqtadagi qiymatini toping.

41. $W=x^2+y^2$, $x=\cos t$, $y=\sin t$; $t_0=\pi$.

42. $W=\frac{x}{z}+\frac{y}{z}$, $x=\cos^2 t$, $y=\sin^2 t$, $z=\frac{1}{t}$; $t_0=3$.

43. $W=2ye^x-\ln z$, $x=\ln(t^2+1)$, $y=arctgt$, $z=e^t$; $t_0=1$.

44. Agar a va b - o'zgarmas sonlar, $w = u^3 + thu + \cos u$ va $u = ax + by$ bo'lsa, $a\frac{\partial w}{\partial y} = b\frac{\partial w}{\partial x}$ munosabat o'rinli ekanligini ko'sating.

45. Agar $f(u)$ ixtiyoriy differensiallanuvchi funksiya bo'lsa, u holda, $\varphi(x,y) = yf(x^2 - y^2)$ funksiya, $y^2\frac{\partial \varphi}{\partial x} + xy\frac{\partial \varphi}{\partial y} = x\varphi$ tenglamani qanoatlantirishini isbotlang.

46. Agar $f(u)$ ixtiyoriy differensiallanuvchi funksiya bo'lsa, u holda, $\varphi(x,y) = xy + xf\left(\frac{y}{x}\right)$ funksiya, $x\frac{\partial \varphi}{\partial x} + y\frac{\partial \varphi}{\partial y} = xy + \varphi$ tenglamani qanoatlantirishini isbotlang.

47. Agar $f(u)$ ixtiyoriy differensiallanuvchi funksiya bo'lsa, u holda, $\varphi(x,y) = \sin x + f(\sin y - \sin x)$ funksiya, $\cos y\frac{\partial \varphi}{\partial x} + \cos x\frac{\partial \varphi}{\partial y} = xy + \varphi$ tenglamani qanoatlantirishini isbotlang.

Quyidagi funksiyaning M_0 nuqtada $\overrightarrow{M_0M}$ yo'nalish bo'yicha hosilasini toping.

48. $f(x,y) = 5x + 10x^2y + y^5$, $M_0(1,2)$, $M(5,-1)$

49. $f(x,y) = xy^2z^3$, $M_0(3,2,1)$, $M(7,5,1)$

50. $f(x,y,z) = \arcsin\frac{z}{\sqrt{x^2 + y^2}}$, $M_0(1,1,1)$, $M(1,5,4)$

51. Ushbu $f(x,y) = 3x^4 + y^3 + xy$ funksiyaning $M_0(1,2)$ nuqtada, Ox o'q bilan 135° burchak tashkil qilgan nurning yo'nalishi bo'yicha hosilasini toping.

52. Ushbu $f(x,y) = arctg\frac{y}{x}$ funksiyaning $x^2 + y^2 = 2x$ aylananing $M_0\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ nuqtasiga o'tkazilgan tashqi normalning yo'nalishi bo'yicha hosilasini toping.

53. Quyidagi $f(x,y,z) = \ln(e^x + e^y + e^z)$ funksiyaning $M_0(0,0,0)$ nuqtada, Ox, Oy, Oz koordinatalar o'qlari bilan, mos ravishda, $\frac{\pi}{3}, \frac{\pi}{4}$ va $\frac{\pi}{3}$ burchaklarni tashkil qilgan nurning yo'nalishi bo'yicha hosilasini toping.

Quyidagi skalyar maydonning berilgan nuqtadagi gradiyentini toping.

54. $u(x,y) = x^2 - 2xy + 3y - 1$, $gradu\,|_{M(x,y)} = ?$

55. $u(x,y) = 5x^2y - 3xy^3 + y^4$, $gradu\,|_{M(x,y)} = ?$

56. $u = x^2 + y^2$, $gradu|_{(3,2)} = ?$

Quyidagi berilgan funksiyalarning differensialini toping.

22. $u=2x^4-3x^2y^2+x^3y$. **23.** $u=\left(y^3+2x^2+3\right)^4$.

24. $u=\dfrac{y}{x}+\dfrac{x}{y}$. **25.** $u=-\dfrac{x}{\sqrt{x^2+y^2}}$.

26. $u=a^{-\frac{y}{x}}$. **27.** $u=\ln\left(x+\sqrt{x^2+y^2}\right)$.

28. $u=\ln\sin\dfrac{x+1}{\sqrt{y}}$. **29.** $u=arctg\dfrac{x+y}{x-y}$.

Quyidagi funksiyalarning berilgan nuqtalardagi differensialini toping.

30. $u=\dfrac{x^2-y^2}{x^2+y^2}$ $a)$ $(1;1)$; $b)$ $(0;1)$. **31.** $u=\sqrt{xy+\dfrac{x}{y}}$, $(2;1)$

32. $u=\cos(xy+xz)$, $M(x,y,z)$ va $N\left(1;\dfrac{\pi}{6};\dfrac{\pi}{6}\right)$ nuqtalarda.

33. $u=e^{xy}\cdot M(x,y)$ va $O(0;0)$ nuqtada.

34. $u=x^y$, $M(x,y)$ va $M_0(2;3)$nuqtalarda.

35. $u=x\ln(xy)$, $M(x,y)$ va $M_0(-1;-1)$ nuqtalarda.

36. $u=\dfrac{x}{x^2+y^2+z^2}$, $M(1,0,1)$. **37.** $u=arctg\dfrac{xy}{z^2}$, $M(3,2,1)$.

Quyidagi berilgan $f(u)$, $f(u,v)$, $f(u,v,w)$ funksiyalarni differensiallanuvchi va ularning f_u, f_v, f_w hosilalari aniq deb faraz qilib, quyidagi φ funksiyaning differensialini toping.

38. $\varphi=f(u)$, $u=xy+\dfrac{y^2}{x}$.

39. $\varphi=f(u,v,w)$, $u=x^2+y^2+z^2$, $v=x+y+z$, $w=xyz$.

40. Agar $W=\sin(xy+\pi), x=e^t$ va $y=\ln(t+1)$ bo'lsa, $t=0$ da $\dfrac{dw}{dt}$ hosilani hisoblang.

Quyidagi misollarda: $a)$zanjir qoidasidan foydalanib; $b)$ bevosita t bo'yicha differensialab, $\dfrac{dW}{dt}$ ni t ning funksiyasi sifatida ifodalang, so'ngra $\dfrac{dW}{dt}$ ning berilgan $t=t_0$ nuqtadagi qiymatini toping.

41. $W=x^2+y^2$, $x=\cos t$, $y=\sin t$; $t_0=\pi$.

42. $W=\dfrac{x}{z}+\dfrac{y}{z}$, $x=\cos^2 t$, $y=\sin^2 t$, $z=\dfrac{1}{t}$; $t_0=3$.

43. $W=2ye^x-\ln z$, $x=\ln\left(t^2+1\right)$, $y=arctgt$, $z=e^t$; $t_0=1$.

44. Agar a va b - o'zgarmas sonlar, $w=u^3+thu+\cos u$ va $u=ax+by$ bo'lsa, $a\frac{\partial w}{\partial y}=b\frac{\partial w}{\partial x}$ munosabat o'rinli ekanligini ko'sating.

45. Agar $f(u)$ ixtiyoriy differensiallanuvchi funksiya bo'lsa, u holda, $\varphi(x,y)=yf\left(x^2-y^2\right)$ funksiya, $y^2\frac{\partial\varphi}{\partial x}+xy\frac{\partial\varphi}{\partial y}=x\varphi$ tenglamani qanoatlantirishini isbotlang.

46. Agar $f(u)$ ixtiyoriy differensiallanuvchi funksiya bo'lsa, u holda, $\varphi(x,y)=xy+xf\left(\frac{y}{x}\right)$ funksiya, $x\frac{\partial\varphi}{\partial x}+y\frac{\partial\varphi}{\partial y}=xy+\varphi$ tenglamani qanoatlantirishini isbotlang.

47. Agar $f(u)$ ixtiyoriy differensiallanuvchi funksiya bo'lsa, u holda, $\varphi(x,y)=\sin x+f(\sin y-\sin x)$ funksiya, $\cos y\frac{\partial\varphi}{\partial x}+\cos x\frac{\partial\varphi}{\partial y}=xy+\varphi$ tenglamani qanoatlantirishini isbotlang.

Quyidagi funksiyaning M_0 nuqtada $\overrightarrow{M_0M}$ yo'nalish bo'yicha hosilasini toping.

48. $f(x,y)=5x+10x^2y+y^5,\ M_0(1,2),\ M(5,-1)$

49. $f(x,y)=xy^2z^3,\ M_0(3,2,1),\ M(7,5,1)$

50. $f(x,y,z)=\arcsin\frac{z}{\sqrt{x^2+y^2}},\ M_0(1,1,1),\ M(1,5,4)$

51. Ushbu $f(x,y)=3x^4+y^4+xy$ funksiyaning $M_0(1,2)$ nuqtada, Ox o'q bilan 135° burchak tashkil qilgan nurning yo'nalishi bo'yicha hosilasini toping.

52. Ushbu $f(x,y)=arctg\frac{y}{x}$ funksiyaning $x^2+y^2=2x$ aylananing $M_0\left(\frac{1}{2},\frac{\sqrt{3}}{2}\right)$ nuqtasiga o'tkazilgan tashqi normalning yo'nalishi bo'yicha hosilasini toping.

53. Quyidagi $f(x,y,z)=\ln\left(e^x+e^y+e^z\right)$ funksiyaning $M_0(0,0,0)$ nuqtada, Ox, Oy, Oz koordinatalar o'qlari bilan, mos ravishda, $\frac{\pi}{3}, \frac{\pi}{4}$ va $\frac{\pi}{3}$ burchaklarni tashkil qilgan nurning yo'nalishi bo'yicha hosilasini toping.

Quyidagi skalyar maydonning berilgan nuqtadagi gradiyentini toping.

54. $u(x,y)=x^2-2xy+3y-1,\quad gradu\big|_{M(x,y)}=?$

55. $u(x,y)=5x^2y-3xy^3+y^4,\ gradu\big|_{M(x,y)}=?$

56. $u=x^2+y^2,\ gradu\big|_{(1,2)}=?$

57. $u=\sqrt{4+x^2+y^2}$. $grad u|_{(2,1)}=?$

58. $u=\operatorname{arctg}\frac{y}{x}$. $grad u|_{(1,1)}=?$

59. $u=arctg\,\frac{x}{y}$ skalyar maydonning $(1;1)$ va $(-1;-1)$ nuqtalardagi gradiyentlari orasidagi burchakni toping.

60-65 misollarda funksiyaning orttirmasini uning differensialiga almashtirib, quyida berilgan ifodalarni taqribiy hisoblang.

60. $(1,02)^{4,05}$. **61.** $\sqrt{8,04^2+6,03^2}$. **62.** $(1,02)^3\cdot(0,97)^2$

63. $\sin 32^0\cos 59^0$. **64.** $\ln(0,9^3+0,99^3)$ **65.** $\sqrt{2,03^2+5e^{0,02}}$.

Mustaqil yechish uchun misollarning javoblari

1. $u_x'=2x+6xy^3,\ u_y'=3y^2+9x^2+y^2$. **2.** $u_x'=\frac{2x-y}{y^2},\ u_y'=\frac{xy-2x^2}{y^3}$

3. $u_x'=yz+\frac{1}{yz},\ u_y'=xz-\frac{x}{y^2z},\ u_z=xy-\frac{x}{yz^2}$. **4.** $u_x'=y\cdot\cos(xy+yz)$,

$u_y'=(x+z)\cos(xy+yz),\ \ u_z'=y\cdot\cos(xy+yz)$. **5.** $u_x'=\frac{e^{x/y}}{\cos^2(x+y)}+tg(x+y)\cdot e^{x/y}\cdot\frac{1}{y}$,

$u_y'=\frac{e^{x/y}}{\cos^2(x+y)}+tg(x+y)e^{x/y}\left(-\frac{x}{y^2}\right)$. **6.** $u_x=\frac{1}{y}\cos\frac{x}{y}\cos\frac{y}{x}+\frac{y}{x^2}\sin\frac{x}{y}\cdot\sin\frac{y}{x}$,

$u_y'=-\frac{x}{y^2}\cos\frac{x}{y}\cos\frac{y}{x}-\frac{1}{x}\sin\frac{x}{y}\sin\frac{y}{x}$. **7.** $u_x=e^x(x\sin y+\sin y+\cos y)$,

$u_y'=e^x(x\cos y-\sin y)$. **8.** $u_x'=yx^{y-1},\ u_y=x^y\cdot\ln x$. **9.** $u_x'=z\left(\frac{y}{x}\right)^{z-1}\cdot\left(-\frac{y}{x^2}\right)=-\frac{z}{x}\left(\frac{y}{x}\right)^z$,

$u_y'=\frac{z}{y}\left(\frac{y}{x}\right)^z,\ u_z'=\left(\frac{y}{x}\right)^z\cdot\ln\frac{y}{x}$. **10.** $u_x'=-\frac{2}{\sqrt{x^2+y^2}},\ u_y'=\frac{2x}{y\sqrt{x^2+y^2}}$.

11. $u_x(1;1)=1,\ \ u_y'(1;1)=-2$. **12.** $u_x'(1;2)=\frac{1}{3},\ \ u_y'(1;2)=-\frac{1}{6}$.

13. $u_x'(1;1)=1-\pi,\ u_y(1;1)=1-\pi$. **14.** $u_x'(1;-1)=2,\ u_y(1;-1)=1$

15. $u_x'(0;0)=0,\ \ u_y'=(0;0)=0$. Funksiya $O(0;0)$ nuqtada differensiallanuvchi emas. **16.** $u_x(0;0),\ u_y(0;0)$lar mavjud emas, $u(x,y)$ funksiya $0(0;0)$ nuqtada differensiallanuvchi. **17.** $u_x'(0;0)=u_y(0;0)=0$; $u(x,y)$ funksiya $O(0;0)$ nuqtada differensiallanuvchi. **18.** $u_x'(0,0)=u_y(0,0)=0$; $u(x,y)$ funksiya $0(0;0)$ nuqtada diferensiallanuvchi emas. **19.** $u_x(0,0)=u_y'(0,0)=0$; $u(x,y)$ funksiya $O(0;0)$ nuqtada differensiallanuvchi. **20.** $u'_x(0,0)=u'_y(0,0)$; $u(x,y)$ funksiya $O(0;0)$ nuqtada differensiallanuvchi. **21.** $u'_x(0,0)=u'_y(0,0)$, $u(x,y)$ funksiya $O(0;0)$ nuqtada differensiallanuvchi. **22.** $(8x^3-6xy^2+3x^2y)dx+(x^3-6x^2y)dy$.

23. $4(y^3+2x^2y+3)\cdot(4xydx+(3y^2+2x^2)dy)$. **24.** $\frac{x^2-y^2}{xy}\left(\frac{dx}{x}-\frac{dy}{y}\right)$

25. $y(x^2+y^2)^{-3/2}(ydx-xdy)$. **26.** $a^{-y/x}\frac{\ln a}{x^2}(ydx-xdy)$. **27.** $\frac{1}{\sqrt{y}}\left(dx+\frac{ydy}{x+\sqrt{x^2+y^2}}\right)$.

28. $\frac{1}{\sqrt{y}}ctg\frac{x+1}{\sqrt{y}}\left(dx-\frac{x+1}{2y}dy\right)$. **29.** $\frac{xdy-ydx}{x^2+y^2}$. **30.** $a)\ dx-dy,\ b)\ 0$. **31.** $\frac{1}{2}dx$

32. $du|_N=-\sin x(y+z)\cdot[(y+z)dx+xdy+xdz]$, $du|_N=-\frac{\sqrt{3}}{2}\left(\frac{\pi}{3}dx+dy+dz\right)$.

33. $du|_M=e^{xy}(ydx+xdy)$, $du|_O=0$. **34.** $du|_M=x^y\left(\frac{y}{x}dx+\ln x\,dy\right)$.

$du|_{M_0}=12dx+8\ln 2\,dy$. **35.** $du|_M=(1+\ln xy)dx+\frac{x}{y}dy$, $du|_{M_0}=dx+dy$ **36.** $du|_M=-\frac{1}{2}dz$.

37. $du|_M=\frac{1}{37}(2dx+3dy-12dz)$. **38.** $d\varphi=\left(y-\frac{y^2}{x^2}\right)f'_u dx+\left(x+\frac{2y}{x}\right)f'_u dy$.

39. $d\varphi=(2xf'_u+f'_v+yzf'_w)dx+(2yf'_u+f'_v+xzf'_w)dy+(2zf'_u+f'_v+yxf'_w)dz$.

40. $\left.\frac{dw}{dt}\right|_{t=0}=-1$. **41.** $\frac{dW}{dt}=0$, $\left.\frac{dW}{dt}\right|_{t_0=\pi}=0$. **42.** $\frac{dW}{dt}=1$, $\left.\frac{dW}{dt}\right|_{t_0=2}=1$.

43. $\frac{dW}{dt}=arctgt+1$, $\left.\frac{dW}{dt}\right|_{t_0=1}=\pi+1$. **48.** -18. **49.** $\frac{52}{5}$. **50.** $\frac{1}{5}$. **51.** $-\frac{\sqrt{2}}{2}$. **52.** $\frac{\sqrt{3}}{2}$.

53. $\frac{2+\sqrt{2}}{6}$. **54.** $gradu\,|_{M(x,y)}=2(x-y)\vec{i}+(3-2x)\vec{j}$.

55. $gradu\,|_{M(x,y)}=(10xy-3y^3)\vec{i}+(4y^3-9xy^2)\vec{j}$ **56.** $gradu|_{(1;2)}=6\vec{i}+4\vec{j}$.

57. $gradu|_{(2;1)}=\frac{2}{3}\vec{i}+\frac{1}{3}\vec{j}$. **58.** $gradu|_{(1;1)}=-\frac{1}{2}\vec{i}+\frac{1}{2}\vec{j}$. **59.** $\varphi=\pi$. **60.** 1,08. **61.** 10,05.

62. 1,00. **63.** $-0,03$. **64.** 0,273. **65.** 3,037.

10.3-amaliy mashg'ulot.
Ko'p o'zgaruvchili funksiyaning yuqori tartibli xususiy hosilalari va differensiallari

1-misol. $z=e^{xy}$ fuksiyaning d^2z ikkinchi tartibli to'liq differensialini toping.

Yechilishi. Berilgan funksiyadan x va y o'zgaruvchilar bo'yicha xususiy hosilalarni topamiz:

$$\frac{\partial z}{\partial x}=\left(e^{xy}\right)'_x=e^{xy}y,\quad \frac{\partial z}{\partial y}=\left(e^{xy}\right)'_y=e^{xy}x.$$

Ikkinchi tartibli xususiy hosilalarni topamiz:

$$\frac{\partial^2 z}{\partial x^2}=e^{xy}\cdot y^2,\quad \frac{\partial^2 z}{\partial y^2}=e^{xy}\cdot x^2,$$

$$\frac{\partial^2 z}{\partial x\partial y}=e^{xy}+e^{xy}\cdot xy=(1+xy)e^{xy},\quad \frac{\partial^2 z}{\partial y\partial x}=e^{xy}+e^{xy}\cdot xy=(1+xy)e^{xy}.$$

Oxirgi ikki ifodani solishtirib, ularning o'zaro teng ekanligiga ishonch hosil qilamiz. Topilgan ikkinchi tartibli xususiy hosilalarni $d^2z=z'_{x^2}dx^2+2z'_{xy}dxdy+z'_{y^2}dy^2$ formulaga keltirib qo'yamiz:

$$d^2z=y^2e^{xy}dx^2+2(1+xy)e^{xy}dxdy+x^2e^{xy}dy^2.$$

Mustaqil yechish uchun misollar

Quyidagi funksiyalarning ko'rsatilgan tartibdagi xususiy hosilalarini toping.

1. $u=\sin xy\cdot\ \dfrac{\partial^3 u}{\partial x^2\partial y}=?,\ \dfrac{\partial^3 u}{\partial x\partial y^2}=?.$ **2.** $u=x^4\cos y+y^4\cos x,\ \dfrac{\partial^8 u}{\partial x^4\partial y^4}=?$

3. $u=\sin x\cos 2y,\ \dfrac{\partial^{10} u}{\partial x^4\partial y^6}=?.$ **4.** $u=x^m y^n,\ \dfrac{\partial^{m+n} u}{\partial x^m\partial y^n}?.$

5. $u=(x^2+y)^{10}tgx,\ \dfrac{\partial^{10} u}{\partial x\partial y^9}=?.$ **6.** $u=\dfrac{x+y}{x-y},\ \dfrac{\partial^{m+n} u}{\partial x^m\partial y^n}=?,\ \dfrac{\partial^n u}{\partial y^n}=?.$

7. $u=\ln\dfrac{1}{\sqrt{(x-\xi)^2+(y-\eta)^2}},\ \dfrac{\partial^4 u}{\partial x\partial y\partial\xi\partial\eta}=?$

8. $u=(x^2+y^2)e^{x+y},\ \dfrac{\partial^{m+n} u}{\partial x^m\partial y^n}=?$ **9.** $u=f(x,y)=e^x\cdot\sin y,\ f^{(m+n)}_{x^m y^n}(0,0)=?$

Quyidagi funksiyalarning ko'rsatilgan nuqtalardagi ikkinchi tartibli xususiy hosilalarini toping.

10. $u=\dfrac{x}{x+y},\ (1;0)$ **11.** $u=y^2(1-e^x),\ (0;1)$

12. $u\ln(x^2+y),\ (0;1)$ **13.** $u=y\sin\dfrac{y}{x},\ (2;\pi)$

14. $u=arc\sin\dfrac{x}{\sqrt{x^2+y^2}},\ (1,-1)$ **15.** $u=y+\dfrac{x}{y},\ (1;1)$

16. $u=x+xy-5x^3+\ln(x^2+1),\ (1;1)$

17. $f(x,y)=\begin{cases} xy\dfrac{x^2-y^2}{x^2+y^2}, & (x,y)\neq(0,0)\ bo'lganda, \\ 0, & (x,y)=(0,0)\ bo'lganda \end{cases}$

funksiya (0,0) nuqta uzluksiz ekanligi ma'lum (ko'rsating!). U holda, uning $f_{xy}(0,0)$ va $f_{yx}(0,0)$ xususiy hosilalarini toping.

Quyida berilgan $f(x,y)$ funksiyaning ko'rsatilgan nuqtada ikkinchi tartibli differensialini toping:

18. $u=f(x,y)=e^{xy},\ (1,-1)$.

19. $u=f(x,y)=\dfrac{x}{y}e^{x^2},\ (0;1)$.

20. $u=f(x,y)=x\cos xy.\ \left(\dfrac{\pi}{2};-1\right)$.

21. $u=f(x,y)=arctg(x^2-2y),\ (1;0)$.

22. $u=(\sin x)^{\cos y},\ \left(\dfrac{\pi}{6};\dfrac{\pi}{2}\right)$

23. $u=e^{xyz},\ (1,1,1)$.

Quyidagi funksiyalarning ko'rsatilgan tartibdagi differensial-larini toping.

24. $u=x^3+y^3-3xy(x-y),\ d^3u=?$

25. $u=\sin(x^2+y^2),\ d^3u=?$

26. $u=\ln(x^x y^y z^z),\ d^4u=?$

27. $u=e^{ax+by},\ d^nu=?$

28. $u=X(y)\cdot Y(v),\ d^nu=?.$

29. $u=\sin x\, chy,\ \Delta u=\dfrac{\partial^2u}{\partial x^2}+\dfrac{\partial^2u}{\partial y^2}-?$

Mustaqil yechish uchun misollarning javoblari

1. $u'''_{xyx}=-2y\sin xy-xy^2\cos xy,\ u'''_{yxx}=-2x\sin xy-x^2y\cos xy.$

2. $\dfrac{\partial^8u}{\partial x^4\partial y^4}=24(\cos y+\cos x)$. **3.** $\dfrac{\partial^{10}u}{\partial x^4\partial y^6}=-2^6\sin x\cdot\cos 2y$. **4.** $\dfrac{\partial^{m+n}u}{\partial x^m\partial y^n}=m!n!.$

5. $\dfrac{\partial^{10}u}{\partial x\partial y^9}=10!\left(2xtgx+\dfrac{c^2+y}{\cos^2x}\right)$. **6.** $\dfrac{\partial^{m+n}u}{\partial x^m\partial y^n}=\dfrac{2(-1)^m(n+m-1)!(nx+my)}{(x-y)^{m+n+1}}$,

$\dfrac{\partial^nu}{\partial y^n}=\dfrac{2x\,n!}{(x-y)^{n+1}}$. **7.** $\dfrac{\partial^4u}{\partial x\partial y\partial\xi\partial\eta}=-\dfrac{6}{r^4}+\dfrac{48(x-\xi)^2(y-\eta)^2}{r^8},\ r=\sqrt{(x-\xi)^2+(y-\eta)^2}$

8. $\dfrac{\partial^{m+n}u}{\partial x^m\partial y^n}=e^{x+y}[x^2+y^2+2(mx+ny)+m(m-1)+n(n-1)]$. **9.** $\sin\dfrac{n\pi}{2}$.

10. $\dfrac{\partial^2u}{\partial x^2}=0,\ \dfrac{\partial^2u}{\partial x\partial y}=1,\ \dfrac{\partial^2u}{\partial y^2}=2$. **11.** $\dfrac{\partial^2u}{\partial x^2}=2,\ \dfrac{\partial^2u}{\partial x\partial y}=-2,\ \dfrac{\partial^2u}{\partial y^2}=0$.

12. $\dfrac{\partial^2u}{\partial x^2}=2,\ \dfrac{\partial^2u}{\partial x\partial y}=0,\ \dfrac{\partial^2u}{\partial y^2}=-1$. **13.** $\dfrac{\partial^2u}{\partial x^2}=-\dfrac{\pi^3}{16},\ \dfrac{\partial^2u}{\partial x\partial y}=\dfrac{\pi^2}{8},\ \dfrac{\partial^2u}{\partial y^2}=-\dfrac{\pi}{4}.$

14. $\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 u}{\partial y^2}=\frac{1}{2}, \frac{\partial^2 u}{\partial x \partial y}=0$. **15.** $\frac{\partial^2 u}{\partial x^2}=0, \frac{\partial^2 u}{\partial y^2}=2, \frac{\partial^2 u}{\partial x \partial y}=\frac{\partial^2 u}{\partial y \partial x}=-1$.

16. $\frac{\partial^2 u}{\partial x^2}=-30, \frac{\partial^2 u}{\partial y^2}=0, \frac{\partial^2 u}{\partial x \partial y}=\frac{\partial^2 u}{\partial y \partial x}=1$. **17.** $f_{xy}(0,0)=-1;\ f_{yx}(0,0)=1$.

18. $d^2u=e^{-1}(dx^2+dy^2)$. **19.** $2dx\ d^2u=-2dxdy$. **20.** $u=-2(dx^2-\pi dxdy)$.

21. $d^2u=-dx^2+4dxdy-2dy^2$. **22.** $d^2u=-2\sqrt{3}\,dxdy+\ln^2 2\ dy^2$. **23.** $d^2u=e[dx+dy+dz]^2+2[dxdy+dydz+dzdx]$. **24.** $d^3u=6(dx^3-3dx^2dy+3dxdy^2+dy^3)$.

25. $d^3u=-8(xdx+ydy)^3\cos(x^2+y^2)-12(xdx+ydy)(dx^2+dy^2)\sin(x^2+y^2)$.

26. $d^4u=2\left(\frac{dx^4}{x^3}+\frac{dy^4}{y^3}+\frac{dz^4}{z^3}\right)$. **27.** $d^nu=e^{ax+by}(adx+bdy)^n$.

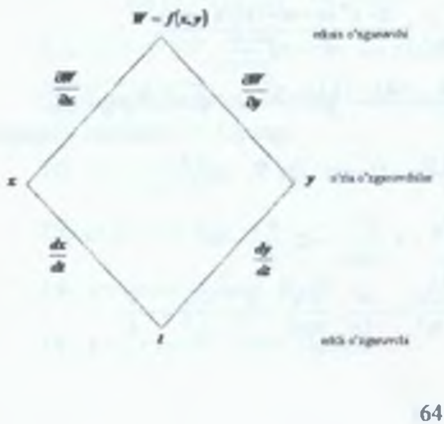
28. $d^nu=\sum_{k=0}^{n} C_n^k X^{(n-k)}(x)\,Y^{(k)}(y)dx^{n-k}dy^k$. **29.** $\Delta u=0$.

10.4-amaliy mashg'ulot.
Ko'p o'zgaruvchili murakkab va oshkormas funksiyalarni differensiallash

1. Ko'p o'zgaruvchili murakkab funksiyaning hosilasi. *Agar* $W=f(x,y)$ diferensiallanuvchi funksiya, x va y lar esa, t erkli o'zgaruvchining diferensiallanuvchi funksiyalari bo'lsa, u holda, W *funksiya ham* t erkli o'zgaruvchining diferensiallanuvchi funksiyasi bo'ladi va

$$\frac{dW}{dt}=\frac{\partial W}{\partial x}\frac{dx}{dt}+\frac{\partial W}{\partial y}\frac{dy}{dt}$$

formula o'rinli.



Bu tasdiqning «daraxt diagrammasi» quyidagicha: $\frac{dW}{dt}$ ni topish uchun, W dan boshlab, har bir yo'l bo'yicha pastga qarab harakat qilinib, yo'lda uchragan hosilalar ko'paytirilib, so'ngra ular qo'shiladi: $\frac{dW}{dt}=\frac{\partial W}{\partial x}\frac{dx}{dt}+\frac{\partial W}{\partial y}\frac{dy}{dt}$.

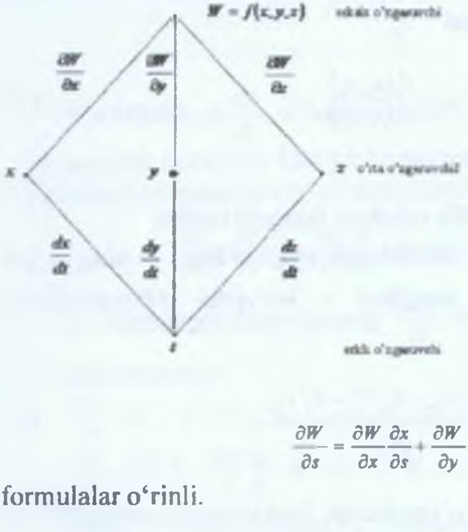
Uch o'zgaruvchili murakkab funksiya uchun zanjir qoidasi: Agar $W=f(x,y,z)$

diferensiallanuvchi funksiya, x, y va z lar esa, t erkli o‘zgaruvchining diferensiallanuvchi funksiyalari bo‘lsa, u holda, W funksiya ham t erkli o‘zgaruvchining diferensiallanuvchi funksiyasi bo‘ladi va

$$\frac{dW}{dt}=\frac{\partial W}{\partial x}\frac{dx}{dt}+\frac{\partial W}{\partial y}\frac{dy}{dt}+\frac{\partial W}{\partial z}\frac{dz}{dt}$$

formula o‘rinli.

Bu tasdiqning «daraxt diagrammasi» quyidagicha: $\frac{dW}{dt}$ ni topish uchun, W dan boshlab har bir yo‘l bo‘yicha pastga qarab harakat qilinib, yo‘lda uchragan hosilalar ko‘paytirilib, so‘ngra ular qo‘shiladi:



$$\frac{dW}{dt}=\frac{\partial W}{\partial x}\frac{dx}{dt}+\frac{\partial W}{\partial y}\frac{dy}{dt}+\frac{\partial W}{\partial z}\frac{dz}{dt}.$$

Ikkita erkli o‘zgaruvchi va uchta o‘rta o‘zgaruvchilar uchun zanjir qoidasi: Agar $W=f(x,y,z)$ $x=g(r,s)$, $y=h(r,s)$ va $z=k(r,s)$ diferensiallanuvchi funksiyalar bo‘lsa, u holda, W funksiya ham r va s erkli o‘zgaruvchilarga nisbatan xususiy hosilalarga ega bo‘ladi va ular uchun,

$$\frac{\partial W}{\partial r}=\frac{\partial W}{\partial x}\frac{\partial x}{\partial r}+\frac{\partial W}{\partial y}\frac{\partial y}{\partial r}+\frac{\partial W}{\partial z}\frac{\partial z}{\partial r},$$

$$\frac{\partial W}{\partial s}=\frac{\partial W}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial W}{\partial y}\frac{\partial y}{\partial s}+\frac{\partial W}{\partial z}\frac{\partial z}{\partial s}$$

formulalar o‘rinli.

2. Oshkormas funksiyaning hosilasi.

1-teorema. $F(x,y)$ funksiya $M_0(x_0,y_0)\in R^2$ nuqtaning biror $U_{h,k}((x_0,y_0))=\{(x,y)\in R^2: x_0-h<x<x_0+h,\ y_0-k<y<y_0+k\}\ (h>0,\ k>0)$ atrofida berilgan bo‘lib, u quyidagi shartlarni qanoatlantirsin:

1) $U_{h,k}((x_0y_0))$ da uzluksiz;

2) $U_{h,k}((x_0y_0))$ da uzluksiz $F'_x(x,y)$, $F'_y(x,y)$ xususiy hosilalarga ega va $F'_y(x_0y_0)\neq 0$;

3) $F(x_0,y_0)=0$.

U holda, $M_0(x_0, y_0)$ nuqtaning shunday
$$U_{\delta,\varepsilon}((x_0,y_0)) = \{(x,y)\in R^2 : x_0-\delta < x < x_0+\delta,\ y_0-\varepsilon < y < y_0+\varepsilon\}\ (0<\delta<h,\ 0<\varepsilon<k)$$
atrofi topiladiki:

a) $\forall x \in (x_0-\delta,\ x_0+\delta)$ uchun $F(x,y)=0$ tenglama yagona, y yechimga $(y\in(y_0-\varepsilon,\ y_0+\varepsilon))$ ega, ya'ni, $F(x,y)=0$ tenglama yordamida, $x\to y : F(x,y)=0$ oshkormas funksiya aniqlanadi;

b) $x=x_0$ bo'lganda, $y=y_0$ unga mos keldi;

c) $x\to y : F(x,y)=0$ oshkormas ko'rinishda aniqlangan funksiya $(x_0-\delta, x_0+\delta)$ oraliqda uzluksiz bo'ladi;

d) oshkormas ko'rinishdagi funksiya $(x_0-\delta, x_0+\delta)$ oraliqda uzluksiz hosilaga ega bo'ladi va uning hosilasi

$$y'_x = -\frac{F_x(x_0,y_0)}{F_y(x_0,y_0)} \tag{1}$$

formula bo'yicha hisoblanadi.

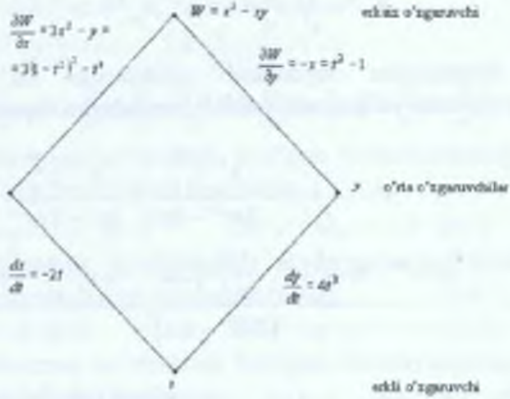
$F(x,y)$ funksiya, $U_{\delta,\varepsilon}((x_0,y_0))$ atrofda uzluksiz ikkinchi tartibli $F''_{x^2}(x,y)$, $F''_{xy}(x,y)$, $F''_{y^2}(x,y)$ hususiy hosilalarga ega bo'lsin. y ning x ga bog'liqligini e'tiborga olib, (1) tenglikni x bo'yicha differensiallab, quyidagini topamiz:

$$y'' = \frac{2F'_xF'_yF''_{xy} - F'^2_yF''_{x^2} - F'^2_xF''_{y^2}}{(F'_y)^3}. \tag{2}$$

Xuddi shunday, oshkormas ko'rinishdagi funksiyaning uchinchi va hokazo tartibdagi hosilalari topiladi.

Mustaqil yechish uchun misollar

1-misol. Ushbu $W=x^3-xy,\ x=1-t^2,\ y=t^4$ murakkab funksiyaning *a*) zanjir qoidasidan foydalanib; *b*) bevosita t bo'yicha differensiallab, $\frac{dW}{dt}$ ni t ning funksiyasi sifatida ifodalang, so'ngra $\frac{dW}{dt}$ ning berilgan $t=1$ nuqtadagi qiymatini toping.



Yechilishi. *a*) $\dfrac{dW}{dt}$ ni topish uchun, «daraxt diagrammasi»ga asosan, W dan boshlab, har bir yo'l bo'yicha pastga qarab harakat qilib, yo'lda uchragan hosilalarni ko'paytiramiz, so'ngra ularni qo'shamiz:

$$\frac{dW}{dt}=\frac{\partial W}{\partial x}\frac{dx}{dt}+\frac{\partial W}{\partial y}\frac{dy}{dt}=\left(3-6t^2+2t^4\right)(-2t)+\left(t^2-1\right)4t^3=2t\left(4t^2-3\right)$$

b) Murakkab funksiyaning $\dfrac{dW}{dt}$ hosilasini $\dfrac{dW}{dt}=\dfrac{\partial W}{\partial x}\dfrac{dx}{dt}+\dfrac{\partial W}{\partial y}\dfrac{dy}{dt}$ formula bo'yicha topamiz:

$$\frac{\partial W}{\partial x}=3x^2-y=3\left(1-t^2\right)^2-t^4=3-6t^2+2t^4,\quad \frac{\partial W}{\partial y}=-x=t^2-1,\quad \frac{dx}{dt}=-2t,\quad \frac{dy}{dt}=4t^3,$$

$$\frac{dW}{dt}=\frac{\partial W}{\partial x}\frac{dx}{dt}+\frac{\partial W}{\partial y}\frac{dy}{dt}=\left(3-6t^2+2t^4\right)(-2t)+\left(t^2-1\right)4t^3=2t\left(4t^2-3\right)$$

Endi $\dfrac{dW}{dt}$ ning berilgan $t=1$ nuqtadagi qiymatini topamiz:

$$\left.\frac{dW}{dt}\right|_{t=1}=\left.2t\left(4t^2-3\right)\right|_{t=1}=2.$$

2–misol. Ushbu $xe^{2y}-y\ln x-8=0$ oshkormas ko'rinishda berilgan funksiyaning birinchi va ikkinchi tartibli hosilalarini toping.

Yechilishi. $F(x,y)=xe^{2y}-y\ln x-8$ funksiya, $x>0$ yarim tekislikda aniqlangan, uzluksiz. Uning xususiy hosilalari mavjud va uzluksiz:

$$F_x'=e^{2y}-\frac{y}{x},\; F_y'=2xe^{2y}-\ln x,$$

$$F_{x^2}'' = \frac{y}{x^2},\ F_{y^2}'' = 4xe^{2y},\ F_{xy}'' = 2e^{2y} - \frac{1}{x}.$$

(1)-(2) formuladan foydalanib oshkormas ko'rinishda berilgan funksiyaning birinchi va ikkinchi tartibli hosilalarini topamiz:

$$y' = -\frac{F_x'}{F_y'} = -\frac{e^{2y} - \frac{y}{x}}{2xe^{2y} - \ln x} = \frac{e^{2y} - \frac{y}{x}}{\ln x - 2xe^{2y}}.$$

$$y'' = \frac{2\left(e^{2y} - \frac{y}{x}\right)\left(2xe^{2y} - \ln x\right)\left(2e^{2y} - \frac{1}{x}\right) - \left(2xe^{2y} - \ln x\right)^2 \frac{y}{x^2} - \left(e^{2y} - \frac{y}{x}\right)^2 4xe^{2y}}{\left(2xe^{2y} - \ln x\right)^3}.$$

1. Ushbu $z = x^2 + xy$, $x = 1 - t^2, y = t^4$ murakkab funksiyaning xususiy hosilasini toping.

2. Ushbu $z = e^{x-3y}$, $x = \sin t,\ y = t^2$ murakkab funksiyaning xususiy hosilasini toping.

3. Ushbu $z = e^x y^2$, $x = u^2 - v^2$, $y = u \cdot v$ murakkab funksiyaning xususiy hosilasini toping.

4. Ushbu $z = x^2 + y^2$, $x = u + v$, $y = u - v$ murakkab funksiyaning xususiy hosilasini toping.

Quyidagi misollarda: a)zanjir qoidasidan foydalanib; b) bevosita t bo'yicha differensialab, $\frac{dW}{dt}$ ni t ning funksiyasi sifatida ifodalang, so'ngra $\frac{dW}{dt}$ ning berilgan $t = t_0$ nuqtadagi qiymatini toping.

5. $W = x^2 + y^2$, $x = \cos t$, $y = \sin t$; $t_0 = \pi$.

6. $W = \frac{x}{z} + \frac{y}{z}$, $x = \cos^2 t$, $y = \sin^2 t$, $z = \frac{1}{t}$; $t_0 = 3$.

7. $W = 2ye^x - \ln z$, $x = \ln(t^2 + 1)$, $y = arctgt$, $z = e^t$; $t_0 = 1$.

8. Agar $W = (x + y + z)^2$, $x = r - s$, $y = \cos(r + s)$, $z = \sin(r + s)$ bo'lsa, $\left.\frac{\partial W}{\partial r}\right|_{(r,s)=(1,-1)}$ toping.

9. Agar $W = x^2 + \frac{y}{x}$, $x = u - 2v + 1$, $y = 2u + v - 2$ bo'lsa, $\left.\frac{\partial W}{\partial u}\right|_{(u,v)=(0,0)}$ ni toping.

Quyidagi oshkormas ko'rinishda berilgan funksiyalarning $\frac{dy}{dx}$ hosilalarini hisoblang.

10. $x^3y - y^3x - 16 = 0$ **11.** $x^5 + y^5 - 2xy - 3 = 0$.

12. $x^2 + y^2 + \ln(x^2 + y^2) - 1 = 0$ **13.** $\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$.

10. $\frac{dy}{dx} = \frac{3x^2y - y^3}{3xy^2 - x^3}$. **11.** $\frac{dy}{dx} = \frac{2y - 5x^4}{5y^4 - 2x}$. **12.** $\frac{dy}{dx} = -\frac{x}{y}$. **13.** $\frac{dy}{dx} = -\frac{b^2x}{a^2y}$.

Quyidagi oshkormas ko‘rinishda berilgan funksiyalarning berilgan A nuqtada f_x, f_y xususiy hosilalarini hisoblang.

15. $u^3 - 2u^2x + uxy - 2 = 0,\ A(1;1)$ **14.** $u^3 + 3uxy + 1 = 0,\ A(0;1)$

Quyidagi oshkormas ko‘rinishdagi funksiyalarning birinchi tartibli hosilasining berilgan nuqtadagi qiymatini toping.

16. $x^3 - 2y^2 + xy = 0,\ (1;1)$. **17.** $x^2 + xy + y^2 - 7 = 0,\ (1;2)$.

Quyidagi oshkormas ko‘rinishda berilgan funksiyalarning birinchi va ikkinchi tartibli hosilalarini hisoblang.

18. $x^2 - y^2 - 4 = 0$. **19.** $1 + xy - \ln(e^{xy} + e^{-xy}) = 0$.

20. $2\cos(x - 2y) - 2y + x = 0$. **21.** $x^3 + y^3 - 3xy = 0$.

Quyidagi oshkormas ko‘rinishdagi funksiyalarning birinchi tartibli hosilasining berilgan nuqtadagi qiymatini toping.

22. $z^3 - xy + yz + y^3 - 2 = 0,\ (1;1;1)$.

23. $\sin(x + y) + \sin(y + z) + \sin(x + z) = 0,\ (\pi, \pi, \pi)$.

Quyidagi oshkormas ko‘rinishda berilgan funksiyalarning birinchi tartibli xususiy hosilalarini va to‘liq differensiallarini hisoblang.

24. $x^2 + y^2 + z^2 - 6x = 0$. **25.** $x^2 + y^2 + z^2 - 2xz = a^2$.

26. $z^2 - xy = 0$. **27.** $z^3 + 3x^2z - 2xy = 0$.

Quyidagi oshkormas ko‘rinishda berilgan funksiyalarning birinchi va ikkichi tartibli to‘liq differensiallarini hisoblang.

28. $x^2 + y^2 + z^2 - 2z = 0$. **29.** $z^3 - 3xyz = a^3$. **30.** 3) $x - z\ln\frac{x}{y} = 0$.

Mustaqil yechish uchun misollarning javoblari

1. $\frac{dz}{dt} = -6t^5 + 8t^3 - 4t$. **2.** $\frac{dz}{dt} = e^{\sin t - 3t^2}(\cos t - 6t)$.

3. $\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u} = 2uv^2(u^2 + 1)e^{u^2 - v^2}$, $\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v} = 2e^{u^2 - v^2}u^2v(1 - v^2)$.

4. $\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u} = 4u$, $\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v} = 4v$.

5. $\frac{dW}{dt} = 0,\ \left.\frac{dW}{dt}\right| = 0$. **6.** $\frac{dW}{dt} = 1,\ \left.\frac{dW}{dt}\right| = 1$. **7.** $\frac{dW}{dt} = \mathrm{arctg}\,t + 1,\ \left.\frac{dW}{dt}\right| = \pi + 1$.

8. $\left.\dfrac{\partial W}{\partial r}\right|_{(r,s)=(1,-1)}=12$. 9. $\left.\dfrac{\partial W}{\partial u}\right|_{(u,v)=(0,0)}=-7$. 10. $\dfrac{dy}{dx}=\dfrac{3x^2y-y^3}{3xy^2-x^3}$. 11. $\dfrac{dy}{dx}=\dfrac{2y-5x^4}{5y^4-2x}$.

12. $\dfrac{dy}{dx}=-\dfrac{x}{y}$. 13. $\dfrac{dy}{dx}=-\dfrac{b^2x}{a^2y}$. 14. $f'_x(1;1)=\dfrac{6}{5}$, $f'_y(1;1)=-\dfrac{2}{5}$. 15. $f'_x(0;1)=1$, $f'_y(0;1)=0$.

16. $\dfrac{4}{3}$. 17. $-\dfrac{4}{5}$. 18. $y_x=\dfrac{x}{y}$, $y''_{x^2}=\dfrac{y^2-x^2}{y^3}$. 19. $y'_x=-\dfrac{x}{y}$, $y''_{x^2}=\dfrac{2y}{x^2}$. 20. $y'_x=\dfrac{1}{2}$, $y''_{x^2}=0$.

21. $y'_x=\dfrac{x^2-y}{x-y^2}$, $y''_{x^2}=\dfrac{(x^2-y)(y^2-x)+2x(y^2-x)^2+2y(x^2-y)^2}{(x-y^2)^3}$.

22. $\left.\dfrac{\partial z}{\partial x}\right|_{(1,1,1)}=\dfrac{1}{4}$, $\left.\dfrac{\partial z}{\partial y}\right|_{(1,1,1)}=-\dfrac{3}{4}$. 23. $\left.\dfrac{\partial z}{\partial x}\right|_{(x,x,x)}=-1$, $\left.\dfrac{\partial z}{\partial y}\right|_{(x,x,x)}=-1$.

24. $\dfrac{\partial z}{\partial x}=\dfrac{3-x}{z}$, $\dfrac{\partial z}{\partial y}=\dfrac{-y}{z}$, $dz=\dfrac{1}{x}[(3-x)dx-ydy]$.

25. $\dfrac{\partial z}{\partial x}=1$, $\dfrac{\partial z}{\partial y}=\dfrac{y}{x-z}$, $dz=dx+\dfrac{y}{x-z}dy$ 26. $\dfrac{\partial z}{\partial x}=\dfrac{y}{2z}$, $\dfrac{\partial z}{\partial y}=\dfrac{x}{2z}$, $dz=\dfrac{ydx+xdy}{2z}$.

27. $\dfrac{\partial z}{\partial x}=\dfrac{2y-6xz}{3(x^2+y^2)}$, $\dfrac{\partial z}{\partial y}=\dfrac{2x}{3(x^2+y^2)}$, $dz=\dfrac{(2y-6xz)dx+2xdy}{3(x^2+y^2)}$

28. $dz=\dfrac{xdx+ydy}{1-z}$, $d^2z=\dfrac{1-z+x^2}{(1-z)^2}dx^2++\dfrac{2xy}{(1-z)^3}dxdy+\dfrac{1-z+y^2}{(1-z)^2}dy^2$. 29. $dz=\dfrac{yzdx+xzdy}{z^2-xy}$,

$$d^2z=\frac{-2xy^3z}{(z-xy)^3}dx^2+2\frac{z(z^4-2xyz^2-x^2y^2)}{(z-xy)^3}dxdy+\frac{2x^3yz}{(z-xy)^3}dy^2.$$

30. $dz=\dfrac{yzdx+z^2dy}{y(x+z)}$, $d^2z=-\dfrac{z^2(ydx-xdy)}{(z-xy)^3}$.

10.5-amaliy mashg'ulot.
Ko'p o'zgaruvchili funksiyaning ekstremumlari

1-misol. $z=x^3-y^3-3xy$ funksiyani ekstremumga tekshiring.

Yechilishi. Berilgan funksiyadan x va y o'zgaruvchilar bo'yicha xususiy hosilalarni topamiz:

$$\frac{\partial z}{\partial x}=3x^2-3y,\quad \frac{\partial z}{\partial y}=-3y^2-3x.$$

So'ngra stasionar nuqtalarni aniqlaymiz:

$$\begin{cases}\dfrac{\partial z}{\partial x}=0,\\[2mm] \dfrac{\partial z}{\partial y}=0,\end{cases}\Rightarrow\begin{cases}x^2-y=0,\\ y^2+x=0\end{cases}$$

Bu sistemaning yechimlari $(0,0)$ va $(-1,1)$ bo'ladi. Bu nuqtalarda x va y o'zgaruvchilar bo'yicha ikkinchi tartibli xususiy hosilalarni topamiz:

$$\frac{\partial^2 z(x,y)}{\partial x^2} = 6x, \quad \frac{\partial^2 z(0,0)}{\partial x^2} = 0, \quad \frac{\partial^2 z(-1;1)}{\partial x^2} = -6 < 0,$$
$$\frac{\partial^2 z(x,y)}{\partial y^2} = -6y, \quad \frac{\partial^2 z(0,0)}{\partial y^2} = 0, \quad \frac{\partial^2 z(-1;1)}{\partial y^2} = -6 < 0,$$
$$\frac{\partial^2 z(x,y)}{\partial x \partial y} = -3, \quad \frac{\partial^2 z(0,0)}{\partial x \partial y} = -3, \quad \frac{\partial^2 z(-1,1)}{\partial x \partial y} = -3.$$

Endi ekstremumning yetarli shartidan foydalanib, lokal ekstremum nuqtalarni topamiz. $(0,0)$ nuqtada

$$\Delta(0;0) = \frac{\partial^2 z(0,0)}{\partial x^2}\frac{\partial^2 z(0,0)}{\partial y^2} - \left(\frac{\partial^2 z(0,0)}{\partial x \partial y}\right)^2 = 0 \cdot 0 - (-3)^2 = -3 < 0,$$

demak, bu nuqtada lokal ekstremum yo‘q. $(-1,1)$ nuqtada $\frac{\partial^2 z(-1;1)}{\partial x^2} = -6 < 0$ va

$$\Delta(-1;1) = \frac{\partial^2 z(-1;1)}{\partial x^2}\frac{\partial^2 z(-1,1)}{\partial y^2} - \left(\frac{\partial^2 z(-1,1)}{\partial x \partial y}\right)^2 = -6(-6) - (-3)^2 = 27 > 0.$$

Demak, ekstremumning yetarli shartiga asosan, $(-1,1)$ nuqta - funksiyaning lokal maksimum nuqtasi bo‘ladi. Bu $(-1,1)$ nuqtadagi funksiyaning qiymati $z_{max} = z(-1,1) = 1$.

Mustaqil yechish uchun misollar

Quyidagi ikki o‘zgaruvchili funksiyalarni ekstremumga tekshiring:

1. $u = -x^2 - y^2$. **2.** $u = 2x^2 + xy + y^2$.

3. $u = x^3 - 3xy - 3y$. **4.** $u = x^4 + y^4 - 4xy$.

5. $u = -2x^2 + xy - 2y^2 + 6x + 6y$. **6.** $u = x^3 - 9xy + y^3$.

7. $u = x^3 + 6xy + 8y^3 - 1$. **8.** $u = xy(1 - x - y)$.

9. $u = x^2 - 3xy + y^2 - 4x + 5y + 6$. **10.** $u = x^2 + y^2 - 8x - 2$.

11. $u = x^2 + xy - y^2 - 3x - 6y$. **12.** $u = 3x^2 - x^3 + 3y^2 + 4y$.

13. $u = 3x^2 - y^2 + 4y + 5$. **14.** $u = x^2 + xy + 2y^2 - x + y$.

15. $u = -x^2 - xy - y^2 + 3x + 6y$. **16.** $u = (x + y^2)e^{x/2}$.

17. $u = x^3 - 3axy + y^3$. **18.** $u = x^2 + xy + y^2 - 4\ln x - 10\ln y$.

19. $u = \sin x + \sin y + \sin(x+y)$, bunda $0 \le x \le \frac{\pi}{2}$, $0 \le y \le \frac{\pi}{2}$.

20. $u = xe^{y+\sin y}$. **21.** $u = \frac{8}{x} + \frac{x}{y} + y$.

22. $f(x,y) = x^2 - xy + y^2 + 2x + 2y - 4$.

23. $f(x,y)=2x^3+3xy+2y^3$. 24. $f(x,y)=x^3+y^3+3x^2-3y^2$.

Quyidagi uch o'zgaruvchili funksiyalarni ekstremumga tekshiring:

25. $u=x^2+y^2+z^2-4x+6y-2z$. 26. $u=x^2+y^2+z^2-xy+x-2z$.

27. $u=x^2+y^2+(z+1)^2-xy+x$. 28. $u=x^3+y^2+z^2+6xy-4z$.

29. $u=xyz(16-x-y-2z)$. 30. $u=\frac{226}{x}+\frac{x^2}{y}+\frac{y^2}{z}+z^2$.

31. $u=\frac{1}{z}+\frac{z}{y}+\frac{y}{x}+x+1$. 32. $u=x^{2/3}+y^{2/3}+z^{2/3}$.

Quyidagi ikki o'zgaruvchili funksiyalarni shartli ekstremumga tekshiring:

33. $u=xy,\ x+y-2=0$. 34. $u=x^2+y^2,\ x+y-1=0$.

35. $u=x^2+y^2,\ 3x+4y-12=0$. 36. $u=xy,\ 2x+3y-5=0$.

37. $u=xy^2,\ x+2y-1=0$. 38. $u=x^2+y^2-xy+x+y-4,\ x+y+3=0$.

39. $u=\cos^2 x+\cos^2 y,\ \ x-y-\frac{\pi}{4}=0$. 40. $u=5-3x-4y,\ x^2+y^2=25$.

41. $u=1-4x-8y,\ x^2-8y^2=8$. 42. $u=x^2+xy+y^2,\ x^2+y^2=1$.

Quyidagi uch o'zgaruvchili funksiyalarni shartli ekstremumga tekshiring.

43. $u=2x^2+3y^2 4z^2,\ x+y+z-13=0$.

44. $u=xy^2z^3,\ x+y+z-12=0,\ x>0, y>0, z>0$.

45. $u=x-2y+2z,\ x^2+y^2+z^2-9=0$.

46. $u=xy+2xz+2yz,\ xyz=108$.

47. $u=x^2+y^2+z^2,\ \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,\ a>0, b>0, c>0$.

48. $f(x,y)=x^3+y^2$ funksiyaning $x^2+y^2=1$ aylanadagi ekstremum qiymatlarini toping.

49. $f(x,y)=x^2+3y^2+2y$ funksiyaning $x^2+y^2\le 1$ doiradagi ekstremum qiymatlarini toping.

50. $f(x,y,z)=x-y+z$ funksiyaning $x^2+y^2+z^2=1$ birlik sferadagi ekstremum qiymatlarini toping.

51. $f(x,y,z)=x(y+z)$ funksiyaning $x^2+y^2=1$ to'g'ri doiraviy konus va $xz=1$ giperbolik silindrlarning kesishish chizig'idagi ekstremum qiymatlarini toping.

Quyidagi funksiyalarning ko'rsatilgan D to'plamda eng katta va eng kichik qiymatlarini toping.

52. $u = x^3 - 3xy + y^3,\ D = \{(x,y) \in R^2 : 0 \le x \le 2\ \ -1 \le y \le 2\}$

53. $u = x - 2y + 5,\ D = \{(x,y) \in R^2 : x \ge 0,\ y \ge 0,\ x + y \le 1\}$

54. $u = x^2 - 4x - y^2,\ D = \{(x,y) \in R^2 : x^2 + y^2 \le 9\}$

55. $u = xy(4 - x - y),\ D = \{(x,y) \in R^2 : x \ge 0,\ y \ge 0,\ x + y \le 8\}$

56. $u = x^3 - 9xy + y^3 + 27,\ D = \{(x,y) \in R^2 : 0 \le x \le 4\ \ 0 \le y \le 4\}$

57. $u = \sin x + \sin y + \sin(x + y),\ D = \{(x,y) \in R^2 : 0 \le x \le \pi/2\ \ 0 \le y \le \pi/2\}$

Quyidagi funksiyalarning ko'rsatilgan D to'plamda eng katta va eng kichik qiymatlarini toping:

58. $u = xy + x + y,\ D = \{(x,y) \in R^2 : -2 \le x \le 2\ \ -2 \le y \le 2\}$

59. $u = x^3 - 6xy + 8y^3 + 1,\ D = \{(x,y) \in R^2 : 0 \le x \le 2\ \ -1 \le y \le 1\}$

60. $u = 3 + 2xy,\ D = \{(x,y) \in R^2 : -4 \le x^2 + y^2 \le 9\}$

61. $u = x^4 - y^4,\ D = \{(x,y) \in R^2 : x^2 + y^2 \le 9\}$

62. $u = x^2 + y^2,\ D = \{(x,y) \in R^2 : (x - \sqrt{2})^2 + (y - \sqrt{2})^2 \le 9\}$

63. $u = \cos x \cos y \cos(x + y),\ D = \{(x,y) \in R^2 : 0 \le x \le \pi\ \ 0 \le y \le \pi\}$

Berilgan funksiyalarning berilgan R sohada maksimum va minimum qiymatlarini toping.

64. $f(x,y) = x^2 + xy + y^2 - 3x + 3y,\ R$: birinchi kvadrantda $x + y = 4$ to'g'ri chiziq bilan kesilgan uchburchakli soha.

65. $f(x,y) = y^2 - xy - 3y + 2x,$ R: $x = \pm 2$ va $y = \pm 2$ to'g'ri chiziqlar bilan chegaralangan kvadratik soha.

66. $f(x,y) = x^2 - y^2 - 2x + 4y,$ R: pastdan Ox o'q, yuqoridan $y = x + 2$ to'g'ri chiziq va o'ngdan $x = 2$ to'g'ri chiziq bilan chegaralangan uchburchakli soha.

67. $f(x,y) = x^3 + y^3 + 3x^2 - 3y^2,$ R: $x = \pm 1$ va $y = \pm 1$ to'g'ri chiziqlar bilan chegaralangan kvadratik soha.

Mustaqil yechish uchun misollarning javoblari

1. $u_{max} = u(0,0) = 0$ **2.** $u_{min} = u(0,0) = 0.$ **3.** $(-1; 1)$ stasionar nuqtada ekstremum yo'q. **4.** $u_{min} = u(1; 1) = u(-1; -1) = -2,$ $O(0; 0)$ stasionar nuqtada ekstremum yo'q. **5.** $u_{max} = u(2; 2) = 12$ **6.** $u_{min} = u(3;3) = -27,$ $O(0; 0)$ stasionar nuqtada ekstremum yo'q. **7.** $u_{max} = u(-1; -0,5) = 0.$ **8.** $u_{min} = u\left(\frac{1}{3}; \frac{1}{3}\right) = \frac{1}{27}$ **9.** $\left(\frac{7}{5}; -\frac{2}{5}\right)$ stasionar nuqtada ekstremum yo'q. **10.** $u_{min} = u(4; 0) = -18.$ **11.** Ekstremum yo'q.

12. $u_{min}=u\left(0;-\frac{2}{3}\right)=-\frac{4}{3}$. $\left(2;-\frac{2}{3}\right)$ stasionar nuqtada ekstremum yo‘q. **13.** Ekstremum yo‘q. **14.** $u_{min}=u(1;-1)=0$. **15.** $u_{max}=u(0;3)=9$. **16.** $u_{min}=u(-2;0)=-\frac{2}{e}$. **17.** $a<0$ da $u_{max}=u(a;a)=-a^3$, $a>0$ da $u_{min}=u(a;a)=-a^3$. **18.** $u_{min}=u(1;2)=7-10\ln 2$. **19.** $u_{max}=u(\frac{\pi}{3},\frac{\pi}{3})=\frac{3}{2}\sqrt{3}$. **20.** Ekstremum yo‘q. **21.** $u=u_{min}(4;2)=6$. **22.** $f_{min}=f(-2,-2)=-8$. **23.** $f_{max}=f\left(-\frac{1}{2},-\frac{1}{2}\right)=\frac{1}{4}$. **24.** $f_{min}=f(0,2)=-4$, $f_{max}=f(-2,0)=4$. **25.** $u_{min}=u(2;-3;1)=-14$. **26.** $u_{min}=u\left(-\frac{2}{3};-\frac{1}{3};1\right)=-\frac{4}{3}$. **27.** $u_{min}=u\left(-\frac{2}{3};-\frac{1}{3};-1\right)=-\frac{1}{3}$. **28.** $u_{min}=u(6;-18;2)=-112$. **29.** $u_{max}=u(4;4;2)=128$. **30.** $u_{min}=u(8;4;2)=60$. **31.** $u_{min}=u(1;1;1)=5$, $u_{max}=u(-1;1;-1)=-3$. **32.** $u_{min}=u(0;0;0)=0$. **33.** $u_{max}=u(1;1)=1$. **34.** $u_{min}=u(0,5;\ 0,5)=0,5$. **35.** $u_{max}=u\left(\frac{36}{25};\frac{48}{25}\right)=\frac{144}{25}$. **36.** $u_{max}=u\left(\frac{5}{4};\frac{5}{6}\right)=\frac{25}{24}$. **37.** $u_{max}=u(1,0)=0$ $u_{min}=u\left(\frac{1}{3};\frac{1}{3}\right)=\frac{1}{27}$. **38.** $u_{min}=u\left(-\frac{3}{2};-\frac{3}{2}\right)=-\frac{19}{4}$. **39.** $u_{min}=u\left(\frac{5\pi}{8}+\pi k;\ \frac{3\pi}{8}+\pi k\right)=1-\frac{\sqrt{2}}{2}$, $u_{max}=u\left(\frac{\pi}{8}+\pi k;\ -\frac{\pi}{8}+\pi k\right)=1+\frac{\sqrt{2}}{2}$, $k\in Z$. **40.** $u=u_{min}(3;4)=-20$, $u=u_{max}(-3;-4)=30$. **41.** $u=u_{min}(-4;1)=9$, $u=u_{max}(4;-1)=-7$. **42.** $u=u_{min}\left(\pm\frac{\sqrt{2}}{2};\mp\frac{\sqrt{2}}{2}\right)=\frac{1}{2}$, $u=u_{max}\left(\pm\frac{\sqrt{2}}{2};\pm\frac{\sqrt{2}}{2}\right)=\frac{3}{2}$. **43.** $u_{max}=u(6;4;3)=156$. **44.** $u_{max}=u(2;4;6)=6912$. **45.** $u_{max}=u(1;-2;2)=9$, $u_{min}=u(-1;2;-2)=-9$. **46.** $u_{max}=u(6;6;3)=108$. **47.** $u_{max}=u(\pm a,0;0)=a^2$, $u_{min}=u(0;0;\pm c)=c^2$. **48.** $f_{max}=f(0,\pm 1)=f(1,0)=1$, $f_{min}=f(-1,0)=-1$. **49.** $f_{max}=f(0,1)=5$, $f_{min}=f(0,-1/3)=-\frac{1}{3}$. **50.** $f_{max}=f\left(\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)=\sqrt{3}$, $f_{min}=f\left(\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)=-\sqrt{3}$. **51.** $\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},\sqrt{2}\right)$ va $\left(-\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}},-\sqrt{2}\right)$- maksimum nuqtalari, ularda funksiya $\frac{3}{2}$ qiymat qabul qilidi; $\left(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},-\sqrt{2}\right)$ va $\left(\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}},\sqrt{2}\right)$- minimum nuqtalari, ularda funksiya $\frac{1}{2}$ qiymat qabul qiladi. **52.** $u_{e\ kich}=u(1;1)=u(0;-1)=-1$, $u_{e\ kat}=u(2;-1)=13$. **53.** $u_{e\ kat}=u(1;0)=6$ **54.** $u_{e\ kich}=u(1;-2\sqrt{2})=u(1;2\sqrt{2})=-11$, $u_{e\ kat}=u(-3;0)=21$. **55.** $u_{e\ kich}=u(4;4)=-64$, $u_{e\ kat}=u\left(\frac{4}{3};\frac{4}{3}\right)=\frac{64}{27}$.

56. $u_{e.\,kich} = u(3;\,3) = 0,\ u_{e.\,kat} = u(4;\,0) = u(0;\,4) = 91$. **57.** $u_{e.\,kat} = u(\pi/3;\,\pi/3) = \frac{3}{2}\sqrt{3}$

58. $u_{e.\,kich} = -6,\ u_{e.\,kat} = 14$ **59.** $u_{e.\,kich} = -7,\ u_{e.\,kat} = 9+4\sqrt{2}$ **60.** $u_{e.\,kich} = -6,\ u_{e.\,kat} = 12$

61. $u_{e.\,kich} = -81,\ u_{e.\,kat} = 81$. **62.** $u_{e.\,kich} = 0,\ u_{e.\,kat} = 25$. **63.** $u_{e.\,kich} = -\frac{1}{8},\ u_{e.\,kat} = 1$.

64. $f_{max} = f(0,4) = 28;\ f_{min} = f\left(\frac{3}{2},0\right) = -\frac{9}{4}$. **65.** $f_{max} = f(2,-2),\ \ f_{min} = f\left(-2;\frac{1}{2}\right) = -\frac{17}{4}$.

66. $f_{max} = f(-2,0) = 8,\ f_{min} = f(1,0) = -1$. **67.** $f_{max} = f(1,0) = 4,\ f_{min} = f(0,-1) = -4$.

10-bob bo'yicha amaliy mashg'ulotlarni mustahkamlash uchun nazorat topshiriqlari

10.1-masala. Quyidagi funksiyaning aniqlanish sohasini toping.

10.1.1. $z = \sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}$. **10.1.2.** $z = \ln(y^2 - 4x + 8)$

10.1.3. $z = \frac{1}{R^2 - x^2 - y^2}$. **10.1.4.** $z = \sqrt{x+y} + \sqrt{x-y}$

10.1.5. $z = \frac{1}{\sqrt{x^2 + y^2 - 1}}$. **10.1.6.** $z = \sqrt{(x^2 + y^2 - 1)(4 - x^2 - y^2)}$

10.1.7. $z = \frac{1}{\sqrt{x+y}} + \frac{1}{\sqrt{x-y}}$. **10.1.5.** $z = \arcsin\frac{y-1}{x}$

10.1.9. $z = \sqrt{\frac{x^2 + y^2 - x}{2x - x^2 - y^2}}$ **10.1.10.** $z = \sqrt{x - \sqrt{y}}$.

10.1.11. $z = \arcsin\frac{y}{x}$. **10.1.12.** $z = \arcsin\frac{x}{y^2} + \arcsin(1-y)$.

10.1.13. $z = \sqrt{\frac{x^2 + 2x + y^2}{x^2 - 2x + y^2}}$. **10.1.14.** $u = \arccos\frac{z}{\sqrt{x^2 + y^2}}$.

10.1.15. $u = \ln(-1 - x^2 - y^2 + z^2)$. **10.1.16.** $z = \sqrt{x \cdot \sin y}$.

10.1.17. $z = \ln[x\ln(y - x)]$. **10.1.15.** $z = \frac{1}{\sqrt{x}} + \frac{1}{\sqrt{y}} + \frac{1}{\sqrt{z}}$.

10.1.19. $z = \frac{\sqrt{4x - y^2}}{\ln(1 - x^2 - y^2)}$. **10.1.20.** $z = \arcsin\frac{x}{y^2} + \arcsin(1-y)$.

10.1.21. $z = \sqrt{1 - \frac{x^2}{4} - \frac{y^2}{9}}$. **10.1.22.** $z = \ln(y^2 - 4x + 5)$

10.1.23. $z=\dfrac{1}{16-x^2-y^2}$. **10.1.24.** $z=\sqrt{2x+y}+\sqrt{2x-y}$

10.1.25. $z=\dfrac{1}{\sqrt{x^2+y^2-4}}$.

10.1.26-misol. Quyidagi $z=\dfrac{\ln xy}{\sqrt{4-x^2-y^2}}$ funksiyaning aniqlanish sohasini toping.

Yechilishi ([3], 2-q., 33-34 betlar; [9], 2-t., 4- bo'lim; [30], 14- bo'lim). Berilgan $z=\dfrac{\ln xy}{\sqrt{4-x^2-y^2}}$ funksiyada logarifm va arifmetik ildiz bo'lganligi uchun,

$$xy>0 \quad \text{yoki} \begin{cases} x>0, \\ y>0, \end{cases} \begin{cases} x<0, \\ y<0, \end{cases} \tag{1}$$

birinchi va uchinchi chorakda, hamda

$$4-x^2-y^2>0 \text{ yoki } x^2+y^2<4 \ , \tag{2}$$

markazi $(0,0)$ nuqtada, radusi $R=2$ bo'lgan ochiq doira ichida, ifodalar aniqlangan bo'ladi. (1) va (2) tengsizlikni bir vaqtda qanoatlantiruvchi nuqtalar to'plamida berilgan funksiyaning aniqlanish sohasi

$$D(z)=\left\{(x;\,y)\in R^2: 0<x<2,\ 0<y<\sqrt{4-x^2}\right\}\cup$$
$$\cup\left\{(x;\,y)\in R^2: -2<x<0,\ -\sqrt{4-x^2}<y<0\right\}$$

bo'ladi.

10.2- masala. Quyidagi funksiyaning karrali limitlarini hisoblang.

10.2.1. $\lim\limits_{\substack{x\to0\\y\to0}}\dfrac{\sin xy}{\sqrt{x^2+y^2}}$. **10.2.2.** $\lim\limits_{\substack{x\to\infty\\y\to\infty}}\dfrac{x+y}{-xy+y^2}$.

10.2.3. $\lim\limits_{\substack{x\to\infty\\y\to3}}\left(1+\dfrac{1}{x}\right)^{\frac{x^2}{x+y}}$. **10.2.4.** $\lim\limits_{\substack{x\to\infty\\y\to\infty}}\left(\dfrac{xy}{x^2+y^2}\right)^{x^2}$.

10.2.5. $\lim\limits_{\substack{x\to\infty\\y\to\infty}}\dfrac{x^2+y^2}{x^4+y^4}$. **10.2.6.** $\lim\limits_{\substack{x\to0\\y\to0}}\left(x^2+y^2\right)^{x^2y^2}$.

10.2.7. $\lim\limits_{\substack{x\to\infty\\y\to0}}\left(1+\dfrac{2}{x}\right)^{\frac{x^2}{x+y}}$. **10.2.5.** $\lim\limits_{\substack{x\to\infty\\y\to\infty}}\left(x^2+y^2\right)e^{-(x+y)}$.

10.2.9. $\lim\limits_{\substack{x\to 1\\ y\to 0}} \frac{\ln\left(x+e^{y}\right)}{\sqrt{x^2+y^2}}$. **10.2.10.** $\lim\limits_{\substack{x\to 0\\ y\to 0}} \frac{\sin\left(x^4y^2\right)}{\left(x^2+y^2\right)}$.

10.2.11. $\lim\limits_{\substack{x\to 0\\ y\to 0}} \frac{e^{\frac{1}{x^4+y^4}}}{x^4+y^4}$. **10.2.12.** $\lim\limits_{\substack{x\to 0\\ y\to 0}} \left(1+x^2y^2\right)^{\frac{1}{x^2+y^2}}$.

«Funksiya limiti mavjud bo'lmasligining ikki yo'l qoidasi» ni tadbiq qilib, yaqinlashishning (intilishning) har xil yo'llarini qarab chiqib, quyidagi berilgan $f(x,y)$ funksiyaning $(x,y)\to(0,0)$ da limitga ega bo'lmasligini ko'rsating.

10.2.13. $f(x,y)=\frac{-x}{\sqrt{x^2+y^2}}$. **10.2.14.** $f(x,y)=\frac{x^4}{x^4+x^2}$.

10.2.15. $f(x,y)=\frac{x^2+y}{x^2-y}$. **10.2.16.** $f(x,y)=\frac{x^4-x^2}{x^4+x^2}$.

10.2.17. $f(x,y)=\frac{xy}{|xy|}$. **10.2.15.** $f(x,y)=\frac{x-1}{x+y}$.

10.2.19. $f(x,y)=\frac{x+y}{x-y}$. **10.2.20.** $f(x,y)=\frac{x^2}{x^2-y}$.

Quyidagi funksiyaning karrali limitlarini hisoblang.

10.2.21. $\lim\limits_{\substack{x\to 0\\ y\to 0}} \frac{\sin 2xy}{\sqrt{x^2+y^2}}$. **10.2.22.** $\lim\limits_{\substack{x\to \infty\\ y\to \infty}} \frac{3x+5y}{-3xy+2y^2}$.

10.2.23. $\lim\limits_{\substack{x\to \infty\\ y\to 2}} \left(1+\frac{1}{x}\right)^{\frac{x^2}{x+y}}$. **10.2.24.** $\lim\limits_{\substack{x\to \infty\\ y\to \infty}} \left(\frac{2xy}{x^2+y^2}\right)^{2x^2}$.

10.2.25. $\lim\limits_{\substack{x\to \infty\\ y\to \infty}} \frac{x^2+y^2}{x^3+y^3}$.

10.2.26-misol. Ushbu $\lim\limits_{\substack{x\to 0\\ y\to 0}} \frac{x^2y}{x^2+y^2}$ limitini toping.

Yechilishi ([3], 2-q., 35-37 betlar; [9], 2-t., 4- bo'lim; [30], 14- bo'lim). Ushbu $x=r\cos\varphi$, $y=r\sin\varphi$ qutb koordinatalar sistemasiga o'tib, ba'zi bir limitlarni hisoblash qulay. Ravshanki, agar $x\to 0, y\to 0$ da $r\to 0$. U holda

$$\lim\limits_{\substack{x\to 0\\ y\to 0}} \frac{x^2y}{x^2+y^2}=\lim_{r\to 0}\frac{r^3\cos\varphi\sin\varphi}{r^2\left(\cos^2\varphi+\sin^2\varphi\right)}=\lim_{r\to 0} r\cos\varphi\sin\varphi=0.$$

10.3-masala. $\lim\limits_{x\to x_0}\lim\limits_{y\to y_0} f(x,y)$ va $\lim\limits_{y\to y_0}\lim\limits_{x\to x_0} f(x,y)$ takroriy limitlarni hisoblang.

10.3.1. $f(x,y)=\sin\dfrac{\pi x}{2x+y}$; $x_0=\infty,\ y_0=\infty.$

10.3.2. $f(x,y)=\dfrac{x^2+xy+y^2}{x^2-xy+y^2}$; $x_0=y_0=0.$

10.3.3. $f(x,y)=\log_x(x+y)$; $x_0=1,\ y_0=0.$

10.3.4. $f(x,y)=\dfrac{\sin(x+y)}{2x+3y}$; $x_0=y_0=0.$

10.3.5. $f(x,y)=\dfrac{\cos x-\cos y}{x^2+y^2}$; $x_0=y_0=0.$

10.3.6. $f(x,y)=\dfrac{\sin|x|-\sin|y|}{\sqrt{x^2+y^2}}$; $x_0=y_0=0.$

10.3.7. $f(x,y)=\dfrac{\sin 3x-tg2y}{6x+3y}$; $x_0=y_0=0.$

10.3.8. $f(x,y)=\dfrac{x^2+y^2}{x^2+y^4}$; $x_0=y_0=\infty.$

10.3.9. $f(x,y)=\dfrac{x^y}{1+x^y}$, $x_0=\infty,\ y_0=+0.$

10.3.10. $f(x,y)=\sin\dfrac{\pi x}{2x+y}$; $x_0=\infty,\ y_0=\infty.$

10.3.11. $f(x,y)=\dfrac{1}{xy}tg\dfrac{xy}{1+xy}$; $x_0=0,\ y_0=\infty.$

10.3.12. $f(x,y)=\begin{cases}\left(1+\dfrac{1}{x+y}\right)^{x+y}, x+y\neq 0.\\ 1,\ x+y=0;\ x_0=y_0=\infty.\end{cases}$

10.3.13. $f(x,y)=\begin{cases}\dfrac{x-y+x^2+y^2}{x+y}, & x\neq -y,\\ 0, & x=-y, \qquad x_0=y_0=0.\end{cases}$

10.3.14. $f(x,y)=\dfrac{\sin x+\sin y}{x+y}$; $x_0=y_0=0$

10.3.15. $f(x,y)=\dfrac{x^2\cdot\sin\dfrac{1}{x}+y}{x+y}$; $x_0=y_0=0.$

10.3.16. $f(x,y)=\begin{cases}\dfrac{x^2+y^2}{|x|-|y|}, & |x|\neq|y|,\\ 0, & |x|=|y|;\end{cases}\quad x_0=y_0=0.$

10.3.17. $f(x,y)=\dfrac{\ln\left(x+e^y\right)}{\sqrt{x^2+y^2}};\ x_0=1,\ y_0=0.$

10.3.18. $f(x,y)=\dfrac{5-\sqrt{25-xy}}{xy};\ x_0=y_0=0.$

10.3.19. $f(x,y)=\dfrac{\sqrt{1+x^2y^2}-1}{x^2+y^2};\ x_0=y_0=0.$

10.3.20. $f(x,y)=\dfrac{\ln(x+y)}{y};\ x_0=1,\ y_0=0.$

10.3.21. $f(x,y)=\sin\dfrac{\pi x}{4x+3y};\ x_0=\infty,\ y_0=\infty.$

10.3.22. $f(x,y)=\dfrac{x^2+2xy+y^2}{x^2-2xy+y^2};\ x_0=y_0=0.$

10.3.23. $f(x,y)=\log_x(2x+3y); x_0=1, y_0=0.$

10.3.24. $f(x,y)=\dfrac{\sin(x+y)}{3x+5y};\ x_0=y_0=0.$

10.3.25. $f(x,y)=\dfrac{\cos 2x-\cos 2y}{x^2+y^2};\ x_0=y_0=0.$

10.3.21-misol. $\lim\limits_{y\to0}\lim\limits_{x\to0}\sin\dfrac{\pi(x+y)}{2x+3y}$ va $\lim\limits_{x\to0}\lim\limits_{y\to0}\sin\dfrac{\pi(x+y)}{2x+3y}$ takroriy limitlarni hisoblang.

Yechilishi ([3], 2-q., 38-39 betlar; [9], 2-t., 4- bo'lim; [30], 14- bo'lim). Har bir tayinlangan y va x uchun, mos ravishda takroriy limitlarni hisoblaymiz:

$$\lim_{y\to0}\lim_{x\to0}\sin\frac{\pi(x+y)}{2x+3y}=\lim_{y\to0}\sin\frac{\pi y}{3y}=\frac{\sqrt{3}}{2},\quad \lim_{x\to0}\lim_{y\to0}\sin\frac{\pi(x+y)}{2x+3y}=\lim_{x\to0}\sin\frac{\pi x}{2x}=1.$$

10.4-masala. Xususiy hosilaning ta'rifidan foydalanib, $f(x,y)$ funksiyaning berilgan nuqtadagi xususiy hosilalarini toping.

10.4.1. $f(x,\ y)=\dfrac{x^2}{y+1},\quad f'_x(3,\ 2)=?\ f'_y(3,\ 2)=?$

10.4.2. $f(x,\ y)=e^{-x}\sin y,\quad f'_x(1,\ 2)=?\ f'_y(1,\ 2)=?$

10.4.3 $f(x, y)=e^{-x}\sin y$ funksiyaning $f_x'(1, 2)$, $f_y'(1, 2)$ xususiy hosilalarini $\Delta x=0{,}1$, $\Delta y=0{,}1$ bo'lganda hisoblang.

10.4.4 $z=y\ln(x^2-y^2)$ funksiya $\frac{1}{x}\cdot\frac{\partial z}{\partial x}+\frac{1}{y}\cdot\frac{\partial z}{\partial y}=\frac{z}{y^2}$ tenglamani qanoatlantirishini isbotlang.

10.4.5 $z=y^{\frac{z}{x}}\sin\left(\frac{y}{x}\right)$ funksiya $x^2\frac{\partial z}{\partial x}+xy\frac{\partial z}{\partial y}=yz$ tenglamani qanoatlantirishini isbotlang.

Quyida berilgan $f(x,y,z)$ funksiyaning $f_x'(x,y,z)$, $f_y'(x,y,z)$ va $f_z'(x,y,z)$ xususiy hosilalarini toping.

10.4.6. $f(x,y,z)=z\,\mathrm{arctg}\,\frac{x+y}{1-xy}$. **10.4.7.** $f(x,y,z)=\arcsin\frac{xz}{\sqrt{x^2+y^2}}$

10.4.8. $f(x,y,z)=\frac{1}{\sqrt{x^2+y^2+z^2}}$. **10.4.9.** $f(x,y,z)=x^{y/z}$.

10.4.10. $f(x,y,z)=x^{y^z}$. **10.4.11.** $f(x,y,z)=\frac{xz}{\sqrt{x^2+y^2}}$.

10.4.12. $f(x,y,z)=2y\sqrt{x}+3y^2\sqrt[3]{z^2}$. **10.4.13.** $f(x,y,z)=e^{\frac{x}{y}}+e^{-\frac{z}{y}}$.

10.4.14. $f(x,y,z)=e^{xyz}\cdot\sin xy$.

10.4.15. $f(x,y,z)=(x-y)(x-z)(y-z)$. **10.4.16.** $f(x,y,z)=x^{zy}$.

10.4.17. $f(x,y,z)=\mathrm{tg}\frac{x^2}{yz}$. **10.4.18.** $f(x,y,z)=\frac{x^2}{y^2}-\frac{xz}{y}$.

10.4.19. $f(x,y,z)=e^{(x^2+y^2+z^2)^2}$. **10.4.20.** $f(x,y,z)=\frac{\cos xz}{y}$.

Xususiy hosilaning ta'rifidan foydalanib, $f(x,y)$ funksiyaning berilgan nuqtadagi xususiy hosilalarini toping.

10.4.21. $f(x, y)=\frac{x^2}{y^2+x}$, $f_x'(1, 2)=?$ $f_y'(1, 2)=?$

10.4.22. $f(x, y)=e^{-x+y}\sin y$, $f_x'(1, 1)=?$ $f_y'(1, 1)=?$

10.4.23 $f(x, y)=e^{-x}\sin xy$ funksiyaning $f_x'(1, 2)$, $f_y'(1, 2)$ xususiy hosilalarini $\Delta x=0{,}1$, $\Delta y=0{,}1$ bo'lganda hisoblang.

10.4.24 $z=\ln(x^2+y^2)$ funksiya $\frac{1}{x}\cdot\frac{\partial z}{\partial x}-\frac{1}{y}\cdot\frac{\partial z}{\partial y}=0$ tenglamani qanoatlantirishini isbotlang.

10.4.25 $z=\sin 2\frac{y}{x}$ funksiya $\frac{x}{y}\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}=0$ tenglamani qanoatlantirishini isbotlang.

10.4.26-misol. $f(x,y,z)=\operatorname{arctg}(xyz)$ funksiyaning $f_x'(x,y,z)$, $f_y'(x,y,z)$ va $f_z'(x,y,z)$ xususiy hosilalarini toping.

Yechilishi ([3], 2-q., 59-63 betlar; [9], 2-t., 5- bo'lim; [30], 14- bo'lim). Ushbu $(\operatorname{arctg}u)'=\frac{u'}{1+u^2}$ formulaga asosan, $f_x'(x,y,z)$, $f_y'(x,y,z)$ va $f_z'(x,y,z)$ xususiy hosilalarni topamiz:

$$f_x'(x,y,z)=(\operatorname{arctg}(xyz))_x'=\frac{yz}{1+(xyz)^2},$$

$$f_y'(x,y,z)=\frac{xz}{1+(xyz)^2},\quad f_z'(x,y,z)=\frac{xy}{1+(xyz)^2}.$$

Maple tizimidan foydalanib, misolning javobini tekshirish:

> f:=arctan(x*y*z): Diff(f,x)=simplify(diff(f,x));

$$\frac{\partial}{\partial x}\arctan(x\,y\,z)=\frac{y\,z}{1+x^2\,y^2\,z^2}$$

> Diff(f,y)=simplify(diff(f,y));

$$\frac{\partial}{\partial y}\arctan(x\,y\,z)=\frac{x\,z}{1+x^2\,y^2\,z^2}$$

> Diff(f,z)=simplify(diff(f,z));

$$\frac{\partial}{\partial z}\arctan(x\,y\,z)=\frac{x\,y}{1+x^2\,y^2\,z^2}$$

10.5-masala. Berilgan funksiyaning ko'rsatilgan tartibdagi xususiy hosilasi va differensialini toping.

10.5.1 $f(x,y)=\frac{x+y}{x-y}$, $\frac{\partial^{m+n}f}{\partial x^m\partial x^n}$.

10.5.2 $f(x,y)=x^m y^n$; $\frac{\partial^{m+n}f}{\partial x^m\partial y^n}$.

10.5.3. $f(x,y)=e^{2x}\sin+e^x\cos\frac{y}{2}$; $\frac{\partial^{m+n}f}{\partial x^m\partial x^n}$.

10.5.4 $f(x,y,z)=e^{xyz}$ $\frac{\partial^3 f}{\partial x\partial y\partial z}$.

10.5.5 $f(x,y)=\sin x\cdot\cos 2y;\quad \frac{\partial^{10} f}{\partial x^4 \partial y^6}$.

10.5.6. $f(x,y)=\arcsin\frac{x}{\sqrt{x^2+y^2}};\ \frac{\partial^2 f}{\partial x^2},\frac{\partial^2 f}{\partial y^2}$.

10.5.7 $f(x,y)=(x^2+y)^{10}tgx;\frac{\partial^{10} f}{\partial x\partial y^9}$.

10.5.8 $f(x,y)=\sin xy;\quad \frac{\partial^3 f}{\partial x^2\partial y},\quad \frac{\partial^3 f}{\partial x\partial y^2}$.

10.5.9 $f(x,y)=\sqrt{x^2+y^2}e^{xy};d^2 f$.

10.5.10 $f(x,y,z)=(\frac{x}{y})^2;d^2 f$.

10.5.11 $u=f(x+y,z^2);d^2u$.

10.5.12 $u=f(xy)\cdot g(x\cdot z);d^2u$.

10.5.13 $u=f(\sin x+\cos y);d^2u$.

10.5.14 $u=f(xy,x^2+y^2);\ d^2u.$

10.5.15 $u=f(2x+3y+tz);d^nu$.

5. 3.5.16 $u=f(2x,3y,2z);d^nu$.

10.5.17. $f(x,y,z)=\sqrt[z]{\frac{x}{y}},\ df(1,1,1)$ va $d^2 f(1,1,1)$.

Funksiyaning orttirmasini uning differensialiga almashtirib, quyida berilgan ifodalarni taqribiy hisoblang.

10.5.18. $1{,}002\cdot 2{,}003^2\cdot 3{,}004^3$. **10.5.19** $\sqrt{1{,}02^3+1{,}97^3}$.

10.5.20 $\sin 29^0\cdot tg46^0$.

Berilgan funksiyaning ko'rsatilgan tartibdagi xususiy hosilasi va differensialini toping.

10.5.21 $f(x,y)=\frac{x+y}{x-y},\ \frac{\partial^{10} f}{\partial x^4\partial y^6}$.

10.5.22 $f(x,y)=x^4y^3;\frac{\partial^7 f}{\partial x^4\partial y^3}$.

10.5.23. $f(x,y)=e^{2x}\sin+e^x\cos\frac{y}{2};\frac{\partial^8 f}{\partial x^5\partial x^3}$.

10.5.24 $f(x,y,z)=e^{xyz}\quad \frac{\partial^4 f}{\partial x\partial y\partial z^2}$.

10.5.25 $f(x,y)=\sin 3x\cdot\cos 2y;\quad \dfrac{\partial^7 f}{\partial x^4\partial y^3}$.

10.5.26-misol. $z=e^{x^2y^2}$ fuksiyaning d^2z ikkinchi tartibli to‘liq differensialini toping.

Yechilishi ([3], 2-q., 88-92 betlar; [9], 2-t., 5- bo‘lim; [30], 14- bo‘lim). Berilgan funksiyadan x va y o‘zgaruvchilar bo‘yicha xususiy hosilalarni topamiz:

$$\frac{\partial z}{\partial x}=\left(e^{x^2y^2}\right)'_x=e^{x^2y^2}2xy^2,\quad \frac{\partial z}{\partial y}=\left(e^{x^2y^2}\right)'_y=e^{x^2y^2}2x^2y.$$

Ikkinchi tartibli xususiy hosilalarni topamiz:

$$\frac{\partial^2 z}{\partial x^2}=e^{x^2y^2}\cdot 4x^2y^4+e^{x^2y^2}\cdot 2y^2=2y^2e^{x^2y^2}(2x^2y^2+1),$$

$$\frac{\partial^2 z}{\partial y^2}=e^{x^2y^2}\cdot 4x^4y^2+e^{x^2y^2}\cdot 2x^2=2x^2e^{x^2y^2}(2x^2y^2+1),$$

$$\frac{\partial^2 z}{\partial x\partial y}=e^{x^2y^2}\cdot 4x^3y^3+e^{x^2y^2}\cdot 4xy=4xye^{x^2y^2}(x^2y^2+1),$$

$$\frac{\partial^2 z}{\partial y\partial x}=e^{x^2y^2}\cdot 4x^3y^3+e^{x^2y^2}\cdot 4xy=4xye^{x^2y^2}(x^2y^2+1).$$

Oxirgi ikki ifodani solishtirib, ularning o‘zaro teng ekanligiga ishonch hosil qilamiz. Topilgan ikkinchi tartibli xususiy hosilalarni

$$d^2z=z''_{x^2}dx^2+2z''_{xy}dxdy+z''_{y^2}dy^2$$

formulaga keltirib qo‘yamiz:

$$d^2z=2y^2e^{x^2y^2}\left(2x^2y^2+1\right)dx^2+8xye^{x^2y^2}\left(x^2y^2+1\right)dxdy+2x^2e^{x^2y^2}\left(2x^2y^2+1\right)dy^2.$$

Maple tizimidan foydalanib, misolning javobini tekshirish:

```
> restart; f:=exp(x^2*y^2):
> Diff(f,x$2)=simplify(diff(f,x$2));
```

$$\frac{\partial^2}{\partial x^2}\left(e^{(x^2y^2)}\right)=2y^2e^{(x^2y^2)}(1+2x^2y^2)$$

```
> Diff(f,y$2)=simplify(diff(f,y$2));
```

$$\frac{\partial^2}{\partial y^2}\left(e^{(x^2y^2)}\right)=2x^2e^{(x^2y^2)}(1+2x^2y^2)$$

```
> Diff(f,x,y)=diff(f,x,y);
```

$$\frac{\partial^2}{\partial y\,\partial x}\left(e^{(x^2y^2)}\right)=4xye^{(x^2y^2)}+4x^3y^3e^{(x^2y^2)}$$

10.6-masala. Quyidagi funksiyani ekstremumga tekshiring.

10.6.1. $z = x^2 + (y-1)^2$. **10.6.2.** $u = x^2 + xy + y^2 + \frac{1}{x} + \frac{1}{y}$.

10.6.3. $z = x^2 - (y-1)^2$. **10.6.4.** $u = -x^2 - xy - y^2 + x + y$.

10.6.5. $u = x^3 + y^3 - 3axy$. **10.6.6.** $u = x^4 + y^4 - 36xy$.

10.6.7. $z = x^2 - xy + y^2 - 2x + y$. **10.6.8.** $z = x^2 - 2xy^2 + y^4 - y^5$.

10.6.9. $z = 2x^4 + y^4 - x^2 - zy^2$.

10.6.10. $z = xy + \frac{50}{x} + \frac{20}{y} \quad (x > 0, y > 0)$.

10.6.11. $z = e^{2x}(x + y^2 + 2y)$. **10.6.12.** $z = 3x^2y + y^3 - 18x - 30y$.

10.6.13. $u = xy + yz + zx$. **10.6.14.** $z = (x^2 + y^2)e^{-(x^2+y^2)}$

10.6.15. $z = 4 - (x^2 + y^2)^2 / 3$.

10.6.16. $u = x^2 + 2y^2 + z^2 - 2x + 4y - 6z + 1$.

10.6.17. $u = 2x^2 + y^2 + z^2 - 2xy + 4z - x$.

10.6.18. $u = x^3 + xy + y^2 - 2zx + 2z^2 + 3y - 1$.

10.6.19. $z = 1 - \sqrt{x^2 + y^2}$. **10.6.20.** $z = (x - y + 1)^2$.

10.6.21. $z = (x-1)^2 + (y-1)^2$. **10.6.22.** $u = x^2 + 2xy + y^2$.

10.6.23. $z = x^2 + (y+1)^2$.

10.6.24. $u = -x^2 - 2xy - y^2 + 2x + 2y$.

10.6.25. $u = x^3 + y^3 - 6xy$.

10.6.26-misol . Ushbu

$$z = 2x^3 + xy^2 + 5x^2 + y^2 + 1$$

funksiyaning ekstremumlarini toping.

Yechilishi. Berilgan funksiyaning xususiy hosilalarini topamiz:

$$z'_x = 6x^2 + y^2 + 10x, \; z'_y = 2xy + 2y.$$

Stasionar nuqtalarni topish uchun ushbu

$$\begin{cases} z'_x = 0, \\ z'_y = 0 \end{cases} \Rightarrow \begin{cases} 6x^2 + y^2 + 10x = 0, \\ 2xy + 2y = 0. \end{cases}$$

tenglamalar sistemasini yechamiz. Quyidagi $P_1(0;0)$, $P_2\left(-\frac{5}{3};0\right)$, $P_3(-1;2)$, $P_4(-1;-2)$ nuqtalar sistemaning yechimi bo‘lib, berilgan funksiyaning stasionar nuqtalar bo‘ladi. Endi berilgan funksiyaning ikkinchi tartibli xususiy hosilalarini topamiz:

$$z''_{x^2}=12x+10;\ z''_{xy}=2y;\ z''_{y^2}=2x+2.$$

Ekstremumining yetarli shartiga asosan, har bir stasionar nuqtani ekstremumga tekshiramiz.

1) $P_1(0;0)$ nuqtada

$$z''_{x^2}(P_1)=a_{11}=10;\ z''_{xy}(P_1)=a_{12}=0;\ z''_{y^2}(P_1)=a_{22}=2;\ \Delta(P_1)=a_{11}a_{22}-a_{12}^2=20$$

bo‘lad. Demak, ekstremumning yetarli shartiga asosan, $\Delta(P_1)=20>0$; $a_{11}=10>0$; $a_{22}=2>0$ bo‘lganligi sababli, berilgan funksiya $P_1(0;0)$ nuqtada minimumga erishadi:

$$z_{min}=z(P_1)=1.$$

2) $P_2\left(-\frac{5}{3};0\right)$ nuqtada

$$z''_{x^2}(P_2)=a_{11}=-10;\ z''_{xy}(P_2)=a_{12}=0;\ z''_{y^2}(P_2)=a_{22}=-\frac{4}{3};\ \Delta(P_2)=\frac{40}{3}$$

bo‘ladi. Demak, ekstremumning yetarli shartiga asosan, $\Delta(P_2)>0$; $a_{11}=-10<0$; $a_{22}=-\frac{4}{3}<0$ bo‘lganligi sababli, berilgan funksiya $P_2\left(-\frac{5}{3};0\right)$ nuqtada maksimumga erishadi:

$$z_{max}=z(P_2)=5\frac{17}{27}.$$

3) $P_3(-1;2)$ nuqtada

$$z''_{x^2}(P_3)=a_{11}=-2;\ z''_{xy}(P_3)=a_{12}=4;\ z''_{y^2}(P_3)=a_{22}=0;\ \Delta(P_3)=-16$$

bo'ladi. Demak, ekstremumning yetarli shartiga asosan, $\Delta(P_3)<0$ bo'lganligi sababli, berilgan funksiya $P_3(-1;2)$ nuqtada ekstremumga erishmaydi.

4) $P_4(-1;-2)$ nuqtada

$$z''_{x^2}(P_4)=a_{11}=-2;\ z''_{xy}(P_4)=a_{12}=-4;\ z''_{y^2}(P_4)=a_{22}=0;\ \Delta(P_4)=-16$$

bo'ladi. Demak, ekstremumning yetarli shartiga asosan, $\Delta(P_4)<0$ bo'lganligi sababli, berilgan funksiya $P_4(-1;-2)$ nuqtada ekstremumga erishmaydi.

10.7-masala. Berilgan funksiyaning ko'rsatilgan to'plamdagi eng katta va eng kichik qiymatlarini toping.

10.7.1. $u=xy-x^2y-\dfrac{xy^2}{2}$, $\quad 0\le x\le 1,\ 0\le y\le 2.$

10.7.2. $u=x^2+3y^2-x+18y-4$, $\quad 0\le x\le 1,\ 0\le y\le 1.$

10.7.3. $u=x^3+3y^2-3xy$, $\quad 0\le x\le 2,\ 0\le y\le 1.$

10.7.4. $u=\dfrac{xy}{2}-\dfrac{x^2y}{6}-\dfrac{xy^2}{8}$, $\quad x\ge 0, y\ge 0,\ \dfrac{x}{3}+\dfrac{y}{4}\le 1.$

10.7.5. $u=x^6+y^6-3x^2+6xy-3y^2$, $\quad 0\le y\le x\le 2.$

10.7.6. $u=\cos x\cos y\cdot\cos(x+y)$, $\quad 0\le x\le\pi,\ 0\le y\le\pi.$

10.7.7. $u=(x-y^2)\sqrt[3]{(1-x)^2}$, $\quad y^2\le x\le 2.$

10.7.5. $u=x^3+y^3-9xy+2x+27$, $\quad 0\le x\le 6\ \ 0\le y\le 6.$

10.7.9. $u=x^4+y^4-2x^2+4xy-2y^2$, $0\le x\le 2,\ 0\le y\le 2.$

10.7.10. $u=xy+yz+zx$, $\quad x^2+y^2+z^2\le 9.$

10.7.11. $u=x+y+z$, $\quad x^2+y^2\le z\le 1.$

10.7.12. $u=2\sin x+2\sin y+\sin(x+y)$, $\quad 0\le x\le\dfrac{\pi}{2},\ 0\le y\le\dfrac{\pi}{2}.$

10.7.13. $z=x^2+y^2$, $\quad x^2+y^2\le 4.$

10.7.14. $z=x^2+2xy-4x+8y$, $\quad x=0,\ x=1,\ y=0,\ y=2.$

10.7.15. $z=e^{-x^2-y^2}(2x^2+3yz)$, $x^2+y^2\le 4.$

10.7.16. $z=x-2y-3$, $\ 0\le x\le 1,\ 0\le y\le 1.$

10.7.17. $z=x^2+y^2-12x+16y$, $\ x^2+y^2\le 25.$

10.7.15. $z=x^2-xy+y^2$, $\ |x|+|y|\le 1.$

10.7.19. $u = x^2 + 2y^2 + 3z^2,\ x^2 + y^2 + z^2 \le 100.$

10.7.20. $u = (x + y + z)e^{-(x+2y+2z)},\ x > 0,\ y > 0,\ z > 0.$

10.7.21. $u = xy - 2x^2y, \qquad 0 \le x \le 1, 0 \le y \le 2.$

10.7.22. $u = x^2 + 5y^2 + x + 18y - 4, \quad 0 \le x \le 1, 0 \le y \le 1.$

10.7.23. $u = x^3 + 3y^2 - 4xy, \qquad 0 \le x \le 2, 0 \le y \le 1.$

10.7.24. $u = \dfrac{xy}{4} - \dfrac{x^2y}{2} - \dfrac{xy^2}{4}, \qquad x \ge 0, y \ge 0, \dfrac{x}{3} + \dfrac{y}{4} \le 1.$

10.7.25. $u = x^3 + y^3 - 3x^2 + 4xy - 3y^2, \quad 0 \le y \le x \le 2.$

10.7.26-misol. $z = x^2 + y^2 - xy + x + y$ funksiyaning $x = 0,\ y = 0,$ $x + y = -3$ chiziqlar bilan chegaralangan sohadagi eng katta va eng kichik qiymatlarini toping.

Yechilishi ([30], 14- bo'lim). Oxy tekislikda $\overline{D}$ sohaga tegishli stasionar nuqtalarni aniqlaymiz:

$$\left.\begin{aligned} \frac{\partial z}{\partial x} &= 2x - y + 1 = 0, \\ \frac{\partial z}{\partial y} &= 2y - x + 1 = 0, \end{aligned}\right|$$

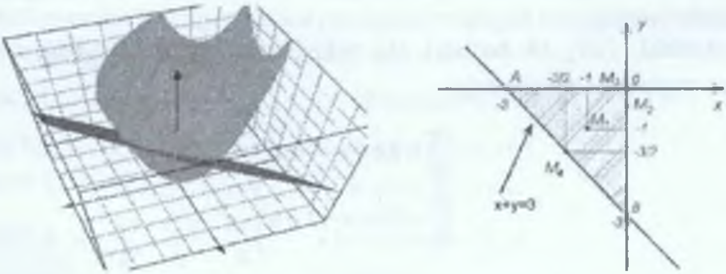
bundan $x = -1,\ y = -1.$ $M(-1,-1)$ nuqtani hosil qildik, bu nuqtada $z_1 = z(-1,-1) = -1.$

Berilgan funksiyani sohaning chegarasida tekshiramiz. OB to'g'ri chiziqda $x = 0$ bo'lib, $z = y^2 + y$ tenglamaga ega bo'lamiz va bu tenglama $[-3,0]$ kesmada bir o'zgaruvchili funksiyaning eng katta va eng kichik qiymatini topish masalasiga keladi. $z'_y = 2y + 1$ ni topib, uni nolga tenglaymiz: $2y + 1 = 0,$ bundan $y = -\dfrac{1}{2};\ z''_{yy} = 2$ bo'lgani uchun, $M_2\left(0, -\dfrac{1}{2}\right)$ - shartli lokal minimum nuqtasiga ega bo'lamiz va unda $z_2 = z\left(0, -\dfrac{1}{2}\right) = -\dfrac{1}{4}$ qiymatni hosil qilamiz. OB kesmaning uchlarida: $z_3 = z(0, -3) = 6, \qquad z_4 = z(0,0) = 0.$

OA to'g'ri chiziqda, $y = 0$ bo'lib, $z = x^2 + x$ ni hosil qilamiz. $z'_x = 2x + 1 = 0,\ x = -\dfrac{1}{2},\ z''_{xx} = 2,$ ya'ni $M_3\left(-\dfrac{1}{2}, 0\right)$ lokal minimum nuqtasi bo'lib, unda $z_5 = z\left(-\dfrac{1}{2}, 0\right) = -\dfrac{1}{4}.$ A nuqtada: $z_6 = z(-3,0) = 6.$

AB kesmaning tenglamasi $x+y=-3$ bo‘lib, undan $y=-x-3;\ z=3x^2+9x+6,\ z'_x=6x+9,\ x=-\frac{3}{2},\ M_4\left(-\frac{3}{2},-\frac{3}{2}\right)$ stasionar nuqtaga ega bo‘ldik: $z_7=z\left(-\frac{3}{2},-\frac{3}{2}\right)=-\frac{3}{4}$. Funksiyaning AB kesma uchlaridagi qiymatlari yuqorida aniqlangan edi.

Berilgan z funksiyaning topilgan barcha qiymatlarini solishtirib, funksiya $A(-3;0)$ va $B(0;-3)$ nuqtalarda, eng katta $z_{\text{eng katta}}=6$ va $M_1(-1,-1)$ stasionar nuqtada, eng kichik $z_{\text{eng kichik}}=-1$ qiymatlarga erishishini aniqlaymiz (1-chizma).



1-chizma

10.8-masala. Quyidagi skalar maydonning berilgan nuqtadagi gradiyenti, berilgan yo‘nalishdagi hosilasini toping.

10.8.1. $u(x,y)=x^2-2xy+3y-1,\ grad u\,\big|_{M(x,y)}=?$

10.8.2. $u(x,y)=5x^2y-3xy^3+y^4,\ grad u\,\big|_{M(x,y)}=?$

10.8.3. $u=x^2+y^2,\ grad u\big|_{(3;\,2)}=?$

10.8.4. $u=\sqrt{4+x^2+y^2},\ grad u\big|_{(2;1)}?$

10.8.5. $u=\operatorname{arctg}\frac{y}{x},\qquad grad u\big|_{(x_0,y_0)}=?$

10.8.6. $u=\arcsin\frac{x}{x+y}$ skalyar maydoning $(1,1)$, $(3,4)$ nuqtalardagi gradiyentlari orasidagi burchakni toping.

10.8.7. $z_1=\sqrt{x^2+y^2},\quad z_2=x-3y+\sqrt{3xy}$ funksiyalarning $(3,4)$ nuqtadagi gradiyentlari orasidagi burchakni toping.

10.8.8. $grad(\varphi\psi)=\varphi\, grad\psi+\psi\, grad\varphi$ tenglikni isbotlang.

10.8.9. $z=\varphi(u,v),\ u=\psi(x,y),\ v=\zeta(x,y)$ funksiyalar berilganda $grad\, z=\frac{\partial\varphi}{\partial u}gradu+\frac{\partial\varphi}{\partial v}gradv$ tenglikning to'g'riligini ko'rsating.

10.8.10. $u=\ln(x^2+y^2+z^2)$ funksiyaning $u=2\ln 2-\ln(gradu)^2$ munosabatni qanoatlantirishini ko'rsating.

10.8.11. $z=x^3-3x^2y+3xy^2+1$ funksiyaning $M(3;1)$ nuqtadagi shu nuqtadan $N(6;5)$ nuqtaga qarab yo'nalgan, yo'nalish bo'yicha hosilasini toping.

10.8.12. $z=arctgxy$ funksiyaning (1,1) nuqtadagi, shu nuqtadan birinchi chorak bissektrissasi yo'nalishi bo'yicha hosilasini toping.

10.8.13. $z=x^2y^2-xy^3-3y-1$ funksiyaning (2,1) nuqtadagi shu nuqtadan koordinatalar boshiga qarab yo'nalgan, yo'nalish bo'yicha hosilasini toping.

10.8.14. $z=\ln(e^x+\ln^y)$ funksiyaning koorinatalar boshidagi abssissa o'qi bilan φ burchak tashkil qilgan nurning yo'nalishi bo'yicha hosilasini toping.

10.8.15. $f(x,y)=x^3-y^3$ funksiyaning (2,-1) nuqtadagi $\bar{n}=(1,-2)$ yo'nalish bo'yicha hosilasini toping

10.8.16. $f(x,y)=xe^y$ funksiyaning (3,0) nuqtadagi $\vec{n}=(4,-3)$ yo'nalish bo'yicha hosilasini toping.

10.8.17. $f(x,y,z)=x^2+y^2-z^2$ funksiyaning (2,3,4) nuqtadagi $\vec{n}=(2,-2,1)$ yo'nalish bo'yicha hosilasini toping.

10.8.18. $f(x,y,z)=e^{x-z}\cos y$ funksiyaning (1;0;4) nuqtadagi $\vec{n}=(1;1;1)$ yo'nalish bo'yicha hosilasini toping.

10.8.19. $f(x,y,z)=3x^2y^2+2yz$ funksiyaning (-1,0,4) nuqtadagi $\vec{n}=(-1,3,3)$ yo'nalish bo'yicha hosilasini toping.

10.8.20. $f(x,y,z)=x^2-y^2-3$ egri chiziqqa (2,1) nuqtada o'tkazilgan normal vektorni toping.

10.8.21. $u(x,y)=x^3-2xy+3y^3-1,\ gradu\,\big|_{M(2,1)}=?$

10.8.22. $u(x,y)=5x^4y-3xy^3+y^3,\ gradu\,\big|_{M(x,y)}=?$

10.8.23. $u=x^2+y^2+2xy,\ gradu\big|_{(1;2)}=?$

10.8.24. $u=\sqrt{4+x^2+y^2},\ gradu\big|_{(2,1)}?$

10.8.25. $u = \operatorname{arctg}\frac{y}{x}$, $\quad gradu\big|_{(1,1)} = ?$

10.8.26-misol. $u = \operatorname{arctg} xy$ skalar maydonning $y = x^2$ parabolaga tegishli nuqtadagi hosilasini shu parabolalar yo'nalishi bo'yicha hisoblang (abssissalar o'qining o'sishi yo'nalishida).

Yechilishi ([9], 2-t., 5- bo'lim; [30], 14- bo'lim). $y = x^2$ parabolaga tegishli $P(1, 1)$ nuqtadagi $\vec{l}$ yo'nalish, unga shu nuqtaga o'tkazilgan urinmaning yo'nalishi bilan ustma-ust tushadi.

Parabolaga tegishli $P(1, 1)$ nuqtadagi $\vec{l}$ urinma Ox o'q bilan α burchak tashkil qilsin. Hosilaning geometrik ma'nosiga ko'ra, quyidagiga ega bo'lamiz: $y' = 2x$, $tg\alpha = y'\big|_{x=1} = 2$.

Urinmaning yo'naltiruvchi kosunuslarini topamiz:

$$\cos\alpha = \frac{1}{\sqrt{1 + tg^2\alpha}} = \frac{1}{\sqrt{5}}, \quad \cos\beta = \sin\alpha = \frac{tg\alpha}{\sqrt{1 + tg^2\alpha}} = \frac{2}{\sqrt{5}}.$$

Berilgan $u = \operatorname{arctg} xy$ skalar maydonning $P(1, 1)$ nuqtadagi xususiy hosilalarining qiymatlarini hisoblaymiz:

$$\left.\frac{\partial u}{\partial x}\right|_P = \left.\frac{y}{1 + (xy)^2}\right|_P = \frac{1}{2}, \quad \left.\frac{\partial u}{\partial y}\right|_P = \left.\frac{x}{1 + (xy)^2}\right|_P = \frac{1}{2}.$$

Topilgan qiymatlarni, $\frac{\partial u(x_0, y_0)}{\partial \ell} = \frac{\partial f(x_0, y_0)}{\partial x}\cos\alpha + \frac{\partial f(x_0, y_0)}{\partial y}\sin\alpha$

formulaga keltirib qo'yamiz: $\frac{\partial u}{\partial l} = \frac{1}{2}\frac{1}{\sqrt{5}} + \frac{1}{2}\frac{2}{\sqrt{5}} = \frac{3}{2\sqrt{5}}$.

11-BOB. SONLI QATORLAR

11.1- §. Sonli qatorlar. Yaqinlanuvchi qatorlarning asosiy xossalari

11.1. Qator tushunchasi va uning yaqinlashuvchanligi. Ushbu

$$a_1, a_2, \ldots, a_n, \ldots$$

cheksiz sonlar ketma – ketligi berilgan bo'lib, bu sonlardan

$$a_1 + a_2 + \ldots + a_n + \ldots \qquad (11.1.1)$$

belgini tuzamiz va uni cheksiz *(sonli) qator* deb ataymiz. (11.1.1) qator ko'p hollarda, qisqacha $\sum_{n=1}^{\infty} a_n$ simvol orqali yoziladi, bunda $a_1, a_2, \ldots, a_n, \ldots$ lar qatorning *hadlari*, a_n esa, qatorning umumiy hadi deyiladi.

Qatorning hadlaridan quyidagicha, yangi sonlar ketma – ketligi $\{S_n\}$ ni tuzamiz:

$$\left.\begin{array}{l} S_1 = a_1, \\ S_2 = a_1 + a_2, \\ S_n = a_1 + a_2 + a_3 + \ldots + a_n, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{array}\right|$$

$\{S_n\}$ sonlar ketma ketligi (11.1.1) qatorning qismiy yig'indilar ketma – ketligi deyiladi. Jumladan, $S_n = \sum_{k=1}^{n} a_k$ yig'indiga (11.1.1) qatorning n-qismiy yig'indisi deyiladi.

$\sum_{n=1}^{\infty} a_n$ qator qismiy yig'indilar ketma – ketligi $\{S_n\}$ ning $n \to \infty$ dagi limiti S $\left(\lim\limits_{n\to\infty} S_n = S\right)$ ga qatorning *yig'indisi* deyiladi va u

$$S = a_1 + a_2 + a_3 + \ldots + a_n + \ldots\ldots = \sum_{n=1}^{\infty} a_n$$

kabi belgilanadi.

11.1.2-ta'rif. Agar S – chekli bo'lsa, qator yaqinlashuvchi, cheksiz, ya'ni $S=\infty$, $S=\pm\infty$ yoki $\lim\limits_{n\to\infty} S_n$ – mavjud bo'lmasa, qator *uzoqlashuvchi* deyiladi.

Shunday qilib, har bir qator ikkita ketma – ketlik bilan bir qiymatli aniqlanadi. Haqiqatan ham, agar qator hadlari ketma – ketligi a_n berilganda, qatorning qismiy yig'indilar ketma – ketligi $S_n = a_1 + a_2 + ... + a_n$, $n = 1,2,3,....$ formula bo'yicha topiladi.

Agar qatorning qismiy yig'indilar ketma – ketligi $\{S_n\}$ berilgan bo'lsa, qatorning hadlari $a_1 = S_1$, $a_m = S_n - S_{n-1}$, $n = 2,3......$ formula bilan aniqlanadi.

11.1.3-teorema. Agar $\forall n \in N$ uchun $a_n = b_n - b_{n+1}$ tenglik o'rinli bo'lib, $\exists \lim\limits_{n\to\infty} b_n = b$ – chekli bo'lsa, u holda (11.1.1) qator yaqinlashuvchi bo'lib, uning yig'indisi $S = b_1 - b$ ga teng bo'ladi.

Isboti. Teoremaning shartlariga ko'ra,

$$S_n = \sum_{k=1}^{n} a_k = \sum_{k=1}^{n}(b_k - b_{k+1}) = b_1 - b_2 + b_2 - b_3 + ... + b_n - b_{n+1},$$

$\lim\limits_{n\to\infty} b_{n+1} = b$, $S = b_1 - b$ bo'ladi.■

11.1.4-teorema (qator yaqinlashishining zaruriy sharti). Agar (11.1.1) qator yaqinlanuvchi bo'lsa, u holda

$$\lim_{n\to\infty} a_n = 0 \qquad (11.1.5)$$

bo'ladi.

Isboti. Agar (11.1.1) qator yaqinlanuvchi bo'lsa, u holda uning xususiy yig'indilari ketma-ketligi $S_n(n = 1,2,.....)$ va $S_{n-1}(n = 2,3,.....)$ bir xil limit S ga ega bo'ladi, ya'ni $\lim\limits_{n\to\infty} S_{n-1} = S$. Ravshanki, $a_n = S_n - S_{n-1}, n = 2,3,......$ Bundan,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}(S_n - S_{n-1}) = \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1} = 0.■$$

11.1.6-eslatma. (11.1.5) shart, qator yaqinlashishi uchun yetarli shart bo'lolmaydi.

11.2. Yaqinlashuvchi qatorlarning asosiy xossalari

1⁰. Agar (11.1.1) va $\sum\limits_{n=1}^{\infty} b_n$ qatorlar yaqinlashuvchi bo'lib, ularning yig'indilari mos ravishda S va σ bo'lsa, u holda $\forall \lambda, \mu \in R$ uchun

$$\sum_{n=1}^{\infty}(\lambda a_n + \mu b_n)$$

qator ham yaqinlashuvchi va uning yig'indisi $A = \lambda S + \mu\sigma$ bo'ladi.

Agar (11.1.1) qatorning dastlabki m ta hadi olib tashlansa, u holda hosil bo'lgan, ushbu

$$a_{m+1} + a_{m+2} + ... + a_{m+k} + ... = \sum_{n=m+1}^{\infty} a_n \qquad (11.2.1)$$

qatorga (11.1.1) qatorning $m-$ *qoldig'i* deyiladi va uni $r_m = \sum_{n=m+1}^{\infty} a_n$ kabi belgilaymiz.

2⁰. Agar (11.1.1) qator yaqinlashuvchi bo'lsa, u holda uning ixtiyoriy qoldig'i (11.2.1) ham yaqinlashuvchi bo'ladi va aksincha (11.2.1) qoldiq qator yaqinlashuvchi bo'lsa, (11.1.1) qator ham yaqinlashuvchi.

11.2.2-natija (11.1.1) qator yaqinlashuvchi bo'lsa, uning qoldig'i r_m, $m \to \infty$ da nolga intiladi.

Haqiqatan ham, (11.1.1) qator yaqinlashuvchi va uning yig'indisi S bo'lsa, u holda $S = S_m + r_m$ bo'ladi, bundan $r_m = S - S_m$ bo'lib,

$$\lim_{n\to\infty} r_m = S - S = 0$$

bo'ladi.

11.2.3-eslatma. 2^0-ga asosan, qatorning chekli sondagi hadlarni olib tashlash yoki unga chekli sondagi hadlarni qo'shish bilan qatorning yaqinlashish xarakterli o'zgarmaydi.

3⁰. Agar (11.1.1) qator yaqinlashuvchi bo'lsa, uning hadlarining o'rnilarini o'zgartirmasdan gruppalab tuzilgan

$$\sum_{j=1}^{\infty} b_j \qquad (11.2.4)$$

qator ham yaqinlashuvchi va uning yig'indisi (11.1.1) qatorning yig'indisiga teng bo'ladi.

11.2.5-eslatma. 3^0 – xossasaning teskarisi o'rinli emas, ya'ni (11.2.4) qatorning yaqinlashishidan, (11.1.1) qatorning yaqinlashishi kelib chiqmaydi.

11.3. Musbat sonli qatorlar va ularning yaqinlashish sharti. Ushbu

$$\sum_{n=1}^{\infty} a_n \qquad (11.3.1)$$

qatorning hamma hadlari manfiy bo'lmasin, ya'ni $\forall n \in N$ uchun $a_n \geq 0$ bo'lsin.

11.3.2-teorema. (11.3.1) qatorning yaqinlashuvchi bo'lishi uchun uning qismiy yig'indilari ketma – ketligi $\{S_n\}$ ning yuqoridan chegaralangan, ya'ni

$$\exists M > 0 : \forall n \in N \to S_n = \sum_{k=1}^{n} a_k \leq M \qquad (11.3.3)$$

bo'lishi zarur va yetarli.

Isboti. Zaruriyligi. Ravshanki, $\{S_n\}$- o'suvchi ketma-ketlik shartga ko'ra, $S_n - S_{n-1} = a_n \geq 0,\ n \geq 1$ bo'ladi.

Agar hadlari manfiy bo'lmagan (11.3.1) qator yaqinlashuvchi bo'lsa, chekli $\lim\limits_{n\to\infty} S_n = S$ mavjud bo'ladi, u holda o'suvchi ketma – ketlikning limiti haqidagi teoremaga asoson, $\forall n \in N \to S_n \leq S$, ya'ni (11.3.3) shart bajariladi.

Yetarliligi. (11.3.1) qator (11.3.3) shartni qanoatlantirsin, u holda $\{S_n\}$ o'suvchi ketma – ketlik yuqoridan chegaralangan bo'ladi.

Demak, u chekli limitga ega, ya'ni (11.1.1) qator yaqinlashuvchi bo'ladi. ■

11.3.4-teorema. Agar

$$\sum_{n=1}^{\infty} a_n,\ a_n \geq 0,\ n = 1,2,\ldots\ldots \qquad (11.3.5)$$

qatorning hadlari monoton kamayuvchi, ya'ni $a_1 \geq a_2 \geq \ldots\ldots \geq a_n \geq \ldots$ bo'lsa, u holda (11.3.5) qator bilan

$$\sum_{k=0}^{\infty} 2^k a_{2k} = a_1 + 2a_2 + 4a_4 + \ldots\ldots \qquad (11.3.6)$$

qator bir vaqtda yaqinlashuvchi yoki bir vaqtda uzoqlashuvchi bo'ladi.

11.4. Qatorlarni taqqoslash haqidagi teorema

11.4.1-teorema. Agar $\forall n \in N$ uchun

$$0 \leq a_n \leq b_n \qquad (11.4.2)$$

shart bajarilsa, u holda

$$\sum_{n=1}^{\infty} b_n \qquad (11.4.3)$$

qatorning yaqinlashuvchiligidan (11.3.1) qatorning yaqinlashuvchanligi, (11.3.1) qatorning uzoqlashuvchiligidan esa, (11.4.3) qatorning uzoqlashuvchiligi kelib chiqadi.

11.4.4-eslatma. Agar (11.4.2) shart biror m – (belgilangan) nomerdan boshlab, ya'ni $\forall n \geq m$ uchun bajarilganda ham teorema o'z kuchini saqlaydi.

11.4.5-natija.

$$b_n \neq 0,\ n = 1,2,\dots \text{ va } \lim_{n\to\infty} \frac{a_n}{b_n} = k \qquad (11.4.6)$$

bo'lsin, u holda:

1) agar (11.4.3) qator yaqinlashuvchi va $k < \infty$ bo'lsa, u holda (11.3.31) qator ham yaqinlashuvchi bo'ladi;

2) agar (11.4.3) qator uzoqlashuvchi va $k > 0$ bo'lsa, u holda (11.3.1) qator ham uzoqlashuvchi bo'ladi.

11.4.7-eslatma. Agar $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = k$ limit mavjud bo'lib, $0 < k < \infty$ bo'lsin, (11.3.1) va (11.4.3) qatorlar bir vaqtda yaqinlashuvchi yoki bir vaqtda uzoqlashuvchi bo'ladi.

11.4.8-eslatma. Agar $n \geq n_0$ uchun $a_n > 0, b_n > 0$ bo'lib, $n \to \infty$ da $a_n \sim b_n$, ya'ni $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = 1$ bo'lsa, (11.3.1) va (11.4.3) qatorlar bir vaqtda yaqinlashuvchi yoki bir vaqtda uzoqlashuvchi bo'ladi.

11.4.9-natija. Agar (11.3.1) va (11.4.3) qatorning hadlari $k \geq m$ uchun $a_k > 0, b_k > 0$ bo'lib

$$\frac{a_{k+1}}{a_k} \leq \frac{b_{k+1}}{b_n} \qquad (11.4.10)$$

shartni qanoatlantirsa, u holda (11.4.3) qatorning yaqinlashishdan (11.3.1) qatorning yaqinlashishi, (11.3.1) qatorning uzoqlashishidan (11.4.3) qatorning uzoqlauiiui kelib chiqadi.

11.2- §. Musbat sonli qatorlar uchun yaqinlashish alomatlari

$$\sum_{n=1}^{\infty} a_n \; (a_n \geq 0,\; n = 1,2,.....)$$

qator berilgan bo'lsin. Yuqoridagi 11.4.1-teoremadagi

$$\sum_{n=1}^{\infty} b_n \, (b_n \geq 0, n = 1,2.....)$$

qator vazifasida standart yaqinlashuvchi va uzoqlanuvchi ushbu

$$\sum_{n=1}^{\infty} q^{n-1} (0 < q < 1),\; \sum 1 = 1 + 1 + ... + 1 + ...$$

qatorlarni olib, ularni berilgan qator bilan taqqoslash natijasida quyidagi yaqinlashish alomatlari Dalamber va Koshi alomatlarini hosil qilamiz.

11.5. Dalamber alomati.

11.5.1-teorema. Ushbu

$$\sum_{n-1}^{\infty} a_n \; (a_n > 0,\; n = 1,2,.....) \qquad (11.5.2)$$

(11.5.2) qatorning hadlari $\forall n \in N$ uchun $a_n > 0$ shartni qanoatlantirsin. U holda; a) agar shunday $q(0 < q < 1)$ soni va n_0 nomer topilib, $\forall n \geq n_0$ uchun

$$D_n = \frac{a_{n+1}}{a_n} \leq q \qquad (11.5.3)$$

tengsizlik o'rinli bo'lsa, (11.5.2) qator yaqinlashuvchi bo'ladi;

b) agar shunday n_0 nomer mavjud bo'lib,

$$\forall n \geq n_0 \text{ uchun } D_n = \frac{a_{n+1}}{a_n} \geq 1 \qquad (11.5.4)$$

tengsizlik bajarilsa, (11.5.2) qator uzoqlashuvchi bo'ladi.

Isboti. a) $0 < q < 1$ va (11.5.3) shart o'rinli bo'lsin. U holda

$$a_{n_0+1} \leq a_{n_0} q,$$

$$a_{n_0+2} \leq a_{n_0+1} q \leq a_{n_0} q^2,$$

....

$$a_{n_0+p} \leq a_{n_0+p-1} q \leq .... \leq a_{n_0} q^p,\; p \in N,$$

....

Ravshanki, $\sum_{p=1}^{\infty} a_{n_0} q^p$ (bunda $0<q<1$) qator yaqinlashuvchi, u holda

11.4.1- taqqoslash teoremasiga asosan, $\sum_{p=1}^{\infty} a_{n_0+p}$ qator ham yaqinlashuvchi

bo'ladi. Bundan (11.5.2) qatorning yaqinlashuchiligi $(a_n>0,\ n=1,2,......)$ kelib chiqadi.

b) (11.5.4) shartdan,

$$a_{n_0+1} \ge a_{n_0},\ a_{n_0+2} \ge a_{n_0+2} \ge a_{n_0+1} \ge a_{n_0},\ a_{n_0+3} \ge a_{n_0},....,$$

bundan

$$\forall p \in N \text{ uchun } a_{n_0+p} \ge a_{n_0} > 0 \qquad (11.5.5)$$

Demak, $\sum_{p=1}^{\infty} a_{n+p}$ qator va u bilan birga (11.5.2) qator uzoqlashuvchi

bo'ladi, chunki (11.5.5) shartga asosan, $\lim_{n\to\infty} a_n \ne 0$, ya'ni qator yaqinlashishining zaruriy shart bajarilmaydi.■

11.5.6-natija (Dalamber alomatining limitik ko'rinishi). Ushbu

$$\sum_{n=1}^{\infty} a_n,\ a_n>0,\ n=1,2,.......$$

qator berilgan bo'lsin.

Agar

$$\exists \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lambda \qquad (11.5.7)$$

bo'lib, $\lambda<1$ bo'lsa, (11.5.2) qator yaqinlashuvchi, $\lambda>1$ bo'lsa, (11.5.2) qator uzoqlashuvchi bo'ladi.

11.6. Koshi alomati

11.6.1-teorema. Agar: a)

$$\sum_{n=1}^{\infty} a_n,\ a_n \ge 0,\ \forall n \in N \qquad (11.6.2)$$

qator berilgan bo'lib, shunday q $(0<q<1)$ va n_0 nomer mavjud bo'lib, $\forall n \ge n_0$ lar uchun

$$\sqrt[n]{a_n} \le q \qquad (11.6.3)$$

tengsizlik o'rinli bo'lsa, (11.6.2) qator yaqinlashuvchi bo'ladi:

b) agar shunday n_0 nomer mavjud bo‘lib, $\forall n \geq n_0$ lar uchun

$$\sqrt[n]{a_n} \geq 1 \qquad (11.6.4)$$

tengsizlik bajarilsa, (11.6.2) qator uzoqlashuvchi bo‘ladi.

Isboti. a) (11.6.3) shartdan $\forall n \geq n_0$ lar uchun $a_n \leq q^n$ tengsizlik bajariladi. $\sum_{n=n_0}^{\infty} q^n$ $(0 < q < 1)$ qator yaqinlashuvchi bo‘lganligi uchun, 11.4.1 – teoremaga ko‘ra, berilgan (11.6.2) qator ham yaqinlashuvchi bo‘ladi.

b) (11.6.4) shartdan, $a_n \geq 1$ tengsizlik $\forall n \geq n_0$ lar uchun bajariladi. $\sum 1 = 1+1+....+1+....$ qator uzoqlashuvchi bo‘lgani uchun, yana 11.4.1–teoremaga asosak, (11.6.2) qator ham uzoqlashuvchi bo‘ladi. ■

11.6.5-natija (Koshi alomatining limitik ko‘rinishi). Agar

$$\exists \lim_{n\to+\infty} \sqrt[n]{a_n} = \lambda \qquad (11.6.6)$$

bo‘lib, $\lambda < 1$ bo‘lganda, (11.6.2) qator yaqinlashuvchi, $\lambda > 1$ bo‘lganda esa, (11.6.2) qator uzoqlashuvchi bo‘ladi.

11.6.7-eslatma. Agar (11.5.7) shart yoki (11.6.6) shart $\lambda = 1$ da bajarilsa, (11.5.2) qator yaqinlashuvchi ham, uzoqlashuvchi ham bo‘lishi mumkin, ya’ni Dalamber (Koshi) alomati $\lambda = 1$ da (11.5.2) qatorning yaqinlashish, uzoqlashish masalasini hal qilolmaydi. Masalan, $\sum_{n=1}^{\infty} \frac{1}{n}$ va $\sum_{n=1}^{\infty} \frac{1}{n^2}$ qatorlar (11.5.7) va (11.6.6) shartlarni qanoatlantiradi, lekin birinchi qator uzoqlashuvchi, ikkinchi qator esa, yaqinlashuvchi.

11.6.8-eslatma. (11.5.7) limitning mavjudligidan (11.6.6.) limitning mavjudligi kelib chiqadi va ular bir – biriga teng bo‘ladi. Buning aksi o‘rinli emas.

11.7. Raabe alomati. Biz yuqorida berilgan musbat qatorni taqqoslash uchun geometrik progressiyani olib, Dalamber va Koshi alomatlarini keltirib chiqardik. Endi, ularga nisbatan «sezgirroq» alomatini keltirib chiqarish uchun berilgan qatorni taqqoslash maqsadida, geometrik progressiyaga nisbatan «sekinroq» yaqinlashadigan, uzoqlashadigan standart $\sum_{n=1}^{\infty} \frac{1}{n^s}$ qatorni olib, quyidagi Rabe alomatini keltirib chiqaramiz.

11.7.1-teorema (Raabe alomati). Agar n ning hamma qiymatida yoki biror n_0 qiymatidan boshlab

$$n\left(\frac{a_n}{a_{n+1}}-1\right)\geq q>1 \quad \left(n\left(\frac{a_n}{a_{n+1}}-1\right)\leq 1\right) \qquad (11.7.2)$$

tengsizlik bajarilsa (11.5.2) qator yaqinlashuvchi (uzoqlashuvchi) bo'ladi.

11.7.3-natija. Agar $R_n = n\left(\frac{a_n}{a_{n+1}}-1\right)$ ifoda chekli yoki cheksiz $\lim\limits_{n\to+\infty} R_n = R$ limitga ega bo'lsa, u holda $R>1$ bo'lganda, (11.5.2) qator yaqinlashuvchi, $R<1$ bo'lganda esa, (11.5.2) qator uzoqlashuvchi bo'ladi.

Dalamber alomati bilan Raabe alomati solishtirilganda, Raabe alomati, Dalamber alomatiga qaraganda kuchliroq (segirroq) alomat ekanligini ko'rish mumkin.

Agar $\lim\limits_{\lambda\to\infty} D_n = \lim\limits_{n\to+\infty}\frac{a_{n+1}}{a_n}=\lambda$ mavjud va $\lambda\neq 1$ bo'lsa, u holda $\lim\limits_{n\to+\infty} R_n = \lim\limits_{n\to+\infty} n\left(\frac{a_n}{a_{n+1}}-1\right)=R$ mavjud va $\lambda<1$ bo'lganda, $R=+\infty, \lambda>1$ da $R=-\infty$ teng bo'ladi.

Shunday qilib, agar Dalamber alomati berilgan qatorning yaqinlashish xarakterini aniqlasa, Raabe alomati ham albatta qatorning yaqinlashish xarakterini aniqlaydi.

Bundan tashqari Raabe alomati R ning $R=\pm\infty$ qiymatida va boshqa ($R=1$dan tashqari) qiymatlarda ham qatorning yaqinlashish xarakterini aniqlaydi. Dalamber alomati $\lambda=1$ bo'lganda qatorning yaqinlashish xarakterini aniqlamaydi, lekin Raabe alomati bu holda ham qatorning yaqinlashish xarakterini aniqlaydi.

Raabe alomatida ham $R=1$ bo'lganda, qatorning yaqinlashish xarakterini aniqlab bo'lmaydi. Bu holda, Raabe alomatiga nisbatan murakkabroq «sezgirroq» alomatlarga murojaat qilishga to'g'ri keladi.

11.8. Makloren – Koshining integral alomati.

Ushbu

$$\sum_{n=1}^{\infty} a_n \qquad (11.8.1)$$

musbat hadli qator berilgan bo'lsin. $[1,\infty)$ oraliqda aniqlangan, manfiy bo'lmagan, uzluksiz, o'smaydigan $f(x)$ funksiya uchun $f(n)=a_n$ $(n=1,2,....)$ bo'lsin. Bu holda berilgan qatorning ko'rinishi

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} f(n)$$

shaklida bo'ladi. Ravshanki, $f(x)$ funksiya uchun boshlang'ich funksiya mavjud: $F(x) = \int f(x)dx$ $(F'(x) = f(x))$. Shartga ko'ra, $F'(x) = f(x) > 0$ uchun $F(x)$ o'suvchi bo'ladi.

Demak, u $x \to \infty$ da chekli yoki cheksiz L limitga ega ([15], 1 t.,47 b. qar.), ya'ni $\exists \lim_{x \to \infty} F(x) = L$.

11.8.2-teorema(Makloren – Koshining integral alomati). Agar $F(x)$ boshlang'ich funksiyaning limiti L mavjud va chekli bo'lsa, berilgan (11.8.1) qator yaqinlashuvchi, L cheksiz bo'lsa, (11.8.1) qator uzoqlashuvchi bo'ladi.

11.8.3-eslatma. n – nomerning boshlang'ich qiymati 1 ning o'rniga boshqa ixtiyoriy natural n_0 soni olish mumkin, u holda $f(x)$ funksiyani aniqlangan deb qarash kerak bo'ladi.

11.8.4-eslatma. 11.8.2-teoremani 1742 yilda geometric shakilda Makloren isbotlagan. 1827 yilda Koshi tomonidan qaytadan isbot qilingan.

11.3-§. Ixtiyoriy ishorali qatorlar va ularning yaqinlashuvchiligi

Biz yuqorida musbat hadli qatorlarning yaqinlashiuvchilik masalasiga to'xtaldik, ya'ni musbat hadli qatorlarning yaqinlashish sharti yordamida taqqoslash teoremalarini keltirib, bu teoremalarga asoslanib, musbat qatorlarning yaqinlashish alomatlarini o'rgandik. Endi ixtiyoriy ishorali qatorlarning yaqinlashuvchiligi masalasi bilan shug'ullanamiz.

Ixtiyoriy ishorali qotor berilganda, uning yaqinlashish masalasini o'rganish, uni musbat sonli qatorlarga keltirib o'rganiladi. Agar berilgan qatorning hamma hadlari musbat bo'lmasdan, biror nomerdan boshlab musbat bo'lsa, u holda uni tekshirish masalasini chekli sondagi boshlang'ich hadlarini (11.3.2- teoremaga q.) olib tashlab, musbat hadli qatorlarni tekshirishga olib kelinadi. Agar qator hadlarining ishorasi manfiy yoki biror nomerdan boshlab hamma hadlari ishorasi manfiy bo'lsa, bu holda ham, hamma hadlarining ishorasini o'zgartirish bilan yoki boshlang'ich chekli sondagi hadlarini olib tashlash bilan uni tekshirish musbat qatorni tekshirishga olib kelinadi. Agar qatorning cheksiz ko'p hadlari musbat ishorali, cheksiz ko'p hadlari manfiy ishorali bo'lsa, u holda qatorni tekshirish uchun yangi tushunchalar: absolyut yaqinlashishuvchi qator hamda shartli yaaqinlauvchi qator tushunchalarini kiritishga to'g'ri keladi.

11.9. Ixtiyoriy ishorali qatorning yaqinlashuvchiligi haqidagi teorema. Ixtiyoriy ishorali

$$\sum_{n=1}^{\infty} a_n \tag{11.9.1}$$

qator berilgan bo'lsin.

11.9.2-teorema (Koshi teoremasi). Ixtiyoriy ishorali $\sum_{n=1}^{\infty} a_n$ qatorning yaqinlashuvchi bo'lishi uchun $\forall\varepsilon > 0$ son olinganda ham shunday $n_0 \in \mathbf{N}$ son mavjud bo'lib, barcha $n > n_0$ va $p \in \mathbf{N}$ lar uchun

$$|a_{n+1} + a_{n+2} + ... + a_{n+p}| < \varepsilon \tag{11.9.3}$$

tengsizligining bajarilishi zarur va yetarli.

11.9.2-teoremaga ixtiyoriy qator yaqinlashishining *Koshi kriteriysi* deyiladi.

11.9.4-eslatma. Agar (11.9.3) shart bajarilmasa, ya'ni $\exists\varepsilon_0 > 0 : \forall k \in N \;\; \exists n \geq k \;\; \exists p \in N :$

$$|S_{n+p} - S_n| = |a_{n+1} + a_{n+2} + ... + a_{n+p}| \geq \varepsilon_0 \tag{11.9.5}$$

tengsizlik o'rinli bo'lsa, (11.9.1) qator uzoqlashuvchi bo'ladi.

11.9.6-misol. Ushbu $\sum_{n=1}^{\infty} \frac{1}{n}$ garmonik qatorning uzoqlashuvchi ekanligi isbotlang.

Yechilishi. $\forall k \in N$ uchun $n = k, \;\; p = k$ deb olsak, u holda

$$\sum_{k=n+1}^{n+p} a_k = \frac{1}{k+1} + ... + \frac{1}{2k} > k \cdot \frac{1}{2k} = \frac{1}{2} = \varepsilon_0$$

(11.9.5) shartga ko'ra, berilgan qator uzoqlashuvchi. ■

11.10. Absolyut va shartli yaqinlashuvchi qatolar. Ixtiyoriy ishorali (11.9.1) qator berilgan bo'lsin. Bu qator hadlarining absolyut qiymatlaridan ushbu

$$\sum_{n=1}^{\infty}|a_n| = |a_1| + |a_2| + ... + |a_n| + ... \qquad (11.10.1)$$

qatorni tuzamiz.

11.10.2-ta'rif. Agar $\sum_{n=1}^{\infty}|a_n|$ qator yaqinlashuvchi bo'lsa, u holda (11.9.1) qator *absolyut yaqinlashuvchi* deyiladi.

11.10.3-ta'rif. Agar (11.9.1) qator yaqinlashuvchi bo'lib, $\sum_{n=1}^{\infty}|a_n|$qator uzoqlashuvchi bo'lsa, u holda $\sum_{n=1}^{\infty}a_n$ qator *shartli yaqinlashuvchi* deyiladi.

(11.9.1) qatorning absolyut yaqinlashuvchiligini aniqlash uchun (11.10.1) qatorga 11.2-§ dagi o'rganilgan Dalamber va Koshi alomatlarini qo'llash mumkin.

Dalamber alomati. (11.9.1) $(a_n \neq 0, n = 1,2,..)$ qator berilgan bo'lib, uning hadlari uchun $(a_m \neq 0, n = 1,2,...)$ $\lim\limits_{n\to+\infty}\frac{|a_{n+1}|}{|a_n|} = d$ limit mavjud bo'lsin. U holda:

1) $d < 1$ bo'lganda, (11.9.1) qator yaqinlashuvchi;

2) $d > 1$ bo'lganda esa, (11.9.1) qator uzoqlashuvchi bo'ladi.

Koshi alomati. (11.9.1) qator berilgan bo'lib, uning hadlari uchun $\lim\limits_{n\to+\infty}\sqrt[n]{|a_n|} = k$ limit mavjud bo'lsin. U holda:

1) $k < 1$ bo'lganda, (11.9.1) qator yaqinlashuvchi;

2) $k > 1$ bo'lganda esa, (11.9.1) qator uzoqlashuvchi bo'ladi.

Demak, Koshi va Dalamber yaqinlashish alomatlari berilgan qatorning absolyut yaqinlashishini aniqlashda yetarli bo'lar yekan. Lekin shartli yaqinlashuvchi qatorlar uchun ularni qo'llab bo'lmaydi. Shartli yaqinlashuvchi qatorlar uchun yaqinlashish alomatlarini aniqlash, ancha muhim masalalardandir. Quyida biz shunday alomatlardan bazilari bilan tanishamiz.

Navbatdagi yaqinlashish alomatini ushbu

$$\sum_{k=1}^{\infty}a_k b_k \qquad (11.10.2)$$

maxsus ko'rinishdagi qatorlar uchun keltiramiz, bunda a_k va b_k lar haqiqiy sonlar bo'lib, ulardan biri ishora saqlasa, ikkinchisi turli ishorali qiymatlar qabul qilishi mumkin.

11.10.3-teorema (Dirixle-Abel alomati). Agar $\{a_k\}$ ketma-ketliklardan tuzilgan

$$\sum_{k=1}^{\infty} a_k \tag{11.10.4}$$

qatorning qismiy yig'indilari chegaralangan, ya'ni

$$\left|\sum_{k=1}^{n} a_k\right| \le M \tag{11.10.5}$$

bo'lsa va $\{b_k\}$ ketma-ketlik monoton kamayib, ya'ni

$$b_k \ge b_{k+1} \quad (k = 1,2,...) \tag{11.10.6}$$

va nolga intilsa, u holda (11.10.2) qator yaqinlashadi.

Leybnis alomati

Agar barcha b_k $(k = 1,2,...)$ lar musbat sonlar bo'lsa, u holda ushbu

$$\sum_{k=1}^{\infty} (-1)^{k-1} b_k = b_1 - b_2 + b_3 - b_4 + ... \tag{11.10.7}$$

qatorga *ishorasi almashinuvchi* qator deyiladi.

11.10.8-teorema (Leybnis alomati). Agar b_k musbat sonlar ketma-ketligi monoton bo'lib nolga intilsa, (11.10.7) - ishorasi almashinuvchi qator yaqinlashuvchi bo'ladi.

Isboti. Agar $a_k = (-1)^{k-1}, S_n = \sum_{k=1}^{n} a_k = \sum_{k=1}^{n} (-1)^{k-1}$ deb belgilasak, u holda, $S_1 = 1$, $S_2 = 0$ umumiy holda $S_{2n-1} = 1, S_{2n} = 0,\ n = 1,2,3...$ bajariladi.

Demak, S_n yig'indilar ketma-ketligi chegaralangan.

Shunday qilib 11.10.3-teoremaning shartlari bajarilar ekan. Bundan esa, 11.10.7 qatorning yaqinlashuvchiligi kelib chiqadi. ■

11.10.9-eslatma. 11.10.8-teoremaning shartini qanoatlantiruvchi qatorni Leybnis qatori deb ataymiz.

Absolyut yaqinlashuvchi qatorlarning xossalari

11.10.10-teorema. Agar qator absolyut yaqinlashuvchi bo'lsa, u holda u yaqinlashuvchi bo'ladi

11.10.11-teorema. Agar (11.9.1) qator absolyut yaqinlashuvchi bo'lib, $\{b_n\}$ ketma-ketlik chegaralangan bo'lsa, u holda

$$\sum_{n=1}^{\infty} a_n b_n = a_1 b_1 + a_2 b_2 + \ldots + a_n b_n + \ldots$$

qator absolyut yaqinlashuvchi bo'ladi.

11.10.11-teorema. $\sum_{k=1}^{\infty} a_k$ va $\sum_{k=1}^{\infty} b_k$ qatorlar absolyut yaqinlashuvchi bo'lsin. U holda, bu qatorlar hadlarining(takrorlanmasdan) mumkin bo'lgan o'zaro hamma ko'paytmasidan tuzilgan ushbu

$$\sum_{j=1}^{\infty} a_{m_i} b_{m_j}$$

qator ham absolyut yaqinlashuvchi bo'ladi va uning yig'indisi

$$\sum_{j=1}^{\infty} a_{k_j} b_{m_j} = \left(\sum_{k=1}^{\infty} a_k\right) \cdot \left(\sum_{k=1}^{\infty} b_k\right)$$

teng bo'ladi.

11.10.13-teorema (Riman). Agar $\sum_{k=1}^{\infty} a_k$ qator shartli yaqinlashsa, u holda istalgan haqiqiy A soni uchun bu qator hadlari o'rnini shunday almashtirish mumkinki, natijada hosil bo'lgan qator yaqinlashuvchi bo'lib, uning yig'indisi shu A soniga teng bo'ladi.

11-bob bo'yicha nazariy materiallarni mustahkamlash uchun topshiriqlar

11.1. Sonli qatorlarning ta'rifi ([1], 1-t., 7- bo'lim; [3], 1-q., 341 bet; [12], 1-q., 380-381 betlar; [10], 2-q.,7-8 betlar; [5], 2-t., 257-258 betlar; [9], 2-t., 1- bo'lim, [30], 11- bo'lim).

11.2. Qatorning qismiy yig'indilar ketma-ketligi ([1], 1-t., 7- bo'lim; [3], 1-q., 341-342 betlar; [12], 1-q., 380-381 betlar; [10], 2-q., 7-8 betlar; [5], 2-t., 257 bet; [9], 2-t., 1- bo'lim, [30], 11- bo'lim).

11.3. Qatorning yaqinlashish va uzoqlashuvchiligi ([1], 1-t., 7- bo'lim; [3], 1-q., 342-343 betlar; [12], 1-q., 380-381betlar; [10], 2-q., 7-8 betlar; [5], 2-t.,257-258 betlar; [9], 2-t., 1- bo'lim, [30], 11- bo'lim).

11.4. Qator yig'indisining ta'rifi ([1], 1-t., 7- bo'lim; [3], 1-q., 342 bet; [12], 1-q., 380-381betlar; [10], 2-q., 7-8 betlar; [5], 2-t., 257 bet; [9], 2-t., 1- bo'lim, [30], 11- bo'lim).

11.5. Qatorning qoldig'i ([1], 1-t., 7- bo'lim; [3], 1-q., 343 bet; [12], 1-q., 384 bet; [10], 2-q., 257-bet; [5], 2-t., 260-261betlar; [9], 2-t., 1- bo'lim, [30], 11- bo'lim).

11.6. Qator yaqinlashishining zaruriy sharti ([1], 1-t., 7- bo'lim; [3], 1-q., 344-345 betlar; [12], 1-q., 387-bet; [10], 2-q.,262-bet; [5], 2-t., 262-bet; [9], 2-t., 1- bo'lim, [30], 11- bo'lim).

11.7. Yaqinlashuvchi qatorlarning sodda xossalari ([1], 1-t., 7- bo'lim; [3], 1-q., 343-346 betlar; [12], 1-q., 384-387 betlar; [10], 2-q., 41-44 betlar; [5], 2-t., 261-262 betlar; [9], 2-t., 1- bo'lim, [30], 11- bo'lim).

11.8. Umumlashgan garmonik qator deb qanday qatorga aytiladi va u qaysi hollarda yaqinlashuvchi bo'ladi ([1], 1-t., 7- bo'lim; [3], 1-q., 344-345 betlar; [5], 2-t., 263-264 betlar; [9], 2-t., 1- bo'lim, [30], 11- bo'lim).

11.6. Geometrik $\sum_{n=1}^{\infty} q^n$ qator qaysi holda yaqinlashuvchi bo'ladi ([1], 1-t., 7- bo'lim; [3], 1-q., 343 bet; [12], 1-q.,282 bet; [5], 2-t., 258-259 betlar; [9], 2-t., 1- bo'lim, [30], 11- bo'lim).

11.7. Qator yaqinlashishi uchun Koshi kritereysi. ([1], 1-t., 7- bo'lim; [3], 1-q, 346-348 betlar; [12], 1-q., 401-402 betlar; [5], 2-t., 293-294 betlar; [9], 2-t., 1- bo'lim, [30], 11- bo'lim).

11.8. Musbat sonli qatorlarning yaqinlashishi uchun zaruriy va yetarli shart. ([1], 1-t., 7- bo'lim, [12], 1-q., 388-389 betlar; [10], 2-q., 12-13 betlar; [5], 2-t., 262-263 betlar; [9], 2-t., 1- bo'lim, [30], 11- bo'lim).

11.9. Musbat sonli qatorlarni taqqoslash teoremalari (1-3 teoremalar) ([1], 1-t., 7- bo'lim; [3], 1-q., 348-349 betlar; [12], 1-q., 389-393 betlar; [10], 2-q., 13-15 betlar; [5], 2-t., 264-266 betlar; [9], 2-t., 1- bo'lim, [30], 11- bo'lim).

11.10. Musbat sonli qatorlar uchun yaqinlashish alomatlari: Dalamber ([3], 1-q., 355-356 betlar; [12], 1-q., 395-396 betlar, [5], 2-t., 271-272 betlar), Koshi([3], 1-q., 353-355 betlar; [12], 1-q., 393-395 betlar, [5], 2-t., 270-271 betlar), Raabe ([3], 1-q., 359-361 betlar; [12], 1-q., 397-399 betlar, [5], 2-t.,272-274 betlar), Koshi - Makloren ([3], 1-q., 358-359 betlar; [12], 1-q., 399-401 betlar, [5], 2-t., 281-285 betlar), Kummer([5], 2-t., 277-279 betlar), Bertrana([5], 2-t., 279 bet), alomatlari; [1], 1-t., 7- bo'lim, [9], 2-t., 1- bo'lim, [30], 11- bo'lim).

11.11. $\sum_{n=1}^{\infty} a_n \left(a_n \geq 0, n = 1,2,....\right)$ qator bilan $\sum_{k=1}^{\infty} 2^k a_{2^k}$ qatorlarning bir vaqtda yaqinlashishi va uzoqlashishi haqidagi teorema ([5], 2-t., 288-289 betlar, [1], 1-t., 7- bo'lim, [9], 1-t.,1- bo'lim).

11.12. Ixtiyoriy ishorali qatorlar va ularning yaqinlashuvchiligi. Leybnis tipidagi qatorlarning yaqinlashuvchiligi. ([12], 1-q., 401, 404-406 -betlar; [5], 2-t., 293, 302-303 betlar, [1], 1-t., 7- bo'lim).

11.13. Abel va Dirixle alomatlari ([3], 2-q., 6-9 betlar; [10], 2-q., 36-38 betlar, [5], 2-t., 307-309 betlar).

11.14. Absalyut va shartli yaqinlashuvchi qatorlar. Absalyut yaqinlashuvchi qatorlarning xossalari. ([3], 1-q., 362-363 betlar; [12], 1-q., 402-405 betlar; [10], 2-q., 28-30 betlar; [5], 2-t., 394-396 betlar, [1], 1-t., 7- bo'lim).

11.15. Riman teoremasi. ([12], 1-q.,411-bet; [10], 2-q., 31-33 betlar; [5], 2-t., 317-320 betlar).

11.1-amaliy mashg'ulot.
Sonli qatorlar. Musbat xadli qatorlar

1-misol. Quyidagi sonly qatorni yaqinlashishga tekshiring:

$$\frac{1}{5} + \frac{1}{45} + \cdots + \frac{1}{16n^2 - 8n - 3} + \cdots$$

Yechilishi. Ushbu $16n^2-8n-3$ kvadrat uchhadni ko'paytuvchilarga ajratamiz:

$$16n^2-8n-3=16n^2+4n-12n-3=(4n-3)(4n+1).$$

Berilgan sonly qatorning qismiy yig'indisini yuqoridagi tenglikdan foydalanib quyidagi ko'rinishda yozamiz:

$$S_n=\sum_{k=1}^{n}a_k=\sum_{k=1}^{n}\frac{1}{(4k-3)(4k+1)}=$$
$$=\frac{1}{4}\sum_{k=1}^{n}\left(\frac{1}{4k-3}-\frac{1}{4k+1}\right)=\frac{1}{4}\left(1-\frac{1}{4n+1}\right).$$

Endi limitga o'tamiz: $\lim\limits_{n\to\infty}S_n=\lim\limits_{n\to\infty}\frac{1}{4}\left(1-\frac{1}{4n+1}\right)=\frac{1}{4}$.

Demak, 11.1.2-ta'rifga ko'ra, berilgan sonly qator yaqinlashuvchi ekan.

Misolni Matlab 7.9 dasturidan foydalanib, yechish:

>> *syms n*

>> $symsum(1/(16*n\text{^}2-8*n-3),n,1,inf)$

ans = 1/4

Misolni Mathcad 14 dasturidan foydalanib, yechish:

$$\sum_{n=1}^{\infty}\frac{1}{16n^2-8n-3}\to\frac{1}{4};$$

Misolni mathematica 8 dasturidan foydalanib, yechish:

In[1]: = Sum[1/(16 * n^2 − 8 * n − 3), {n, 1, Infinity}]

Out[1] = $\frac{1}{4}$.

2-misol. Quyidagi qatorning uzoqlashuvchi ekanligini isbotlang:

$$0{,}07+\sqrt{0{,}07}+\sqrt[3]{0{,}07}+\cdots+\sqrt[n]{0{,}07}+\cdots$$

Yechilishi. Dastlab quyidagi munosabatni keltiramiz:

$$\lim_{n\to\infty}\sqrt[n]{0{,}07}=\lim_{n\to\infty}(0{,}07)^{\frac{1}{n}}=1\neq 0$$

Ushbu munosabatdan qatorlar yaqinlashishining zaruriy shartiga ko'ra, $\sum_{n\geq 1}\sqrt[n]{0{,}07}$ qatorning uzoqlashuvchi ekanligini ko'rish mumkin.

Misolni Matlab 7.9 dasturidan foydalanib, yechish:

>> *syms n*

$>> limit(0{,}07\text{^}(1/n), inf)$
$ans = 1$

Misolni Mathcad 14 dasturidan foydalanib yechish:

$$\lim_{n\to\infty} \sqrt[n]{0{,}07} \to 1.0$$

Misolni mathematica 8 dasturidan foydalanib yechish:

$In[2]:= Limit[(0.07)\text{^}(1/n), n \to Infinity]$
$Out[2] = 1.$

Misolni Maple 15 dasturidan foydalanib yechish:

$$\lim_{n\to\infty} \sqrt[n]{0.07}$$

1.

3-misol. Quyidagi musbat hadli qatorni yaqinlashishga tekshiring:

$$\sum_{n\geq 1} \frac{2\cdot 7\cdot 12\cdot \ldots \cdot (5n-3)}{5\cdot 9\cdot 13\cdot \ldots \cdot (4n+1)}$$

Yechimi.a) Ushbu qatorni 11.5 banddagi Dalamber alomatiga ko'ra, tekshiramiz:

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{2\cdot 7\cdot 12\cdot \ldots \cdot (5n+2)}{5\cdot 9\cdot 13\cdot \ldots \cdot (4n+5)}\cdot\frac{5\cdot 9\cdot 13\cdot \ldots \cdot (4n+1)}{2\cdot 7\cdot 12\cdot \ldots \cdot (5n-3)} =$$
$$= \lim_{n\to\infty}\frac{5n+2}{4n+5} = \frac{5}{4} > 1;$$

Demak, berilgan qator uzoqlashuvchi ekan.

Misolni Mathcad 14 dasturidan foydalanib yechish:

$$a(n) := \frac{\prod_{k=1}^{n}(5\cdot k-3)}{\prod_{k=1}^{n}(4\cdot k+1)} \quad \lim_{n\to\infty}\frac{a_{n+1}}{a_n} \to \frac{5}{4}$$

Misolni mathematica 8 dasturidan foydalanib yechish:

$In[10] := a[n\_] := Product[5 * k - 3, \{k, 1, n\}]/ Product[4 * k + 1, \{k, 1, n\}]$
$In[11] := Limit[a[n+1]/a[n], n \to Infinity]$

$$Out[11] = \frac{5}{4}$$

Misolni Maple 15 dasturidan foydalanib yechish:

$$a := n \to \frac{\prod_{k=1}^{n}(5\cdot k-3)}{\prod_{k=1}^{n}(4\cdot k+1)}$$

$$n \to \frac{\prod_{k=1}^{n}(5\cdot k-3)}{\prod_{k=1}^{n}(4\cdot k+1)}$$

$$\lim_{n\to\infty}\frac{a(n+1)}{a(n)}$$

$$\frac{5}{4}$$

4-misol. $f(x)=e^x$ funksiyaning x o'zgaruvchiga nisbatan yoyilgan qatoridan foydalanib, $\sqrt{e}$ sonini o'n minglar xonasigacha aniqlikda hisoblang.

Yechilishi. n –darajagacha Makleron formulasidan foydalanib, $f(x)$ funksiyani darajali qatorga yoyamiz:

$$e^x = 1+\frac{x}{1!}+\frac{x^2}{2!}+\cdots+\frac{x^n}{n!}+\frac{x^{n+1}}{(n+1)!}\cdot e^{\theta x},\quad \theta\in(0;1);$$

$x=\frac{1}{2}$ deb olsak,

$$e^{\frac{1}{2}} = 1+\frac{1}{2\cdot 1!}+\frac{1}{2^2\cdot 2}+\cdots+\frac{1}{2^n\cdot n!}+\frac{1}{2^{n+1}\cdot(n+1)!}\cdot e^{\frac{\theta}{2}},\quad \theta\in(0;1).$$

Demak,

$$R_n\left(\frac{1}{2}\right)=\frac{1}{2^{n+1}\cdot(n+1)!}\cdot e^{\frac{\theta}{2}}\underset{\theta\in(0;1)}{<}\frac{1}{2^{n+1}\cdot(n+1)!}\cdot e^{\frac{1}{2}}<\frac{1}{2^{n+1}\cdot(n+1)!}\cdot 2=$$
$$=\frac{1}{2^n\cdot(n+1)!}.$$

$\sqrt{e}$ sonini o'n minglar xonasigacha hisoblashimiz kerakligi uchun:

$$\frac{1}{2^n\cdot(n+1)!}<\frac{1}{10^6}\Leftrightarrow 2^n\cdot(n+1)!<10^6\Leftrightarrow n\geq 7,$$

ya'ni, $n\geq 7$ da $\sqrt{e}$ sonini o'n minglar xonasigacha hisoblashimiz mumkin, demak:

$$\sqrt{e}\cong 1+\frac{1}{2\cdot 1!}+\frac{1}{2^2\cdot 2!}+\cdots+\frac{1}{2^7\cdot 7!}=1{,}6487345.$$

Ushbu misolni Matlab 7.9 paketi yordamida tekshiramiz:

$>> vpa(exp(1/2),7)$

$ans =$

1.648721

Misolni Mathcad 14 dasturidan foydalanib yechish:

$\sqrt{e}=1.6487213$

Misolni mathematica 8 dasturidan foydalanib yechish:

$$In[100] = SetAccuracy[E\wedge(1/2),7]$$
$$Out[100] = 1.648721$$

Misolni Maple 15 dasturidan foydalanib yechish:

$$Digits := 7$$
$$7$$
$$evalf(\sqrt{e})$$
$$1.648721$$

5-misol. $\sum_{n=1}^{\infty}\frac{3+(-1)^n}{2^{n+2}}$ Quyidagi musbat sonli qatorlarni taqqoslash teoremalari yordamida yaqinlashishga tekshiring.

Yechilishi. Ravshanki $2 \le 3+(-1)^n \le 4$, $0 < a_n = \frac{3+(-1)^n}{2^{n+2}} \le \frac{1}{2^n} = b_n$.

Ma'lumki, $\sum_{n=1}^{\infty}\frac{1}{2^n}\left(q=\frac{1}{2^n}<1\right)$ geometrik qator yaqinlashuvchi bo'lgani uchun 11.4.1-teoremaga ko'ra, berilgan qator yaqinlashuvchi.

6-misol. $\sum_{n=1}^{\infty}\frac{1\cdot3\cdot5\ldots(2n-1)}{3^n\cdot(n+1)}$ qatorni Dalamber alomatidan foydalanib yaqinlashishga tekshiring.

Yechilishi. Berilgan qatorning umumiy hadiga ko'ra,

$$\frac{a_{n+1}}{a_n} = \frac{1\cdot3\cdot5\ldots(2n-1)\cdot(2n+1)}{3^{n+1}\cdot1\cdot2\cdot3\ldots n\cdot(n+1)(n+2)}\cdot\frac{3^n\cdot(n+1)}{1\cdot3\cdot5\ldots(2n-1)}$$

nisbatni tuzib, uning $n\to\infty$ dagi limitini topamiz:

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{(2n+1)}{3\cdot(n+2)} = \frac{2}{3} < 1.$$

Demak, berilgan qator Dalamber alomatiga ko'ra yaqinlashuvchi.

7-misol. $\sum_{n=1}^{\infty}q^{n-1}$ qatorning yaqinlashuvchilikka tekshiring.

Yechilishi. Ma'lumki, bu qatorning n-qismiy yig'indisi $(q\neq1)$

$$S_n = \sum_{k=1}^{n}q^{k-1} = \frac{1-q^n}{1-q} = \frac{1}{1-q} - \frac{q^n}{1-q}.$$

Agar, $|q|<1$ bo'lsa, $\lim_{n\to\infty}S_n = \frac{1}{1-q}$, $S = \frac{1}{1-q}$ – chekli, demak, ta'rifga ko'ra, berilgan qator yaqinlashuvchi bo'ladi. $q\ge1$ bo'lganda, geometrik qator uzoqlashuvchi bo'ladi, chunki uning yig'indisi aniq ishorali cheksiz bo'ladi. q ning qolgan qiymatlarida limit mavjud bo'lmaydi. Masalan, $q=-1$ bo'lsa:

$$1-1+1-1+...=1+(-1)+1+(-1)+.....$$

bu qatorning qismiy yig'indilari ketma – ketligi limitga ega emas, chunki $S_{2k}=0,\ k=1,2,...,S_{2k+1}=1,\ k=0,1,2,....$ ■

8-misol. Ushbu $\sum_{n=1}^{\infty}\frac{1}{(\alpha+n)(\alpha+n+1)},\ (\alpha\neq-1,-2,-3,..........)$ qatorning yaqinlanshuvchiligini ko'rsating va uning yig'indisini toping.

Yechilishi. $a_n=\frac{1}{(\alpha+n)(\alpha+n+1)}=b_n-b_{n+1},\ b_n=\frac{1}{\alpha+n},\quad \forall n\in N$

11.1.3-teoremaning hamma shartlari bajariladi, haqiqatan ham

$$a_n=b_n+1,\ \lim_{n\to\infty}b_n=\lim_{n\to\infty}\frac{1}{\alpha+n}=0,\ b_1=\frac{1}{\alpha+1},\ S=b_1-b=\frac{1}{a+1}.$$ ■

9-misol. Ushbu

$$\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$$

qatorning uzoqlashuvchi ekanligini isbotlang.

Yechilishi.

$$S_n=\sum_{k=1}^{n}\frac{1}{\sqrt{n}}=\frac{1}{1}+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+...+\frac{1}{\sqrt{n}}\geq n\cdot\frac{1}{\sqrt{n}}=\sqrt{n}$$

bundan, $\lim_{n\to\infty}S_n=+\infty$.

Demak, 11.1.2-ta'rifga ko'ra, berilgan qator uzoqlashuvchi. ■

10-misol. Ushbu

$$\sum_{n=1}^{\infty}\frac{1}{n^p}$$

umumlashgan garmonik qatorni yaqinlashuvchilikga tekshiring.

Yechilishi. $p\leq0$ bo'lganda, berilgan qatorning uzoqlashuvchiligi 11.3.2 – teoremadan kelib chiqadi, $p>1$ bo'lganda, 11.3.4 – teoremani qo'llash mumkin, chunki $\frac{1}{n^p}$ ionoton kamayuvchi.

$$\sum_{k=0}^{\infty}2^k\frac{1}{(2^k)^p}=\sum 2^{k(1-p)}.$$

Bunda, $1-p<0$ bo'lganda, $2^{k(1-p)}<1$ bo'ladi. $q=2^{(1-p)}$ desak, u holda $\sum_{k=1}^{\infty} q^k$ –cheksiz kamayuvchi progressiya hosil bo'ladi. Shuning uchun bu qator yaqinlashuvchi va $\sum_{k=0}^{\infty} q^k = \frac{1}{1-q}$.

Agar $p<1$ bo'lsa, $2^{(1-p)}>1$ bo'ladi. Bu holda, $\sum_{k=1}^{\infty} q^k$ - geometrik progressiya uzoqlashuvchi bo'ladi.

Shunday qilib, umumlashgan garmonik qator $p>1$ bo'lganda yaqinlashuvchi, $p\le 1$ bo'lganda uzoqlashuvchi bo'ladi. ■

11-misol. $\sum_{n=1}^{\infty} \frac{5+3\cdot(-1)^n}{2^{n+3}}$ qatorni yaqinlashuvchilikka tengshiring.

Yechilishi. $\forall n\in N$ uchun $2\le 5+3\cdot(-1)^n \le 8$ tengsizlik o'rinli bo'ladi. U holda

$$0<a_n=\frac{5+3\cdot(-1)^n}{2^{n+3}}\le \frac{1}{2^n}=b_n$$

bo'ladi. Ma'lumki, $\sum_{n=1}^{\infty} \frac{1}{2^n}$ – geometrik qator $\left(q=\frac{1}{2}<1\right)$ yaqinlashuvchi. Demak, 11.4.3 – teoremaga ko'ra, berilgan qator yaqinlashuvchi bo'ladi. ■

12-misol $\sum_{n=1}^{\infty} a_n$, bunda $a_n=\sqrt[3]{\frac{2n+1}{2n-1}}-\sqrt{\frac{n-1}{n+1}}$ qatorni yaqinlashuvchilikka tekshiring.

Yechilishi. Asimptotik formuladan foydalanib, quyidagilarga ega bo'lamiz: Ma'lumki, $t\to 0$ da, $(1+t)^\alpha = 1+\alpha t+o(t)$, bu formulaga asosan, $n\to\infty$

$$\sqrt[3]{\frac{2n+1}{2n-1}}=\left(1+\frac{2}{2n-1}\right)^{1/3}=1+\frac{2}{3(n-1)}+o\left(\frac{1}{n}\right)=1+\frac{1}{3n}+o\left(\frac{1}{n}\right),$$

$$\sqrt{\frac{n-1}{n+1}}=\left(1-\frac{2}{n+1}\right)^{1/2}=1+\frac{1}{2}\left(-\frac{2}{n+1}\right)+o\left(\frac{1}{n}\right)=1-\frac{1}{n}+o\left(\frac{1}{n}\right),$$

bundan, $n\to\infty$ da $a_n\sim\frac{4}{3n}$.

Demak, $\sum_{n=1}^{n} \frac{1}{n}$ – garmonik qator uzoqlashuvchi bo'lganligi uchun berilgan qator uzoqlashuvchi bo'ladi. ■

13-misol. Ushbu $\sum_{n=1}^{\infty}\frac{n!}{n^n}$ qatorni Dalamber alomatidan foydalanib, yaqinlashuvchilikka tekshiring.

Yechilishi.

$$D_n=\frac{a_{n+1}}{a_n}=\frac{(n+1)!}{(n+1)^{n+1}}:\frac{n!}{n^n}=\left(\frac{n}{n+1}\right)^n=\left(1+\frac{1}{n}\right)^{-n}.$$

Bundan,

$$\lim_{n\to\infty}D_n=\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{-n}=\frac{1}{e}<1.$$

Demak, 11.5.6-natijaga asosan, berilgan qator yaqinlashuvchi. ■

14-misol. Ushbu $\sum_{n=2}^{\infty}\frac{1}{(\ln n)^n}$ qatorning yaqinlashuvchiligini Koshi alomatiga ko‘ra, isbotlang.

Yechilishi. Ushbu $\sqrt[n]{\frac{1}{(\ln n)^n}}$ ifodani tuzamiz.

$$\lim_{n\to\infty}\sqrt[n]{\frac{1}{(\ln n)^n}}=\lim_{n\to\infty}\frac{1}{\ln n}=0,\quad \lambda=0.$$

Demak, 11.6.5- natijaga asosan, berilgan qator yaqinlashuvchi. ■

15-misol Ushbu

$$\sum_{n=1}^{\infty}\frac{n!x^n}{(x+a_1)(2x+a_2).....(nx+a_n)}$$

qatorni yaqinlashishga tekshiring, bunda $x>0$, a_n – musbat, chekli limitga ega.

Yechilishi.

$$D_n=\frac{a_{n+1}}{a_n}=\frac{(n+1)x}{(n+1)x+a_{n+1}},\quad \lim_{n\to\infty}D_n=\lim_{n\to\infty}\frac{(n+1)x}{(n+1)x+a_{n+1}}=1,\quad \lambda=1.$$

Demak, berilgan qatorni yaqinlashishga tekshirishga Dalamber alomatini qo‘llab bo‘lmaydi. Endi $R_n=n\left(\frac{a_n}{a_{n+1}}-1\right)$ ifodani tuzamiz:

$$R_n=\frac{n\cdot a_{n+1}}{(n+1)x},\quad \lim_{n\to\infty}R_n=\lim_{n\to\infty}\frac{n\cdot a_{n+1}}{(n+1)x}=\frac{a}{x}.$$

Demak, Raabe alomatiga ko‘ra, $x<a$ da, berilgan qator yaqinlashuvchi, $x>a$ da esa uzoqlashuvchi bo‘ladi. $x=a$ da, qatorning yaqinlashish xarakterini aniqlab bo‘lmaydi. ■

16-misol. Ushbu

$$\sum_{n=3}^{\infty}\frac{1}{n\ln n\ln(\ln n)}$$

qator yaqinlashuvchilikga tekshiring.

Yechilishi. Berilgan qatorning umumiy hadi $a_n=\frac{1}{n\ln n\ln(\ln n)}$, u holda $f(x)=\frac{1}{n\ln n\ln(\ln x)}$ $[3,\infty)$ da aniqlangan, uzluksiz, o'smovchi va manfiy bo'lmagan funksiya. $f(x)$ funksiyaning boshlang'ich funksiyasi esa,

$$F(x)=\int_3^x\frac{dt}{t\ln t\ln\ln t}=\ln\ln\ln x-\ln\ln\ln 3$$

bo'lib, $x\to\infty$ da $F(x)\to\infty$.

Demak, Makloren – Koshining integral alomatiga ko'ra, berilgan qator uzoqlashuvchi. ■

17-misol. Ushbu $\sum_{n=1}^{\infty}n^3e^{-n^4}$ qatorni yaqinlashuvchilikga tekshiring.

Yechilishi. $f(x)=x^3e^{-x^4}$, $x\geq1$ da kamayuvchi va manfiy bo'lmagan funksiya. $f(x)=x^3e^{-x^4}$ funksiya uchun bolang'ich funksiya

$$F(x)=\int_1^x t^3e^{-t^4}dt=\frac{1}{4}\left(e^1-e^{x^4}\right)$$

$x\to\infty$ da $F(x)$ ning limiti mavjud va chekli, ya'ni $\lim_{x\to\infty}F(x)=\frac{1}{4e}$.

Demak, Makloren – Koshining integral alomatiga ko'ra, berilgan qator yaqinlashuvchi. ■

Mustaqil yechish uchun misollar

Quyidagi qatorlarning yaqinlashuvchiligini ko'rsating va yig'indisini toping:

1. $\frac{1}{1\cdot2}+\frac{1}{2\cdot3}+\ldots+\frac{1}{n\cdot(n+1)}+\ldots$.

2. $\frac{1}{1\cdot4}+\frac{1}{4\cdot7}+\ldots+\frac{1}{(3n-2)\cdot(3n+1)}+\ldots$.

3. $\frac{1}{1\cdot2\cdot3\cdot4}+\frac{1}{2\cdot3\cdot4\cdot5}+\ldots+\frac{1}{n(n+1)(n+2)(n+3)}+\ldots$.

4. $\frac{3}{1^2\cdot 2^2}+\frac{5}{2^2\cdot 3^2}+\ldots+\frac{2n+1}{n^2\cdot(n+1)^2}+\ldots$

5. $\left(\frac{1}{10}+\frac{2}{10^2}+\frac{5}{10^3}\right)+\left(\frac{1}{10^2}+\frac{2}{10^3}+\frac{5}{10^4}\right)+\ldots+\left(\frac{1}{10^n}+\frac{2}{10^{n+1}}+\frac{5}{10^{n+2}}\right)+\ldots$

6. $\frac{1}{1\cdot(1+m)}+\frac{1}{2\cdot(2+m)}+\ldots+\frac{1}{n\cdot(n+m)}+\ldots\quad(m\in N)$.

Quyidagi qatorlar uchun qator yaqinlashuvchiligining zaruriy sharti bajarilmasligini ko‘rsating:

7. $\sum\limits_{n=1}^{\infty}\left(\frac{3n^3-2}{3n^3+4}\right)^{n^3}$. 8. $\sum\limits_{n=1}^{\infty}(n^2+2)\ln\frac{n^2+1}{n^2}$. 9. $\sum\limits_{n=2}^{\infty}\frac{1}{\sqrt[n]{\ln n}}$.

10. $\sum\limits_{n=1}^{\infty}\sin n\alpha$, bunda $\alpha\neq\pi m,\ m\in Z$. 11. $\sum\limits_{n=1}^{\infty}\frac{n^{n+\frac{1}{n}}}{\left(n+\frac{1}{n}\right)^n}$. 12. $\sum\limits_{n=1}^{\infty}\sqrt[n]{0{,}002}$.

Quyidagi qatorlarning S_n qismiy yig‘indilari ketma-ketligini va S yig‘indisini toping:

13. $\sum\limits_{n=1}^{\infty}\left(\frac{3}{2^{n-1}}+\frac{(-1)^{n-1}}{2\cdot 3^{n-1}}\right)$. 14. $\sum\limits_{n=1}^{\infty}\frac{1}{5}\left(\frac{1}{5n-2}-\frac{1}{5n+3}\right)$.

15. $\sum\limits_{n=2}^{\infty}\frac{1}{2}\left(\frac{1}{n-1}-\frac{1}{n+1}\right)$. 16. $\sum\limits_{n=1}^{\infty}\frac{2n+1}{n^2(n+1)^2}$.

17. $\sum\limits_{n=1}^{\infty}\frac{n-\sqrt{n^2-1}}{\sqrt{n(n+1)}}$. 18. $\sum\limits_{n=1}^{\infty}\frac{1}{(4n+5)(4n+9)}$.

Taqqoslash teoremalaridan foydalanib, qatorlarni yaqinlashishga tekshiring:

19. $\sum\limits_{n=1}^{\infty}\frac{5+3(-1)^{n+1}}{3^n}$. 20. $\sum\limits_{n=1}^{\infty}\frac{\sin^4 3n}{n\sqrt{n}}$. 21. $\sum\limits_{n=1}^{\infty}\frac{n^3}{e^n}$.

22. $\sum\limits_{n=1}^{\infty}\frac{arctgn}{n^2+1}$. 23. $\sum\limits_{n=1}^{\infty}\frac{\cos\frac{\pi}{4n}}{\sqrt[5]{2n^5-1}}$. 24. $\sum\limits_{n=1}^{\infty}\frac{n^{n-1}}{(2n^2+n+1)^{\frac{n+1}{2}}}$.

25. $\sum\limits_{n=1}^{\infty}\frac{n^5}{2^{n+}3^n}$. 26. $\sum\limits_{n=1}^{\infty}\frac{\sin^2 n}{2^{n+1}}$.

6.27. $\sum\limits_{n=1}^{\infty}\frac{1}{n^2-4n+5}$. 6.28. $\sum\limits_{n=1}^{\infty}\frac{1}{\sqrt{n^2+2n}}$.

Dalamber alomatidan foydalanib, quyidagi qatorlarni yaqinlashishga tekshiring: $\sum\limits_{n=1}^{\infty} a_n$.

29. $a_n = \dfrac{n^{12}}{(n+2)!}$. 30. $a_n = \dfrac{n^4}{4^n}$.

31. $a_n = \dfrac{n!a^n}{n^n}$, $a \neq e$, $a > 0$. 32. $a_n = \dfrac{3 \cdot 6 \ldots (3n)}{(n+1)!} \arcsin\dfrac{1}{2^n}$.

33. $a_n = \dfrac{(2n)!}{(n!)^2}$. 34. $a_n = \dfrac{n!(2n+1)!}{(3n)!}$.

35. $a_n = \dfrac{2 \cdot 5 \ldots (3n+2)}{2^n \cdot (n+1)!}$. 36. $a_n = \dfrac{1 \cdot 3 \cdot 5 \ldots (2n-1)}{2 \cdot 5 \cdot 8 \ldots (3n-1)}$.

37. $a_n = \dfrac{n!}{2^n + 1}$. 38. $a_n = \dfrac{\ln^{101} n}{4!}$, $n \geq 2$.

Koshi alomatidan foydalanib, quyidagi qatorlarni yaqinlashishga tekshiring: $\sum\limits_{n=1}^{\infty} a_n$.

39. $a_n = \dfrac{1}{(\ln n)^n}$, $n \geq 2$. 40. $a_n = \left(\dfrac{4}{n}\right)^n$.

41. $a_n = \left(\dfrac{n^2+5}{n^2+6}\right)^{n^3}$. 42. $a_n = 3^{n+1}\left(\dfrac{n}{n+1}\right)^{n^2}$.

43. $a_n = \left(\dfrac{6n+4}{5n-3}\right)^{\frac{n}{2}}\left(\dfrac{5}{6}\right)^{\frac{2n}{3}}$. 44. $a_n = \dfrac{n^n}{(\ln(n+1))^{n/2}}$.

45. $a_n = \left(\dfrac{2n-1}{2n+1}\right)^{n(n-1)}$. 46. $a_n = \dfrac{3^n}{n(\sqrt{2})^n}$.

47. $a_n = \left(\dfrac{n}{3n-1}\right)^{2n-1}$. 48. $a_n = \dfrac{2^{n-1}}{n^n}$.

Koshining integral alomatidan foydalanib, qatorlarni yaqinlashishga tekshiring:

49. $\sum\limits_{n=1}^{\infty} \dfrac{1}{n(1+\ln n)}$. 50. $\sum\limits_{n=1}^{\infty} \dfrac{1}{\sqrt{6n+5}}$. 51. $\sum\limits_{n=1}^{\infty} \dfrac{5}{4+n^2}$.

52. $\sum\limits_{n=1}^{\infty} \left(\dfrac{1+n}{1+n^2}\right)^2$. 53. $\sum\limits_{n=1}^{\infty} \dfrac{\ln(n+1)}{(n+1)^2}$. 54. $\sum\limits_{n=2}^{\infty} \dfrac{1}{\sqrt{n}} \ln\dfrac{n+1}{n-1}$.

55. $\sum_{n=1}^{\infty} ne^{-n^2}$. 56. $\sum_{n=1}^{\infty} \frac{e^{-\sqrt{n}}}{\sqrt{n}}$. 57. $\sum_{n=2}^{\infty} \frac{1}{n\ln^{\alpha+1} n}$, $(\alpha > 0)$.

Mustaqil yechish uchun misollarning javoblari

1. $S = 1$. **2.** $S = \frac{1}{3}$. **3.** $S = \frac{1}{18}$. **4.** $S = 1$. **5.** $S = \frac{5}{36}$.

6. $S = \frac{1}{m}\left(1 + \frac{1}{2} + ... + \frac{1}{m}\right)$. **13.** $S_n = \frac{51}{8} - \frac{3}{2^{n+1}} + \frac{(-1)^{n-1}}{8 \cdot 3^{n-1}}$, $S = \frac{51}{8}$.

14. $S_n = \frac{1}{5}\left(\frac{1}{3} - \frac{1}{5n+3}\right)$, $S = \frac{1}{15}$. **15.** $S_n = \frac{3}{4} - \frac{1}{2}\left(\frac{1}{n+1} + \frac{1}{n+2}\right)$, $S = \frac{3}{4}$.

16. $S_n = 1 - \frac{1}{(n+1)^2}$, $S = 1$. **17.** $S_n = \sqrt{\frac{n}{n+1}}$, $S = 1$.

18. $S_n = \frac{1}{4}\left(\frac{1}{9} - \frac{1}{4n+9}\right)$, $S = \frac{1}{36}$.

19. Yaqinlashuvchi. **20.** Yaqinlashuvchi. **21.** Yaqinlashuvchi.
6.**22.** Yaqinlashuvchi. **23.** Uzoqlashuvchi. **24.** Yaqinlashuvchi.
25. Yaqinlashuvchi. **26.** Yaqinlashuvchi. **27.** Yaqinlashuvchi.
28. Uzoqlashuvchi. **29.** Yaqinlashuvchi. **30.** Yaqinlashuvchi.
31. Uzoqlashuvchi. **32.** Uzoqlashuvchi. **33.** Uzoqlashuvchi.
34. Yaqinlashuvchi. **35.** Uzoqlashuvchi. **36.** Yaqinlashuvchi.
37. Uzoqlashuvchi. **38.** Yaqinlashuvchi. **39.** Yaqinlashuvchi.
40. Yaqinlashuvchi. **41.** Yaqinlashuvchi. **42.** Uzoqlashuvchi
43. Yaqinlashuvchi. **44.** Ixtiyoriy α lar uchun yaqinlashuvchi.
45. Yaqinlashuvchi. **46.** Uzoqlashuvchi. **47.** Yaqinlashuvchi.
48. Yaqinlashuvchi. **49.** Uzoqlashuvchi. **50.** Uzoqlashuvchi.
51. Yaqinlashuvchi. **52.** Yaqinlashuvchi. **53.** Yaqinlashuvchi.
54. Yaqinlashuvchi. **55.** Yaqinlashuvchi. **56.** Yaqinlashuvchi.
57. Yaqinlashuvchi.

11.2-amaliy mashg'ulot.
Ixtiyoriy ishorali qatorlar va ularning yaqinlashuvchiligi

1-misol. $\sum_{n=2}^{\infty} (-1)^n \frac{\ln n}{n}$. qatorni absolyut va shartli yaqinlashuvchilikga tekshiring.

Yechilishi. 1) Berilgan qator hadlarining absolyut qiymatlaridan tuzilgan ushbu $\sum_{n=2}^{\infty}\frac{\ln n}{n}$ qatorni yaqinlashishga tekshiramiz. Bu qatorining umumiy hadi $a_n=\frac{\ln n}{n}=f(n)$ da $n=x$ $f(x)=\frac{\ln x}{x}$ funksiya $[2;+\infty)$ da musbat, uzluksiz $f(x)=\frac{\ln x}{x}$ funksiyani monotonlikka tekshiramiz:

$$f'(x)=\left(\frac{\ln x}{x}\right)'=\frac{1-\ln x}{x^2}$$

Agar $x>e$ bo'lsa, $f'(x)<0$ bo'ladi, ya'ni $f(x)$ funksiya monoton kamayuvchi. Demak, $\frac{\ln x}{x}$ funksiya Makloren Koshi alomatining hamma shartlarini qanoatlantiradi. Shuning uchun, $\sum_{n=2}^{\infty}\frac{\ln n}{n}$ qatorga Makloren - Koshining integral alomatini qo'llaymiz:

$$F(x)=\int\frac{\ln x}{x}dx=\frac{\ln^2 x}{2}\underset{x\to\infty}{\to}+\infty$$

bo'lgani uchun $\sum_{n=2}^{\infty}\frac{\ln n}{n}$ - qator uzoqlashuvchi.

2) Endi $\sum_{n=2}^{\infty}(-1)^n\cdot\frac{\ln n}{n}$ qatorni yaqinlashishiga tekshiramiz. Ravshanki bu qatorning umumiy hadi Leybnis teoremasining hamma shartlarini qanoatlantiradi, ya'ni $c_n=(-1)^n\cdot\frac{\ln n}{n}$ absolyut qiymati bo'yicha monoton kamayuchi va $n\to\infty$ bu qator shartli yaqinlashuvchi.

2-misol. Ushbu

$$\sum_{n=1}^{\infty}a_n\cos nx \tag{1}$$

qatorni yaqinlashishga tekshiring, bunda $\{a_n\}$ - monoton bo'lib, $\lim_{n\to\infty}a_n=0$

Yechilishi. Ma'lumki,

$$\sum_{k=1}^{\infty}\cos kx=\frac{\sin\frac{nx}{2}\cos\frac{(n+1)x}{2}}{\sin\frac{x}{2}},\quad x\neq 2\pi m, m\in Z.$$

bundan $\left|\sum_{k=1}^{n}\cos kx\right|\leq\frac{1}{\left|\sin\frac{x}{2}\right|}$, $x=2\pi m$, $m\in Z$, shartga ko'ra $\{a_n\}$ monoton bo'lib nolga intiladi.

Demak, berilgan (1) qator $x \neq 2\pi m$, $m \in Z$ lar uchun yaqinlashuvchi bo'ladi. Xususiy holda,

$$\sum_{n=1}^{\infty} \frac{\cos 2nx}{n^{\alpha}}, \ \alpha > 0$$

qator $x \neq \pi m$, $m \in Z$ lar uchun yaqinlashuvchi bo'ladi.■

3-misol. Ushbu

$$\sum_{n=1}^{\infty} \frac{\cos \frac{n\pi}{5}}{\sqrt{n} \ln\left(1+\frac{1}{n}\right)^{-n}}$$

qatorni yaqinlashishga tekshiring.

Yechilishi. $\left\{\left(1+\frac{1}{n}\right)^{-n}\right\}$-monoton kamayuchi va chegaralangan,

$\sum_{n=1}^{\infty} \frac{\cos \frac{n\pi}{5}}{\sqrt{n} \ln(n+1)}$ qator esa, yaqinlashuvchi.

Demak, berilgan qator Abel alomatiga asosan yaqinlashuvchi.■

4-misol. $\sum_{n=2}^{\infty} \frac{\ln^{100} n}{n} \cdot \sin \frac{n\pi}{4}$ qatorni yaqinlashishga tekshiring.

Yechilishi. Ma'lumki, $S_n = \sum_{k=2}^{n} \sin \frac{k\pi}{4}$ chegeralangan:

$$\left|\sum_{k=1}^{n} \sin \frac{k\pi}{4}\right| < \frac{1}{\sin \frac{\pi}{8}}.$$

$\{b_k\} = \{\frac{\ln^{100} k}{k}\}$ ketma-ketlik istalgancha katta n larda nolga monoton intiladi.

Haqiqatdan ham

$$\lim_{x\to+\infty} x^{-1} \ln^{100} x = 100 \lim_{x\to+\infty} x^{-1} \ln^{99} x = 100 \cdot 99 \cdot \lim_{x\to+\infty} x^{-1} \ln^{98} x = \ldots =$$

$$= 100! \lim_{x\to+\infty} \frac{1}{x} = 0, \quad (x^{-1} \ln^{100} x)' < 0, \ x > e^{100}.$$

Demak, Drixle-Abel alomatiga asosan, berilgan qator yaqinlashuvchi.■

Mustaqil yechish uchun misollar

Quyidagi qatorlarning absolyut yaqinlashuvchiligini isbotlang:

1. $\sum\limits_{n=1}^{\infty}(-1)^n\dfrac{1}{ne^{\sqrt{n}}}$. **2.** $\sum\limits_{n=1}^{\infty}(-1)^n\dfrac{n^5}{2^n+3^n}$. **3.** $\sum\limits_{n=1}^{\infty}(-1)^n\ln\left(1+\sin^2\dfrac{\pi}{n}\right)$.

4. $\sum\limits_{n=1}^{\infty}(-1)^n\sqrt[3]{n}\,arctg\dfrac{2n+1}{n^3+2}$. **5.** $\sum\limits_{n=1}^{\infty}\dfrac{(-n)^n}{(2n)!}$. **6.** $\sum\limits_{n=1}^{\infty}(-1)^n\dfrac{n^2}{\left(3+\dfrac{1}{n}\right)^n}$.

7. $\sum\limits_{n=1}^{\infty}(-1)^{\frac{n(n-1)}{2}}\dfrac{n^{100}}{2^n}$. **8.** $\sum\limits_{n=1}^{\infty}\dfrac{\sin\dfrac{n\pi}{4}}{n^3+\sin\dfrac{\pi n}{4}}$. **9.** $\sum\limits_{n=1}^{\infty}\left(\dfrac{1}{n\sin\dfrac{1}{n}}-\cos\dfrac{1}{n}\right)\cos\pi n$.

Ishorasi almashinuvchi qatorlarning absolyut, shartli yaqinlashishini yoki uzoqlashishini tekshiring:

10. $\sum\limits_{n=2}^{\infty}(-1)^n\dfrac{\ln n}{n^2+1}$. **11.** $\sum\limits_{n=1}^{\infty}(-1)^n\dfrac{(2n-1)!!}{(2n)!!}$.

12. $\sum\limits_{n=2}^{\infty}(-1)^n\dfrac{\ln n}{3n+2}$. **13.** $\sum\limits_{n=1}^{\infty}(-1)^n\dfrac{\sqrt{n+2}}{2^{n^2}}$.

14. $\sum\limits_{n=3}^{\infty}(-1)^n\dfrac{\sqrt{n+1}-\sqrt{n-2}}{n}$. **15.** $\sum\limits_{n=1}^{\infty}(-1)^n\left[\dfrac{(2n)!!}{(2n-1)!!}\right]^{-2}$.

Quyidagi qatorlarni Dirixle va Abel alomatlari bo'yicha yaqinlashishga tekshiring:

16. $\sum\limits_{n=1}^{\infty}(-1)^n\dfrac{\sin^2 n}{n}$. **17.** $\sum\limits_{n=1}^{\infty}\dfrac{\sin nx}{n^{\alpha}},\ \alpha>0$.

18. $\sum\limits_{n=2}^{\infty}\dfrac{\cos\dfrac{\pi n^2}{n+1}}{\ln^2 n}$. **19.** $\sum\limits_{n=1}^{\infty}\dfrac{\sin n\sin n^2}{n}$.

Quyidagi qatorlarni yaqinlashuvchi ekanligini isbotlang va ularning yig'indisini toping

20. $1-\dfrac{3}{2}+\dfrac{5}{4}-\dfrac{7}{8}+\ldots$. **21.** $1+\dfrac{1}{2}-\dfrac{1}{4}+\dfrac{1}{8}+\dfrac{1}{16}-\dfrac{1}{32}+\ldots$.

22. $1-\dfrac{1}{2}+\dfrac{1}{3}-\dfrac{1}{4}+\dfrac{1}{5}-\dfrac{1}{6}+\ldots$.

Mustaqil yechish uchun misollarning javoblari

10. Absolyut yaqinlashuvchi. **11.** Shartli yaqinlashuvchi. **11.** Shartli yaqinlashuvchi. **13.** Absolyut yaqinlashuvchi. **14.** Shartli yaqinlashuvchi. **15.** Uzoqlashuvchi. **16.** Yaqinlashuvchi. **17.** $\forall x \in R$-lar uchun yaqinlashuvchi. **18.** Yaqinlashuvchi. **19.** Yaqinlashuvchi. **20.** $\frac{2}{9}$. **21.** $\frac{10}{3}$. **22.** $\ln 2$.

11-bob bo'yicha amaliy mashg'ulotlarni mustahkamlash uchun nazorat topshiriqlari

11.1-masala. Quyidagi qatorlarning: 1) n ta hadlarning yig'indisi S_n ni ; 2) qator yaqinlashishining ta'rifigi ko'ra ularning yaqinlashishini isbotlang; 3) qatorning yig'indisini toping.

11.1.1. $\sum\limits_{n=1}^{\infty}\frac{1}{(2n+5)(2n+7)}$. **11.1.2.** $\sum\limits_{n=1}^{\infty}\frac{1}{n(n+1)(n+2)}$.

11.1.3. $\sum\limits_{n=2}^{\infty}\frac{2}{4n^2-9}$. **11.1.4.** $\sum\limits_{n=3}^{\infty}\frac{2n+1}{n(n^2-1)}$.

11.1.5. $\sum\limits_{n=2}^{\infty}\frac{24}{9n^2-12n-5}$. **11.1.6.** $\sum\limits_{n=1}^{\infty}\frac{7}{49n^2-7n-12}$.

11.1.7. $\sum\limits_{n=1}^{\infty}\frac{4}{4n^2+4n-3}$. **11.1.8.** $\sum\limits_{n=1}^{\infty}\frac{4-5n}{n(n-1)(n-2)}$.

11.1.9. $\sum\limits_{n=1}^{\infty}\frac{5n+3}{n(n+1)(n+3)}$. **11.1.10.** $\sum\limits_{n=1}^{\infty}\frac{1}{n(n^2-4)}$.

11.1.11. $\sum\limits_{n=0}^{\infty}\left(\frac{7}{3^n}+\frac{1}{4^n}\right)$. **11.1.12.** $\sum\limits_{n=1}^{\infty}\left(\frac{1}{\sqrt{n}}-\frac{1}{\sqrt{n+1}}\right)$.

11.1.13. $\sum\limits_{n=0}^{\infty}\left(\frac{1}{2^n}+\frac{(-1)^n}{5^n}\right)$. **11.1.14.** $\sum\limits_{n=1}^{\infty}\frac{2n+1}{n^2(n+1)^2}$.

11.1.15. $\sum\limits_{n=1}^{\infty}\frac{n+6}{(n+3)(n+2)n}$. **11.1.16.** $\sum\limits_{n=1}^{\infty}\frac{3^n+5^5}{15^n}$.

11.1.17. $\sum\limits_{n=1}^{\infty}\frac{5^n-3^n}{15^n}$. **11.1.18.** $\sum\limits_{n=1}^{\infty}\frac{4^n-3^n}{12^n}$.

11.1.16. $\sum\limits_{n=1}^{\infty}\frac{2^n+5^n}{10^n}$. **11.1.20.** $\sum\limits_{n=1}^{\infty}\frac{3^n+4^n}{12^n}$.

11.1.21. $\sum\limits_{n=1}^{\infty}\frac{3^n+6^n}{18^n}$. **11.1.22.** $\sum\limits_{n=1}^{\infty}\frac{15^n-10^n}{25^n}$.

11.1.23. $\sum_{n=1}^{\infty}\frac{8^n-3^n}{24^n}$. **11.1.24.** $\sum_{n=1}^{\infty}\frac{2^n+7^n}{14^n}$.

11.1.25. $\sum_{n=1}^{\infty}\frac{8^n+4^n}{16^n}$. **11.1.26.** $\sum_{n=0}^{\infty}\left(\frac{5}{2^n}+\frac{1}{3^n}\right)$.

Yechilishi ([2], 8-bo'lim, [9], 1-bo'lim). 1) Berilgan qatorning S_n – qismiy yig'indisini tuzamiz va uni hisoblaymiz.

$$S_n=\left(\frac{5}{1}+\frac{1}{1}\right)+\left(\frac{5}{2}+\frac{1}{3}\right)+\left(\frac{5}{4}+\frac{1}{9}\right)+....+\left(\frac{5}{2^n}+\frac{1}{3^n}\right)=$$

$$=\left(5+\frac{5}{2}+\frac{5}{2^2}+...+\frac{5}{2^n}\right)+\left(1+\frac{1}{3}+\frac{1}{3^2}+....+\frac{1}{3^n}\right)=$$

$$=5\left(1+\frac{1}{2}+\frac{1}{2^2}+...+\frac{1}{2^n}\right)+\left(1+\frac{1}{3}+\frac{1}{2^2}+....+\frac{1}{3^n}\right)=$$

$$=5\cdot\frac{1-\frac{1}{2^{n+1}}}{1-\frac{1}{2}}+\frac{1-\frac{1}{3^{n+1}}}{1-\frac{1}{3}}=10+\frac{3}{2}-\frac{1}{2^n}-\frac{1}{2\cdot 3^n}=\frac{23}{2}-\left(\frac{1}{2^n}+\frac{1}{2\cdot 3^n}\right)$$

Shunday qilib, $S_n=\frac{23}{2}-\frac{1}{2}\left(\frac{1}{2^{n-1}}+\frac{1}{3^n}\right)$.

2) Qator yaqinlashishining ta'rifigi ko'ra, ularning yaqinlashishini isbotlaymiz:

$$\lim_{n\to\infty}S_n=\lim_{n\to\infty}\left[\frac{23}{2}-\frac{1}{2}\left(\frac{1}{2^{n-1}}+\frac{1}{3^n}\right)\right]=\frac{23}{2}-\frac{1}{2}\lim_{n\to\infty}\left(\frac{1}{2^{n-1}}+\frac{1}{3^n}\right)=\frac{23}{2}.$$

Demak, $S=\frac{23}{2}$ chekli bo'lgani uchun berilgan qator yaqinlashuvchi.

3) $S=\frac{23}{2}$ berilgan qatorning yig'indisi bo'ladi.

Maple tizimidan foydalanib, misolning javobini tekshirish:

> sum(5/2^n+1/3^n, n=0..infinity);

$$\frac{23}{2}$$

11.2-masala. Quyidagi qatorlar uchun qator yaqinlashishining zaruriy sharti bajarilmasligini ko'rsating.

11.2.1. $\sum_{n=1}^{\infty}\left(n^2+2\right)\ln\frac{n^2+1}{n^2}$. **11.2.2.** $\sum_{n=1}^{\infty}\left(\frac{n+2}{n+3}\right)^n$.

11.2.3. $\sum_{n=1}^{\infty}\frac{-n}{2n+5}$. **11.2.4.** $\sum_{n=1}^{\infty}\ln\frac{1}{n}$.

11.2.5. $\sum_{n=1}^{\infty}\left(1-\frac{1}{n}\right)^{n}$.

11.2.6. $\sum_{n=1}^{\infty}\left(\frac{3n^3-2}{3n^3+4}\right)^{n^2}$.

11.2.7. $\sum_{n=1}^{\infty} n\cdot\ln\left(\frac{1+n}{n}\right)$.

11.2.8. $\sum_{n=1}^{\infty}\left(n^2+9\right)\arcsin\frac{1}{n^2+5}$.

11.2.9. $\sum_{n=1}^{\infty} n\cdot tg\frac{1}{n}$.

11.2.10. $\sum_{n=1}^{\infty} n\cdot sh\frac{1}{n}$.

11.2.11. $\sum_{n=1}^{\infty}\frac{1}{\sqrt[n]{\ln(n+1)}}$.

11.2.12. $\sum_{n=1}^{\infty}\sqrt[n]{0,(23)}$.

11.2.13. $\sum_{n=1}^{\infty} n\left(2^{\frac{1}{n}}-1\right)$

11.2.14. $\sum_{n=1}^{\infty}\frac{n}{\sqrt{n}-\sqrt{n+1}}$.

11.2.15. $\sum_{n=1}^{\infty}\frac{n^2}{\log_2(n+1)}$.

11.2.16. $\sum_{n=1}^{\infty} n^{\frac{1}{\sqrt{n}}}$

11.2.17. $\sum_{n=1}^{\infty}\frac{\sqrt[3]{n}}{\ln^2(n+1)}$.

11.2.18. $\sum_{n=1}^{\infty}\sqrt[n]{n+2}$.

11.2.19. $\sum_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^{\sqrt{n}}$.

11.2.20. $\sum_{n=1}^{\infty}\frac{\sqrt[n]{8}-1}{\sqrt[n]{16}-1}$.

11.2.21. $\sum_{n=1}^{\infty}\left(n^3+3\right)\ln\frac{n^3+1}{n^3}$.

11.2.22. $\sum_{n=1}^{\infty}\left(\frac{n^2+2}{n^2+3}\right)^{n^2}$.

11.2.23. $\sum_{n=1}^{\infty}\frac{2n}{3n+7}$.

11.2.24. $\sum_{n=1}^{\infty}\ln\frac{1}{2n+1}$.

11.2.25. $\sum_{n=1}^{\infty}\left(1+\frac{1}{2n}\right)^{2n}$.

11.2.26. $\sum_{n=1}^{\infty}\left(\frac{2n^2-5}{2n^2+3}\right)^{n^2}$.

Yechilishi ([2], 8-bo‘lim, [9], 1-bo‘lim). Berilgan qatorning umumy hadi $a_n=\left(\frac{2n^2-5}{2n^2+3}\right)^{n^2}$. buni quyidagicha yozib olamiz:

$$a_n=\frac{\left(1-\frac{5}{2n^2}\right)^{n^2}}{\left(1+\frac{3}{2n^2}\right)^{n^2}}$$

Ikkinchi ajoyib limitdan foydalanib, $a_n$ning $n\to\infty$dagi limitini topamiz:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{\left(1-\frac{5}{2n^2}\right)^{n^2}}{\left(1+\frac{3}{2n^2}\right)^{n^2}} = \frac{e^{5/2}}{e^{3/2}} = e^{-4} \neq 0.$$

Qator yaqinlashishining zaruriy sharti bajarilmyadi. Demek, berilgan qator uzoqlashuvchi.

Maple tizimidan foydalanib, misolning javobini tekshirish:

> sum((2*n^2-5)/(2*n^2+3), n=1..infinity);

$$\infty$$

11.3– masala. Qo‘yidagi qatorlarni Dalamber alomatidan foydalanib yaqinlashishga tekshiring.

11.3.1. $\sum\limits_{n=1}^{\infty}\frac{1}{(2n+5)(2n+7)}$.

11.3.2. $\sum\limits_{n=1}^{\infty}\frac{1}{n(n+1)(n+2)}$.

11.3.3. $\sum\limits_{n=1}^{\infty}\frac{n^5}{5^n}$.

11.3.4. $\sum\limits_{n=1}^{\infty}\frac{n^4}{4^n}$.

11.3.5. $\sum\limits_{n=1}^{\infty}\frac{n!\,a^n}{n^n}$, $a\neq e, a>0$.

11.3.6. $\sum\limits_{n=1}^{\infty}\frac{(2n)!}{(n!)^2}$

11.3.7. $\sum\limits_{n=1}^{\infty}\frac{(3n+1)!}{8^n\cdot n^2}$.

11.3.8. $\sum\limits_{n=1}^{\infty}\frac{n!(2n+1)!}{(3n)!}$

11.3.9. $\sum\limits_{n=1}^{\infty}\frac{n^n}{n!}$.

11.3.10. $\sum\limits_{n=1}^{\infty} n^2\cdot\sin\frac{\pi}{2^n}$.

11.3.11. $\sum\limits_{n=1}^{\infty}\frac{1\cdot 3\ldots(2n-1)}{3^n\cdot n!}$.

11.3.12. $\sum\limits_{n=1}^{\infty} n\cdot tg\frac{\pi}{2^n+1}$

11.3.13. $\sum\limits_{n=1}^{\infty}\frac{(n+1)!}{2^n\cdot n!}$.

11.3.14. $\sum\limits_{n=1}^{\infty}\frac{n!}{2^n+1}$

11.3.15. $\sum\limits_{n=1}^{\infty}\frac{\ln^{101} n}{4!}$.

11.3.16. $\sum\limits_{n=1}^{\infty}\frac{2^n\cdot n!}{n=1}$.

11.3.17. $\sum\limits_{n=1}^{\infty}\frac{3^n\cdot n!}{n^n}$.

11.3.18. $\sum\limits_{n=1}^{\infty}\left(\sqrt{2}-\sqrt[n+1]{2}\right)$

11.3.16. $\sum\limits_{n=1}^{\infty}\frac{n}{(n-1)!}$.

11.3.20. $\sum\limits_{n=1}^{\infty}\frac{1\cdot 3\cdot 5\ldots(2n-1)}{2^{2n}\cdot(n-1)!}$

11.3.21. $\sum\limits_{n=1}^{\infty}\frac{n^2}{7^n}$.

11.3.22. $\sum\limits_{n=1}^{\infty}\frac{n^6}{6^n}$.

11.3.23. $\sum\limits_{n=1}^{\infty} \frac{n!}{n^n}$. **11.3.24.** $\sum\limits_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$

11.3.25. $\sum\limits_{n=1}^{\infty} \frac{(2n+1)!}{5^n \cdot n^3}$. **11.3.26.** $\sum\limits_{n=1}^{\infty} \frac{1\cdot 3\cdot 5 ... (2n-1)}{3^n \cdot (n+1)}$

Yechilishi ([30], 11-bo'lim, [9], 1-bo'lim). Berilgan qatorning umumiy hadiga ko'ra,

$$\frac{a_{n+1}}{a_n} = \frac{1\cdot 3\cdot 5 ... (2n-1)\cdot(2n+1)}{3^{n+1}\cdot 1\cdot 2\cdot 3 ... n\cdot(n+1)(n+2)}\cdot\frac{3^n\cdot(n+1)!}{1\cdot 3\cdot 5 ... (2n-1)}$$

nisbatni tuzib, uning $n \to \infty$ dagi limitini topamiz:

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{(2n+1)}{3\cdot(n+2)} = \frac{2}{3} < 1.$$

Demak, berilgan qator Dalamber alomatiga ko'ra yaqinlashuvchi.

Maple tizimidan foydalanib, misolning javobini tekshirish:

```
> sum((1*3*5*7 (2*n-1))/(3^n)*(n+1), n=1..infinity );
```

$$\frac{525}{4}.$$

11.4-masala. Quyidagi qatorlarni Koshi alomatidan foydalanib yaqinlashishga tekshiring.

11.4.1. $\sum\limits_{n=1}^{\infty}\frac{1}{(2n+5)(2n+7)}$. **11.4.2.** $\sum\limits_{n=1}^{\infty}\frac{1}{n(n+1)(n+2)}$.

11.4.3. $\sum\limits_{n=1}^{\infty}\left(\frac{n-1}{2n+1}\right)^n$. **11.4.4.** $\sum\limits_{n=1}^{\infty}\frac{1}{n+1}\cdot\left(1+\frac{1}{n}\right)^{n^2}$

11.4.5. $\sum\limits_{n=1}^{\infty}\frac{1}{3^n}\left(1+\frac{1}{n}\right)^{n^2}$. **11.4.6.** $\sum\limits_{n=1}^{\infty}\sqrt{n}\cdot\left(\frac{n}{4n-3}\right)^{2n}$

11.4.7. $\sum\limits_{n=1}^{\infty} n\cdot\left(\arcsin\frac{1}{n}\right)^n$. **11.4.8.** $\sum\limits_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^{n^2}$.

11.4.9. $\sum\limits_{n=1}^{\infty} 3^{n+1}\cdot\left(\frac{n}{n+1}\right)^{n^2}$. **11.4.10.** $\sum\limits_{n=1}^{\infty}\frac{2^{n-1}}{n^n}$.

11.4.11. $\sum\limits_{n=1}^{\infty}\left(\frac{n}{3n-1}\right)^{2n-1}$. **11.4.12.** $\sum\limits_{n=1}^{\infty}\left(\frac{n^2+5}{n^2+6}\right)^{n^2}$

11.4.13. $\sum\limits_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^{n^2}\cdot\frac{1}{4^n}$. **11.4.14.** $\sum\limits_{n=1}^{\infty}\left(\frac{4n-3}{5n+4}\right)^{n^2}$

11.4.15. $\sum\limits_{n=1}^{\infty}\frac{1}{3^n}\cdot\left(\frac{n}{n+1}\right)^{-n^2}$ **11.4.16.** $\sum\limits_{n=1}^{\infty}\left(\frac{2n+2}{3n-1}\right)^{n}\cdot(n+1)^3$

11.4.17. $\sum\limits_{n=1}^{\infty}\frac{n^5\cdot 3^n}{(2n+1)^n}$. **11.4.18.** $\sum\limits_{n=1}^{\infty}\left(\frac{2n+1}{3n-2}\right)^{n^2}$.

11.4.19. $\sum\limits_{n=1}^{\infty}\left(\frac{n}{3n-1}\right)^{n^3}$. **11.4.20.** $\sum\limits_{n=1}^{\infty}\frac{n^{n+2}}{\left(2n^2+1\right)^{n/2}}$.

11.4.21. $\sum\limits_{n=1}^{\infty}\left(\frac{n-1}{2n+1}\right)^{n^2}$. **11.4.22.** $\sum\limits_{n=1}^{\infty}\left(1+\frac{1}{2n}\right)^{n^2}$

11.4.23. $\sum\limits_{n=1}^{\infty}\frac{1}{5^n}\left(1+\frac{1}{n}\right)^{n^2}$. **11.4.24.** $\sum\limits_{n=1}^{\infty}\sqrt{n}\cdot\left(\frac{n}{2n+1}\right)^{2n}$

11.4.25. $\sum\limits_{n=1}^{\infty}n\cdot\left(arctg\frac{1}{n}\right)^{n}$. **11.4.26.** $\sum\limits_{n=1}^{\infty}\left(\frac{2n+3}{n+1}\right)^{n^2}$.

Yechilishi ([9], 1-bo'lim, [30], 11-bo'lim). Berilgan qatorning umumiy hadidan $K_n=\sqrt[n]{a_n}$, bunda $a_n=\left(\frac{2n+3}{n+1}\right)^{n^2}$ ifodani tuzib, $n\to\infty$ da uning limitini hisoblaymiz.

$$\lim_{n\to\infty}K_n=\lim_{n\to\infty}\left(\frac{2n+3}{n+1}\right)^{n}=\lim_{n\to\infty}2^n\left(\frac{n+\frac{3}{2}}{n+1}\right)^{n}=$$

$$=\lim_{n\to\infty}2^n\cdot\left(1+\frac{\frac{1}{2}}{n+1}\right)^{n}=\lim_{n\to\infty}2^n\left[\left(1+\frac{\frac{1}{2}}{n+1}\right)^{2n+2}\right]^{\frac{n}{2n+2}}=$$

$$=\lim_{n\to\infty}2^n\cdot e^{\frac{n}{2n+2}}=+\infty$$

Demak, berilgan qator Koshi alomatiga ko'ra uzoqlashuvchi.

11.5-masala. Qo'yidagi qatorlarni Makloren – Koshining integral alomatidan foydalanib yaqinlashishga tekshiring.

11.5.1. $\sum_{n=1}^{\infty} \frac{1}{(2n+1)\ln^3(2n+1)}$.

11.5.2. $\sum_{n=1}^{\infty} \frac{1}{(3n+4)\ln^2(3n+4)}$.

11.5.3. $\sum_{n=2}^{\infty} \frac{n^2}{(n^3+1)\ln n}$.

11.5.4. $\sum_{n=1}^{\infty} \frac{1}{(n/3-1)\ln^2(n/2)}$.

11.5.5. $\sum_{n=2}^{\infty} \frac{1}{(3n-1)\ln n}$.

11.5.6. $\sum_{n=1}^{\infty} \frac{1}{(n+3)\ln(3n+1)}$.

11.5.7. $\sum_{n=2}^{\infty} \frac{n}{(n^2-3)\ln^2 n}$.

11.5.8. $\sum_{n=2}^{\infty} \frac{n}{(n^2-1)\ln n}$.

11.5.9. $\sum_{n=1}^{\infty} \frac{1}{\sqrt[7]{(3+7n)^{10}}}$.

11.5.10. $\sum_{n=1}^{\infty} \frac{1}{\sqrt[5]{(3n-1)^4}}$.

11.5.11. $\sum_{n=1}^{\infty} \frac{1}{(n+3)\ln^2(n+3)}$.

11.5.12. $\sum_{n=1}^{\infty} \frac{1}{\sqrt[4]{(4n+5)^3}}$.

11.5.13. $\sum_{n=1}^{\infty} \frac{5+n}{25+n^2}$.

11.5.14. $\sum_{n=1}^{\infty} \frac{2+n}{4+n^2-n}$.

11.5.15. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{(4n-3)^3}}$.

11.5.16. $\sum_{n=1}^{\infty} n\, e^{-n^2}$.

11.5.17. $\sum_{n=1}^{\infty} \frac{3n}{(n^2-2)\ln(2n)}$.

11.5.18. $\sum_{n=2}^{\infty} \frac{1}{n\ln^{5\alpha+1} n}$, $(\alpha>0)$.

11.5.19. $\sum_{n=1}^{\infty} \frac{e^{-\sqrt{n}}}{\sqrt{n}}$.

11.5.20. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}\cdot\ln\frac{n+1}{n-1}$.

11.5.21. $\sum_{n=2}^{\infty} \frac{2}{n\ln^2 n}$.

11.5.22. $\sum_{n=1}^{\infty} \frac{n}{5(3n^2+1)}$.

11.5.23. $\sum_{n=1}^{\infty} \frac{1}{\sqrt[7]{(5+3n)^9}}$.

11.5.24. $\sum_{n=1}^{\infty} \frac{1}{\sqrt[5]{(2n+1)^3}}$.

11.5.25. $\sum_{n=1}^{\infty} n^2 e^{-n^3}$.

11.5.26. $\sum_{n=2}^{\infty} \frac{1}{n\ln^{\alpha+1} n}$, $(\alpha>0)$.

Yechilishi. ([9], 1-bo'lim, [30], 11-bo'lim) $a_n = f(n) = \frac{1}{n\ln^{\alpha+1} n}$;

$f(x) = \frac{1}{x\ln^{\alpha+1} x}$ funksiya $x \geq 2$ bo'lganda uzluksiz musbat kamayuvchi bo'lgani uchun. Makloren – Koshining integral belgisiga asosan,

$F(x)=\int \frac{dx}{x\ln^{\alpha+1} x}$ funksiyaning $x \to +\infty$ dagi limitini hisoblaymiz. Buning uchun avvalo $\int \frac{dx}{x\ln^{\alpha+1} x}$ - integralni hisoblaymiz:

$$\int \frac{dx}{x\ln^{\alpha-1} x} = \int \frac{d\ln x}{\ln^{\alpha+1} x} = -\frac{1}{\alpha \ln^2 x}. \quad \lim_{x\to\infty} F(x) = -\lim_{x\to\infty} \frac{1}{\alpha \ln^2 x} = 0.$$

Demak, Makloren – Koshining integral alomatiga ko'ra, berilgan qator yaqinlashukvchi bo'ladi.

12-BOB. FUNKSIONAL KETMA-KETLIKLAR VA QATORLAR

12.1-§. Funksional ketma-ketliklar, funksional qatorlar va ularning yaqinlashuvchiligi

12.1. Funksional ketma-ketliklar va ularning yaqinlashuvchiligi. Elementlari biror $X \subset R$ to'plamda aniqlangan

$$f_1(x), f_2(x), ..., f_n(x), ... \qquad (12.1.1)$$

funksiyalar ketma-ketligi berilgan bo'lsin.

Agar har bir n songa biror qoida bo'yicha X to'plamda aniqlangan $f_n(x)$ funksiyani mos qo'yilgan bo'lsa, u holda nomerlangan $f_1(x), f_2(x), ..., f_n(x), ...$ funksiyalar to'plami *funksional ketma-ketlik deyiladi* va qisqacha $\{f_n(x)\}$ kabi belgilanadi. Umumiy holda $\{f_n(x)\}$ ketma-ketlik turli hadlarining aniqlanish sohasi, umuman aytganda, turlicha bo'lishi ham mumkin. Biz buyerda X sifatida shu sohalarning umumiy qismini olamiz. (12.1.1) ketma ketlikdagi $f_n(x)$ funksiya shu ketma-ketlikning umumiy hadi deyiladi. X to'plamdan x_0 $(x_0 \in X)$ nuqtani olib, (12.1.1) ketma-ketlik har bir hadining shu nuqtadagi qiymatini hisoblab, natijada

$$f_1(x_0), f_2(x_0), ..., f_n(x_0), ...$$

sonlar ketma-ketligini hosil qilamiz.

12.1.2-ta'rif. Agar $\{f_n(x_0)\}$ sonlar ketma-ketligi yaqinlashuvchi (uzoqlashuvchi) bo'lsa, $\{f_n(x)\}$ funksional ketma-ketlik x_0 nuqtada *yaqinlashuvchi (uzoqlashuvchi)* deyiladi.

12.1.3-ta'rif. Agar $\{f_n(x)\}$ funksional ketma-ketlik X to'plamning har bir nuqtasida yaqinlashuvchi (uzoqlashuvchi) bo'lsa, u X to'plamda *yaqinlashuvchi (uzoqlashuvchi)* deyiladi.

12.1.4-eslatma. $\{f_n(x)\}$ funksional ketma-ketlikning yaqinlashish sohasi $\{f_n(x)\}$ funksional ketma-ketlikning aniqlanish sohasiga teng yoki uning bir qismi, yoki bo'sh to'plam ham bo'lishi mumkin.

Faraz qilaylik, $\{f_n(x)\}$ funksional ketma-ketlik $X \subset R$ to'plamda yaqinlashuvchi bo'lsin. U holda $\forall x_0 \in X$ uchun unga mos kelgan,

$$f_1(x_0), f_2(x_0), \ldots, f_n(x_0), \ldots$$

ketma-ketlik chekli limitga ega bo'ladi, ya'ni $\lim\limits_{n\to\infty} f_n(x_0) = f(x_0)$.

Agar X to'plamdan olingan har bir x ga, unga mos kelgan $f_1(x), f_2(x), \ldots, f_n(x), \ldots$ ketma-ketlikning limitini mos qo'ysak, ya'ni $f : x \to \lim\limits_{n\to\infty} f_n(x)$, unda X to'plamda aniqlangan biror $f(x)$ funksiya hosil bo'ladi. $f(x)$ funksiya $\{f_n(x)\}$ funksional ketma-ketlikning limit funksiyasi deb ataladi va uni

$$\lim_{n\to\infty} f_n(x) = f(x) \; (x \in X) \qquad (12.1.5)$$

kabi yozamiz yoki qisqacha $f_n(x) \xrightarrow{X} f(x)$ deb belgilaymiz. (12.1.5) ni "ε" tilida quyidagicha ham yozish mumkin:

$$\forall \varepsilon > 0 \;\; \exists n_0 = n_0(\varepsilon, x) \in N : \;\; \forall n \geq n_0, \;\; \forall x \in X \;\; \Rightarrow \;\; |f_n(x) - f(x)| < \varepsilon.$$

Bu yaqinlashishni "nuqtaviy" yaqinlashish deb ham yuritiladi.

12.1.6-misol. Ushbu $\{f_n(x)\} = \left\{\dfrac{n^2+1}{n^2+x^2}\right\}$ funksional ketma-ketlikning limit funksiyasini toping.

Yechilishi. Berilgan ketma-ketlikning hamma hadlari $X = R$ da aniqlangan. Shuning uchun bu ketma-ketlikning aniqlanish sohasi $X = R$ dan iborat. Umumiy hadi $f_n(x) = \dfrac{n^2+1}{n^2+x^2}$. Endi limit funksiyani topamiz

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{1+\dfrac{1}{n^2}}{1+\dfrac{x^2}{n^2}} = 1.$$

Demak, berilgan funksional $\left\{\dfrac{n^2+1}{n^2+x_2}\right\}$ ketma-ketlikning limit funksiyasi $f(x) = 1$ bo'ladi, ya'ni $\dfrac{n^2+1}{n^2+x^2} \xrightarrow{R} 1$.

12.2. Funksional qatorlar va ularning yaqinlashuvchiligi. Biror X($X \subset R$) to'plamda $u_1(x), u_2(x),..., u_n(x),...$ funksiyalar ketma- ketligi berilgan bo'lsin.

12.2.1-ta'rif. Ushbu

$$u_1(x)+u_2(x)+...+u_n(x)+...$$

ifodaga *funksional qator* deyiladi va u $\sum_{n=1}^{\infty} u_n(x)$ kabi belgilanadi:

$$u_1(x)+u_2(x)+...+u_n(x)+...=\sum_{n=1}^{\infty} u_n(x). \qquad (12.2.2)$$

Bunda $u_1(x), u_2(x),...,u_n(x),...$ lar qatorning hadlari, $u_n(x)$ esa, funksional qatorning umumiy hadi deb ataladi. (12.2.2) funksional qatorning hadlaridan tuzilgan ushbu

$$\begin{aligned} &S_1(x)=u_1(x) \\ &S_2(x)=u_1(x)+u_2(x) \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ &S_n(x)=u_1(x)+u_2(x)+...+u_n(x) \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned} \qquad (12.2.3)$$

yig'indilar ketma-ketligi (12.2.2) funksional qatorning qismiy yig'indilari ketma-ketligi deyiladi va u $\{S_n(x)\}$ kabi belgilanadi.

12.2.4-eslatma. Ushbu $\sum_{n=1}^{\infty} u_n(x)$ funksional qator turli hadlarining aniqlanish sohalari (to'plamlari), umuman aytganda, turlicha bo'ladi. Biz bu yerda X to'plam sifatida shu sohalarning umumiy qismini tushunamiz.

X to'plamdan $x_0 (x_0 \in X)$ nuqtani olib, (12.2.2) funksional qator har bir $u_n(x)$ $(n=1,2,...)$ hadlarining shu nuqtadagi qiymatini hisoblab, ushbu

$$\sum_{n=1}^{\infty} u_n(x_0)=u_1(x_0)+u_2(x_0)+...+u_n(x_0)+... \qquad (12.2.5)$$

sonli qatorni hosil qilamiz.

12.2.6-ta'rif. Agar (12.2.5) sonli qator yaqinlashuvchi (uzoqlashuvchi) bo'lsa, (12.2.2) funksional qator x_0 nuqtada *yaqinlashuvchi (uzoqlashuvchi) deyiladi.*

12.2.7-ta'rif. Agar (12.2.2) funksional qator X to'plamning har bir nuqtasida yaqinlashuvchi (uzoqlashuvchi) bo'lsa, (12.2.2) funksional qator X to'plamda *yaqinlashuvchi (uzoqlashuvchi)* deyiladi. Bu holda (12.2.2) funksional qator *"nuqtaviy" yaqinlashish* deyiladi.

Faraz qilaylik, (12.2.2) funksional qator X to'plamda yaqinlashuvchi bo'lsin. U holda $\forall x_0 \in X$ uchun unga mos kelgan (12.2.5) sonli qator yaqinlashuvchi bo'ladi va uning yig'indisi biror S_0 songa teng bo'ladi. Agar X to'plamdan olingan har bir x ga, unga mos kelgan

$$\sum_{n=1}^{\infty} u_n(x) = u_1(x) + u_2(x) + \ldots + u_n(x) + \ldots$$

qatorning yig'indisini mos qo'ysak, u holda X to'plamda aniqlangan biror $S(x)$ funksiya hosil bo'ladi. Bu $S(x)$ funksiya $\sum\limits_{n=1}^{\infty} u_n(x)$ funksional qatorning yig'indisi deyiladi va u

$$S(x) = \sum_{n=1}^{\infty} u_n(x)$$

kabi yoziladi.

Sonli qatorlarning yaqinlashish (uzoqlashish) ta'rifiga asosan, funksional qatorning x_0 nuqtadagi yaqinlashish (uzoqlashish) ta'rifini quyidagicha ham berish mumkin.

12.2.8-ta'rif. Agar $n \to \infty$ da (12.2.3) funksional ketma-ketlik x_0 nuqtada yaqinlashuvchi (uzoqlashuvchi) bo'lsa, (12.2.2) funksional qator x_0 nuqtada *yaqinlashuchi (uzoqlashuvchi)* deyiladi.

Agar $n \to \infty$ da $\{S_n(x)\}$ funksional ketma-ketlik X to'plamda $S(x)$ limit funksiyaga ega bo'lsa, ya'ni

$$\lim_{n\to\infty} S_n(x) = S(x) \text{ yoki } n \to \infty \text{ da } S_n(x) \xrightarrow{X} S(x)$$

bo'lsa, $S(x)$ funksiya (12.2.2) qatorning *yig'indisi* deyiladi.

12.2.9-ta'rif. Agar

$$\sum_{n=1}^{\infty} |u_n(x)| = |u_1(x)| + |u_2(x)| + \ldots + |u_n(x)| + \ldots \qquad (12.2.10)$$

funksional qator $x=x_0$ nuqtada yaqinlashuvchi bo'lsa, (12.2.2) funksional qator x_0 *nuqtada absolyut yaqinlashuvchi* deyiladi.

12.2.11-ta'rif. Agar X to'plamning har bir nuqtasida (12.2.10) qator yaqinlashuvchi bo'lsa, (12.2.2) funksional qator X to'plamda *absolyut yaqinlashuvchi* deb ataladi.

Agar $x=x_0$ nuqtada (12.2.10) qator uzoqlashuvchi bo'lib, (12.2.2) qator yaqinlashuvchi bo'lsa, (12.2.2) qator $x=x_0$ nuqtada *shartli yaqinlashuvchi* deyiladi.

(12.2.2) va (12.2.10) qatorlar yaqinlashadigan nuqtalar to'plami mos ravishda (12.2.2) qatorning *yaqinlashish va absolyut yaqinlashish sohasi* deyiladi.

12.2.12-eslatma. Berilgan (12.2.2) funksional qatorning yaqinlashish va absolyut yaqinlashish sohasini topishda sonli qatorlar mavzusida ko'rib o'tilgan Dalamber va Koshi alomatlaridan foydalanish mumkin.

12.2.12 -misol. Ushbu

$$\sum_{n=1}^{\infty}\frac{nx}{(1+x)(1+2x)\cdots(1+nx)}$$

funksional qatorning $X=(0,\infty)$ da yig'indisini toping.

Yechilishi. 11.1.3-teoremaga ko'ra, funksional qatorning umumiy hadini quyidagi ko'rinishda ifodalaymiz:

$$u_n(x)=\frac{nx}{(1+x)(1+2x)\cdots(1+nx)}=$$

$$=\frac{1}{(1+x)(1+2x)\cdots(1+(n-1)x)}-\frac{1}{(1+x)(1+2x)\cdots(1+(n-1)x)(1+nx)}.$$

U holda funksional qatorning qismiy yig'indisi

$$S_n(x)=1-\frac{nx}{(1+x)(1+2x)\cdots(1+nx)}$$

bo'ladi. Endi $n\to\infty$ da limitga o'tamiz:

$$\lim_{n\to\infty}S_n(x)=\lim_{n\to\infty}\left(1-\frac{1}{(1+x)(1+2x)\cdots(1+nx)}\right)=1.$$

Demak, berilgan funksional qatorning yig'indisi $S(x)=1$ bo'ladi.

12.2-§. Funksional ketma-ketlik va qatorlarning tekis yaqinlashuvchiligi

12.3. Funksional ketma-ketliklarning tekis yaqinlashuvchiligi. Ushbu

$$f_1(x), f_2(x), ..., f_n(x), ... \qquad (12.3.1)$$

funksional ketma-ketlik X $(X \subseteq R)$ to'plamda yaqinlashuvchi va uning limit funksiyasi $f(x)$ bo'lsin.

12.3.2-ta'rif. Agar $\forall \varepsilon > 0$ son olganda ham $\exists n_0(\varepsilon) \in N$ nomer topilib, $\forall n > n_0(\varepsilon)$ va $\forall x \in X$ lar uchun bir vaqtda

$$|f_n(x) - f(x)| < \varepsilon$$

tengsizlik bajarilsa, $\{f_n(x)\}$ funksional ketma-ketlik X to'plamda $f(x)$*ga tekis yaqinlashadi* deyiladi va u qisqacha $n \to \infty$ da

$$f_n(x) \overset{X}{\rightrightarrows} f(x)$$

kabi belgilanadi.

12.3.3-eslatma. 12.3.2-ta'rifdagi n_0 natural son faqat ε ga bog'liq bo'lib, x larga bog'liq bo'lmaydi. «Nuqtaviy» yaqinlashish (12.1.5) da esa, n_0 natural son ε hamda x ga bog'liq bo'ladi. Shunday qilib, tekis yaqinlashishdan «nuqtaviy» yaqinlashish kelib chiqadi. Buning aksi o'rinli emas.

12.3.4-ta'rif. $\forall m \in N$ olinganda ham, $\exists \varepsilon_0 > 0$, $\exists n \geq m$ va $x_0 \in X$ mavjud bo'lib,

$$|f_n(x_0) - f(x_0)| \geq \varepsilon_0$$

tengsizlik bajarilsa, $\{f_n(x)\}$ funksional ketma-ketlik X to'plamda $f(x)$ funksiyaga *tekis yaqinlashmaydi* deyiladi va u qisqacha

$$f_n(x) \overset{X}{\not\rightrightarrows} f(x)$$

kabi belgilanadi.

12.3.5-ta'rif. Agar $n\to\infty$ da $f_n(x)\overset{X}{\to}f(x)$ bo'lib, lekin $f_n(x)\overset{X}{\not\rightrightarrows}f(x)$ bo'lsa, $\{f_n(x)\}$ketma-ketlik X da $f(x)$ funksiyaga *tekis yaqinlashmaydi (notekis yaqinlashadi)* deyiladi.

Xususiy holda, agar

$$f_n(x)\overset{X}{\to}f(x) \text{ va } \exists\varepsilon_0>0,\ \forall m\in N\ \exists n\ge m \text{ va } \exists x_n\in X$$
$$|f_n(x_n)-f(x_n)|\ge\varepsilon_0 \tag{12.3.6}$$

shart bajarilsa, $\{f_n(x)\}$ ketma-ketlik X da $f(x)$ ga tekis yaqinlashmaydi deyiladi.

12.3.6-misol. $f_n(x)=x^n,\ 0\le x<1$. Bu $\{f_n(x)\}$ ketma-ketlik $\forall x\in[0,1)$ uchun $n\to\infty$ da $f_n(x)\overset{[0,1)}{\to}0$. Lekin, u $\forall x\in[0,1)$ da $f(x)=0$ limit funksiyaga tekis yaqinlashmaydi. Haqiqatan ham,

$$\sup_{x\in[0,1)}|x^n-0|=1,\ f_n\overset{[0,1)}{\not\rightrightarrows}0.$$

Berilgan ketma-ketlik $[0,q]\ (0<q<1)$ kesmada $f(x)=0$ funksiyaga tekis yaqinlashadi. Haqiqatan ham,

$$\sup_{x\in[0,q]}|x^n-0|=q^n\underset{n\to\infty}{\to}0.$$

Tekis yaqinlashuvchi funksional ketma-ketliklar quyidagi xossalarga ega.

1⁰. Agar $\{f_n(x)\}$ va $\{g_n(x)\}$ funksional ketma-ketliklar

$$n\to\infty \text{ da } f_n\overset{X}{\rightrightarrows}f(x),\ g_n\overset{X}{\rightrightarrows}g(x)$$

bo'lsa, u holda $\{\lambda f_n(x)+\mu g_n(x)\}$ (bunda λ,μ ixtiyoriy haqiqiy sonlar) funksional ketma-ketlik ham

$$n\to\infty \text{ da } \lambda f_n(x)+\mu g_n(x)\overset{X}{\rightrightarrows}\lambda f(x)+\mu g(x)$$

bo'ladi.

2^0. Agar $\{f_n(x)\}$ funksional ketma-ketlik $f_n(x)\overset{X}{\rightrightarrows} f(x)$ bo'lib, $g(x)$ funksiya esa X to'plamda chegaralangan bo'lsa, ya'ni

$$\exists M>0:\ \forall x\in X \text{ uchun } |g(x)|\le M$$

bo'lsa, u holda $\{g(x)f_n(x)\}$ ketma-ketlik ham $n\to\infty$ da $g(x)f_n(x)\overset{X}{\rightrightarrows} g(x)f(x)$ bo'ladi.

12.3.7-teorema (tekis yaqinlashish sharti). (12.3.1) funksional ketma-ketlik X to'plamda $f(x)$ga tekis yaqinlashishi uchun

$$\lim_{n\to\infty}\sup_{x\in X}|f_n(x)-f(x)|=0 \qquad (12.3.8)$$

shartning bajarilishi zarur va yetarli, ya'ni

$$n\to\infty \text{ da } f_n(x)\overset{X}{\rightrightarrows} f(x) \Leftrightarrow \lim_{n\to\infty}\sup_{x\in X}|f_n(x)-f(x)|=0\,.$$

12.3.9-t eorema. (12.3.1) funksional ketma-ketlik X to'plamda $f(x)$ ga tekis yaqinlashishi uchun shunday $\{a_n\}$ sonli ketma-ketlik (bunda $\lim\limits_{n\to\infty} a_n=0$) va shunday m nomer mavjud bo'lib, barcha $n>m$ va barcha $x\in X$ lar uchun

$$|f_n(x)-f(x)|\le a_n \qquad (12.3.10)$$

tengsizlikning bajarilishi zarur va yetarli.

12.3.11-natija. $n\to\infty$ da $f_n(x)\overset{X}{\not\rightrightarrows} f(x)$ bo'lishi uchun $\exists x_n\in X$: $n\to\infty$ da

$$|f_n(x_n)-f(x_n)|\not\to 0 \qquad (12.3.12)$$

bo'lishi zarur va yetarli.

12.3.13-misol. Ushbu

$$\{f_n(x)\}=\left\{\frac{\cos n\sqrt{x}}{n}\right\}$$

funksional ketma-ketlikning $X=[0;+\infty)$ to'plamda tekis yaqinlashuvchi ekanligini ko'rsating.

Yechilishi. Bu ketma-ketlikning limit funksiyasi

$$f(x)=\lim_{n\to\infty}\frac{\cos n\sqrt{x}}{n}=0$$

bo'lib, $X=[0;+\infty)$ da yaqinlashuvchi bo'ladi. Endi yaqinlashish xarakterini aniqlaymiz. $\forall\varepsilon>0$ son olinganda ham $m=\left[\frac{1}{\varepsilon}\right]$ deyilsa, unda barcha $n>m$ va $\forall x\in X$ va lar uchun

$$|f_n(x)-f(x)|=\left|\frac{\cos n\sqrt{x}}{n}-0\right|=\left|\frac{\cos n\sqrt{x}}{n}\right|\le\frac{1}{n}<\frac{1}{m+1}<\varepsilon$$

bo'ladi. Yuqoridagi **12.3.2**-ta'rif va **12.3.7**-teoremaga ko'ra, $\{f_n(x)\}=\left\{\frac{\cos n\sqrt{x}}{n}\right\}$ ketma-ketlik $f(x)=0$ limit funksiyaga tekis yaqinlashadi:

$$\frac{\cos n\sqrt{x}}{n}\overset{X}{\underset{\to}{\longrightarrow}}0.$$

12.3.14-teorema (funksional ketma-ketlikning tekis yaqinlashishi uchun Koshi kriteriysi). $\{f_n(x)\}$ funksional ketma-ketlik X to'plamda limit funksiyaga ega bo'lishi va unga tekis yaqinlashishi uchun, ixtiyoriy $\varepsilon>0$ uchun x ga bog'liq bo'lmagan shunday nomer $m(\varepsilon)$ mavjud bo'lib, $n>m$ bo'lganda va istalgan $p\in N$ da x ning X dagi hamma qiymatlari uchun bir vaqtning o'zida

$$|f_{n+p}(x)-f_n(x)|<\varepsilon$$

tengsizlikning o'rinli bo'lishi zarur va yetarli.

12.3.14-teoremadagi *Koshi shartini*, qisqacha, kvantor belgisidan foydalanib, quyidagicha yozish mumkin:

$$\forall\varepsilon>0\ \exists m(\varepsilon):\ \forall n\ge m,\forall p\in N,\forall x\in X:|f_{n+p}(x)-f_n(x)|<\varepsilon. \quad (12.3.15)$$

Agar Koshi sharti bajarilmasa, ya'ni

$$\exists\varepsilon_0>0:\forall k\in N,\ \exists n\geq k\ \exists p\in N\ \exists\bar{x}\in X:\left|f_{n+p}(\bar{x})-f_n(\bar{x})\right|\geq\varepsilon_0$$

bo'lsa, u holda $\{f_n(x)\}$ funksional ketma-ketlik X to'plamda $f(x)$ funksiyaga *notekis yaqinlashuvchi* deyiladi.

Xususiy holda, agar

$$\exists\varepsilon_0>0:\exists m\in N\ \forall n\geq m\ \exists p\in N\ \exists x_n\in X:\left|f_{n+p}(x_n)-f(x_n)\right|\geq\varepsilon_0$$

bo'lsa, $\{f_n(x)\}$ ketma-ketlik X to'plamda $f(x)$ funksiyaga tekis yaqinlashuvchi bo'lmaydi.

12.3.16-misol. Ushbu

$$\{f_n(x)\}=\{\frac{\cos\sqrt{nx}}{\sqrt{n+2x}}\}$$

funksional ketma-ketlikning $X=[0;+\infty)$ to'plamda tekis yaqinlashuvchiligini Koshi teoremasidan foydalanib ko'rsating.

Yechilishi. $\{f_n(x)\}=\{\frac{\cos\sqrt{nx}}{\sqrt{n+2x}}\}$ ketma-ketlik uchun $[0;+\infty)$ da (12.3.15) shartni bajarilishini ko'rsatamiz:

$$\left|f_{n+p}(x)-f_n(x)\right|=\left|\frac{\cos\sqrt{(n+p)x}}{\sqrt{n+p+2x}}-\frac{\cos\sqrt{nx}}{\sqrt{n+2x}}\right|\leq\left|\frac{\cos\sqrt{(n+p)x}}{\sqrt{n+p+2x}}\right|+\left|\frac{\cos\sqrt{nx}}{\sqrt{n+2x}}\right|\leq$$

$$\leq\sup_{x\in X}\left|\frac{\cos\sqrt{(n+p)x}}{\sqrt{n+p+2x}}\right|+\sup_{x\in X}\left|\frac{\cos\sqrt{(nx}}{\sqrt{n+2x}}\right|=\frac{1}{\sqrt{n+p}}+\frac{1}{\sqrt{n}}.$$

Agar $\forall\varepsilon>0$ songa ko'ra natural m soni $m=\left[\frac{4}{\varepsilon^2}\right]+1$ deb olinsa, u holda barcha $n>m$ va barcha $p\in N$ lar uchun

$$\frac{1}{\sqrt{n+p}}+\frac{1}{\sqrt{n}}<\frac{2}{\sqrt{m}}<\varepsilon$$

bo'ladi.

Demak, $\forall\varepsilon>0$ son olinganda ham shunday natural m son mavjudki, $\forall n>m,\forall p\in N$ va $\forall x\in[0;\infty)$ lar uchun

$$\left|f_{n+p}(x)-f_n(x)\right|<\varepsilon$$

tengsizlik o'rinli bo'ladi. Bu esa, $\left\{\dfrac{\cos\sqrt{nx}}{\sqrt{n+2x}}\right\}$ ketma-ketlikning, **12.3.14**- Koshi teoremasiga ko'ra, X da tekis yaqinlashuvchi ekanligini ko'rsatadi.

12.4. Funksional qatorlarning tekis yaqinlashuvchiligi. Ushbu

$$\sum_{n=1}^{\infty}u_n(x)=u_1(x)+u_2(x)+\cdots+u_n(x)+\cdots \qquad (12.4.1)$$

funksional qator $X(X\subseteq R)$ to'plamda yaqinlashuvchi va uning yig'indisi $S(x)$ bo'lsin, ya'ni

$$\lim_{n\to\infty}S_n(x)=S(x)=\sum_{n=1}^{\infty}u_n(x).$$

12.4.2-ta'rif. Agar (12.4.1) funksional qatorning $\{S_n(x)\}$ qismiy yig'indilari ketma-ketligi X to'plamda $S(x)$ ga tekis yaqinlashsa, (12.4.1) funksional qator X to'plamda $S(x)$ga *tekis yaqinlashadi* deyiladi va u qisqacha

$$S_n(x)\overset{X}{\underset{\to}{\to}}S(x) \qquad (12.4.3)$$

kabi belgilanadi.

12.4.4-eslatma. Funksional qatorlarning tekis yaqinlashuvchiligi (yaqinlashmovchiligi) tushunchasi ham ularning oddiy yaqinlashuvchiligi singari, funksional ketma-ketliklarning tekis yaqinlashuvchilik (yaqinlashmovchiligi) tushunchasi orqali kiritiladi.

12.4.2-ta'rifni qisqacha, kvantor belgisidan foydalanib, quyidagicha yozish mumkin:

$$\forall\varepsilon>0\exists m(\varepsilon):\forall n>m\forall x\in X\to\left|S_n(x)-S(x)\right|<\varepsilon. \qquad (12.4.5)$$

12.4.6-ta'rif. (12.4.1) qatorning dastlabki n ta hadini tashlab yuborgandan so'ng, hosil bo'lgan ushbu

$$r_n(x)=u_{n+1}(x)+u_{n+2}(x)+\cdots=\sum_{k=n+1}^{\infty}u_k(x)$$

qatorga (12.4.1) funksional qatorning n ta hadidan keyingi *qoldig'i* deyiladi.
Bunda

$$r_n(x)=S(x)-S_n(x)$$

bo'ladi. U holda (12.4.3) shartni quyidagi ko'rinishda ifodalash mumkin:

$$r_n(x)\underset{\rightarrow}{\overset{X}{\rightarrow}}0. \qquad (12.4.7)$$

(12.4.3) va (12.4.7) shartlar teng kuchli.

12.4.8-t a'rif. Agar X to'plamda $S_n(x)$ ketma-ketlikning limit funksiyasi mavjud bo'lsa va (12.4.5) shart bajarilmasa, ya'ni

$$\exists\varepsilon_0>0:\forall k\in N\exists n\geq k\exists\bar{x}\in X\rightarrow|S_n(\bar{x})-S(\bar{x})|\geq\varepsilon_0$$

bo'lsa, $S_n(x)$ ketma-ketlik X to'plamda $S(x)$ ga *notekis yaqinlashadi* deyiladi.

12.4.9-misol. Ushbu

$$\sum_{n=0}^{\infty}(e^{-nx}-e^{-(n+1)x})$$

funksional qatorni, ta'rifga ko'ra, $a)X=(0;\infty)$; $b)X_1=[\delta;\infty),\delta>0$ sohalarda tekis yaqinlashishga tekshiring.

Yechilishi. Berilgan qatorning qismiy yig'indisini topamiz:

$$S_n(x)=(1-e^{-x})\cdot(1+e^{-x}+e^{-2x}+\cdots+e^{-(n-1)x})=$$
$$=(1-e^{-x})\frac{(1-e^{-nx})}{1-e^{-x}}=1-e^{-nx}.$$

Bundan

$$\lim_{n\to\infty}S_n(x)=1,\ S(x)=1.$$

a) Funksional qatorni ta'rifga ko'ra, tekis yaqinlashishga tekshirish uchun $r_n(x) = S(x) - S_n(x)$ ayirmani qaraymiz:

$$|r_n(x)| = |1 + e^{-nx} - 1| = e^{-nx},$$

$\forall\varepsilon > 0$ son olganda $m = \left[\frac{1}{x}\ln\frac{1}{\varepsilon}\right] + 1$ $(x \neq 0)$ deyilsa, u holda $n > m$ lar uchun

$$|r_n(x)| = |S_n(x) - S(x)| = e^{-nx} < \varepsilon \qquad (12.4.10)$$

tengsizlik bajariladi.

Agar $x = 0$ bo'lsa, ravshanki $\forall n \in N$ lar uchun $S_n(0) = 0$; $S(0) = 1$ bo'lib, $|r_n(0)| = |S_n(0) - S(0)| = 1$ bo'ladi. m natural son $\varepsilon > 0$ va $x\ (0 < x < \infty)$ larga bog'liq bo'lib, u barcha x lar uchun umumiy bo'la olmaydi, chunki $m = \left[\frac{1}{x}\ln\frac{1}{\varepsilon}\right] + 1$ ning $(0;\infty)$ da x bo'yicha maksimumi chekli son bo'lmaydi, ya'ni $\forall n \in N$ son olsak ham $\exists\varepsilon_0 > 0\left(\varepsilon_0 = \frac{1}{e^2}\right)$ va $\exists\, x_n = \frac{1}{n} \in (0,\infty)$ nuqta topiladiki,

$$\left|S\left(\frac{1}{n}\right) - S_n\left(\frac{1}{n}\right)\right| = e^{-1} > \varepsilon_0$$

bo'ladi.

Demak, berilgan funksional qator, **12.4.8**-ta'rifga ko'ra, $X = (0;\infty)$ sohada notekis yaqinlashuvchi bo'ladi.

b) $\forall\varepsilon > 0$ son olinganda ham $m_1 = \max\limits_{x \in [\delta;\infty)}\left\{\left[\frac{1}{x}\ln\frac{1}{\varepsilon}\right] + 1\right\} = \left[\frac{1}{\delta}\ln\frac{1}{\varepsilon}\right] + 1$ deb olinsa, $\forall n > m_1$ va $\forall x \in [\delta;\infty)$ lar uchun birdaniga (12.4.10) tengsizlik bajariladi. Shunday qilib, (12.4.5) shartga asosan, berilgan funksional qator $[\delta,\infty)$ sohada x ga nisbatan tekis yaqinlashuvchi bo'ladi, ya'ni $S_n(x) \overset{[\delta;+\infty)}{\rightrightarrows} 1$.

Funksional qator tekis yaqinlashish ta'rifidan hamda funksional ketma-ketlikning tekis yaqinlashish kriteriysidan quyidagi funksional qator tekis yaqinlashish va kriteriysi kelib chiqadi.

6.4.11-teorema. (12.4.1) funkional qatorning X da tekis yaqinlashishi uchun

$$\lim_{n\to\infty}\sup_{x\in X}|r_n(x)|=0 \qquad (12.4.12)$$

shartning bajarilishi zarur va yetarlidir.

12.4.12-teorema (funksional qatorning tekis yaqinlashishi uchun Koshi kriteriysi). (12.4.1) funksional qator X da tekis yaqinlashishi uchun $\forall\varepsilon>0$ son olinganda ham $\exists m(\varepsilon)(m\in N)$ nomer topilib, $\forall n\geq m(\varepsilon)$, barcha butun $p\geq 0$ sonlar va $\forall x\in X$ lar uchun

$$|S_{n+p}-S_n|=\left|\sum_{k=n+1}^{n+p}u_k(x)\right|<\varepsilon \qquad (12.4.14)$$

shartning bajarilishi zarur va yetarli.

Teoremaning isboti funksional ketma-ketlikning tekis yaqinlashish kriteriysi, ya'ni 12.3.14-teoremani funksional qator qismiy yig'indisiga qo'llash natijasida kelib chiqadi.

12.4.15-teorema (zaruriy shart). Agar (12.4.1) funksional qator X da tekis yaqinlashuvchi bo'lsa, u holda uning umumiy hadi $u_n(x)\ (n=1,2,...)$ $u_n(x)\overset{X}{\underset{\rightarrow}{\rightarrow}}0$ bo'ladi.

Isboti. Shartga ko'ra, (12.4.1) funksional qator X to'plamda tekis yaqinlashuvchi bo'lgani uchun (12.4.14) shart bajariladi. (12.4.14) shartda $p=1$ deb olinsa, u holda

$$\forall\varepsilon>0\exists m(\varepsilon)\in N:\forall n>m\forall x\in X\rightarrow|u_{n+1}(x)|<\varepsilon$$

tengsizlik bajariladi, ya'ni $n\to\infty$da $u_n(x)\overset{X}{\underset{\rightarrow}{\rightarrow}}0$. ■

12.4.16-eslatma. Koshi kriteriysidan, ya'ni **12.4.12**-teoremadan xususiy holda, $p=0$ bo'lganda, **12.4.15**-teorema kelib chiqadi.

Agar 12.4.12-teoremaning shartlari bajarilmasa, ya'ni

$$\exists\varepsilon_0>0\ \forall m\in N\exists n\geq m\exists p\in N\exists\bar{x}\in X\rightarrow\left|\sum_{k=n+1}^{n+p}u_k(x)\right|\geq\varepsilon_0$$

bo'lsa, (12.4.1) funksional qator X da tekis yaqinlashuvchi bo'lmaydi.

Xususiy holda, agar

$$\exists\varepsilon_0 > 0\ \exists n_0 \in N\ \forall n \geq N_0\ \exists x_n \in X \rightarrow |u_n(x_n)| \geq \varepsilon_0$$

bajarilsa, u holda (12.4.1) funksional qator X da tekis yaqinlashuvchi bo'lmaydi.

12.4.17-teorema (taqqoslash alomati). $u_k(x)$ va $v_k(x)$ funksiyalar X to'plamda aniqlangan bo'lib, $\forall x \in X$ va $\forall k \in N$ uchun

$$|u_k(x)| \leq v_k(x)$$

tengsizlik o'rinli bo'lib, $\sum\limits_{k=1}^{\infty} v_k(x)$ funksional qator X to'plamda tekis yaqinlashuvchi bo'lsa, u holda $\sum\limits_{k=1}^{\infty} u_k(x)$ funksional qator ham X to'plamda absolyut va tekis yaqinlashuvchi bo'ladi.

Bu teoremadan xususiy holda quyidagi teorema kelib chiqadi.

12.4.18-teorema (Veyershtrass alomati). Agar (12.4.1) funksional qatorning har bir hadi X da aniqlangan bo'lib, $\forall x \in X$ va $\forall n > n_0$ uchun

$$|u_n(x)| \leq c_n \tag{12.4.19}$$

tengsizlikni qanoatlantirsa va

$$\sum_{n=1}^{\infty} c_n = c_1 + c_2 + \ldots + c_n + \ldots \tag{12.4.20}$$

sonli qator yaqinlashuvchi bo'lsa, u holda (12.4.1) funksional qator X da absolyut va tekis yaqinlashuvchi bo'ladi.

Isboti. (12.4.19) shartga asosan, $\forall x \in X$, $\forall n > n_0$ va barcha butun $p \geq 0$ uchun

$$|u_{n+1}(x) + u_{n+2}(x) + \ldots + u_{n+p}(x)| \leq c_{n+1} + c_{n+2} + \ldots + c_{n+p}$$

tengsizlik o'rinli bo'ladi. (12.4.20) sonli qator yaqinlashuvchi bo'lganligi uchun $\exists n_0 \in N$ va $\forall n > n_0$ boshlab

$$c_{n+1}+c_{n+2}+...+c_{n+m}<\varepsilon$$

bo'ladi.

Demak, $\forall\varepsilon>0$ son olinganda ham $\exists n_0(\varepsilon)(n_0\in N)$ nomer topilib, $\forall n\geq n_0(\varepsilon)$, barcha butun $p\geq 0$ sonlar va $\forall x\in X$ lar uchun

$$\left|S_{n+p}-S_n\right|=\left|\sum_{k=n+1}^{n+p}u_k(x)\right|<\varepsilon$$

tengsizlik o'rinli bo'ladi. Bundan **12.4.12**-teoremaga asosan, (12.4.1) funksional qator tekis yaqinlashuvchi bo'ladi.■

12.4.21-natija. Agar (12.4.20) sonli qator yaqinlashuvchi bo'lsa, bunda $c_n=\sup\limits_{x\in R}|u_n(x)|$, (12.4.1) funksional qator tekis yaqinlashuvchi bo'ladi.

12.4.22-eslatma. **12.4.18**-Veyershtrass alomati funksional qatorning tekis yaqinlashishi uchun faqat yetarli shart bo'lib, lekin zaruriy shart bo'la olmaydi.

12.4.23-misol. Ushbu

$$\sum_{n=1}^{\infty}\frac{1}{x^4+n^4}$$

funksional qatorni $X=(-\infty;+\infty)$ da tekis yaqinlashuvchilikka tekshiring.

Yechilishi. Bu funksional qatorning umumiy hadi $u_n(x)=\dfrac{1}{x^4+n^4}$ funksiyadan iborat. X dagi barcha x va $n\in N$ lar uchun

$$|u_n(x)|=\left|\frac{1}{x^4+n^4}\right|\leq\frac{1}{n^4}$$

tengsizlik o'rinli bo'ladi. Bunda

$$\sum_{n=1}^{\infty}\frac{1}{n^4}$$

sonli qator yaqinlashuvchi.

Demak, **12.4.18**-Veyershtrass alomatiga ko'ra, berilgan funksional qator X da absolyut va tekis yaqinlashuvchi bo'ladi.

Quyidagi

$$\sum_{n=1}^{\infty} a_n(x)b_n(x) = a_1(x)b_1(x) + ... + a_n(x)b_n(x) + ... \qquad (12.4.\ 24)$$

funksional qatorni qaraymiz, bunda $a_n(x),\ b_n(x),(n=1,2,...)$ funksiyalar $X(X \subset R)$ da aniqlangan.

12.4.25-teorema (Dirixle alomati). Agar: 1) $\{a_n(x)\}$ funksional ketma–ketlik $\forall x \in X$ lar va $\forall n \in N$ lar uchun monoton bo'lib , $a_{n+1}(x) \le a_n(x)$ yoki $a_{n+1}(x) \ge a_n(x)$ va $a_n(x) \overset{X}{\underset{\to}{\to}} 0$ bo'lsa;

2) $\sum_{n=1}^{\infty} b_n(x)$ qatorning qismiy yig'indilari ketma-ketligi $B_n(x) = \sum_{k=1}^{n} b_k(x)$ $\forall n \in N, \forall x \in X$ larda chegaralangan bo'lsa, ya'ni $\exists M > 0$: $\forall n \in N$, $\forall x \in X$ uchun

$$|B_n(x)| \le M$$

bo'lsa, u holda (12.4. 24) qator X to'plamda tekis yaqinlashuvchi bo'ladi.

12.4.26-teorema (Abel alomati). Agar:

1) $\sum_{n=1}^{\infty} b_n(x)$ qator X to'plamda tekis yaqinlashuvchi bo'lsa, ya'ni $B_n(x) \overset{X}{\underset{\to}{\to}} B(x)$;

2) $\{a_n(x)\}$ ketma–ketlik X to'plamda monoton bo'lsa, ya'ni $\forall n \in N$, $\forall x \in X$ uchun $a_{n+1}(x) \le a_n(x)$ yoki $a_{n+1}(x) \ge a_n(x)$ va $\exists M > 0$: $\forall n \in N$, $\forall x \in X$ lar uchun $|a_n(x)| \le M$ bo'lsa, u holda (12.4.24) funksional qator X da yaqinlashuvchi bo'ladi.

12.4.27-eslatma. Dirixle alomatidan xususiy holda $b_n(x) = (-1)^{n+1}$ Leybnits alomati kelib chiqadi.

12.4.28-misol. Ushbu

$$\sum_{n=1}^{\infty} \frac{\cos nx}{n^{\alpha}}$$

funksional qatorni $X = [\varepsilon, 2\pi - \varepsilon]$, $\varepsilon > 0$ da tekis yaqinlashuvchilikka tekshiring.

Yechilishi. Agar $\alpha > 1$ bo'lsa, Veyershtrass alomatiga ko'ra, berilgan funksional qator X da tekis yaqinlashuvchi bo'ladi.

Haqiqatan ham, $\forall x \in X$ va $\forall n \in N$ lar uchun

$$|\cos nx| \le 1, \quad |u_n(x)| = \left|\frac{\cos nx}{n^\alpha}\right| \le \frac{1}{n^\alpha}$$

bo'lib, $\sum_{n=1}^{\infty} \frac{1}{n^\alpha}$ sonli qator yaqinlashuvchi bo'ladi.

Agar $0 < \alpha < 1$ bo'lsa, $\{a_n\} = \left\{\frac{1}{n^\alpha}\right\}$ ketma–ketlik monoton kamayuvchi va $a_n \overset{X}{\underset{\to}{\to}} 0$. Ushbu $\sum_{n=1}^{\infty} \cos nx$ funksional qatorning qismiy yig'indilari ketma – ketligi uchun

$$\{B_n(x)\} = \left\{\sum_{k=1}^{n} \cos kx\right\} = \left\{\frac{\sin\frac{nx}{2}\cos\frac{(n+1)x}{2}}{\sin\frac{x}{2}}\right\}, \quad x \ne 2\pi m, \ m \in Z$$

formula o'rinli. Bundan $\forall x \in X$ va $\forall n \in N$ lar uchun

$$\left|\sum_{k=1}^{n} \cos kx\right| \le \frac{1}{\left|\sin\frac{x}{2}\right|} \le \frac{1}{\sin\frac{\varepsilon}{2}}, \quad x \ne 2\pi m, \ m \in Z$$

tengsizlik o'rinli bo'ladi. Demak, 12.4.25- teoremga (Dirixle alomati) ko'ra, berilgan qator X da tekis yaqinlashuvchi bo'ladi.

Agar $\alpha \le 0$ bo'lsa, u holda berilgan funksional qator X da uzoqlashuvchi bo'ladi. Haqiqatan ham, har bir berilgan $x \in X$ uchun qator yaqinlashuvchiligining zaruriy sharti bajarilmaydi, ya'ni $u_n(x)$ X to'plamda 0 ga tekis intilmaydi: $u_n(x) \overset{X}{\not\rightrightarrows} 0$.

12.3-§. Funksional qatorlar yig'indisining hamda funksional ketma-ketlik limit funksiyasining funksional xossalari

12.5. Funksional qator yig'indisining uzluksizligi. Funksional qatorning tekis yaqinlashish haqidagi 12.4.2-ta'rifni e'tiborga olganda funksional qatorning barcha xossalarini funksional ketma-ketlikning xossalaridan va aksincha ketma-ketlikning barcha xossalarini funksional qatorning xossalaridan keltirib chiqarish qiyinchilik tug'dirmaydi. Lekin, bu nazariyaning barcha tadbiqlarida ham amaliyotda ham deyarli hamma vaqt funksional qator uchraydi. Shu sababli biz kelsida funksional qator yig'indisining xossalarini ifodalovchi hamma teoremalarni keltiramiz.

X to'plamda yaqinlashuvchi

$$\sum_{n=1}^{\infty} u_n(x) = u_1(x) + u_2(x) + \cdots + u_n(x) + \cdots \qquad (12.5.1)$$

funksional qator berilgan bo'lib, uning yig'indisi $S(x)$ bo'lsin.

12.5.2-teorema. Agar (12.5.1) funksional qatorning har bir hadi $u_n(x)$ $(n = 1,2,\ldots)$ X to'plamda uzluksiz bo'lib, funksional qator X da tekis yaqinlashuvchi bo'lsa, u holda funksional qatorning yig'indisi $S(x)$ ham X to'plamda uzluksiz bo'ladi.

12.5.6-eslatma. 12.5.2-teoremadagi (12.5.1) qatorning X da tekis yaqinlashuvchilik sharti funksional qator yig'indisi $S(x)$ ning uzluksiz bo'lishi uchun yetarli shart bo'ladi, lekin zaruriy shart bo'laolmaydi.

12.5.7-misol. Ushbu

$$\sum_{n=1}^{\infty} [nxe^{-nx} - (n-1)xe^{-(n-1)x}]$$

funksional qatorning $X = [0;1]$ da notekis yaqinlashishini, lekin uning yig'indisi X da uzluksiz funksiya bo'lishini ko'rsating.

Yechilishi. Berilgan funksional qatorning qismiy yig'indisini topamiz:

$$S_n(x) = \sum_{k=1}^{n} [kxe^{-kx} - (k-1)xe^{-(k-1)x}] = xe^{-x} + 2xe^{-2x} - xe^{-x} + \ldots +$$
$$+ nxe^{-nx} - (n-1)xe^{-(n-1)x} = nxe^{-nx}.$$

Endi $n \to \infty$ da $S_n(x)$ funksiyaning limitini hisoblaymiz:

$$S(x) = \lim_{n\to\infty} S_n(x) = \lim_{n\to\infty} \frac{nx}{e^{nx}} = 0 .$$

Demak, berilgan funksional qatorning yig'indisi $S(x)=0$ X da uzluksiz funksiya ekan. Ammo, berilgan qator X da tekis yaqinlashuvchi emas. Haqiqatan ham,

$$\sup_{x\in[0,1]} |S_n(X) - S(x)| = \sup_{x\in[0,1]} |nxe^{-nx}| = \frac{1}{e} \Rightarrow \lim_{n\to\infty} \sup_{x\in[0;1]} |r_n(x)| = e^{-1} \neq 0.$$

Shuning uchun, qator X da o'zining $S(x)$ yig'indisiga tekis yaqinlashmaydi.

12.6. Funksional qatorlarda hadma-had limitga o'tish. Yaqinlashuvchi (12.5.1) funksional qator berilgan bo'lib, uning yig'indisi $S(x)$, x_0 nuqta esa, X to'plamning limit nuqtasi bo'lsin.

12.6.1-teorema. Agar $x \to x_0$ da (12.5.1) funksional qatorning har bir $u_n(x)$ $(n=1,2,...)$ hadi chekli

$$\lim_{x\to x_0} u_n(x) = c_n \quad (n=1,2,...)$$

limitga ega bo'lib, berilgan qator X to'plamda tekis yaqinlashuvchi bo'lsa,

$$\sum_{n=1}^{\infty} c_n = c_1 + c_2 + ... + c_n + ...$$

qator ham yaqinlashuvchi, uning yig'indisi C esa, $S(x)$ ning $x \to x_0$ dagi limitiga

$$\lim_{x\to x_0} S(x) = C$$

teng bo'ladi.

12.6.2-eslatma. 12.6.1-teoremaning shartlari bajarilganda

$$\lim_{x\to x_0} \sum_{n=1}^{\infty} u_n(x) = \sum_{n=1}^{\infty} \lim_{x\to x_0} u_n(x)$$

tenglik o'rinli bo'ladi.

12.6.3-misol. Ushbu

$$\lim_{x\to 1-0}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{x+n-1}\cdot\frac{x^n}{x^n+1}$$

limitni toping.

Yechilishi. a) Berilgan funksional qator $X=[1;\infty)$ to'plamda Abel alomatiga ko'ra, tekis yaqinlashuvchi bo'ladi, ya'ni

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{x+n-1}\overset{X}{\underset{\to}{\to}}S(x);$$

b) barcha $x\in X$ va $\forall n\in N$ lar uchun $a_n(x)=1-\dfrac{1}{x^n+1}\le 1$ va $a_{n+1}(x)\le a_n(x)$ bo'ladi. Bundan tashqari,

$$\lim_{x\to 1-0}u_n(x)=\lim_{x\to 1-0}\frac{(-1)^{n+1}}{x+n-1}\frac{x^n}{x^n+1}=\frac{(-1)^{n+1}}{2n}=c_n,\quad n\in N\ .$$

$\sum\limits_{n=1}^{\infty}c_n=\sum\limits_{n=1}^{\infty}\dfrac{(-1)^{n+1}}{2n}$ yaqinlashuvchi va uning yig'indisi $\dfrac{1}{2}\ln 2$ ga teng.

12.6.1-teoremaning shartlari bajarilayapti. Endi funksional qatorda hadma-had limitga o'tish mumkin:

$$\lim_{x\to 1-0}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{x+n-1}\frac{x^n}{x^n+1}=\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}=\frac{1}{2}\ln 2\,.$$

12.7.Funksional qatorni hadma-had integrallash. Yaqinlashuvchi (12.5.1) funksional qator $X=[a,b]$ segmentda berilgan bo'lib , uning yiindisi $S(x)$ bo'lsin.

12.7.1-teorema. Agar (12.5.1) funksional qatorning har bir $u_n(x)$ hadi $X=[a,b]$ segmentda uzluksiz bo'lib, qatorning o'zi shu segmentda tekis yaqinlashuvchi bo'lsa, u holda

$$\int_a^b u_1(x)dx+\int_a^b u_2(x)dx+...+\int_a^b u_n(x)dx+...$$

qator ham yaqinlashuvchi va

$$\int_a^b \sum_{n=1}^{\infty} u_n(x)dx = \int_a^b S(x)dx \qquad (12.7.2)$$

tenglik o‘rinli bo‘ladi.

12.7.3-eslatma. 12.7.1-teoremada qatorning tekis yaqinlashuvchiligi yetarli shart bo‘lib, lekin zaruriy shart bo‘la olmaydi, ya’ni ba’zan tekis yaqinlashuvchilik sharti bajarilmagan funksional qatorni ham hadma-had integrallash mumkin.

12.7.4-misol. Ushbu

$$\sum_{n=1}^{\infty} \left(x^{\frac{1}{2n+1}} - x^{\frac{1}{2n-1}}\right)$$

funksional qatorni $[0;1]$ da hadma-had integrallash mumkinmi?

Yechilishi. Berilgan funksional qatorning qismiy yig‘indisini topamiz:

$$S_n(x) = (x^{\frac{1}{3}} - x) + (x^{\frac{1}{5}} - x^{\frac{1}{3}}) + \ldots + (x^{\frac{1}{2n-1}} - x^{\frac{1}{2n-3}}) + (x^{\frac{1}{2n+1}} - x^{\frac{1}{2n-1}}) = x^{\frac{1}{2n-1}} - x .$$

Endi $n \to \infty$ da limitga o‘tamiz:

$$S(x) = \lim_{n\to\infty} (x^{\frac{1}{2n+1}} - x) = \begin{cases} 0, x = 0, x = 1 \ bo'lganda, \\ 1 - x, 0 < x < 1 \ bo'lganda. \end{cases}$$

Demak, berilgan funksional qatorning yig‘indisi uzilishga ega bo‘lgan funksiyadir. Shuning uchun berilgan funksional qator uchun $[0;1]$ da tekis yaqinlashuvchilik sharti bajarilmaydi.

Demak, **12.7.1**–teoremani qo‘llash huquqiga ega emasmiz. Lekin, qator yig‘indisini va qatorni hadma-had integrallash mumkin:

$$\int_0^1 S(x)dx = \frac{1}{2}, \quad \sum_{n=1}^{\infty}\int_0^1 (x^{\frac{1}{2n+1}} - x^{\frac{1}{2n-1}})dx = \sum_{n=1}^{\infty}\left(\frac{2n+1}{2(n+1)} - \frac{2n-1}{2n}\right) =$$

$$= \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \frac{1}{2}\sum_{k=1}^{\infty}\left(\frac{1}{k} - \frac{1}{k+1}\right) = \frac{1}{2}.$$

Buni e’tiborga olsak,

$$\int_0^1 S(x)dx = \int_0^1 [\sum_{n=1}^{\infty}(x^{\frac{1}{2n+1}} - x^{\frac{1}{2n-1}})]dx = \sum_{n=1}^{\infty}\int_0^1 (x^{\frac{1}{2n+1}} - x^{\frac{1}{2n-1}})dx$$

tenglik o'rinli bo'ladi.

12.7.5-misol. Ushbu

$$f(x) = \sum_{n=1}^{\infty}\frac{\cos^2 nx}{n(n+1)}$$

funksiyaning R da uzluksizligini isbotlang va $\int_0^{2\pi} f(x)dx$ integralni hisoblang.

Yechilishi. $u_n(x) = \dfrac{\cos^2 nx}{n(n+1)}$ $(n = 1,2,...)$ funksiyalar R da uzluksiz. $\forall x \in R$ va $\forall n \in N$ lar uchun $|u_n(x)| \le \dfrac{1}{n(n+1)}$ tengsizlik o'rinli bo'ladi.

Veyershtrass alomatiga ko'ra, berilgan funksional qator tekis yaqinlashuvchi. **12.5.2**-teoremaga ko'ra, $f(x)$ funksiya R da uzluksiz. Bu yerda **12.7.1**–teoremaning ham hamma shartlari bajariladi. Shuning uchun berilgan funksional qatorni hadma-had integrallash mumkin:

$$\int_0^{2\pi}\sum_{n=1}^{\infty}\frac{\cos^2 nx}{n(n+1)}dx = \int_0^{2\pi}\frac{\cos^2 x}{1\cdot 2}dx + \int_0^{2\pi}\frac{\cos^2 2x}{2\cdot 3}dx + ... + \int_0^{2\pi}\frac{\cos^2 nx}{n(n+1)}dx + ...$$

$$+ ... = \pi(\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + ... + \frac{1}{n(n+1)} + ...) = \pi$$

Demak, $\int_0^{2\pi} f(x)dx = \pi$.

12.8. Funksional qatorni hadma-had differensiallash. Yaqinlashuvchi (12.5.1) funksional qator $[a,b]$ segmentda berilgan bo'lib, $S(x)$ uning yig'indisi bo'lsin.

12.8.1-teorema. Agar (12.5.1) funksional qatorning har bir $u_n(x)$ hadi $[a,b]$ segmentda uzluksiz $u_n'(x)$ hosilaga ega bo'lib,

$$\sum_{n=1}^{\infty} u_n'(x) = u_1'(x) + u_2'(x) + ... + u_n'(x) + ... \qquad (12.8.2)$$

qator $[a,b]$ segmentda tekis yaqinlashuvchi bo'lsa, u holda (12.5.1) qatorning $S(x)$ yig'indisi $[a,b]$ segmentda $S'(x)$ hosilaga ega va

$$S'(x)=(\sum_{n=1}^{\infty} u_n(x))'=\sum_{n=1}^{\infty} u'_n(x)$$

bo'ladi.

12.8.3-eslatma. 12.8.1-teoremadagi funksional qatorning tekis yaqinlashuvchilik sharti yetarli bo'lib, u zaruriy shart emas.

12.8.4-misol. Ushbu

$$\sum_{n=1}^{\infty} arctg\frac{x}{n^4}$$

funksional qatorni $(-\infty;+\infty)$ da hadma-had differensiallash mumkinmi?

Yechilishi. Berilgan qatorning umumiy hadi $u_n(x)=arcrc\frac{x}{n^4}$ $(n\in N)$ $(-\infty;+\infty)$ da uzluksiz $u'_n(x)=\frac{n^4}{n^8+x^2}$ hosilaga ega bo'ladi. $(-\infty;\infty)$ da berilgan funksional qator taqqoslash alomatiga ko'ra ($n\to\infty$ da $arctg\frac{x}{n^4}\sim\frac{x}{n^4}$) yaqinlashuvchi va $S(x)$ yig'indiga ega. Bundan tashqari, hosilalardan tuzilgan

$$\sum_{n=1}^{\infty} u'_n(x)=\sum_{n=1}^{\infty}\frac{n^4}{n^8+x^2}$$

funksional qator $(-\infty;\infty)$ da Veayershtrass alomatiga ko'ra, tekis yaqinlashuvchi bo'ladi.

Demak, berilgan funksional qatorning yig'indisi $(-\infty;+\infty)$ da $S'(x)$ hosilaga ega va

$$S'(x)=(\sum_{n=1}^{\infty} arctg\frac{x}{n^4})'=\sum_{n=1}^{\infty}\frac{n^4}{n^8+x^2}.$$

12.4-§. Darajali qatorlar

12.9. Darajali qator, uning yaqinlashish radiusi va intervali. Ushbu

$$\sum_{n=0}^{\infty} a_n(x-x_0)^n = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \cdots + a_n(x-x_0)^n + \cdots \quad (12.9.1)$$

qatorga darajali qator deyiladi. Bunda $a_0, a_1, a_2, ..., a_n, ...$ o‘zgarmas haqiqiy sonlar darajali qatorning koeffitsiyentlari deyiladi, x_0 esa, ixtiyoriy o‘zgarmas son. (12.9.1) darajali qator ushbu

$$\sum_{n=0}^{\infty} u_n(x)$$

funksional qatorning xususiy holi bo‘lib hisoblanadi:

$$u_n(x) = a_n(x-x_0)^n, \quad n = 0,1,2,...$$

$x - x_0 = t$ belgilash yordamida (12.9.1) darajali qatorni

$$\sum_{n=0}^{\infty} a_n t^n = a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n + \cdots$$

ko‘rinishga keltirish mumkin. Shuning uchun biz bundan keyin ushbu

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots \quad (12.9.2)$$

ko‘rinishdagi qatorni o‘rganish bilan kifoyalanamiz.

12.9.3-teorema (Abel teoremasi). Agar (12.9.2) darajali qator x ning $x = x_0$ $(x_0 \neq 0)$ qiymatiga yaqinlashuvchi bo‘lsa, u holda x ning $|x| < |x_0|$ tengsizlikni qanoatlantiruvchi barcha qiymatlarida (12.9.2) darajali qator absolyut yaqinlashuvchi bo‘ladi.

Isboti. Shartga ko‘ra, ushbu

$$\sum_{n=0}^{\infty} a_n x_0^n = a_0 + a_1 x_0 + a_2 x_0^2 + \cdots + a_n x_0^n + \cdots \quad (12.9.4)$$

qatorning yaqinlashuvchiligidan uning umumiy hadi $\lim\limits_{n\to\infty} a_n x_0^n = 0$ bo‘ladi.

Demak, $\{a_n x_0^n\}$ ketma-ketlik chegaralangan, ya'ni $\exists M > 0, \forall n \in N$ da

$$|a_n x_0^n| \le M \ (n = 0,1,2,...) \qquad (12.9.5)$$

x ning $|x| < |x_0|$ shartni qanoatlantiruvchi ixtiyoriy qiymatini olib ushbu

$$\sum_{n=0}^{\infty} |a_n x^n| = |a_0| + |a_1 x| + |a_2 x^2| + \cdots + |a_n x^n| + \cdots$$

qatorni tuzamiz. Bunda (12.9.5) ni e'tiborga olgan holda :

$$|a_n x^n| = |a_n x_0^n| \left|\frac{x}{x_0}\right|^n \le M \left|\frac{x}{x_0}\right|^n, \qquad (12.9.6)$$

$$|x| < |x_0| \text{ dan } \left|\frac{x}{x_0}\right| = q < 1.$$

Demak,

$$\sum_{n=0}^{\infty} \left|\frac{x}{x_0}\right|^n = \sum_{n=0}^{\infty} q^n$$

geometrik qator yaqinlashuvchi. U holda ushbu $\sum_{n=0}^{\infty} M \left|\frac{x}{x_0}\right|^n$ qator ham yaqinlashuvchi bo'ladi. (12.9.6) ni e'tiborga olgan holda, sonli qatorni solishtirish haqidagi **11.4.1**-teoremaga asosan, (12.9.4) qatorning absolyut yaqinlashuvchi ekanini topamiz.■

12.9.7-natija. Agar (12.9.2) qator x ning $x = x_0$ qiymatida uzoqlashuvchi bo'lsa, u x ning $|x| > |x_0|$ tengsizlikni qanoatlantiruvchi barcha qiymatlarida uzoqlashuvchi bo'ladi.

$\sum_{n=0}^{\infty} a_n x^n$ qator berilgan bo'lsin. Agar R -manfiy bo'lmagan son yoki $+\infty$ bo'lib, u quyidagi shartni qanoatlantirsa: $|x| < R$ tengsizlikni qanoatlantiruvchi barcha x larda qator yaqinlashuvchi, $|x| > R$ tengsizlikni qanoatlantiruvchi barcha x larda qator uzoqlashuvchi bo'lsa, u holda R ga qatorning *yaqinlashish radiusi* deyiladi. $|x| < R$ tengsizlikni qanoatlantiruvchi barcha x lar to'plamiga, ya'ni $(-R, R)$ yoki $K = \{x \in R : |x| < R\}$ ga qatorning *yaqinlashish* *oralig'i* deyiladi. Yaqinlashish oralig'ining chegaralarida, ya'ni $x=\pm R$ da darajali qator yaqinlashuvchi ham, uzoqlashuvchi ham bo'lishi mumkin.

12.10. Darajali qatorning yaqinlashish radiusini uning koeffitsiyentlari orqali ifodalash. (12.9.2) qator berilgan bo'lsin . Ushbu

$$\rho_1=|a_1|, \rho_2=\sqrt{|a_2|}, \rho_3=\sqrt[3]{|a_3|},\dots,\rho_n=\sqrt[n]{|a_n|},\dots$$

ketma-ketlikni qaraymiz. Bu ketma-ketlikning aniq yuqori limiti mavjud va uni ρ bilan belgilaymiz :

$$\rho=\overline{\lim_{n\to\infty}}\rho_n=\overline{\lim_{n\to\infty}}\sqrt[n]{|a_n|}.$$

12.10.1-teorema (Koshi-Adamar). Har qanday (12.9.2) darajali qatorning yaqinlashish radiusi R mavjud bo'lib, u $R=\dfrac{1}{\rho}$ formula orqali topiladi (bunda, $\rho=0$ bo'lsa, $R=+\infty$, agar $\rho=+\infty$ bulsa, $R=0$ bo'ladi).

12.10.2-eslatma. K intervalning chegarasida, ya'ni $x=\pm R$ da (12.9.2) darajali qator yaqinlashuvchi ham, uzoqlashuvchi bo'lishi ham mumkin. K ga nisbatan kichik istalgan $K_1=\{x:|x|\le\rho_1<R\}$ intervalda (12.9.2) qator absolyut va tekis yaqinlashuvchi bo'ladi.

12.10.3-eslatma. Xususiy holda, agar 1) chekli yoki cheksiz

$$\lim_{n\to\infty}\sqrt[n]{|a_n|}$$

mavjud bo'lsa, u holda (12.9.2) qatorning yaqinlashish radiusi R uchun

$$\frac{1}{R}=\lim_{n\to\infty}\sqrt[n]{|a_n|} \qquad (12.10.4)$$

formula o'rinli:

2) chekli yoki cheksiz

$$\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$$

mavjud bo'lsa, u holda (12.9.2) darajali qatorning yaqinlashish radiusi R uchun

$$R = \lim_{n\to+\infty}\left|\frac{a_n}{a_{n+1}}\right| \qquad (12.10.5)$$

formula o'rinli.

12.10.6-eslatma. $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ darajali qatorning yaqinlashish intervali $(x_0-R;x_0+R)$ bo'ladi. Bunda R ushbu $\sum_{n=0}^{\infty} a_n x^n$ qatorning yaqinlashish radiusi.

12.10.7-eslatma. (12.10.4)-(12.10.5) limitlar mavjud bo'lmasligi ham mumkin. Ammo, (12.9.2) darajali qatorning yaqinlashish radiusini hisoblash uchun umumiy formulaga egamiz, ya'ni

$$R = \frac{1}{\overline{\lim\limits_{n\to+\infty}}\sqrt[n]{|a_n|}}. \qquad (12.10.8)$$

(12.10.8) formula *Koshi-Adamar formulasi* deyiladi.

12.10.9-misol. Ushbu

$$\sum_{n=0}^{\infty}\left(\frac{2+(-1)^n}{3}\right)^n x^n$$

darajali qatorning yaqinlashish radiusini toping.

Yechilishi. Berilgan darajali qator umumiy hadining koeffitsiyenti $a_n = \left(\frac{2+(-1)^n}{3}\right)^n$. Endi R radiusni topish uchun $\{K_n\} = \{\sqrt[n]{|a_n|}\}$ ketma-ketlikning limitini topamiz:

$$\lim_{n\to+\infty} K_n = \lim_{n\to+\infty}\sqrt[n]{\left|\left(\frac{2+(-1)^n}{3}\right)^n\right|} = \lim_{n\to+\infty}\frac{2+(-1)^n}{3} = \begin{cases} 1, & n=2k,\ k\in N, \\ \frac{1}{3}, & n=2k-1,\ k\in N. \end{cases}$$

Ravshanki, $\{K_n\}$ ketma-ketlikning limiti mavjud emas. Berilgan $\{K_n\}$ ketma-ketlikning qismiy limitlari mavjud, ya'ni

$$\lim_{k\to\infty} K_{2k} = 1, \quad \lim_{k\to\infty} K_{2k-1} = \frac{1}{3}.$$

$\{K_n\}$ ketma-ketlikning qismiy ketma-ketliklarining limitlari orasida eng kattasi 1, eng kichigi esa $\frac{1}{3}$ ga teng ekanligini ko'rish qiyin emas.

Demak,

$$\overline{\lim_{n\to\infty}} K_n = \overline{\lim_{n\to\infty}} \frac{2+(-1)^n}{3} = 1, \qquad \underline{\lim_{n\to\infty}} K_n = \underline{\lim_{n\to\infty}} \frac{2+(-1)^n}{3} = \frac{1}{3}.$$

12.10.7-eslatmaga ko'ra, berilgan darajali qatorning yaqinlashish radiusi

$$R = \left[\overline{\lim_{n\to\infty}} \left|\left(\frac{2+(-1)^n}{3}\right)^n\right|\right]^{-\frac{1}{n}} = 1$$

bo'ladi.

12.11. Darajali qatorlarning xossalari

1⁰. Agar (12.9.2) qatorning yaqinlashish radiusi R $(R>0)$ bo'lsa, $0<r<R$ tengsizlikni qanoatlantiruvchi shunday r soni topiladiki, (12.9.2) qator $[-r; r]$ da x ga nisbatan tekis yaqinlashuvchi bo'ladi.

2⁰. Agar (12.9.2) qatorning yaqinlashish radiusi R bo'lsa, bu qatorning $S(x)=\sum_{n=0}^{\infty} a_n x^n$ yig'indisi $(-R; R)$ da uzluksiz funksiya bo'ladi.

3⁰. Agar ushbu $\sum_{n=0}^{\infty} a_n x^n$, $\sum_{n=0}^{\infty} b_n x^n$ darajali qatorlar $x=0$ nuqta atrofida bir xil yig'indilarga ega bo'lsa , u holda bu darajali qatorlar aynan bir-biriga teng , ya'ni $a_0 = b_0, a_1 = b_1, ..., a_n = b_n, ...$ bo'ladi.

4⁰. Agar (12.9.2) qatorning yaqinlashish radiusi R bo'lsa, bu qatorni $[a, b]$ $([a, b] \subset (-R; R))$ segmentda hadma -had integrallash mumkin, ya'ni

$$\int_a^b S(x)dx = \int_a^b \sum_{n=0}^{\infty} a_n x^n dx = \sum_{n=0}^{\infty} \int_a^b a_n x^n dx.$$

5^0. Agar (12.9.2) qatorning yaqinlashish radiusi R bo'lsa, bu qatorni $(-R; R)$ da hadma-had differensiallash mumkin, ya'ni

$$\left(\sum_{n=0}^{\infty} a_n x^n\right)' = \sum_{n=1}^{\infty} n a_n x^{n-1}. \qquad (12.11.1)$$

12.11.2-eslatma. Yuqoridagi darajali qatorning xossalariga asosan, ushbu

$$\sum_{n=0}^{\infty} a_n x^n, \quad \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}, \quad \sum_{n=1}^{\infty} n a_n x^{n-1}$$

darajali qatorlar bir xil yaqinlashish radiusiga ega ekanligi kelib chiqadi.

12.11.3-misol. Ushbu

$$1+\sum_{n=1}^{\infty} \frac{(2n-1)!!}{(2n)!!} x^{2n} = \frac{1}{\sqrt{1-x^2}} \quad (-1 < x < 1)$$

darajali qatorni hadma-had integrallashdan foydalanib,

$$1+\sum_{n=1}^{\infty} \frac{(2n-1)!!}{(2n)!!} \frac{1}{2n+1} = \frac{\pi}{2}$$

bo'lishini ko'rsating.

Yechilishi. Bu darajali qatorni 4^0-xossaga asosan, $[0; x]$ $([0; x] \subset (-1; 1), x > 0)$ segmentda hadma-had integrallash mumkin. U holda arksinusning bizga ma'lum bo'lgan yoyilmasiga ega bo'lamiz:

$$\int_0^x \frac{dt}{\sqrt{1-t^2}} = \arcsin x = x + \sum_{n=1}^{\infty} \frac{(2n-1)!!}{(2n)!!} \frac{x^{2n+1}}{2n+1}.$$

Bu hosil bo'lgan darajali qatorning yaqinlashish intervali $(-1; 1)$ bo'ladi. $x = \pm 1$ da

$$x+\sum_{n=1}^{\infty} \frac{(2n-1)!!}{(2n)!!} \frac{x^{2n+1}}{2n+1}.$$

darajali qator

$$1 \pm \sum_{n=1}^{\infty} \frac{(2n-1)!!}{(2n)!!} \frac{1}{2n+1}$$

sonli qatorga aylanadi. Bu qator, Raabe alomatiga ko'ra, yaqinlashuvchi bo'ladi. Unda $x = 1$ da

$$1 + \sum_{n=1}^{\infty} \frac{(2n-1)!!}{(2n)!!} \frac{1}{2n+1} = \arcsin 1 = \frac{\pi}{2}$$

bo'ladi.

12.5-§. Teylor qatori

12.12. Funksiyalarni Teylor qatoriga yoyish. $f(x)$ funksiya x_0 $(x_0 \in R)$ nuqtaning biror

$$U_\delta(x_0) = \{x \in R : x_0 - \delta < x < x_0 + \delta\} \quad (\delta > 0)$$

atrofida berilgan bo'lib,u shu atrofda istalgan tartibdagi hosilaga ega bo'lsa, ushbu

$$f(x_0) + \sum_{n=1}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n \tag{12.12.1}$$

darajali qator, u yaqinlashuvchi bo'ladimi, yaqinlashuvchi bo'lib, uning yig'indisi $f(x)$ funksiyaga teng bo'ladimi yoki yo'qmi, bundan qat'iy nazar, $f(x)$ funksiyaning $x = x_0$ nuqtadagi **Teylor qatori** deyiladi. Bu qator (12.9.1) darajali qatorga o'xshash bo'lib, bunda

$$f(x_0) = a_0, \quad \frac{f'(x_0)}{1!} = a_1, \quad \frac{f''(x_0)}{2!} = a_2, \quad \frac{f'''(x_0)}{3!} = a_3, \ldots, \frac{f^{(n)}(x_0)}{n!} = a_n, \ldots$$

lar *Teylor koeffitsiyentlari* deyiladi.

Xususiy holda, ya'ni $x_0 = 0$ bo'lganda (12.12.1) Teylor qatori

$$f(0) + \sum_{n=1}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

ko'rinishga keladi. Bu qator ko'p hollarda *Makloren qatori* deb yuritiladi.

12.12.2-teorema. $f(x)$ funksiya biror $U_\delta(x_0)$ atrofida istalgan tartibdagi hosilaga ega bo‘lib, (12.12.1) qator uning $x=x_0$ nuqtadagi Teylor qatori bo‘lsin. Bu qator $U_\delta(x_0)$ da $f(x)$ ga yaqinlashishi uchun uning

$$f(x)=f(x_0)+\frac{f^{(1)}(x_0)}{1!}(x-x_0)+\frac{f^{(2)}(x_0)}{2!}(x-x_0)^2+...+\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n+r_n(x).$$

Teylor formulasi qoldiq hadining $\forall x\in U_\delta(x_0)$ da nolga intilishi, ya’ni $\lim\limits_{n\to\infty} r_n(x)=0$ bo‘lishi zarur va yetarli.

Ma’lumki, Teylor formulasi qoldiq hadi:

$a)$ integral ko‘rinishda $\qquad r_n(x)=\frac{1}{n!}\int\limits_{x_0}^{x}(x-t)^n f^{(n+1)}(t)dt;$

$b)$ Lagranj ko‘rinishida $\qquad r_n(x)=\frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1},$

bunda $c=x_0+\theta(x-x_0),\ \ 0<\theta<1;$

c) Koshi ko‘rinishida

$$r_n(x)=\frac{f^{(n+1)}(c)}{n!}(1-\theta)^n(x-x_0)^{n+1},\quad c=x_0+\theta(x-x_0),\quad 0,<\theta<1;$$

d) Peano ko‘rinishida $r_n(x)=o((x-x_0)^n)$ bo‘ladi.

12.12.3-teorema. $f(x)$ funksiya $(x_0-\rho,x_0+\rho)$ $(\rho>0)$ oraliqda darajali qatorga yoyilgan bo‘lsa:

$$f(x)=a_0+a_1(x-x_0)+a_2(x-x_0)^2+...+a_n(x-x_0)^n+...,$$

bu qator $f(x)$ funksiyaning Teylor qatori bo‘ladi, bunda

$$a_0=f(x_0),\ a_1=\frac{f'(x_0)}{1!},\ a_2=\frac{f''(x_0)}{2!},\ a_3\frac{f'''(x_0)}{3!},...,a_n=\frac{f^{(n)}(x_0)}{n!},...$$

12.12.4-teorema. Agar $f(x)$ funksiya biror (x_0-R,x_0+R) intervalda istalgan tartibdagi hosilaga ega bo‘lib, shunday o‘zgarmas $M>0$ son topilsaki, barcha $x\in(x_0-R,x_0+R)$, hamda barcha $n\in N$ lar uchun

$$|f^{(n)}(x)|\leq M$$

bajarilsa, u holda $(x_0 - R, x_0 + R)$ intervalda $f(x)$ funsiya Teylar qatoriga yoyiladi, ya'ni

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n \qquad (12.12.5)$$

bo'ladi.

12.12.6-misol. Ushbu

$$f(x)=\frac{1}{1+x}$$

funksiyani $x=2$ nuqta atrofida Teylor qatoriga yoying.

Yechilishi. Berilgan $f(x)=\frac{1}{1+x}$ funksiyaning istalgan tartibli hosilalarini topamiz, co'ngra uning va hosilalarining $x_0=2$ nuqtadagi qiymatlarini hisoblaymiz:

$$f(x)=\frac{1}{x+1}=(x+1)^{-1},\quad f(2)=\frac{1}{3},$$

$$f^{(1)}(x)=\left(\frac{1}{1+x}\right)'=-\frac{1}{(x+1)^2},\quad f^{(1)}(2)=-\frac{1!}{3^2},$$

$$f^{(2)}(x)=\frac{2!}{(x+1)^3},\quad f^{(2)}(2)=\frac{2!}{3^3},$$

....

$$f^{(n)}(x)=\frac{(-1)^n n!}{(x+1)^{n+1}},\quad f^{(n)}(2)=\frac{(-1)^n n!}{3^{n+1}},$$

....

Topilganlarni (12.12.1) formulaga qo'ysak, $f(x)=\frac{1}{1+x}$ funksiyaning Teylor qatori

$$\frac{1}{3}-\frac{1!}{3^2}(x-2)+\frac{2!}{3^3}\frac{(x-2)^2}{2!}+...+\frac{(-1)^n n!(x-2)^n}{3^{n+1}n!}+...=$$
$$=\sum_{n=0}^{\infty}\frac{(-1)^n}{3^{n+1}}(x-2)^n \qquad (12.12.7)$$

ko'rinishda bo'ladi. Bu qatorning yaqinlashish intervalini Koshi alomatidan foydalanib topamiz:

$$u_n(x)=\frac{(-1)^n}{3^{n+1}}(x-2)^n,\quad \lim_{n\to\infty}\sqrt[n]{|u_n(x)|}=\lim_{n\to\infty}\sqrt[n]{\left|\frac{(-1)^n}{3^{n+1}}(x-2)^n\right|}=\frac{|x-2|}{3}<1.$$

Endi $|x-2|<3$ tengsizlikni yechamiz: $-3<x-2<3\Rightarrow -1<x<5$. (12.16.7) qatorning yaqinlashish intervali $(-1;5)$ bo'ladi. Intervalning chegaralarida, ya'ni $x=-1$ va $x=5$ nuqtalarda mos ravishda

$$1+1+1+\ldots \text{ va } \qquad 1-1+1-1+\ldots$$

uzoqlashuvchi qatorlar hosil bo'ladi.

Demak, (12.12.7) qatorning yaqinlashish sohasi $(-1;5)$ intervaldan iborat bo'ladi. Bu (12.12.7) qator $(-1;5)$ intervalda $f(x)=\frac{1}{1+x}$ funksiyaga yaqinlashishi uchun $f(x)$ funksiya Teylor formulasining qoldiq hadi barcha $x\in(-1;5)$ da nolga intiladi. Haqiqatan ham,

$$|r_n(x)|=\left|\frac{(-1)^{n+1}(x-2)^{n+1}}{(3+\theta(x-2))^{n+2}}\right|\le\frac{1}{3\left(1+\theta\frac{(x-2)}{3}\right)^{n+2}}\xrightarrow[n\to\infty]{}0,$$

bunda $0<\theta<1$.

Shunday qilib, **12.12.2**-teoremaga asosan, (12.12.7) qatorning yig'indisi $\frac{1}{1+x}$ ga teng bo'ladi, ya'ni

$$\frac{1}{1+x}=\sum_{n=0}^{\infty}\frac{(-1)^n}{3^{n+1}}(x-2)^n.$$

12.13. Elementar funksiyalarning Teylor qatorlari. (12.12.1) formulada $x_0=0$ deb, amaliyotda ko'p uchraydigan ba'zi elementar funksiyalarning darajali qatorlari yoyilmalarini keltiramiz:

1. Ko'rsatkichli funksiyalar:

$$e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}\ (-\infty<x<+\infty,\quad R=\infty).$$

$$a^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}\ln^n a,\ a>0,\ a\neq 1\ (-\infty<x<+\infty,\quad R=\infty).$$

2. Trigonometrik funksiyalar:

$$\cos x=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}(-\infty<x<+\infty,\quad R=\infty). \qquad (12.13.1)$$

$$\sin x=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^{2n-1}}{(2n-1)!}(-\infty<x<+\infty,\quad R=\infty). \qquad (12.13.2)$$

3. Darajali funksiyalar:

$$(x+1)^{\alpha}=1+\sum_{n=1}^{\infty}C_{\alpha}^{n}x^n \qquad (12.13.3)$$

bunda

$$C_{\alpha}^{n}=\frac{\alpha(\alpha-1)\cdot\ldots\cdot(\alpha-(n-1))}{n!}.$$

Agar $\alpha\neq 0,\ \alpha\neq n,(n\in N)$ bo‘lsa, (12.13.3) qatorning yaqinlashish radiusi 1 ga teng bo‘ladi.

(12.13.3) formulaning muhim hususiy hollari:

$$\frac{1}{1-x}=\sum_{n=0}^{\infty}x^n\ ,\ (|x|<1,\quad R=1);$$

$$\frac{1}{1+x}=\sum_{n=0}^{\infty}(-1)^n x^n\ ,\ (|x|<1,\quad R=1);$$

$$\frac{1}{1+x^2}=\sum_{n=0}^{\infty}(-1)^n x^{2n}\ ,\ (|x|<1,\quad R=1).$$

4. Logarifmik funksiyalar:

$$\ln(1+x)=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}(-1<x\leq 1,\ R=1)\ ;$$

$$\ln(1-x)=-\sum_{n=1}^{\infty}\frac{x^n}{n}\ (-1\leq x<1,\ R=1).$$

5.Giperbolik funksiyalar:

$$chx=\sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!},\ (-\infty<x<+\infty,\ R=\infty);$$

$$shx=\sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)!},\ (-\infty<x<+\infty,\ R=\infty).$$

6.Teskari trigonometrik funksiyalar:

$$arctgx=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^{2n-1}}{2n-1},\ (|x|\le 1,\ R=1);$$

$$\arcsin x=\sum_{n=0}^{\infty}\frac{(2n+1)!!x^{2n+1}}{(2n)!!(2n+1)},\ (|x|<1, R=1);$$

$$\arccos x=\frac{\pi}{2}-x-\sum_{n=1}^{\infty}\frac{(2n-1)!!x^{2n+1}}{(2n)!!(2n+1)},\ (|x|<1,\ R=1);$$

$$arcctgx=\frac{\pi}{2}+\sum_{n=0}^{\infty}\frac{(-1)^{n+1}x^{2n+1}}{2n+1},\ (|x|<1, R=1).$$

12.13.4-misol. Ushbu $f(x)=\cos^4 x$ funksiyani $x_0=\frac{\pi}{8}$ nuqta atrofida Teylor qatoriga yoying va yaqinlashish radiusini toping.

Yechilishi. Bizga ma'lumki, $\cos^2 x=\frac{1}{2}(1+\cos 2x)$, u holda

$$f(x)=\left[\frac{1}{2}(1+\cos 2x)\right]^2=\frac{3}{8}+\frac{1}{2}\cos 2x+\frac{1}{8}\cos 4x$$

bo'ladi. So'ngra $x=t+\frac{\pi}{8}$ almashtirishni bajaramiz:

$$f(x)=f(t+\frac{\pi}{8})=g(t)=\frac{3}{8}+\frac{\sqrt{2}}{4}(\cos 2t-\sin 2t)-\frac{1}{8}\sin 4t. \qquad (12.13.5)$$

Endi (12.13.1) va (12.13.2) yoyilmalardan foydalanib, $\cos 2t$, $\sin 2t$ va $\sin 4t$ uchun

$$\cos 2t=\sum_{n=0}^{\infty}\frac{(-1)^n 2^{2n}}{(2n)!}t^{2n},\ \sin 2t=\sum_{n=0}^{\infty}\frac{(-1)^n 2^{2n+1}}{(2n+1)!}t^{2n+1},\ \sin 4t=\sum_{n=0}^{\infty}(-1)^n\frac{4^{2n+1}t^{2n+1}}{(2n+1)!}$$

tengliklarga ega bo'lamiz. $\cos 2t$, $\sin 2t$ va $\sin 4t$ funksiyalarning yoyilmalarini (12.13.5) ga keltirib qo'ysak,

$$g(t)=\frac{3}{8}+\frac{\sqrt{2}}{4}\left(\sum_{n=0}^{\infty}\frac{(-1)^n 2^{2n}t^{2n}}{(2n)!}-\sum_{n=0}^{\infty}\frac{(-1)^n 2^{2n+1}t^{2n+1}}{(2n+1)!}\right)-\frac{1}{8}\sum_{n=0}^{\infty}\frac{(-1)^n 4^{2n+1}t^{2n+1}}{(2n+1)!}$$

ifodaga ega bo'lamiz. Endi $t=x-\frac{\pi}{8}$ ekanigini hisobga olsak, natijada

$$f(x)=\frac{3}{8}+\sqrt{2}\left(\sum_{n=0}^{\infty}\frac{(-1)^n 2^{2n-2}}{(2n)!}(x-\frac{\pi}{8})^{2n}-\sum_{n=0}^{\infty}\frac{(-1)^n 2^{2n-1}}{(2n+1)!}(x-\frac{\pi}{8})^{2n+1}\right)-$$
$$-\sum_{n=0}^{\infty}\frac{(-1)^n 2^{4n-1}}{(2n+1)!}(x-\frac{\pi}{8})^{2n+1}$$

bo'ladi. Hosil bo'lgan qatorning yaqinlashish radiusi $R=\infty$.

12-bob bo'yicha nazariy materiallarni mustahkamlash uchun topshiriqlar

12.1. Funksional ketma-ketlik va uning yaqinlashish ta'rifi (nuqtadagi va to'plamdagi) ([3], 2-q.,119-121 betlar; [12], 2-q., 129-131 betlar; [10], 2-q., 67-70 betlar; [5], 2-t., 419-420 betlar; [9], 2-t., 2- bo'lim).

12.2. Funksional qator va uning yaqinlashishi ta'rifi ([3], 2-q.,130-132 betlar; [12], 2-q., 132-134 betlar; [10], 2-q., 67-70 betlar; [5], 2-t., 420-421 betlar; [9], 2-t., 2- bo'lim).

12.3. Funksional ketma-ketlik qismiy yig'indilar ketma-ketligi qanday tuziladi? ([3], 2-q.,119-121 betlar; [12], 2-q., 129-131 betlar; [10], 2-q., 67-70 betlar; [5], 2-t., 419-bet; [9], 2-t., 2- bo'lim).

12.4. Funksional qatorning yig'indisi deb nimaga aytiladi? ([3], 2-q.,130-133 betlar; [12], 2-q., 134-bet; [5], 2-t., 421-bet; [9], 2-t., 2- bo'lim).

12.5. Funksional ketma-ketlikning tekis yaqinlashish shartini keltiring va uni isbotlang ([3], 2-q.,122-128 betlar: [12], 2-q., 136-141 betlar; [5], 2-t., 421-422 betlar; [9], 2-t., 2- bo'lim).

12.6. Funksional qatorning tekis yaqinlashish shartini keltiring va uni isbotlang ([3], 2-q.,133-138 betlar; [12], 2-q., 142-143 betlar; [10], 2-q., 72-bet; [5], 2-t., 422-423 betlar; [9], 2-t., 2- bo'lim).

12.7. Funksional ketma-ketlik yaqinlashishi uchun Koshi kriteriysi. ([3], 2-q.,126-128 betlar; [12], 2-q., 141-142 betlar; [10], 2-q., 72-74 betlar; [5], 2-t., 425-426 betlar).

12.8. Funksional qatorning tekis yaqinlashishi uchun Koshi kriteriysi ([3], 2-q.,136-138 betlar; [12], 2-q., 144-145 betlar; [10], 2-q., 72-74 betlar; [5], 2-t.,426-427 betlar).

12.9. Funksional qator tekis yaqinlashishi uchun Veyershtrass alomati ([3], 2-q.,136-138 betlar; [12], 2-q., 144-145 betlar; [10], 2-q., 74-76 betlar; [5], 2-t., 427-428 betlar).

12.10. Funksional qatorning X to'plamda tekis yaqinlashishi uchun zaruriy va yetarli shartlar ([3], 2-q.,134-135 betlar; [12], 2-q., 144-bet).

12.11. Funksional qatorning X to'plamda tekis yaqinlashishi uchun Direxle va Abel alomatlari ([3], 2-q.,138-140 betlar; [10], 2-q., 79-83 betlar; [5], 2-t., 429-430 betlar).

12.12. Funksional qator yig'indisining uzluksizligi haqidagi teorema ([3], 2-q.,140-142 betlar; [12], 2-q., 146-147 betlar; [10], 2-q., 86 bet; [5], 2-t., 430-431 betlar).

12.13. Funksional ketma-ketlik limit funksiyasining uzluksizligi haqidagi teorema ([3], 2-q.,128-129 betlar; [12], 2-q., 147-148 betlar; [10], 2-q., 86 bet; [5], 2-t., 431-432 betlar).

12.14. Funksional ketma-ketliklar va funksional qatorlarda hadma-had limitga o'tish haqidagi teoremalar ([3], 2-q.,129-130 betlar; [12], 2-q., 148-150 betlar; [10], 2-q., 83-86 betlar; [5], 2-t., 434 – 436, 442-443 betlar).

12.15. Funksional ketma-ketliklar va funksional qatorlarni hadma-had differensiallash ([3], 2-q.,129, 144-146 betlar; [12], 2-q., 154-156 betlar; [10], 2-q., 90-97 betlar; [5], 2-t., 438-441betlar).

12.16. Funksional ketma-ketliklar va funksional qatorlarni hadma-had integrallash. ([3], 2-q.,129, 142-144 betlar; [12], 2-q., 151-154 betlar; [10], 2-q., 87-90 betlar; [5], 2-t., 436-438 betlar).

12.17. Darajali qator. Abel teoremasi ([3], 2-q.,148-150 betlar; [2] 156-158 betlar; [10], 2-q., 102 bet; [5], 2-t., 298-299 betlar; [9], 2-t., 2- bo'lim. [30], 11- bo'lim).

12.18. Darajali qatorning yaqinlashish radiusi va intervali ([3], 2-q.,154-155 betlar; [2] 158-160 betlar; [10], 2-q., 102-103 betlar; [5], 2-t., 444-445 betlar; [9], 2-t., 2- bo'lim, [30], 11- bo'lim).

12.19. Koshi-Adamar ([3], 2-q.,155-157 betlar; [12], 2-q., 161-165 betlar; [10], 2-q., 104 bet; [5], 2-t., 444-445 betlar; [9], 2-t., 2- bo'lim. [30], 11-bo'lim).

12.20. Darajali qatorlarning xossalari ([3], 2-q.,157-163 betlar; [12], 2-q., 165-171 betlar; [10], 2-q., 105-107 betlar; [5], 2-t., 445-447 betlar; [9], 2-t., 2- bo'lim, [30], 11- bo'lim).

12.21. Funksiyalarni Teylor qatoriga yoyish ([3], 2-q.,163-166 betlar; [12], 2-q., 171-175 betlar; [10], 2-q., 107-108 betlar; [5], 2-t., 449-450 betlar; [9], 2-t., 2- bo‘lim, [30], 11- bo‘lim).

12.22. Elementar funksiyalarning Teylor qatorlari. ([3], 2-q.,166-170 betlar; [12], 2-q., 175-178 betlar; [10], 2-q., 108-111 betlar; [9], 2-t., 2- bo‘lim, [30], 11- bo‘lim).

12.1-amaliy mashg‘ulot.
Funksional ketma-ketliklar, funksional qatorlar va ularning yaqinlashuvchiligi

1-misol. Ushbu $\{f_n(x)\}=\left\{n^2\sin\frac{1}{n^2x}\right\}$ funksional ketma-ketlikning yaqinlashish sohasi va limit funksiyasi topilsin.

Yechilishi. Berilgan ketma-ketlikning hamma hadlari $(-\infty;0)\cup(0;\infty)$ to‘plamda aniqlangan. Bu ketma-ketlikning umumiy hadi $f_n(x)=n^2\sin\frac{1}{n^2x}$. Limit funksiyani topamiz: $\forall x\in(-\infty;0)\cup(0;\infty)$ lar uchun

$$\lim_{n\to\infty}f_n(x)=\lim_{n\to\infty}\frac{\sin\frac{1}{n^2x}}{\frac{1}{xn^2}}\cdot\frac{1}{x}=\frac{1}{x}.$$

Demak, berilgan funksional ketma-ketlik $(-\infty;0)\cup(0;\infty)$ da yaqinlashuvchi, uning limit funksiyasi $\frac{1}{x}$ ga teng: $n^2\sin\frac{1}{n^2x}\xrightarrow{(-\infty;0)\cup(0;\infty)}\frac{1}{x}$.

2-misol. Ushbu

$$\{f_n(x)\}=\left\{\sqrt[4]{x+\frac{1}{n}}-\sqrt[4]{x}\right\}$$

funksional ketma-ketlikni $X=[0;\infty)$ da tekis yaqinlashuvchilikka tekshiring.

Yechilishi. Barcha $x\in X$ va $\forall n\in N$ lar uchun

$$x+\frac{1}{n}\le\left(\sqrt{x}+\frac{1}{\sqrt{n}}\right)^2$$

tengsizlik o‘rinli bo‘ladi. U holda

$$0 \le \sqrt[4]{x+\frac{1}{n}}-\sqrt[4]{x}=\frac{\sqrt{x+\frac{1}{n}}-\sqrt{x}}{\sqrt[4]{x+\frac{1}{n}}+\sqrt[4]{x}} \le n^{\frac{1}{4}}\left(\sqrt{(\sqrt{x}+\frac{1}{\sqrt{n}})^2}-\sqrt{x}\right)=\frac{n^{\frac{1}{4}}}{\sqrt{n}}=\frac{1}{\sqrt[4]{n}}=a_n$$

bo‘ladi.

Demak, **12.3.9**-teoremaga asosan, berilgan funksional ketma-ketlik $[0;\infty)$ da $f(x)=0$ limit funksiyaga tekis yaqinlashadi.

3-misol. Ushbu

$$\{f_n(x)\}=\left\{\frac{\sqrt{nx}}{2+nx}\right\}$$

funksional ketma-ketlikning $X=[0;2]$ da notekis yaqinlashuvchiligini ko‘rsating.

Yechilishi. Bu ketma-ketlikning limit funksiyasi

$$f(x)=\lim_{n\to\infty}\frac{\sqrt{nx}}{2+nx}=0$$

bo‘ladi. Endi berilgan ketma-ketlikning $f(x)=0$ limit funksiyaga notekis yaqinlashishini ko‘rsatamiz. $\forall\varepsilon>0$ son olinganda ham m natural son sifatida $m=\left[\frac{1}{\varepsilon^2 x}\right]$, $x\neq 0$ olinsa, u holda $\forall m<n$ uchun

$$|f_n(x)-f(x)|=\left|\frac{\sqrt{nx}}{2+nx}-0\right|=\frac{\sqrt{nx}}{2+nx}<\frac{1}{\sqrt{nx}}\le\frac{1}{\sqrt{(m+1)x}}<\varepsilon$$

bo‘ladi. Ravshanki, $x=0$ da $\forall n\in N$ lar uchun $f_n(0)=f(0)=0$ bo‘ladi. Bu holda m ning x ga bog‘liqligi evaziga, ixtiyoriy natural n son uchun $\varepsilon_0=\frac{1}{4}$ va $x_n=\frac{1}{n}\in(0;2]$ deb olsak,

$$|f_n(x_n)-f(x_n)|=\frac{\sqrt{\frac{1}{n}n}}{2+\frac{1}{n}n}=\frac{1}{3}>\varepsilon_0$$

bo'ladi. Bu esa berilgan $\{f_n(x)\}$ funksional ketma-ketlikning $f(x)=0$ limit funksiyaga notekis yaqinlashishini bildiradi, ya'ni $x\in[0;2]$ ning barcha qiymatlari uchun bir vaqtda yaraydigan m nomer mavjud emas. **12.3.5-** ta'rifga ko'ra berilgan ketma-ketlik $f_n(x)=\dfrac{\sqrt{nx}}{2+nx}\overset{[0;2]}{\not\rightrightarrows}0$.

4-misol. Ushbu

$$\sum_{n}^{\infty}\frac{\sin nx}{3^n}$$

funksional qatorning Koshi kriteriysiga ko'ra, $X=[0,\infty)$ da tekis yaqinlashuvchi ekanligini ko'rsating.

Yechilishi. $S_{n+p}-S_n$ ayirmani tuzamiz va uni baholaymiz:

$$\left|S_{n+p}-S_n\right|=\left|\frac{\sin(n+1)x}{3^{n+1}}+\frac{\sin(n+2)x}{3^{n+2}}+\ldots+\frac{\sin(n+p)x}{3^{n+p}}\right|\le$$

$$\le\frac{1}{3^{n+1}}+\frac{1}{3^{n+2}}+\ldots\frac{1}{3^{n+p}}=$$

$$=\frac{1}{3^{n+1}}\left(1+\frac{1}{3}+\frac{1}{3^2}+\ldots+\frac{1}{3^{p-1}}\right)=\frac{1}{3^{n+1}}\cdot\frac{1-\left(\frac{1}{3}\right)^p}{1-\frac{1}{3}}=\frac{1}{3^n\cdot 2}\left(1-\left(\frac{1}{3}\right)^p\right)<\frac{1}{3^n}.$$

$\forall\varepsilon>0$ son olinganda ham $m=[\log_3\frac{1}{\varepsilon}]+1$ deb olinsa, u holda $\forall x\in[0,\infty)$, $\forall n>m$ va $\forall p$ lar uchun $\left|S_{n+p}-S_n\right|<\varepsilon$ tengsizlik bajariladi.

Demak, Koshi kriteriysiga asosan, berilgan funksional qator $X=[0,\infty)$ da tekis yaqinlashuvchi bo'ladi.

5-misol. Ushbu

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n^3}+\sqrt{x}}\left(1+\frac{x}{n}\right)^n$$

funksional qatorni $X=[0;1]$ da tekis yaqinlashuvchilikka tekshiring.

Yechilishi. Quyidagi belgilashlarni kiritamiz:

$$b_n(x)=\frac{(-1)^n}{\sqrt{n^3}+\sqrt{x}},\quad a_n(x)=\left(1+\frac{x}{n}\right)^n.$$

Veyershtrass alomatiga ko'ra, $\sum_{n=1}^{\infty} b_n(x)$ qator X da tekis yaqinlashuvchi, ya'ni $\forall x \in X$ va $\forall n \in N$ lar uchun

$$|b_n(x)| \leq \frac{1}{n^{\frac{3}{2}}}$$

tengsizlik o'rinli va $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3}}$ sonli qator yaqinlashuvchi.

$\{a_n(x)\} = \left\{(1+\frac{x}{n})^n\right\}$ ketma-ketlik $[0;1]$ da chegaralangan,

$$\left(1+\frac{x}{n}\right)^n \leq (1+\frac{1}{n})^n < e$$

va $\varphi(t) = (1+\frac{x}{t})^t$ funksiya barcha $t \geq 1$ va $\forall x \in X$ lar uchun o'suvchi funksiyadir.

Demak, berilgan funksional qator, **12.4.26**-teoremaga (Abel alomati) ko'ra, $[0;1]$ da tekis yaqinlashuvchi bo'ladi.

6-misol. Ushbu $\sum_{n=1}^{\infty} x^2 e^{-nx}$ funksional qatorning yig'indisini $X = [0;1]$ da uzluksizlikka tekshiring.

Yechilishi. Berilgan funksional qatorning har bir $u_n(x) = x^2 e^{-nx}$ $(n = 1,2,...)$ hadi X da uzluksiz bo'lib, bu funksional qator X da, Veyershtrass alomatiga ko'ra, tekis yaqinlashuvchi bo'ladi. Haqiqatan ham, $\forall n \in N$ va $\forall x \in [0;1]$ lar uchun

$$|u_n(x)| = \left|x^2 e^{-xe}\right| \leq \frac{1}{e^n} = a_n$$

va $\sum_{n=1}^{\infty} \frac{1}{e^n}$ sonli qator yaqinlashuvchi. Haqiqatan ham, $S(x) = \frac{x^2}{e^x - 1}, x > 0$. $x = 0$ bo'lganda $S(0) = 0$.

Demak, **12.5.2**-teoremaga asosan, berilgan funksional qatorning yig'indisi X da uzluksiz funksiya bo'ladi.

Mustaqil yechish uchun misollar

X to'plamda quyidagi $\{f_n(x)\}$ funksional ketma-ketliklarning limit funksiyasi $f(x)$ topilsin.

1. $f_n(x)=\frac{1}{x^n+2n}$, $X=(-\infty;\infty)$.   2. $f_n(x)=\frac{n^3+1}{x^2+n^3}$, $X=(-\infty;\infty)$.

3. $f_n(x)=x^n-4x^{n+3}+3x^{n+4}$, $X=[0;1]$.   4. $f_n(x)=x^4\cos\frac{1}{xn}$, $X=(0;\infty)$.

5. $f_n(x)=\sqrt{x^2+\frac{1}{\sqrt{n}}}$, $X=(-\infty;\infty)$.   6. $f_n(x)=n(x^{\frac{1}{n}}-1)$, $X=[1;3]$.

Quyidagi funksional qatorlarning (absolyut va shartli) yaqinlashish sohalarini toping.

7. $\sum\limits_{n=1}^{\infty}\ln^n x$.   8. $\sum\limits_{n=1}^{\infty}\frac{1}{n^x}$.   9. $\sum\limits_{n=1}^{\infty}\frac{1\cdot3\cdots(2n-1)}{2\cdot4\cdots(2n)}\left(\frac{2x}{1+x^2}\right)^n$

10. $\sum\limits_{n=1}^{\infty}\frac{n^p\sin nx}{1+n^q}$ $(q>0;0<x<\pi)$.   11. $\sum\limits_{n=1}^{\infty}x^n tg\frac{x}{2^n}$. 12. $\sum\limits_{n=1}^{\infty}\frac{\sin nx}{n^2}$.

X da $\{f_n(x)\}$ funksional ketma-ketlikning tekis yaqinlashuvchiligini isbotlang:

12. $f_n(x)=e^{-nx^2}$, $X=[1;\infty)$.   14. $f_n(x)=\frac{x^n}{1+x^n}$, $X=[0;1-\varepsilon], 0<\varepsilon<1$.

15. $f_n(x)=x^{4n}$. $X=[0;\varepsilon]$, $0<\varepsilon<1$.   16. $f_n(x)=\ln(1+\frac{\cos nx}{\sqrt{n+x}})$, $X=[0;+\infty)$.

17. $f_n(x)=\sqrt{x^2+\frac{1}{n}}$, $X=(-\infty;+\infty)$. 18. $f_n(x)=e^{-(x-n)^2}$, $X=[-4;4]$.

Quyidagi funksional лatorlarning ko'rsatilgan oraliqlarda tekis yaqinlashuvchiligini, Veyershtrass alomatidan foydalanib, isbotlang:

19. $\sum\limits_{n=1}^{\infty}\frac{1}{(n+x)^4}$, $X=[0;\infty)$. 20. $\sum\limits_{n=1}^{\infty}\frac{x^4}{2+\sqrt[3]{n^4}x^4}$, $X=(-\infty;+\infty)$.

21. $\sum\limits_{n=1}^{\infty}\frac{\cos^2 2nx}{\sqrt[4]{n^5+x^4}}$, $X=(-\infty;\infty)$. 22. $\sum\limits_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n^4+\ln^2 x}}$, $X=(1;\infty)$.

23. $\sum\limits_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n^3+e^{-3x}}}$, $X=[-3;3)$. 24. $\sum\limits_{n=1}^{\infty}\frac{(-1)^n\sin nx}{3^n+\cos x}$, $X=(-\infty;\infty)$.

Mustaqil yechish uchun misollarning javoblari

1. $f(x)=0$. 2. $f(x)=1$. 3. $f(x)=0$. 4. $f(x)=x^4$. 5. $f(x)=|x|$. 6. $f(x)=\ln x$.

7. $(\frac{1}{e},e)$-absolyut yaqinlashuvchi. 8. $(1,+\infty)$- absolyut yaqinlashuvchi. 9. $x\neq1$ - absolyut yaqinlashuvchi; $x=-1$- shartli yaqinlashuvchi. 10. $q>p+1$-

absolyut yaqinlashuvchi; $p < q \le p+1$ - shartli yaqinlashuvchi. **11.** $(-2,2)$-absolyut yaqinlashuvchi. **12.** $(-\infty,+\infty)$ -absolyut yaqinlashuvchi.

12.2-amaliy mashg'ulot.
Darajali qator, uning yaqinlashish radiusi va intervali

1-misol. Ushbu

$$\sum_{n=1}^{\infty} \frac{(x-1)^n}{n3^n}$$

darajali qatorning yaqinlashish radiusi, yaqinlashish intervali hamda yaqinlashish sohasini toping.

Yechilishi. Berilgan darajali qator n- hadining koeffisiyenti $a_n = \frac{1}{3^n n}$.

Qatorning yaqinlashish radiusini (12.10.5) formulaga asosan topamiz:

$$\rho = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\frac{(n+1)3^{n+1}}{n3^n} = \lim_{n\to\infty} 3(1+\frac{1}{n}) = 3, \quad \rho = 3.$$

12.10.6-eslatmaga ko'ra, $x_0 = 1$, $\rho = 3$, berilgan qatorning yaqinlashish intervali esa $(x_0 - \rho; x_0 + \rho) = (-2;4)$ bo'ladi. Endi qatorning yaqinlashishini yaqinlashish intervalining chegaralarida tekshirib ko'ramiz.

Agar :

a) $x = -2$ bo'lsa, $\sum\limits_{n=1}^{\infty} \frac{(-1)^n 2^n}{3^n n}$ qator hosil bo'ladi. Bu qator Leybnis alomatiga ko'ra yaqinlashuvchi.

b) $x = 4$ bo'lsa, $\sum\limits_{n=1}^{\infty} \frac{1}{n}$ uzoqlashuvchi garmonik qator hosil bo'ladi.

Demak, berilgan darajali qatorning yaqinlashish sohasi $[-2;4)$ yarim intervaldan iborat.

2-misol. Quyidagi berilgan darajali qatorning yaqinlashish radiusi, yaqinlashish intervali va yaqinlashish sohasini toping.

$$\sum_{n=1}^{\infty}\left(\frac{n}{3n-1}\right)^{3n+1} x^n.$$

Yechilishi. Berilgan darajali qatorda $a_n = \left(\frac{n}{3n-1}\right)^{3n+1}$. Darajali qatorning yaqinlashish radiusini (12.10.4) formulaga binoan topamiz:

$$\rho = \frac{1}{\lim\limits_{n\to\infty}\sqrt[n]{a_n}} = \frac{1}{\lim\limits_{n\to\infty}\sqrt[n]{\left(\frac{n}{3n-1}\right)^{3n+1}}} = \lim_{n\to\infty}\frac{1}{\left(\frac{n}{3n-1}\right)^{\frac{3n+1}{n}}} =$$

$$= \lim_{n\to\infty}\frac{3^{\frac{3n+1}{n}}}{\left[\left(1+\frac{1}{3n-1}\right)^{3n-1}\right]^{\frac{3n+1}{n(3n-1)}}} = 3^3 = 27.$$

Demak, darajali qatorning yaqinlashish radiusi $\rho = 27$, yaqinlashish intervali esa, (-27; 27) dan iborat. Endi yaqinlashish intervalining chegaralarida darajali qatorni yaqinlashishga tekshiramiz. $x = 27$ bo'lganda $\sum\limits_{n=1}^{\infty}\left(\frac{n}{3n-1}\right)^{3n+1}(27)^n$ qator hosil bo'ladi. Bu qatorni yaqinlashishga tekshirishda Koshi alomatidan foydalanamiz:

$$\lim_{n\to\infty} K_n = \lim_{n\to\infty}\sqrt[n]{\left(\frac{n}{3n-1}\right)^{3n+1}(27)^n} = 27\lim_{n\to\infty}\left(\frac{n}{3n+1}\right)^{\frac{3n+1}{n}} = \frac{27}{27} = 1,\ k = 1.$$

Mustaqil yechish uchun misollar

Quyidagi darajali qatorning yaqinlashish radiusi, yaqinlashish intervali hamda yaqinlashish sohasini toping:

1. $\sum\limits_{n=1}^{\infty}(-1)^{n-1}\frac{(x-3)^{2n}}{n5^n}$. **2.** $\sum\limits_{n=1}^{\infty}\frac{x^{n-1}}{n3^n\ln n}$. **3.** $\sum\limits_{n=1}^{\infty}\left(\frac{n}{3n-1}\right)^{3n+1}x^n$.

4. $\sum\limits_{n=1}^{\infty}\ln^3\left(1+\frac{1}{\sqrt{n}}\right)x^n$. **5.** $\sum\limits_{n=1}^{\infty}\frac{(x-1)^n}{2^n+3^n}$. **6.** $\sum\limits_{n=1}^{\infty}tg\frac{\pi}{3^n}x^{2n}$.

7. $\sum\limits_{n=1}^{\infty}\frac{5^n+(-3)^n}{n+1}x^n$. **9.** $\sum\limits_{n=1}^{\infty}3^n(n^3+2)(x-1)^{2n}$

Quyidagi qatorlarning yaqinlashish sohalarini toping:

9. $\sum\limits_{n=1}^{\infty}x^{2n+1}\sin\frac{\pi}{2^n}$. **10.** $\sum\limits_{n=1}^{\infty}\frac{\lg^n x}{n}$. **11.** $\sum\limits_{n=1}^{\infty}(\sin(\sqrt{n+1}-\sqrt{n}))(x+1)^n$.

12. $\sum\limits_{n=1}^{\infty}\left(arctg\frac{1}{5^n}\right)(x-5)^n$. **12.** $\sum\limits_{n=1}^{\infty}\frac{2^n n}{n^n}(x-1)^{2n}$.

Quyidagi darajali qatorlarning yig'indilarini hadma-had differensiallash yordamida toping:

14. $x+\frac{x^3}{3}+\frac{x^5}{5}+\ldots$ **15.** $x-\frac{x^3}{3}+\frac{x^5}{5}-\ldots$

Quyidagi darajali qatorlarning yig'indilarini hadma-had integrallash yordamida toping:

16. $x+2x^2+3x^3+\ldots$ **17.** $x+\frac{1}{2}x^2+\frac{1}{3}x^3+\ldots+\frac{1}{n+1}x^{n+1}+\ldots$

Misollarning javoblari

1. $\rho=\sqrt{5}$, $(3-\sqrt{5};\ 3+\sqrt{5})$, $[3-\sqrt{5};\ 3-\sqrt{5}]$. **2.** $\rho=3$, $(-3,3)$, $[-3;3]$. **3.** $\rho=27$, $(-27;27)$. **4.** $\rho=1,(-1;1),[-1;1]$ **5.** $\rho=3$, $(-2,4)$. **6.** $\rho=\sqrt{3},(-\sqrt{3};\sqrt{3})$ **7.** $\rho=0$, $x=0$. **8.** $\rho=\infty$, $(-\infty;\infty)$. **9.** $(-\sqrt{2};\sqrt{2})$. **10.** $[-10^{-1};10]$. **11.** $(-2;0)$. **12.** $(0,6)$ **12.** $\left(1-\sqrt{\frac{e}{2}};1+\sqrt{\frac{e}{2}}\right)$ **14.** $\frac{1}{2}\ln\frac{1+x}{1-x}$ $(|x|<1)$. **15.** $arctgx$ $(|x|\le 1)$ **16.** $\frac{x}{(1-x)^2}$ $(|x|<1)$ **17.** $-\ln|1-x|$ $(|x|<1)$.

12.3-amaliy mashg'ulot.
Teylor qatori. Funksiyalarni darajali qatorlarga yoyish

1-misol. $f(x)=\cos^2 x$ Funksiyalarni $x_0=\frac{\pi}{4}$ nuqta atrofida Teylor qatoriga yoying va yaqinlashish radiusini toping:

Yechilishi. Berilgan funksiyani quyidagi ko'rinishda tasvirlaymiz: $f(x)=\cos^2 x=\frac{1}{2}+\frac{1}{2}\cos 2x$. $x-\frac{\pi}{4}=t$ deb belgilab, $\cos 2x=-\sin 2t$ ekanligini topamiz. Natijada $f(x)=g(t)=\frac{1}{2}-\frac{1}{2}\sin 2t$. Endi $\sin t$ ning makloren qatoriga yoyilmasida foydalanib $g(t)=\frac{1}{2}-\frac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^n 2^{2n+1}}{(2n+1)!}t^{2n+1}$ ni topamiz. Bu yerda

$$f(x)=\frac{1}{2}+\sum_{n=0}^{\infty}\frac{(-1)^{n-1}2^{2n}}{(2n+1)!}\left(x-\frac{\pi}{4}\right)^{2n+1}$$

Hosil bo'lgan qatorning yaqinlashish radiusini tekshirishda $\rho=\lim\limits_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$ formulasidan foydalanamiz:

$$\rho=\lim_{n\to\infty}\left|\frac{2^{2n}}{(2n+1)!}\cdot\frac{(2n+3)!}{2^{2n+2}}\right|=\frac{1}{4}\lim_{n\to\infty}(2n+3)=+\infty.$$

Demak,darajali qatorning yaqinlashish radiusi $\rho=\infty$. Shunday qilib, hosil bo'lgan darajali qator $\forall x\in(-\infty,\infty)$ uchun $f(x)=\cos^2 x$ funksiyaga yaqinlashadi.

Mustaqil yechish uchun misollar

Quyidagi funksiyalarni x ning darajalari bo'yicha darajali qatorga yoying:

1. $y=\sin^3 x$. **2.** $y=\dfrac{x}{6-x-x^2}$. **3.** $y=\ln(1+x+x^2+x^3)$.

4. $y=\dfrac{1}{(1-x^3)^2}$. **5.** $y=\cos^2 x$. **6.** $y=\arcsin x^3$.

7. $y=e^{-x^2}$. **8.** $y=4^x$.

Quyidagi funksiyalarni ko'rsatilgan nuqta atrofida Teylor qatoriga yoying va bu qatorlarning yaqinlashish radiusini toping:

9 $f(x)=\sin\dfrac{2\pi x}{3}$, $x_0=3$. **10.** $f(x)=\dfrac{1}{x^2+4x+7}$, $x_0=-2$

11. $f(x)=e^x$, $x_0=3$. **12.** $f(x)=\sqrt{x}$, $x_0=1$.

12. $f(x)=\dfrac{x}{x^2-5x+6}$, $x_0=5$. **14.** $f(x)=2^x$, $x_0=1$.

Quyida Keltirilgan sonlarni ko'rsatilgan aniqlikda hisoblang:

15. $\cos 18^\circ$, 0,0001 **16.** $e^{\frac{1}{2}}$, 0,00001. **17.** $\ln 0{,}98$, 0,0001

Integral ostidagi funksiyani darajali qatorga yoyish yordamida, berilgan integralni ko'rsatilgan aniqlikda hisoblang:

18. $\int_0^{1/4} e^{-x^2}dx$, 0,001. **19** $\int_0^{0,5}\dfrac{\sin 2x}{x}dx$, 0,001.

Darajali qatorlar yordamida quyida keltirilgan limitlarni hisoblang:

20. $\lim\limits_{x\to 0}\dfrac{x-\operatorname{arctg} x}{x^3}$. **21.** $\lim\limits_{x\to 0}\dfrac{1-\cos x}{e^x-1-x}$.

Mustaqil yechish uchun misollarning javoblari

1. $\frac{3}{4}\sum\limits_{n=0}^{\infty}(-1)^{n+1}\frac{3^{2n}-1}{(2n+1)!}x^{2n+1}$, $(|x|<+\infty)$. **2.** $-\frac{1}{5}\sum\limits_{n=0}^{\infty}\frac{(-1)^n}{3^n}x^n-\frac{1}{5}\sum\limits_{n=0}^{\infty}\frac{1}{2^n}x^n$, $(-2<x<2)$.

3. $\sum\limits_{n=1}^{\infty}\frac{(-1)^{n+1}+\left[1+(-1)^n\right](-1)^{\frac{n}{2}+1}}{n}x^n$, $(-1<x\le 1)$ **4.** $\sum\limits_{n=1}^{\infty}(n+1)\cdot x^{3n}$, $(|x|<1)$

5. $1+\sum\limits_{n=1}^{\infty}(-1)^n\frac{2^{2n-1}}{(2n)!}x^{2n}$, $(|x|<+\infty)$ **6.** $x^3+\sum\limits_{n=1}^{\infty}\frac{(2n-1)!!}{2^n n!(2n+1)}x^{6n+3}$, $(|x|\le 1)$

7. $\sum\limits_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{n!}$, $(|x|<\infty)$ **8.** $\sum\limits_{n=0}^{\infty}\frac{\ln^n 4}{n!}x^n$, $(|x|<+\infty)$

9 $\sum\limits_{n=0}^{\infty}\frac{1}{n!}\left(\frac{2\pi}{3}\right)^{2n+1}(x-3)^{2n+1}$, $R=\infty$. **10.** $\sum\limits_{n=0}^{\infty}\frac{(-1)^n}{3^n}(x+2)^{2n}$, $R=\sqrt{3}$.

11. $e^3\sum\limits_{n=0}^{\infty}\frac{1}{n!}(x-3)^n$, $R=\infty$ **12.** $\sum\limits_{n=1}^{\infty}\frac{(-1)^{n-1}(2n-3)!!}{n!2^n}(x-1)^n$, $R=\frac{5}{4}$.

12. $x\sum\limits_{n=1}^{\infty}\frac{(-1)^n}{2^{n+1}}(x-5)^n-x\sum\limits_{n=1}^{\infty}\frac{(-1)^n}{3^{n+1}}(x-5)^n$, $R=2$ **14.** $2\sum\limits_{n=0}^{\infty}\frac{\ln^n 2}{n!}(x-1)^n$, $R=\infty$.

15. $0,9551$. **16.** $1,648719$. **17.** $\ln 0,98\approx -0,0202$.

18. $0,245$. **110** $0,946$. **20.** $\frac{1}{3}$. **21.** 1.

12-bob bo'yicha amaliy mashg'ulotlarni mustahkamlash uchun nazorat topshiriqlari

12.1-masala. $\{f_n(x)\}$ funksional ketma-ketlikning X to'plamdagi $f(x)$ limit funksiyasini toping.

12.1.1. $f_n(x)=\dfrac{\ln nx}{nx^2}$, $X=[1;+\infty)$.

12.1.2. $f_n(x)=\dfrac{n^3+1}{x^2+n^3}$, $X=(-\infty;+\infty)$.

12.1.3. $f_n(x)=x^n-4x^{n+3}+3x^{n+4}$, $X=[0;1]$.

12.1.4. $f_n(x)=n^{3/2}\left(1-\dfrac{\cos\sqrt[4]{x}}{n}\right)$, $X=[0;+\infty)$.

12.1.5. $f_n(x)=\dfrac{x+xn^3+x^3n^6}{1+x^2n^6}$, $X=[1;+\infty)$.

12.1.6. $f_n(x)=\sqrt{x^2+\dfrac{1}{\sqrt{n}}}$, $X=(-\infty;+\infty)$.

12.1.7. $f_n(x)=n(x^{\frac{1}{n}}-1)$, $X=[1;3]$.

12.1.8. $f_n(x)=n^3x^2e^{-nx}$, $X=[0;+\infty)$.

12.1.9. $f_n(x)=n\,\mathrm{arctg}\,nx^2$, $X=(0;+\infty)$.

12.1.10. $f_n(x)=n\sin\dfrac{\sqrt{x}}{n}$, $X=[0;+\infty)$.

12.1.11. $f_n(x)=x\,\mathrm{arctg}\,nx$, $X=(0;+\infty)$.

12.1.12. $f_n(x)=n\left(\sqrt{x^2+\frac{1}{n}}-x\right),\ X=(0;+\infty).$

12.1.13. $f_n(x)=n[\ln(x+n)-\ln n],\ X=[1;+\infty).$

12.1.14. $f_n(x)=\sqrt[n]{1+x^n+(\frac{x^2}{2})^n},\ X=[0;+\infty).$

12.1.15. $f_n(x)=\dfrac{\sin n\sqrt{x}}{\ln(n+1)},\ X=[0;+\infty).$

12.1.16. $f_n(x)=\dfrac{n}{x^2+n^2}\operatorname{arctg}\sqrt{nx},\ X=[0;+\infty).$

12.1.17. $f_n(x)=\ln\left(3+\dfrac{n^2e^x}{n^4+e^{2x}}\right),\ X=[0;+\infty).$

12.1.18. $f_n(x)=\dfrac{\operatorname{arctg} n^2x}{\sqrt{n^3+x^2}},\ X=(-\infty;+\infty).$

12.1.19. $f_n(x)=\dfrac{nx^2}{x+3n+2},\ X=[0;+\infty).$

12.1.20. $f_n(x)=n\left(x^{\frac{1}{n}}-1\right),\ X=[1;3].$

12.1.21. $f_n(x)=\dfrac{n^5+3}{x^4+n^5},\ X=(-\infty;+\infty).$

12.1.22. $f_n(x)=x^n-6x^{n+1}+9x^{n+2},\ X=[0;1].$

12.1.23. $f_n(x)=\sqrt{x^4+\frac{1}{n^4}},\ X=(-\infty;+\infty).$

12.1.24. $f_n(x)=\dfrac{n^2x^2}{x+6n^2+5},\ X=[0;+\infty).$

12.1.25. $f_n(x)=\dfrac{\operatorname{arctg} n^2x}{\sqrt{n^4+x^2}},\ X=(-\infty;+\infty).$

12.1.26. $f_n(x)=n\left(\sqrt{x+\frac{1}{n}}-\sqrt{x}\right),\ X=[0;+\infty).$

Yechilishi ([2], 11-bo'lim, [9], 2-bo'lim). Berilgan funksional ketma-ketlikni ushbu $f_n(x)=\dfrac{1}{\sqrt{x+\frac{1}{n}}+\sqrt{x}}$ ko'rinishga keltirib, so'ngra uning limit funksiyasini topamiz:

$$\lim_{n\to\infty} f_n(x)=\lim_{n\to\infty}\frac{1}{\sqrt{x+\frac{1}{n}}+\sqrt{x}}=\frac{1}{2\sqrt{x}}.$$

Demak, $f(x)=\dfrac{1}{2\sqrt{x}}$.

12.2-masala. Ko'rsatilgan oraliqda funksional qatorning tekis yaqinlashuvchiligini Veyershtrass alomatidan foydalanib ko'rsating.

12.2.1. $\sum\limits_{n=1}^{\infty}\dfrac{x^n}{n^2},\ X=[-1;1]$.

12.2.2. $\sum\limits_{n=1}^{\infty}\dfrac{1}{(n+x)^2},\ X=[0;+\infty)$.

12.2.3. $\sum\limits_{n=1}^{\infty}\dfrac{x^4}{2+\sqrt[3]{n^4}\cdot x^4},\ X=(-\infty;+\infty)$.

12.2.4. $\sum\limits_{n=1}^{\infty}\dfrac{x^2}{1+n^{3/2}x^2},\ X=(-\infty;+\infty)$.

12.2.5. $\sum\limits_{n=1}^{\infty}\dfrac{arctgnx}{x^4+n\sqrt[3]{n}},\ X=(-\infty;+\infty)$.

12.2.6. $\sum\limits_{n=1}^{\infty}e^{-\sqrt{nx}},\ X=[1;+\infty)$.

12.2.7. $\sum\limits_{n=1}^{\infty}\sin^2\dfrac{\sqrt{x}}{1+n^2x},\ X=[0;+\infty)$.

12.2.8. $\sum\limits_{n=1}^{\infty}x^2e^{-nx},\ X=[0;+\infty)$.

12.2.9. $\sum_{n=1}^{\infty}\frac{\sin nx}{n\sqrt{n}}, \ X=(-\infty;+\infty).$

12.2.10. $\sum_{n=1}^{\infty}\frac{\cos 3nx}{\sqrt{n^3+x^3}}, \ X=[0;+\infty).$

12.2.11. $\sum_{n=1}^{\infty}\frac{(-1)^n}{x+2^n}, \ X=(-2;+\infty).$

12.2.12. $\sum_{n=1}^{\infty}\frac{(-1)^n\sin nx}{3^n+\cos x}, \ X=(-\infty;+\infty).$

12.2.13. $\sum_{n=1}^{\infty}\ln\left(1+\frac{x\ln^3 x}{n^3}\right), \ X=[0;100].$

12.2.14. $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^{2n}}{n}, \ X=(-1;1).$

12.2.15. $\sum_{n=1}^{\infty}\sin\frac{1}{\sqrt{n}}\,arctg\frac{2x}{x^2+n^2}, \ X=(-\infty;+\infty).$

12.2.16. $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{x+\sqrt{n^3}}, \ X=[0;+\infty).$

12.2.17. $\sum_{n=1}^{\infty}\frac{n\cdot\ln(1+nx)}{x^n}, \ X=(2;+\infty).$

12.2.18. $\sum_{n=1}^{\infty}\frac{x^n}{n!}, \ X=(-2;+2).$

12.2.19. $\sum_{n=1}^{\infty}\frac{x^n}{n\cdot 2^n}, \ X=[-\frac{3}{2};\frac{3}{2}].$

12.2.20. $\sum_{n=1}^{\infty}n^3e^{-n^2x}, \ X=(\delta;+\infty), \ \delta>0.$

12.2.21. $\sum_{n=1}^{\infty}\frac{x^{4n}}{n!}, \ X=(-\infty;+\infty).$

12.2.22. $\sum\limits_{n=1}^{\infty}\frac{10^n x^n}{\sqrt{n^3}}$, $X=[-0,1;\ 0,1]$.

12.2.23. $\sum\limits_{n=1}^{\infty}\frac{x^{3n}}{8^n(n^2+1)}$, $X=[-2;2]$.

12.2.24. $\sum\limits_{n=1}^{\infty}\frac{x^n}{1+n^{3/2}}$, $X=[-1;1]$.

12.2.25. $\sum\limits_{n=1}^{\infty}\frac{(x+3)^n}{n\sqrt[3]{n}}$, $X=[-4;-2]$.

12.2.26. $\sum\limits_{n=1}^{\infty}\frac{(x-1)^n}{(3n+1)\cdot 3^n}$, $X=[-1;3]$.

Yechilishi ([2], 11-bo'lim, [9], 2-bo'lim, [30], 11.7-bo'lim). $\forall x\in[-1;3]$ uchun $|u_n(x)|=\frac{|x-1|^n}{(3n+1)\cdot 3^n}\le\frac{2^n}{(3n+1)\cdot 3^n}$ o'rinli. $\sum\limits_{n=1}^{\infty}a_n=\sum\limits_{n=1}^{\infty}\frac{2^n}{(3n+1)\cdot 3^n}$ - sonli qatorni Dalamber alomatidan foydalanib, yaqinlashishga tekshiramiz:

$$a_n=\frac{(\frac{2}{3})^n}{3n+1},\quad a_{n+1}=\frac{(\frac{2}{3})^{n+1}}{3n+4}.$$

$\lim\limits_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim\limits_{n\to\infty}\frac{2}{3}\frac{(3n+1)}{(3n+4)}=\frac{2}{3}<1$ bo'lgani uchun majorant sonli qator yaqinlashuvchi.

Demak, Veyershtrass alomatiga ko'ra, berilgan funksional qator $[-1;3]$ da tekis yaqinlashuvchi.

12.3-masala. Berilgan funksiyalarni x ning darajalari bo'yicha darajali qatorga yoying.

12.3.1. $\sin^3 x$.

12.3.2. $\cos^2 x$.

12.3.3. $\frac{x}{1+x-2x^2}$.

12.3.4. $\frac{1}{(1-x^3)^2}$.

12.3.5. $\frac{x}{(1-x)(1-x^2)}$.

12.3.6. $\arccos(1-2x^2)$.

12.3.7. $\frac{x}{6-x-x^2}$. **12.3.8.** $\frac{1}{1-x-x^2}$.

12.3.9. 4^x. **12.3.7.** 2^x.

12.3.11. $\ln(1+x+x^2+x^3)$. **12.3.12.** $\frac{1}{1+x+x^2+x^3}$.

12.3.13. e^{-x^2}. **12.3.14.** e^{-x^3}.

12.3.15. $\frac{x}{\sqrt{1-2x}}$. **12.3.16.** $\frac{x}{\sqrt{1-2x}}$.

12.3.17. $\frac{x^{10}}{1-x}$. **12.3.18.** $\frac{1}{(1-x^3)^2}$.

12.3.19. $\ln(x+\sqrt{1+x^2})$. **12.3.20.** $e^x(1-x)+e^{-x}(1+x)$.

12.3.21. $\frac{x-2}{x^2-5x+4}$. **7.3.11.22.** $\cos^5 x$.

12.3.23. $\frac{x^4}{4-x^2}$. **7.3.11.24.** $\sin^5 x$.

12.3.25. $\frac{3x-5}{x^2-x-2}$. **12.3.26.** $\frac{3x-5}{x^2-4x+3}$.

Yechilishi ([2], 11-bo'lim, [9], 2-bo'lim, [30], 11.8-bo'lim). Berilgan $\frac{3x-5}{x^2-4x+3}=\frac{3x-5}{(x-1)(x-3)}$ kasrni sodda kasrlar yig'indisini shaklida tasvirlaymiz, ya'ni

$$\frac{3x-5}{(x-1)(x-3)}=\frac{1}{x-1}+\frac{2}{x-3}=-(1-x)^{-1}-\frac{2}{3}(1-\frac{x}{3})^{-1}.$$

So'ngra (12.13.3) formuladan foydalanib, ushbu

$$\frac{3x-5}{(x-1)(x-3)}=-\sum_{n=1}^{\infty}\left(1+\frac{2}{3^{n+1}}\right)x^n,\ \ (|x|<1)$$

ekanligini topamiz.

12.4-masala Funksiyalarni ko'rsatilgan nuqta atrofida Teylor qatoriga yoying va yaqinlashish radiusini toping.

12.4.1. $\frac{1}{x^2-5x+6}$, $x_0=1$. **12.4.2.** $\sqrt{x^3}$, $x_0=1$.

12.4.3. $\frac{1}{x+3}$, $x_0=-2$.

12.4.4. $\frac{1}{2x+5}$, $x_0=3$.

12.4.5. $\frac{1}{(x-3)^2}$, $x_0=1$.

12.4.6. $\sin\frac{\pi x}{4}$, $x_0=2$.

12.4.7. $\sqrt{x}$, $x_0=1$.

12.4.8. $\frac{x}{x^2-5x+6}$, $x_0=5$.

12.4.9. $\ln(5x+3)$, $x_0=\frac{2}{5}$.

12.4.7. $\frac{1}{\sqrt{4+x}}$, $x_0=-3$.

12.4.11. $\frac{1}{\sqrt{x-1}}$, $x_0=2$.

12.4.12. $\frac{1}{x^2-4x+3}$, $x_0=-2$.

12.4.13. $\sin x$, $x_0=a$.

12.4.14. $\frac{1}{(x^2-6x+18)^2}$, $x_0=3$.

12.4.15. $\ln x$, $x_0=1$.

12.4.16. $\frac{1}{\sqrt{x^2-12x+40}}$, $x_0=6$.

12.4.17. $\frac{1}{x^2+4x+7}$, $x_0=-2$.

12.4.18. $\frac{1}{x^2-4x+3}$, $x_0=1$.

12.4.19. $\cos\frac{\pi x}{2}$, $x_0=1$.

12.4.20. $\frac{1}{x}, x_0=3$.

12.4.21. $\frac{1}{x^2-4x+3}$, $x_0=1$.

12.4.22. $\sqrt{x^5}$, $x_0=1$.

12.4.23. $\frac{1}{3x+5}$, $x_0=2$.

12.4.24. $\cos x$, $x_0=\frac{\pi}{4}$.

12.4.25. $\frac{1}{(x+2)^2}$, $x_0=-1$.

12.4.26. $\cos^4 x, x_0=\frac{\pi}{4}$.

12.5-masala. Differensial tenglamaning berilgan boshlang'ich shartlarni qanoatlantiruvchi yechimining x bo'yicha yoyilmasini toping.

12.5.1. $y'=xy+e^y, y(0)=0$.

12.5.2. $y'-y=0$, $y(0)=1$.

12.5.3. $(1+x^2)y'-1=0, y(0)=1$.

12.5.4. $y'=x^2y^2+1, y(0)=1$.

12.5.5. $y'+xy=0, y(0)=1.$

12.5.6. $(1-x^2)y''-xy'=0, y(0)=0, y'(0)=1.$

12.5.7. $y'=x^2-y^2, y(0)=\frac{1}{2}.$

12.5.8. $y'=x^3+y^2, y(0)=\frac{1}{2}.$

12.5.9. $y'=x+y^2, y(0)=-1.$

12.5.10. $y'=x+x^2+y^2, y(0)=1.$

12.5.11. $y'=2\cos x-xy^2, y(0)=1.$

12.5.12. $y'=e^x-y^2, y(0)=0.$

12.5.13. $y'=x+y+y^2, y(0)=1.$

12.5.14. $y'=x+y^2, y(0)=1$

12.5.15. $y''=x+y^2, y(0)=0, y'(0)=1.$

12.5.16. $y''=\frac{y'}{y}-\frac{1}{x}, y(1)=1, y'(1)=0.$

12.5.17. $y'=x^2y^2+y\sin x, y(0)=\frac{1}{2}.$

12.5.18. $y'=x^2y^2+ye^x, y(0)=\frac{1}{3}.$

12.5.19. $(1-x^2)y''-5xy'-4y=0, y(0)=1, y'(0)=1.$

12.5.20. $y'=e^{3x}+2xy^2, y(0)=1.$

12.5.21. $y'=x\sin x-y^2, y(0)=1.$

12.5.22. $y'=xe^x-y^2=0, y(0)=1.$

12.5.23. $(1+x^2)y'-1=0, y(0)=1.$

12.5.24. $y'=e^{\sin x}+x, y(0)=0.$

12.5.25. $y''+xy=0, y(0)=1, y'(0)=0.$

12.5.26. $y'=y\cos x+2\cos y, y(0)=0.$

Yechilishi ([30], 11,10-bo'lim). Berilgan differensial tenglamaning yechimini ushbu $y=y(0)x+y''(0)\frac{x^2}{2!}+y'''(0)\cdot\frac{x^3}{3!}+...$ Makloren qatori ko'rinishida izlaymiz. Shartga ko'ra $x=0$ da $y(0)=0$; ekanini e'tiborga olib, berilgan differensial tenglamadan $y'(0), y''(0), y'''(0)$ larni topamiz. Buning uchun berilgan differensial tenglamani ketma – ket ikki marta differensiallaymiz: $y'(0)=2$;

$$y'' = y'\cos x - y\sin x - 2\sin y \cdot y';$$
$$y''' = y''\cos x - y'\sin x - y'\sin x - 2\cos y \cdot (y')^2 - 2\sin y \cdot y'',$$

bundan $x=0$ da $y''(0)=2$, $y'''(0)=-6$ topamiz. Topilgan hosilalarning qiymatini izlanuvchi qatorga qo'yib, berilgan differensial tenglamaning taqribiy yechimini topamiz. $y=2x+x^2-x^3$.

13-BOB.
FURYE QATORLARI. FURYE INTEGRALI

13.1-§. Furye qatorining ta'rifi

13.1. Ba'zi asosiy tushunchalar. 1. ***Davriy funksiya tushunchasi va uning asosiy xossalari***. Tabiatda va texnikada shunday xossalar uchraydiki, ular ma'lum bir t vaqt o'tgandan so'ng o'zlarining boshlang'ich holatlarini takrorlaydi. Bularga osmon jismlarining aylanma harakati, har xil asbob va mashinalarning tebrenma harakati misol bo'la oladi. Bunday hodisalarni matematikada *davriy* deb ataluvchi funksiyalar yoradmida o'rganiladi.

13.1.1-ta'rif. Agar $f(x)$ $(-\infty < x < \infty)$ funksiya uchun shunday T $(T \neq 0)$ son mavjud bo'lib, barcha $x \in (-\infty;\infty)$ lar uchun

$$f(x+T) = f(x)$$

tenglik o'rinli bo'lsa,u holda $f(x)$ *davriy funksiya*, T esa, uning *davri* deyiladi.

13.1.1-ta'rifdan ko'rinadiki, $f(x)$ davriy funksiyaning grafigini biror $[a, a+T]$ $(a \neq 0)$ oralig'ida yasab so'ngra uni butun o'qqa davriylik qonuniga asosan davom ettirish yetarli.

1^0. Agar T son $f(x)$ funksiyaning davri bo'lsa, u holda ixtiyoriy butun (musbat yoki manfiy) k uchun $T_1 = kT$ ham $f(x)$ funksiyaning davri bo'ladi, yani

$$f(x+T_1) = f(x).$$

2^0.Agar T_1 va T_2 sonlar $f(x)$ funksiya uchun davr bo'lsa, $T_1 \pm T_2$ ham $f(x)$ funksiyaning davri bo'ladi.

3^0.Agar $f(x)$ funksiya barcha $x \in (-\infty;\infty)$ lar uchun o'zgarmas bo'lsa, u davriy funksiya bo'lib, uning davri T son bo'ladi.

4^0. Agar $f(x)$ funksiya $(f(x) \neq 0)$ uzliksiz davriy funksiya bolsa,u vaqtda u yeng kichik musbat T davrga ega bo'ladi va u funksiyaning davri deb qabul qilinadi.

2. *Funksiyani davriy davom yettirish.* $f(x)$ funksiya $(a,b]$ intervalda aniqlangan bo'lib, bu funksiya yordamida yordamchi quyidagi

$$f^*(x) = f(x-(b-a)m),\ x \in (a+m(b-a), b+m(b-a)$$

$(m = 0, \pm 1, \pm 2, \pm 3, ...)$ funksiyani tuzamiz.

Ravshanki, $f^*(x)$ funksiya $(-\infty;\infty)$ kesmada davriy funksiya bo'lib, uning davri $T_0 = b-a$ ga teng bo'ladi. Bu bajarilgan jarayonni funksiyani *davriy davom yettirish* deyiladi.

Agar $f(x)$ funksiya $(a,b]$ da uzliksiz bo'lib va

$$f(a+0) = \lim_{x \to a+0} f(x) = f(b)$$

bo'lsa, u holda davom yettirilgan $f^*(x)$ funksiya $(-\infty;\infty)$ da uzliksiz bo'ladi.

Agar $f(x)$ funksiya $[a,b)$ oraliqda berilgan bo'lsa, uni davriy davom yettirish ham yuqoridagi singari bo'ladi:

$$f^*(x) = f(x-(b-a)m),\ x \in [a+m(b-a), b+m(b-a))\ (m = 0, \pm 1, \pm 2, \pm 3, ...)$$

Agar $f(x)$ funksiya (a,b) intervalda berilgan bo'lsa, uni $(-\infty;\infty)/\{a+m(b-a);\ m = 0, \pm 1, \pm 2, \pm 3, ...\} = X^*$ to'plamga davriy davom yettirish mumkin:

$$f^*(x) = f(x-(b-a)m),\ x \in (a+m(b-a), b+m(b-a))\ (m = 0, \pm 1, \pm 2, \pm 3, ...)$$

Agar $f(x)$ funksiya $[a,b]$ da berilgan bo'lsa, uni $(-\infty;\infty)$ ga umuman aytaganda, ikki xil davriy davom yettirish mumkin:

$$f^*(x) = f(x-(b-a)m),\ x \in (a+m(b-a), b+m(b-a))\ (m = 0, \pm 1, \pm 2, \pm 3, ...),$$

$$f^{**}(x) = f(x-(b-a)m),\ x \in [a+m(b-a), b+m(b-a))\ (m = 0, \pm 1, \pm 2, ...).$$

5^0. 1) agar $f_1(x), f_2(x), f_3(x), f_4(x), ... f_n(x)$ funksiyalarning har biri T davrga ega bo'lsa, u holda $(-\infty;\infty)$ da $F(x) = \sum_{k=1}^{n} f_k(x)$ funksiya ham T davrga ega bo'ladi.

2) agar $f_1(x), f_2(x), f_3(x), f_4(x), ... f_n(x)$. funksiyalarning har biri T davrga ega bo'lsa, u holda $F(x)=\prod_{k=1}^{n} f_k(x)$ funksiya ham T davrga ega bo'ladi.

3) agar $f_1(x)$ va $f_2(x)$ funksiyalar bir xil T davrga ega bo'lsa, u holda $\frac{f_1(x)}{f_2(x)}$ yoki $\frac{f_2(x)}{f_1(x)}$ funksiyalar ham T davrga ega bo'ladi.

13.1.2-eslatma. Bu xossadan, $f_1(x)$ va $f_2(x)$ funksiyalar uchun eng kichik T davr bo'linma funksiya uchun ham eng kichik davr bo'ladi degan natija kelib chiqmasligini qayd qilamiz. Masalan, $\sin x$ va $\cos x$ funksiyalarning har biri 2π davrga ega bo'lgani uchun tgx va $ctgx$ uchun ham 2π davr bo'ladi,lekin keyingi ikki funksiya uchun 2π dan kichik π son ham davr bo'ladi.

4) agar $f_1(x)$ va $f_2(x)$ funksiyalar mos ravishda o'lchovdosh , ya'ni

$$\frac{T_1}{T_2}=\frac{n_1}{n_2} \tag{13.1.3}$$

(bunda n_1 va n_2 –biror butun sonlar) bo'lgan T_1 va T_2 davrlarga ega bo'lsa, u holda $T=n_2T_1=n_1T_2$ son har ikki funksiya uchun davr bo'ladi. O'lchovdosh, ya'ni (13.1.3) ni qanoatlantiruvchi T_1 va T_2 davrlarga ega bo'lgan ikkita funksiya yig'indisi, ayirmasi va ko'paytmasi davriy funksiya bo'lib, $T=n_2T_1=n_1T_2$ ular uchun davr bo'ladi.

Masalan, $\sin x$ va tgx funksiya uchun davrlari mos ravishda $T_1=2\pi$ va $T_2=\pi$ bo'lib, $\frac{T_1}{T_2}=\frac{2}{1}$ yoki $\frac{T_2}{T_1}=\frac{1}{2}$ bo'lgani uchun $T=2\pi$ har ikkalasi uchun davr bo'ladi, undan tashqari, 2π son $\sin x+tgx, tgx\cdot\sin x, \cos x, \sec x$ funksiyalar uchun ham davr bo'ladi.

5^0. Agar T davrga ega bo'lgan $f(x)$ funksiya o'zining aniqlanish sohasida differinsiallanuvchi bo'lsa, u holda uning $f'(x)$ hosilasi ham T davrga ega bo'ladi.

6^0. Agar $f(x)$ funksiya T davrga ega va integrallanuvchi bo'lib, $\int_0^T f(x)dx=0$ bo'lsa, u holda $F(x)=\int_0^x f(t)dt$ funksiya ham T davrga ega bo'ladi.

7^0. Agar $f(x)$ funksiya davrli integrallannuvchi bo'lsa, u holda $\forall a \in R$ uchun ushbu

$$\int_a^{a+T} f(x)dx = \int_0^T f(x)dx$$

tenglik o'rinli bo'ladi.

13.2. Garmonikalar superpozitsiyasi. Berilgan funksiyani Furye trigonimetrik qatoriga yoyish. Fizika va texnika fanlarida uchraydigan eng sodda davriy funksiya moddiy nuqtaning biror F kuch ta'siaridagi tebranma harakatini ifadalovchi ushbu

$$G(t) = A\sin(\omega t + \varphi) \qquad (13.2.1)$$

funksiyadir. Odatda bu funksiyani *garmonika* deb yuritiladi. Bu funksiya uchun $T = \frac{2\pi}{\omega}$ son davr bo'ladi.

Haqiqatan ham,

$$A\sin[\omega(t + \frac{2\pi}{\omega}) + \varphi] = A\sin[(\omega t + \varphi) + 2\pi] = A\sin(\omega t + \varphi), \ G(t+T) = G(t)$$

bu funksiyaning eng katta qiymati $|G(t)| = A > 0$ bo'lib, u tebranuvchi moddiy nuqtaning boshlang'ich holatidan, eng katta uzoqlashishni ifodalaydi. Bu uzoqlashish mexanikada tebranish *amplitudasi* deyiladi. $\omega = \frac{2\pi}{T}$ son 2π vaqt ichidagi tebranishlar sonini ifodalaydi va u vaqtda *chastota* deb ataldi, φ son moddiy nuqtaning ($t = 0$ da) boshlang'ich $G(0) = A\sin\varphi$ holatini aniqlovchi miqdor bo'lgani uchun uni *boshlang'ich fazo* deyiladi.

(13.2.1) dan xususiy holda:1) $\varphi = 0, A = 1, \omega = 1$ bo'lsa, $G(t) = \cos t$ funksiya hosil bo'ladi,uning grafigi odatda kosinusoida deb ataladi(13.1-chizma).

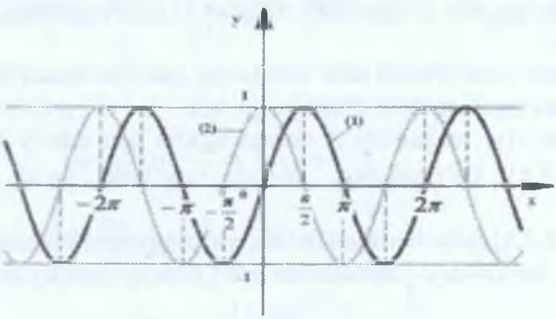
Endi ushbu

$$A_k \sin(\frac{2\pi k}{T}x + \varphi_k), \ k = 1,2, ..., -\infty < x < \infty \qquad (13.2.2)$$

garmonikalar ketma-ketligini qaraylik, bunda $T_k = \frac{T}{k}$ son k-garmonikaning davri, $T = kT_k$ son esa, (13.2.2) garmonikalar ketma ketligining umumiy davri, $\lambda_k = \frac{2\pi k}{T}$ (k = 1,2,...) k-garmonikaning chastotasi. (13.2.2) garmonikalar ketma-ketligining chastotasi $\frac{2\pi}{T}$ songa karrali bo'ladi. (13.2.2) dan

$$f_N(x) = A_0 + \sum_{k=1}^{N} A_k \sin(\frac{2\pi k}{T} x + \varphi_k) \qquad (13.2.3)$$

ifodani tuzamiz. Bu funksiya ham T davrga ega, chunki T son garmonikalar ketma ketligining umumiy davri, A_0 ni 3^0-ga asosan, T davrga ega bo'lgan funksiya deb qarash mumkin.



13.1-chizma. (1) $- y = \sin x$, (2) $- y = \cos x$.

Xuddi shunday yaqinlashuvchi ushbu

$$f(x) = A_0 + \sum_{k=1}^{\infty} A_k \sin(\frac{2\pi k}{T} x + \varphi_k) \qquad (13.2.4)$$

qatorning yig'indsi ham T davrga ega bo'lgan davriy funksiya bo'ladi.

Haqiqatan ham, (13.2.4) qator yaqinlashuvchi bo'lsa, u yig'indiga ega bo'ladi, uni biz $f(x)$ deb belgilaylik

$$f(x+T) = A_0 + \sum_{k=1}^{\infty} A_k \sin[\frac{2\pi k}{T}(x+T) + \varphi_k] =$$

$$= A_0 + \sum_{k=1}^{\infty} A_k \sin[(\frac{2\pi k}{T} x + \varphi_k) + 2\pi k] = A_0 + \sum_{k=1}^{\infty} A_k \sin(\frac{2\pi k}{T} x + \varphi_k) = f(x).$$

Agar ushbu

$$A_k \sin(\frac{2\pi k}{T}x+\varphi_k)=A_k \sin\varphi_k \cdot \cos\frac{2\pi k}{T}x + A_k \cos\varphi_k \cdot \sin\frac{2\pi k}{T}x$$

tenglikda $a_k = A_k \sin\varphi_k$, $b_k = A_k \cos\varphi_k$, $T=2l$, $A_0=\frac{a_0}{2}$ deb olsak, u holda (13.2.3) va (13.2.4) ning ko'rinishi quyidagicha bo'ladi:

$$f_N(x)=\frac{a_0}{2}+\sum_{k=1}^{N}(a_k\cos\frac{k\pi}{l}x+b_k\sin\frac{k\pi}{l}x),$$

$$f(x)=\frac{a_0}{2}+\sum_{k=1}^{\infty}(a_k\cos\frac{k\pi}{l}x+b_k\sin\frac{k\pi}{l}x). \qquad (13.2.5)$$

Yuqorida takidlaganimizdek, bu ikki tenglikning o'ng tomonidagi yig'indilarning har biri $2l$ davrlidir. Odatda (13.2.5) qatorga *trigonometrik qator* deyiladi.

Furye qatori nazariyasida ham funksional qatorlar nazariyasidagi singari quyidagi uchta masala muhim ahamiyatga ega:

1) berilgan $f(x)$ funksiyani $2l$ davrga ega bo'lgan davriy funksiyalardan tuzilgan (13.2.5) ko'rinishdagi cheksiz yig'indi, ya'ni qator orqali ifodalaymiz;

2) agar (13.2.5) qator berilgan bo'lsa, u x ning qaysi kesmadagi qymatlari uchun qanday funksiyaga yaqinlashadi yoki qanday funksiyaning yoyilmasi bo'ladi?

3) agar (13.2.5) tenglik o'rinli bo'lsa, qatorning koeffitsiyentlari a_0, a_k, b_k lar qanday topiladi?

Qo'yilgan masalani yechishdan avval, kelgusida muhim ahmiyatga ega bo'lgan trigonometrik fuunksiyalar sinus va kosinuslardan tuzilgan asosiy trigonometrik funksiyalar sistemasi deb ataluvchi sistemaning ortogonallik xossasi bilan tanishamiz.

Asosiy trigonometrik sistema va uning ortogonalligi

Ravshanki (13.2.5) qatorning hadlari

$$\frac{1}{2},\cos\frac{\pi x}{l},\sin\frac{\pi x}{l},\cos\frac{2\pi x}{l},\sin\frac{2\pi x}{l},\ldots\cos\frac{k\pi x}{l},\sin\frac{k\pi x}{l},\ldots \quad (13.2.6)$$

sistema elementlari orqali tuzilgan. Odatda (13.2.6) sistema asosiy *trigongometrik funksiyalar sistemasi* deyiladi. (13.2.5) asosiy trigonometrik sistema $[-l,l]$ kesmada ortogonal bo‘lgan sistemadir, ya’ni uning o‘zaro teng bo‘lmagan ikkita xar hil hadi ko‘paytmasidan $[-l,l]$ kesma bo‘yicha olingan integral nolga teng; har birining kvadratlaridan $[-l,l]$ kesma bo‘yicha olingan integral noldan farqli;

$$
\left.\begin{aligned}
&\int_{-l}^{l}\frac{1}{2}\cos\frac{k\pi x}{l}dx=\frac{1}{2}\frac{1}{k\pi}\sin\frac{k\pi x}{l}\Bigg|_{x=-l}^{x=l}=0\\
1)\ &\int_{-l}^{l}\frac{1}{2}\sin\frac{k\pi x}{l}dx=\frac{1}{2}\frac{1}{k\pi}\cos\frac{k\pi x}{l}\Bigg|_{x=-l}^{x=l}=\\
&=-\int_{-l}^{l}\frac{1}{2}\cos\frac{k\pi x}{l}dx=-\frac{1}{2}\frac{l}{k\pi}\left[(-1)^k-(-1)^k\right]=0
\end{aligned}\right\} \tag{13.2.7}
$$

$$
2)\ \int_{-l}^{l}\frac{1}{2}\cos\frac{k\pi x}{l}\cos\frac{n\pi x}{l}dx=\frac{1}{2}\int_{-l}^{l}\left[\cos\frac{(k+n)\pi x}{l}+\cos\frac{(k-n)\pi x}{l}\right]dx=0,n\neq k\,. \tag{13.2.8}
$$

Xuddi shunday,

$$
\left.\begin{aligned}
&\int_{-l}^{l}\sin\frac{k\pi x}{l}\sin\frac{n\pi x}{l}dx=0,n\neq k\\
&\int_{-l}^{l}\sin\frac{k\pi x}{l}\cos\frac{n\pi x}{l}dx=0,n\leq k
\end{aligned}\right\} \tag{13.2.9}
$$

$$
\left.\begin{aligned}
&\int_{-l}^{l}\cos^2\frac{k\pi}{l}xdx=\int_{-l}^{l}\frac{1+\cos^2\frac{k\pi}{l}x}{2}dx=l,\\
3)\ &\int_{-l}^{l}\sin^2\frac{k\pi}{l}xdx=\int_{-l}^{l}\frac{1-\cos^2\frac{k\pi}{l}x}{2}dx=l,\\
&\int_{-l}^{l}\left(\frac{1}{2}\right)^2dx=\frac{1}{2}
\end{aligned}\right\} \tag{13.2.10}
$$

Endi (13.2.5) yoyilmaning koeffetsiyentlarini topish masalasi bilan shug‘ullanamiz.

Agar (13.2.5) qator $[-l,l]$ kesmada $f(x)$ ga tekis yoki o‘rtacha yaqinlashuvchi bo‘lsa, u holda uni hadına-had integrallash mumkin bo‘ladi. Bu tasdiq, (13.2.5) ni biror integrallanuvchi funksiyaga ko‘paytirgan holda ham o‘rinli bo‘ladi. (13.2.6) sistemaning ortagonalligini e’tiborga olib, (13.2.5) ning ikkala tomonini $[-l,l]$ kesmada hadma–had integrallab, a_0 koeffetsiyentni topamiz:

$$\int_{-l}^{l} f(x)dx = \frac{a_0}{2}\int_{-l}^{l} dx + \sum_{k=1}^{\infty}[a_k \int_{-l}^{l}\cos\frac{k\pi}{l}xdx + b_k \int_{-l}^{l}\sin\frac{k\pi}{l}xdx] = a_0 \cdot l,$$

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)dx. \qquad (13.2.11)$$

$\cos\frac{k\pi}{l}x$ ning oldidagi koeffetsiyent a_k ni topish uchun (13.2.5) ning ikkala tomonini $\cos\frac{n\pi}{l}x$ ga ko'paytirib, uni x bo'yicha $[-l;l]$ kesmada integrallaymiz ((13.2.7)- (13.2.10) larni etiborga olgan holda):

$$\int_{-l}^{l} f(x)\cos\frac{n\pi}{l}xdx = \frac{a_0}{2}\int_{-l}^{l}\cos\frac{n\pi}{l}xdx +$$

$$+\sum_{k=1}^{\infty} a_k \int_{-l}^{l}\cos\frac{k\pi}{l}\cos\frac{n\pi}{l}xdx + \sum_{k=1}^{\infty} b_k \int_{-l}^{l}\sin\frac{k\pi}{l}x\cos\frac{n\pi}{l}xdx =$$

$$= a_n \int_{-l}^{l}\cos^2\frac{n\pi}{l}xdx = a_n l, \quad a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi}{l}xdx \qquad (13.2.12)$$

ni topamiz.

Xuddi shunday $\sin\frac{k\pi}{l}x$ ning oldidagi koiffetsiyent b_k ni topish uchun ((13.2.7)-(13.2.10) larni e'tiborga olgan holda) (13.2.5) ning ikkala tomonini $\sin\frac{n\pi}{l}x$ ga ko'paytirib, x bo'yicha $[-l;l]$ kesmada integrallab,

$$b_k = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi}{l}xdx \qquad (13.2.13)$$

ni topamiz.

13.2.14-ta'rif. Koeffetsiyentlari (13.2.11),(13.2.12) va (13.2.13) formulalar orqali topiladigan ushbu

$$\frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k\cos\frac{k\pi}{l}x + b_k\sin\frac{k\pi}{l}x) \qquad (13.2.15)$$

trigonometrik qatorga $f(x)$ funksiyaning Furye qatori, a_0, a_k, b_k koyefitsentlarni esa uning *Furye koeffitsiyentlari* deyiladi.

(13.2.11),(13.2.12) va (13.2.13) integrallarning mavjud bo'lishi uchun $f(x)$ funksiyaning $[-l,l]$ kesmada integrallanuvchi bo'lishi yetarli. Shuning uchun har bir $[-l,l]$ kesmada integrrallanuvchi $f(x)$ funksiyaga koeffetsiyentlari (13.2.11),(13.2.12) va (13.2.13) formulalar bilan aniqlanadigan (13.2.15) trigonometrik qatorni mos qo'yish mumkin:

$$f(x) \approx \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos\frac{k\pi}{l}x + b_k \sin\frac{k\pi}{l}x). \qquad (13.2.16)$$

Umuman olganda, $f(x)$ funksiyadan integrallanuvchanligidan tashqari boshqa shart talab qilinmasa (13.2.16) da tenglik ishorasini qo'yib bo'lmaydi.

Agarda $n \to \infty$ da $\int_a^b |f_n(x) - f(x)|^2 dx \to 0$ bo'lsa, $\{f_n(x)\}$ ketma-ketlik $[a,b]$ kesmada $f(x)$ funksiyaga *o'rtacha yaqinlashuvchi* deyiladi.

Endi $f(x)$ ga qanday yetarli shart qo'yilganda (13.2.16) da tenglik ishorasini qo'yish mumkin degan masala bilan shug'ullanamiz. Bu masalani qarashdan avval biror berilgan $f(x)$ funksiyaning (13.2.10) ko'rinishdagi trigonometrik qatorini tuzishga doir bir nechta misollarni qaraymiz.

13.2.17-misol. Ushbu $f(x) = \begin{cases} 0, & -\pi < x < 0, \\ x, & 0 \le x \le \pi \end{cases}$ 2π davrga ega bo'lgan funksiyani Furye qatoriga yoying.

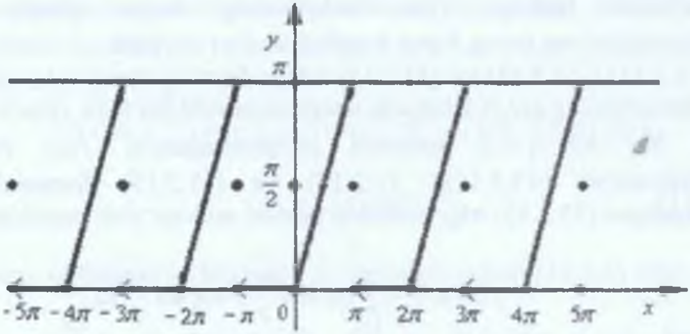
Yechilishi. Berilgan funksiyaning Furye koeffitsiyentlarini topamiz:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\int_0^{\pi} xdx = \frac{1}{\pi}\cdot\frac{x^2}{2}\Big|_0^{\pi} = \frac{\pi}{2},$$

$$a_n = \frac{1}{\pi}\int_0^{\pi} x\cos nxdx = \begin{vmatrix} u = x, dv = \cos nxdx \\ du = dx, v = \frac{1}{n}\sin nx \end{vmatrix} = \frac{1}{\pi}(\frac{x}{n}\sin nx\Big|_0^{\pi} - \int_0^{\pi}\frac{1}{n}\sin nxdx) =$$

$$= \frac{1}{\pi}\frac{1}{n^2}\cos nx\pi\Big|_0^{\pi} = \frac{1}{\pi n^2}((-1)^n - 1), \quad (n \in N),$$

$$b_n = \frac{1}{\pi}\int_0^{\pi} x\sin nxdx = \frac{1}{\pi}(-\frac{x}{n}\cos nx\Big|_0^{\pi} - \frac{1}{n^2}\sin nx\Big|_0^{\pi}) = \cdot\frac{1}{\pi}\frac{\pi}{n}\cos n\pi = \frac{(-1)^{n-1}}{n}, \quad (n \in N)$$



13.2-chizma.

Topilgan koeffitsiyentlarni (13.2.15) qatorga qo'yib

$$\frac{\pi}{4}+\sum_{n=1}^{\infty}\left(-\frac{2}{\pi(2n-1)^2}\cos((2n-1)x)+\frac{(-1)^{n-1}}{n}\sin nx\right)$$

topamiz. Bu qator $x \neq (2n-1)\pi$ nuqtalarda 2π davrli funksiyaga yaqinlashadi. $x=(2n-1)\pi$ nuqtalarda qatorning yig'indisi $\frac{\pi+0}{2}=\frac{\pi}{2}$ ga teng bo'ladi (13.2-chizma).

13.3. Toq va juft funksiyalarni Furye qatoriga yoyish. $f(x)$ funksiya $[-l,l]$ kesmada aniqlangan bo'lsin. $f(x)$ funksiya $\forall x\in[-l,l]$ uchun $f(-x)=f(x)$ shartni qanoatlantirsa, u holda $f(x)$ *juft* funksiya deyiladi, agar $\forall x\in[-l,l]$ uchun $f(-x)=-f(x)$ tenglik o'rinli bo'lsa, $f(x)$ *toq* funksiya deyiladi. Bu ta'riflardan ko'rinadiki, juft funksiyaning grafigi esa, ordinata o'qiga nisbatan semmetrik bo'ladi, toq funksiyaning grafigi esa, koordinata boshiga nisbatan simmetrik bo'ladi.

Agar $f(x)$ funksiya $[-l,l]$ kesmada berilgan ixtiyoriy funksiya bo'lsa, u holda

$$f_1(x)=\frac{f(x)+f(-x)}{2},\qquad f_2(x)=\frac{f(x)-f(-x)}{2}$$

ko'rinishda tuzilgan funksiyalarning birinchisi juft, ikkinchisi esa, toq funksiya bo'ladi.

$[-l,l]$ kesmada berilgan harqanday $f(x)$ funksiyani toq va juft funksiyalar yig'indisi shaklida tasvirlash mumkin, ya'ni

$$f(x)=f_1(x)+f_2(x), \quad x\in[-l,l].$$

Agar $f(x)$ funksiya $[-l,l]$ da integrallanuvchi bo'lsa,u holda

$$\int_{-l}^{l} f(x)dx=\int_{-l}^{0} f(x)dx+\int_{0}^{l} f(x)dx=\int_{0}^{l}[f(x)+f(-x)]dx. \qquad (13.3.1)$$

bunda $\int_{-l}^{0} f(x)dx=-\int_{-l}^{0} f(-x)dx+\int_{0}^{l} f(-x)dx$ (13.3.1) dan

$$\int_{-l}^{l} f(x)dx=\begin{cases} 2\int_{0}^{l} f(x)dx, \text{ agar } f(x) \text{ -juft funksiya bo'lsa,} \\ 0, \text{ agar } f(x)\text{-toq funksiya bo'lsa} \end{cases}$$

$$\frac{1}{2}, \cos\frac{\pi x}{l}, \cos\frac{2\pi x}{l},\ldots\cos\frac{k\pi x}{l},\ldots\text{-juft, } \sin\frac{\pi x}{l}, \sin\frac{2\pi x}{l},\ldots\sin\frac{k\pi x}{l},\ldots$$

-toq funksiyalar.

Agar $f(x)$ $[-l,l]$ kesmada juft funksiya bo'lsa, u holda uning Furye qatori

$$\frac{a_0}{2}+\sum_{k=1}^{\infty} a_k\cos\frac{k\pi}{l}x$$

ko'rinishda bo'ladi.

Agarda $f(x)$ $[-l,l]$ kesmada toq funksiya bo'lsa, uning Furye qatori

$$\sum_{k=1}^{\infty} b_k\sin\frac{k\pi}{l}x$$

ko'rinishda bo'ladi.

Haqiqatan ham, agar $f(x)$ juft funksiya bo'lsa, u holda $f(x)\cos\frac{k\pi}{l}x$- ko'paytma juft, $f(x)\sin\frac{k\pi}{l}x$- ko'paytma toq funksiya bo'ladi.Shuning uchun

$$
\left.\begin{aligned}
a_0 &= \frac{1}{l}\int_{-l}^{l} f(\xi)d\xi = \frac{2}{l}\int_{0}^{l} f(\xi)d\xi, \\
a_k &= \frac{1}{l}\int_{-l}^{l} f(\xi)\cos\frac{k\pi}{l}\xi d\xi = \frac{2}{l}\int_{0}^{l} f(\xi)\cos\frac{k\pi}{l}\xi d\xi, \\
b_k &= \frac{1}{l}\int_{-l}^{l} f(\xi)\sin\frac{k\pi}{l}\xi d\xi = 0
\end{aligned}\right|
$$

Agar $f(x)$-toq funksiya bo'lsa, u holda $f(x)\cos\frac{k\pi}{l}x$-toq, $f(x)\sin\frac{k\pi}{l}x$-juft funksiya bo'ladi. Shuning uchun

$$
a_0 = \frac{1}{l}\int_{-l}^{l} f(\xi)d\xi = 0,\ a_k = \frac{1}{l}\int_{-l}^{l} f(\xi)\cos\frac{k\pi}{l}\xi d\xi = 0,
$$

$$
b_k = \frac{1}{l}\int_{-l}^{l} f(\xi)\sin\frac{k\pi}{l}\xi d\xi = \frac{2}{l}\int_{0}^{l} f(\xi)\sin\frac{k\pi}{l}\xi d\xi .
$$

13.3.2-misol. $f(x) = x$ funksiya $[-l,l]$ oraliqda Furye trigonometrik qatoriga yoyilsin.

Yechilishi. $f(x) = x$ toq funksiya bo'lgani uchun, $a_0 = 0,\ a_n = 0$,

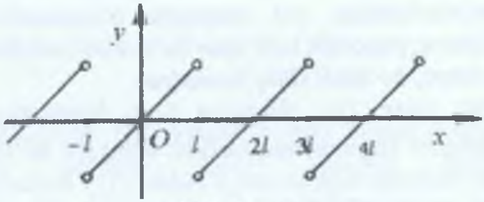
$$
b_n = \frac{2}{l}\int_{0}^{l} f(x)\sin\frac{n\pi}{l}x dx =
$$

$$
= \frac{2}{l}\left\{\left(-\frac{xl}{n\pi}\cos\frac{n\pi}{l}x\right)\Bigg|_{x=0}^{x=l} + \frac{1}{n\pi}\int_{0}^{l}\cos\frac{n\pi}{l}x dx\right\} = \frac{2l}{n\pi}(-1)^{n+1}.
$$

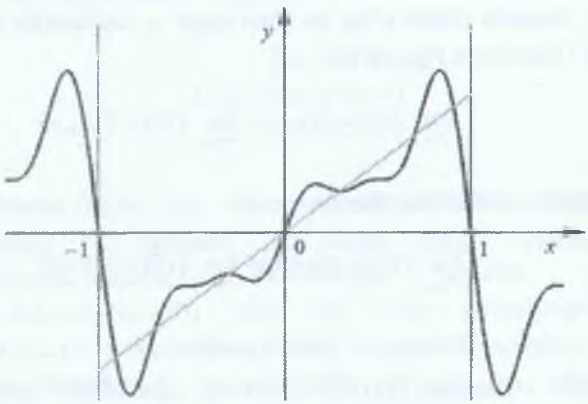
Yuqoridagi asosiy teoremaga asosan,

$$
S(x) = \frac{2l}{n\pi}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}\sin\frac{k\pi x}{l} = \begin{cases} \dfrac{f(x-0)+f(x+0)}{2} = x, -l < x < l \\ \dfrac{f(-l+0)+f(l-0)}{2} = \dfrac{-l+l}{2} = 0, x = \pm l \end{cases}
$$

$f(x)$ va $S(x)$ ning grafigi 13.3-chizmada berilgan. $S(x) - 2l$ davrli funksiya. $-l < x < l$ da $S(x) = x$. $x = (2k+1)l$ $(k = 0, \pm 1, \pm 2, ...)$ nuqtalarda uzilishga ega bo'lib, bu nuqtalardagi qiymati nolga teng, chunki bu nuqtalardagi o'ng va chap limitik qiymatlari nolga teng. 13.4-chizmada $S_5(x)=\frac{2l}{\pi}\sum_{k=1}^{5}\frac{(-1)^{k+1}}{k}\sin\frac{k\pi x}{l}$ ning grafigi berilgan.



13.3-chizma.



13.4-chizma.

13.4. Funksiyani $[-\pi,\pi]$ kesmada Furye qatoriga yoyish. Agar $[-\pi,\pi]$ kesmada berilgan $f(x)$ funksiyani trigonometrik Furye qatoriga yoyish talab qilingan bo'lsa, u holda (13.2.11),(13.2.12) va (13.2.13) formulalarda $l=\pi$ deb, ushbu Furye qatori koeffetsiyentlari va qatoriga

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(\xi)d\xi,\ a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(\xi)\cos\frac{k\pi}{l}\xi d\xi,\ b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(\xi)\sin\frac{k\pi}{l}\xi d\xi,$$

$$f(x)\approx\frac{a_0}{2}+\sum_{k=1}^{\infty}(a_k\cos\frac{k\pi}{l}x+b_k\sin\frac{k\pi}{l}x) \qquad (13.4.1)$$

ega bo‘lamiz. $f(x)$ funksiya umumiy holda $[-\pi,\pi]$ da berilganda (13.4.1) ko‘rinishga $x'=\frac{\pi x}{l}, \varphi(x')=f(\frac{lx}{\pi}), -\pi \le x' \le \pi$ almashtirsh orqali keladi.

13.5.Trigonometrik Furye qatorining yaqinlashishi to‘g‘risida asosiy teorema. Biz endi berilgan funksiya uchun tuzilgan Furye qatorining o‘sha funksiyaga yaqinlashishiga oid masalani o‘rganamiz. Buning uchun matematik analizning yuqorida ko‘rilgan ba’zi tushunchalarini eslab o‘tamiz.

Bo‘lakli uzliksiz, bo‘lakli silliq funksiya.

13.5.1-ta’rif. Agar $f(x)$ funksiya $[a,b]$ kesmaning chekli sondagi nuqtalaridan tashqari hamma nuqtalarida uzluksiz bo‘lib, chekli sondagi nuqtalarida 1-tur uzilishga ega bo‘lsa, u holda $f(x)$ funksiya $[a,b]$ da *bo‘lakli uzluksiz* deyiladi.

13.5.1-ta’rifdan ko‘rinadiki, bunday funksiya $[a,b]$ kesmaga qarashli ixtiyoriy x_0 nuqtada chekli o‘ng va chap (agar x_0 uzliksizlik nuqtasi bo‘lsa o‘zaro teng) limitlarga ega, ya’ni

$$\lim_{x\to x_{0-0}} f(x)=f(x_{0-0}),\ \lim_{x\to x_{0+0}} f(x)=f(x_{0+0})$$

Shuningdek, chetki nuqtalarida

$$\lim_{x\to x_{a+0}} f(x)=f(a+0),\ \lim_{x\to b-0} f(x)=f(b-0)$$

ega bo‘ladi. Masalan, $f_1(x)=\begin{cases}3, -1\le x<0,\\ x^2, 0\le x<2,\\ x^2+1,\ 2\le x\le 3\end{cases}$ funksiya uchta uzluksiz bo‘lakdan iborat bo‘lib, $x=0$ va $x=2$ nuqtalar $f_1(x)$ uchun birinchi tur uzilish nuqtalari bo‘lib,

$$\lim_{x\to -0} f_1(x)=3,\ \lim_{x\to +0} f_1(x)=0,\ \lim_{x\to 2-0} f_1(x)=4,\ \lim_{x\to 2+0} f_1(x)=5.$$

Bundan tashqari, $[-1,3]$ kesmaning chetlarida $\lim\limits_{x\to -1-0} f_1(x)=3,\ \lim\limits_{x\to 3-0} f_1(x)=10$, $[-1,3]$ kesmaning qolgan barcha nuqtalarida $\lim\limits_{x\to x_0} f_1(x)=f(x_0)$ bo‘ladi.

13.5.2-ta’rif. Agar $[a,b]$ da bo‘lakli uzluksiz $f(x)$ funksiya shu kesmaning chekli sondagi nuqtalaridan boshqa barcha nuqtalarida uzliksiz

$f'(x)$ hosilaga ega bo‘lib, shu kesmaning faqat chekli sondagi nuqtalarida chekli o‘ng va chap hosilalarga ega bo‘lsa, u holda $f(x)$ funksiya $[a,b]$ da *bo‘lakli silliq* deyiladi.

Bu 13.5.2-ta'rifdan ravshanki, $[a,b]$ kesmaga qarashli ixtiyoriy x_0 nuqtada

$$\lim_{x\to x_{0-0}} f'(x)=f'(x_0-0),\ \lim_{x\to x_{0-0}} f'(x)=f'(x_0+0)$$

chekli $f'(x_0-0)=f'(x_0+0)$ ikki yoqlama yoki chekli sondagi nuqtalarda $f'(x_0-0)\neq f'(x_0+0)$ bo‘lishi zarur. Bundan tashqari chekli $\lim_{x\to x_{a+0}} f'(x)=f'(a+0)$, $\lim_{x\to b-0} f'(x)=f'(b-0)$ hosilalarag ega bo‘ladi. Yuqoridagi ko'rilgan $f_1(x)$ bo‘lakli silliq funksiyadir.

$$f_1^{/}(x)=\begin{cases}0,-1\le x<0\\ 2x,\ 0\le x<2\\ 2x, 2\le x\le 3\end{cases}$$

13.5.3-eslatma. Agar $f(x)$ funksiya $[a,b]$ da bo‘lakli silliq funksiya bo‘lsa, u holda bu kesmani har doim chekli shunday k ta $[a_0,a_1],[a_1,a_2],[a_2,a_3],...,[a_{k-1},a_k]$ kesmalarga ajratish mumkinki $(a=a_0<a_1<a_2<a_3<...<a_k=b)$, $f(x)$ va $f'(x)$ funksiyalar istalgan $[a_i,a_{i+1}]$ $(i=0,1,2,...,k-1)$ kesmaning ichki nuqtalarida uzliksiz bo‘lib, x o‘ngdan a_i ga ,chapadan a_{i+1} ga intilganda $f(x)$ va $f'(x)$ mos ravishda quyidagi chekli $f(a_i+0), f'(a_i+0)$ va $f(a_{i+1}-0), f'(a_{i+1}-0)$ limitlarga intiladi.

Agar

$$f(a_i)=f(a_i+0), f(a_{i+1})=f(a_{i+1}-0)\ \text{va}\ f'(a_i)=f(a_i+0),$$
$$f'(a_{i+1})=f(a_{i+1}-0)$$

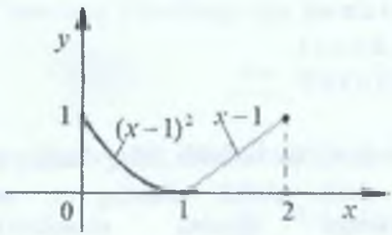
deb qabul qilsak, u holda $f(x)$ va $f'(x)$ funksiyalar $[a_i,a_{i+1}]$ $(i=0,1,2,...,k-1)$ kesmalarning har birida uzliksiz bo‘ladi. $y=f(x)$ funksiya, $[a,b]$ da uzliksiz bo‘lib, uning $f'(x)$ hosilasi o‘sha kesmada bo‘lakli sillliq bo‘lishi ham mumkin. Masalan,

$$f(x)=\begin{cases}(x-1)^2, & 0\le x<1,\\ x-1, & 1\le x\le 2\end{cases}$$

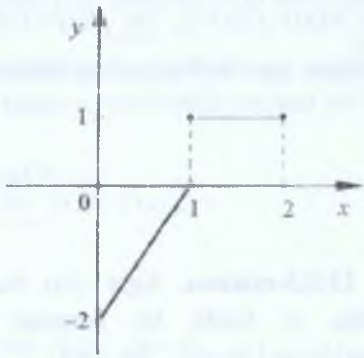
funksiya [0,2] da uzliksiz (13.5-chizma), uning

$$f'(x)=\begin{cases}2(x-1), & 0\le x<1\\ 1, & 1\le x\le 2\end{cases}$$

hosilasi esa, ikkita silliq bo'lakdan iborat (13.6-chizma)



13.5-chizma



13.6 -chizma

13.2-§ . Furye qatorining yaqinlashuvchiligi

13.6. Furye trigonometrik qatorining yaqinlashishiga doir asosiy teorema (Dirixle teoremasi)

13.6.1-teorema. Agar $f(x)$ funksiya $[-l,l]$ kesmada bo'lakli silliq bo'lsa, u holda bu funksiyaning

$$S(x)\approx\frac{a_0}{2}+\sum_{k=1}^{\infty}\left(a_k\cos\frac{k\pi}{l}x+b_k\sin\frac{k\pi}{l}x\right)$$

Furye trigonometrik qatori $[-l,l]$ kesmaga qarashli istalgan x uchun yaqinlashuvchi va uning yig'indisi $S(x)$ uchun quyidagilar o'rinlidir:

1) barcha $x \in (-l, l)$ uchun $S(x) = \frac{f(x-0)+f(x+0)}{2}$ bo'lib, agar x nuqta $f(x)$ ning uzluksizlik nuqtasi bo'lsa, u holda $f(a-0) = f(x+0) = f(x)$, demak, $S(x) = f(x)$ bo'ladi;

2) kesmaning chegaraviy nuqtalarida esa, qator yig'indisi ushbu $S(-l) = S(l) = \frac{f(-l+0)+f(+l-0)}{2}$ tenglik bilan aniqlanadi.

Bu **13.6.1**-teoremani isbotlash uchun zarur bo'lgan asosiy lemmani isbotsiz keltiramiz.

13.6.2-lemma (Asosiy). Agar $f(x)$ funksiya $[a,b]$ kesmada bo'lakli silliq bo'lsa, u holda

$$\left.\begin{aligned} \lim_{\alpha\to 0}\int_a^b f'(x)\sin\alpha x dx = 0, \\ \lim_{\alpha\to 0}\int_a^b f(x)\cos\alpha x dx = 0 \end{aligned}\right\} \tag{13.6.3}$$

bo'ladi.

13.6.4-eslatma. Lemmani yuqorida ko'rilgan funksiyalar sinfidan kengroq sinfdagi funksiyalar uchun ham isbotlash mumkin. Masalan, $[a,b]$ da absolyut integrallanuvchi, ya'ni $\int_a^b |f(x)|dx < \infty$ shartni qanoatlantiruvchi funksiyalar sinfi uchun ham lemma o'rinlidir.

13.6.5-eslatma. Agar $f(x)$ funksiya $2l$ davrli bo'lib, uzunligi $2l$ ga teng bo'lgan kesmada bo'lakli silliq bo'lsa, u holda uning Furye koeffetsiyentlari ,ya'ni a_n va b_n lar n chesizlikga intilganda nolga intiladi, ya'ni

$$\lim_{n\to\infty} a_n = 0, \lim_{n\to\infty} b_n = 0. \tag{13.6.6}$$

Haqiqatan ham (13.2.12) va (13.2.13) formulalarga ko'ra,

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi}{l}x dx\ ,\ b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi}{l}x dx$$

larda $\alpha = \frac{n\pi}{l}$ deb olsak, $n\to\infty$ da $\alpha\to\infty$. U holda asosiy lemmaga asosan, (13.6.9) ning o'rinli ekanligi kelib chiqadi.

Asosiy* 13.6.1-*teoremaning isboti.

Avvalo $f(x)$ funksiyani $2l$ davrli deb, butun OX o'qiga davriy davom yettiramiz. So'ngra $\forall x \in (-\infty,\infty)$ uchun

$$\lim_{n\to+\infty} S_n(x) = \frac{f(x-0)+f(x+0)}{2}$$

ekanligini isbotlaymiz, bunda

$$S_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n}\left(a_k \cos\frac{k\pi}{l}x + b_k \sin\frac{k\pi}{l}x\right) \qquad (13.6.10)$$

$f(x)$ funksiya Furye qatorining $[-l,l]$ kesmadagi qismiy yig'indisi. (13.6.10) ga a_0, a_k, b_k larning (13.2.11),(13.2.12),(13.2.13) lardagi ifodalarni olib kelib qo'yamiz:

$$S_n(x) = \frac{1}{2l}\int_{-l}^{l} f(\xi)d\xi + \frac{1}{l}\sum_{k=1}^{n}\int_{-l}^{l} f(\xi)[\cos\frac{k\pi\xi}{l}x\cos\frac{k\pi}{l}x + \sin\frac{k\pi\xi}{l}x\sin\frac{k\pi}{l}x]d\xi =$$
$$= \frac{1}{l}\int_{-l}^{l} f(\xi)[\frac{1}{2} + \sum_{k=1}^{n}\cos\frac{k\pi(\xi-\alpha)}{l}]d\xi = \frac{1}{l}\int_{-l-x}^{l-x} f(x+z)[\frac{1}{2} + \sum_{k=1}^{n}\cos\frac{k\pi z}{l}]dz. \qquad (13.6.11)$$

$$\sigma_n(z) = \frac{1}{2} + \sum_{k=1}^{n}\cos\frac{k\pi z}{l} \qquad (13.6.12)$$

deb belgilab, uning ko'rinishini quyidagi holga keltiramiz: (13.6.12) ning ikkala tomonini $2\sin\frac{\pi \cdot z}{2l}$ ga ko'paytiramiz, natijada

$$2\sigma_n(z)\sin\frac{\pi z}{2l} = \sin\frac{\pi z}{2l} + \sum_{k=1}^{n} 2\sin\frac{\pi z}{2l}\sin\frac{k\pi z}{l} =$$
$$= \sin\frac{\pi z}{2l} + \sum_{k=1}^{n}[\sin(k+\frac{1}{2})\frac{\pi z}{l} - \sin(k-\frac{1}{2})\frac{\pi z}{l}] = \sin(n+\frac{1}{2})\frac{\pi z}{l},$$

bundan

$$\sigma_n(z) = \frac{\sin(n+\frac{1}{2})\frac{\pi z}{l}}{2\sin\frac{\pi z}{2l}} = \frac{1}{2} + \sum_{k=1}^{n}\cos\frac{k\pi z}{l}\ . \qquad (13.6.13)$$

(13.6.13) ni (13.6.11) ga qo'yish natijasida Furye qatori qismiy yig'indismi uchun:

$$S_n(x)=\frac{1}{l}\int_{-l-x}^{l-x} f(x+z)\frac{\sin(n+\frac{1}{2})\frac{\pi z}{l}}{2\sin\frac{\pi z}{2l}}dz$$

ifodani hosil qilamiz. $f(x)$ funksiya $[-l,l]$ kesmadan tashqariga $2l$ davr bilan davriy davom yettirilgan davriy funksiya. (13.6.13) ga asosan, $\dfrac{\sin(n+\frac{1}{2})\frac{\pi z}{l}}{2\sin\frac{\pi z}{2l}}$ funksiya ham $2l$ davrli funksiya, u holda $f(x+z)\dfrac{\sin(n+\frac{1}{2})\frac{\pi z}{l}}{2\sin\frac{\pi z}{2l}}dz$ funksiya ham $2l$ davrli funksiya. Shunimng uchun bu ko'paytmadan z bo'yicha $2l$ uzunlikka ega bo'lgan ixtiyoriy iraliq bo'yicha olingan integralning qiymati bir xil bo'ladi, ya'ni

$$S_n(x)=\frac{1}{l}\int_{-l}^{l} f(x+z)\frac{\sin(n+\frac{1}{2})\frac{\pi z}{l}}{2\sin\frac{\pi z}{2l}}dz. \qquad (13.6.14)$$

(13.6.14) ning o'ng tomonidagi integralga *Dirixle integrali* deyiladi.

(13.6.13) ni $\frac{1}{l}$ ga ko'paytirib, z bo'yicha $-l$ dan l gacha integrallaymiz, natijada

$$\frac{1}{l}\int_{-l}^{l}\frac{\sin(n+\frac{1}{2})\frac{\pi z}{l}}{2\sin\frac{\pi z}{2l}}dz=\frac{1}{2l}\int_{-l}^{l}dz=1$$

ga ega bo'lamiz, chunki $\int_{-l}^{l}\cos\frac{k\pi}{l}zdz=0,\ k=1,2,\ldots$. $\dfrac{\sin(n+\frac{1}{2})\frac{\pi z}{l}}{2\sin\frac{\pi z}{2l}}$ funksiya juf funksiya bo'lganligi uchun

$$\frac{1}{l}\int_{-l}^{0}\frac{\sin(n+\frac{1}{2})\frac{\pi z}{l}}{2\sin\frac{\pi z}{2l}}dz=\frac{1}{2},\quad \frac{1}{l}\int_{0}^{l}\frac{\sin(n+\frac{1}{2})\frac{\pi z}{l}}{2\sin\frac{\pi z}{2l}}dz=\frac{1}{2}. \tag{13.6.15}$$

(13.6.15) tengliklarning birinchisini $f(x-0)$ ga, ikkinchisini $f(x+0)$ ga ko'paytirib qo'shamiz, natijada

$$\frac{f(x-0)+f(x+0)}{2}=$$
$$=\frac{1}{l}\int_{-l}^{0}f(x-0)\frac{\sin(n+\frac{1}{2})\frac{\pi z}{l}}{2\sin\frac{\pi z}{2l}}dz+\frac{1}{l}\int_{0}^{l}f(x+0)\frac{\sin(n+\frac{1}{2})\frac{\pi z}{l}}{2\sin\frac{\pi z}{2l}}dz. \tag{13.6.16}$$

(13.6.14) dan (13.6.16) ni ayiramiz:

$$S_n(x)-\frac{f(x-0)+f(x+0)}{2}=\frac{1}{l}\int_{-l}^{0}[f(x+z)-f(x-0)]\frac{\sin(n+\frac{1}{2})\frac{\pi z}{l}}{2\sin\frac{\pi z}{2l}}dz+$$
$$+\frac{1}{l}\int_{0}^{l}[f(x+z)-f(x+0)]\frac{\sin(n+\frac{1}{2})\frac{\pi z}{l}}{2\sin\frac{\pi z}{2l}}dz. \tag{13.6.17}$$

Endi (13.6.17) ning o'ng tomondagi ikkala integralning ham $n\to\infty$ da nolga intilishini ko'rsatamiz. Masalan, ikkinchisining nolga intilishini ko'rsatamiz, birinchisining ham nolga intilishi shunday ko'rsatiladi. Buning uchun baholanayotgan integralni I_n deb belgilab uni quyidagicha yozib olamiz:

$$I_n=\frac{1}{l}\int_{0}^{\delta}\frac{f(x+z)-f(x+0)}{z}\cdot\frac{\frac{\pi z}{2l}}{\sin\frac{\pi z}{2l}}\sin(n+\frac{1}{2})\frac{\pi z}{2l}dz+$$
$$+\frac{1}{l}\int_{\delta}^{l}\frac{f(x+z)-f(x+0)}{\sin\frac{\pi z}{2l}}\cdot\sin(n+\frac{1}{2})\frac{\pi z}{2l}dz=I_n'+I_n'' \tag{13.6.18}$$

$(0<\delta<l)$ bo'lsin. $\varepsilon>0$ berilgan. Berilagan ε ga ko'ra δ ni tanlash bilan (13.6.18) dagi birinchi integralni hamma $n=1,2,...$ uchun $\frac{\varepsilon}{2}$ dan kichik yekanligini ko'rsatamiz. Shartga ko'ra $\lim\limits_{z\to 0+0}\frac{f(x+z)-f(x+0)}{z}=f'(x+0)$ $(0<z<\delta)$ bo'lgani uchun

$$\left|\frac{f(x+z)-f(x+0)}{z}\right|<\left|f'(x+0)\right|+1.$$

$\dfrac{\frac{\pi z}{2l}}{\sin\frac{\pi z}{2l}}\xrightarrow[z\to 0]{}1$ bo'lagni uchun, istalgancha kichik $\delta>0$ larda $\forall z\in(0,2\delta)$ va

$n=1,2,\ldots$ uchun $1<\dfrac{\frac{\pi z}{2l}}{\sin\frac{\pi z}{2l}}<2$ bo'ladi. $\forall z$ va $n=1,2,\ldots$ uchun $\left|\sin(n+\frac{1}{2})\frac{\pi z}{l}\right|\le 1.$

Shunday qilib, hamma z va $n=1,2,\ldots$ uchun

$$\left|I_n'\right|\le\frac{1}{\pi}\int_0^{\delta}\left|\frac{f(x+z)-f(x+0)}{2}\right|\cdot\left|\frac{\frac{\pi z}{2l}}{\sin\frac{\pi z}{2l}}\right|\cdot\left|\sin(n+\frac{1}{2})\frac{\pi z}{l}\right|dz\le\frac{2\delta}{\pi}[\left|f'(x+0)\right|+1].$$

Berilgan $\varepsilon>0$ son uchun shunday istalgancha kichik $\delta>0$ tanlash mumkinki,

$$\left|I_n'\right|\le\frac{\varepsilon}{2}. \tag{13.6.19}$$

bo'ladi. Endi tanlangan $\delta>0$ ni belgilab (13.6.18) ning ikkinchi integralini qaraymiz:

$$I_n''=\frac{1}{l}\int_{\delta}^{l}\frac{f(x+z)-f(x+0)}{\sin\frac{\pi z}{2l}}\cdot\sin(n+\frac{1}{2})\frac{\pi z}{2l}dz\cdot$$

$\dfrac{f(x+z)-f(x+0)}{\sin\frac{\pi z}{2l}}$ funksiya $z\in[\delta,l]$ $(\delta>0)$ kesmada bo'lakli silliq funksiya, u holda asosiy lemmaga asosan $\lim\limits_{n\to\infty}I_n''=0$ bo'ladi.

Demak, istalgan $\varepsilon>0$ uchun shunday $N(\varepsilon)$ ko'rsatish mumkinki, barcha $n>N(\varepsilon)$ lar uchun

$$\left|I_n''\right| \le \frac{\varepsilon}{2} \qquad (13.6.20)$$

(13.6.19) va (13.6.20) larni hisobga olgan holda,

$$\left|I_n\right| \le \left|I_n'\right| + \left|I_n''\right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon .$$

Shunday qilib, $\lim\limits_{n\to\infty} I_n = 0$ yeknaligi kelib chiqadi, xuddi shunga o‘xshash $n\to\infty$ da (13.6.17) ning birinchi integralining nolga intilishi ko‘rsatiladi. Nihoyat, (13.6.17) dagi ikki integralning nolga intilishidan

$$\lim_{n\to\infty} S_n(x) = \frac{f(x-0)+f(x+0)}{2} \qquad (13.6.21)$$

kelib chiqadi. Ta’kidlaymizki, $f(x)$ funksiya $[-l,l]$ kesmadan tashqariga ($2l$ davrli) davriy davom yettirilgan bo‘lganligi uchun

$$f(l+0) = f(-l+0), \quad f(-l-0) = f(l-0)$$

bo‘ladi. (13.6.17) da avvalo $x=-l$, so‘ngra $x=l$ desak, u holda (13.6.21) ni ye’tiborga olgan holda

$$\lim_{n\to\infty} S_n(-l) = \lim_{n\to\infty} S_n(l) = \frac{f(-l+0)+f(l-0)}{2} . \blacksquare$$

13.7. $[0,l]$ **da berilgan funksiyani faqat kosinuslar yoki sinuslar bo‘yicha Furye qatoriga yoyish.** $f(x)$ funksiya $[0,l]$ da aniqlangan bo‘lib, u shu kesmada bo‘lakli uzluksiz 206rtho‘lakli silliq bo‘lsin. Uni $[-l,0]$ kesmaqa har xil davom yettirish mumkin, xususiy holda:1) juft ; 2) toq davom ettirish mumkin.

1) -holda $[-\pi,\pi]$ da juft funksiya hosil bo‘ladi. Uning uchun

$$a_0 = \frac{2}{l}\int_0^l f(\xi)d\xi, \quad a_k = \frac{2}{l}\int_0^l f(\xi)\cos\frac{k\pi}{l}\xi d\xi, \quad b_k = 0 \qquad (13.7.1)$$

bo‘ladi, $[-\pi,\pi]$ dagi Furye qatori

$$f(x) \approx \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos\frac{k\pi}{l}\xi \qquad (13.7.2)$$

ko'rinishda bo'ladi.

2)-holda $[-\pi,\pi]$ da toq funksiya hosil bo'lib, uning Furye koeffetsiyentlari,

$$a_0 = 0,\ a_k = 0,\ b_k = \frac{2}{l}\int_0^l f(\xi)\sin\frac{k\pi}{l}\xi d\xi\,. \qquad (13.7.3)$$

Furye qatori esa,

$$f(x) \approx \sum_{k=1}^{\infty} b_k \sin\frac{k\pi}{l}\xi \qquad (13.7.4)$$

ko'rinishda bo'ladi. $(0,l)$ kesmada har ikkala (13.7.2) va (13.7.4) qatorlar $f(x)$ ga yaqinlashadi ($f(x)$ ning uzluksizlik nuqtalarida).

Shunday qilib, $0 \le x \le l$ kesmada berilgan har bir bo'lakli silliq $f(x)$ funksiyani, hoxish bo'yicha faqat kosinuslar bo'yicha koeffetsiyentalari (13.7.1) formula bilan aniqlanadigan (13.7.2) qatorga yoyish yoki faqat sinus bo'yicha koeffetsiyentalari (13.7.3) formula bilan aniqlanadigan (13.7.4) qatorga yoyish mumkin ekan.

13.7.5-misol. $f(x) = x$ funksiyani $[0,l]$ oraliqda sinuslar bo'yicha Furye qatoriga yoying.

Yechilishi. Shartga ko'ra berilgan funksiyani, toqlik qonuniga ko'ra, $[-l,0]$ ga davriy davom ettiramiz, ya'ni $[0,l]$ oraliqda $f(x) = x$ ga teng bo'lgan $2l$ davrli toq davriy funksiyani Furye qatoriga yoyamiz. Hosil bo'lgan funksiya uchun

$$a_k = 0\ (k = 1,2,\ldots),\ b_k = \frac{2}{l}\int_0^l x\sin\frac{k\pi x}{l}dx = -\frac{2l}{k}(-1)^k,\ \ b_n = \frac{2l}{k}(-1)^{k+1}.$$

Shunday qilib $f(x) = x$ funksiyaning $[0,l]$ oraliqda sinuslar bo'yicha yoyilmasi

$$S(x) = \frac{2l}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}\sin\frac{k\pi x}{l} = \begin{cases} \dfrac{f(x-0)+f(x+0)}{2} = x, -l < x < l \\ \dfrac{f(-l+0)+f(l-0)}{2} = \dfrac{-l+l}{2} = 0, x = \pm l \end{cases}$$

ko'rinishda bo'ladi.

13.8. Ortogonal sistemalar bo'yicha Furye qatori. Bessel tengsizligi. $\varphi(x)$ va $\psi(x)$ funksiyalar $[a,b]$ da berilgan va oraliqda ular integrallanuvchi bo'lsin.

13.8.1-ta'rif. Agar

$$\int_a^b \varphi(x)\psi(x)dx = 0$$

bo'lsa, u holda $\varphi(x)$ va $\psi(x)$ funksiyalar $[a,b]$ da *208rthogonal* deyiladi.

$\varphi_1(x), \varphi_2(x), \varphi_3(x), \dots \varphi_n(x), \dots$ funksiyalar sistemasi $[a,b]$ kesmada integarllanuvchi funksiyalar bo'lsin.

13.8.2-ta'rif. Agar

$$\int_a^b \varphi_i(x)\psi_k(x)dx = \begin{cases} 0, & agar\ i \neq k\ bo'\lg anda, \\ > 0, & agar\ i = k\ bo'\lg anda \end{cases} \tag{13.8.3}$$

bo'lsa, u holda bu funksiyalar sistemasi $[a,b]$ kesmada *ortogonal sistema* deyiladi.

Misollar:

1) $$\frac{1}{2}, \cos\frac{\pi x}{l}, \sin\frac{\pi x}{l}, \cos\frac{2\pi x}{l}, \sin\frac{2\pi x}{l}, \dots \cos\frac{k\pi x}{l}, \sin\frac{k\pi x}{l}, \dots \tag{13.8.4}$$

sistema $[-l,l]$ da ortogonal;

2) $$\frac{1}{2}, \cos\frac{\pi x}{l}, \cos\frac{2\pi x}{l}, \dots \cos\frac{k\pi x}{l}, \dots$$

$$\sin\frac{\pi x}{l}, \sin\frac{2\pi x}{l}, \dots \sin\frac{k\pi x}{l}, \dots \tag{13.8.5}$$

sistemalarning har biri $[0,l]$ da ortogonal.

13.8.6-ta'rif. Agar $\varphi(x)$ $[a,b]$ da integrallanuvchi bo'lsa, u holda

$$\|\varphi\| = \left(\int_a^b \varphi^2(x)dx\right)^{1/2}$$

musbat songa $\varphi(x)$ funksiyaning *normasi* deyiladi.

Norma belgisi bo'yicha (13.8.3) ni quyidagicha yozish mumkin:

$$\int_a^b \varphi_i(x)\varphi_k(x)dx = \begin{cases} 0, i \neq k \text{ bo'lsa,} \\ \|\varphi_k\|^2, i = k \text{ bo'lsa.} \end{cases}$$

(13.8.4) asosiy trigonometrik funksiyalarning $[-l,l]$ dagi normasi

$$\left\|\frac{1}{2}\right\| = \sqrt{\frac{l}{2}},\ \left\|\cos\frac{k\pi x}{l}\right\| = \sqrt{l},\ \left\|\sin\frac{k\pi x}{l}\right\| = \sqrt{l}$$

$k = 1,2,\ldots$. (13.8.5) sistemadagi funksiyalarning $[0,l]$ dagi normasi:

$$\left\|\frac{1}{2}\right\| = \sqrt{\frac{l}{2}},\ \left\|\cos\frac{k\pi x}{l}\right\| = \sqrt{\frac{l}{2}},\ \left\|\sin\frac{k\pi x}{l}\right\| = \sqrt{\frac{l}{2}}$$

$f(x)$ funksiya $[a,b]$ da integrallanuvchi bo'lsin va

$$f(x) = \sum_{k=1}^{\infty} a_k\varphi_k(x) \tag{13.8.7}$$

tenglik o'rinli bo'lsin, bunda a_k-o'zgarmas son, $\varphi_k(x)$ $[a,b]$ kesmada ortogonal bo'lgan (13.8.3) dagi funksiyalar sistemasi. Hozircha faraz qilaylik (13.8.7) qator $[a,b]$ kesmada o'zining yig'indisiga tekis yoki o'rtacha yaqinlashuvchi bo'lsin. Bu holda qatorni hadma-had intagrallash mumkin bo'ladi. (13.8.7) tenglikni $\varphi_n(x)$ ga ko'paytirib, x bo'yicha a dan b gacha integrallaymiz:

$$\int_a^b f(x)\varphi_n(x)dx = \sum_{k=1}^{\infty} a_k \int_a^b \varphi_k(x)\varphi_n(x)dx. \tag{13.8.8}$$

(13.8.8) ning o'ng tomonidagi integrallar $k \neq n$ bo'lganda nolga aylanadi (ortogonal bo'lganligi uchun).

Shunday qilib, (13.8.8) dan

$$\int_a^b f(x)\varphi_n(x)dx = a_n\int_a^b \varphi_n^2(x)dx = a_n\|\varphi_n\|^2,\ a_n = \frac{1}{\|\varphi_n\|^2}\int_a^b f(x)\varphi_n(x)dx \tag{13.8.9}$$

ni topamiz. (13.8.9) formula bilan aniqlanadigan a_n son $f(x)$ funksiyaning 209rthogonal sistemalar bo'yicha Furye koeffetsiyenti, koeffetsiyenti (13.8.9) formula bilan topiladigan (13.8.7) qatorni esa, $f(x)$ funksiyaning (13.8.3) 210rthogonal sistemalar bo'yicha *Furye qatori* deyiladi. a_n koeffetsiyentni (13.8.9) formula bilan topishda $f(x)$ dan uning integralanuvchi yekanligini talab qilish yetarli (ta'kidlaymizki $\varphi_k(x)$ $(k=1,2,...)$ funksiyalar $[a,b]$ da integrallanuvchi). U holda $[a,b]$ da integrallanuvchi $f(x)$ funksiyaga (13.8.3) 210rthogonal 210rthogo bo'yicha (13.8.7) Furye qatorini mos qo'yish mumkin bo'lar ekan:

$$f(x)\sim\sum_{k=1}^{\infty}a_k\varphi_k(x), \tag{13.8.10}$$

bunda koeffetsiyent a_n (13.8.9) formula bilan topiladi. (13.8.10) da tenglik ishorasini qo'yish uchun $f(x)$ ga qo'yiladigan yetarli shart $\{\varphi_k(x)\}$ 210rthogonal sistemasining xossalariga bog'liq bo'ladi. Xususiy holda 210rthogonal sistemaning o'rniga asosiy 210rthogonal210ic funksiyalar sistemasi olinganda $f(x)$ funksiyaga qo'yiladigan yetarli shart yuqorida isbot qilingan 13.6.1- asosiy teoremada keltirlgan. Ortogonal sistemalar uchun yuqoridagi asosiy teoremaga o'xshagan teoremani isbot qilish uchun maxsus izlanishni talab etiladi.

13.9. Eng kichik kvadratik chetlanish. Bessel ayniyati. Bessel tengsizlik. (13.8.3) 210rthogonal funksiyalar sistemasining belgilangan

$$\varphi_1(x),\varphi_2(x),\varphi_3(x),...\varphi_n(x)$$

bo'lagining chiziqli kombinatsiayasi

$$\sum_{k=1}^{n}\alpha_k\varphi_k(x) \tag{13.9.1}$$

ni qaraymiz, bunda a_k– ixtiyoriy sonli koeffetsiyent $(k=1,2,...,n)$. (13.9.1) ko'phadga (13.8.3) ortogonal sistemalar bo'yicha *n-tartibli ko'phad* deyiladi. $f(x)$ funksiya $[a,b]$ kesmada integtrallanuvchi bo'lsin. *Quyidagi masalani yechish talab qilinadi*: a_k $(k=1,2,...,n)$ ning qanday qiymatida berilagn ortogonal sistemalar bo'yicha n-tartibli (13.9.1) ko'phad $[a,b]$ kesmada $f(x)$ funksiyadan eng kichik kvadratik chetlanishga ega bo'ladi, ya'ni $a_1,a_2,...,a_n$ larning qanday qiymatida

$$\rho^2(f,\sum_{k=1}^{n}\alpha_k\varphi_k)=\int_a^b[f(x)-\sum_{k=1}^{n}\alpha_k\varphi_k(x)]^2dx=\left\|f-\sum_{k=1}^{n}\alpha_k\varphi_k\right\|^2$$

qiymat o'zining eng kichik absolyut minimumiga erishadi?

$$\rho^2(f,\sum_{k=1}^{n}\alpha_k\varphi_k)=\int_a^b f^2(x)dx-2\sum_{k=1}^{n}\alpha_k\int_a^b f(x)\varphi_k(x)dx+$$

$$+\int_a^b(\sum_{k=1}^{n}\alpha_k\varphi_k(x))^2dx=\int_a^b f^2(x)dx-2\sum_{k=1}^{n}\alpha_k a_k\|\varphi_k\|^2+\sum_{k=1}^{n}\alpha_k^2\|\varphi_k\|^2, \quad (13.9.2)$$

bunda $\int_a^b f(x)\varphi_k(x)dx=a_k\|\varphi_k\|^2$. (13.9.2) dan

$$\rho^2(f,\sum_{k=1}^{n}\alpha_k\varphi_k)=\int_a^b f^2(x)dx-\sum_{k=1}^{n}a_k^2\|\varphi_k\|^2+\sum_{k=1}^{n}(\alpha_k-a_k)^2\|\varphi_k\|^2. \quad (13.9.3)$$

(13.9.3) ning o'ng tomonidan keyingi yig'indi α_k ga bog'liq. Yig'indi musbat bo'lganligi uchun u o'zining aniq quyi chegarasiga $\alpha_k=a_n$ bo'lganda yerishadi. Shuning uchun $\rho^2(f,\sum_{k=1}^{n}\alpha_k\varphi_k)$-kvadratik chetlanishning absolyut minimumi:

$$\min\rho^2(f,\sum_{k=1}^{n}\alpha_k\varphi_k)^2=\int_a^b[f(x)-\sum_{k=1}^{n}\alpha_k\varphi_k(x)]^2dx=$$

$$=\int_a^b f^2(x)dx-\sum_{k=1}^{n}a_k^2\|\varphi_k\|^2. \quad (13.9.4)$$

Ushbu

$$\sum_{k=1}^{n}\alpha_k\varphi_k(x)$$

ko'phad (bunda a_k $f(x)$ funksiyaning $\{\varphi_k(x)\}$ ortogonal sistemalar bo'yicha Furye koyeffitsenti) $f(x)$ funksiyaning $\{\varphi_k(x)\}$ ortogonal sistemalar bo'yicha *Furye ko'phadi* deyiladi.

Shunday qilib, $[a,b]$ da $\{\varphi_k(x)\}$ ortogonal sistemalar bo'yicha hamma n-darajali ko'phadlar ichida $f(x)$ dan yeng kichik kvadratik chetlanishga ega bo'ladigani Furye ko'phadi bo'lib hisoblanar ekan. (13.9.4) tenglikni *Bessel ayniyati* deyiladi. (13.9.4) ning o'ng tomoni manfiy emas, shuning uchun undan

$$\sum_{k=1}^{n} a_k^2\|\varphi_k\|^2 \le \int_a^b f^2(x)dx \qquad (13.9.5)$$

tengsizlikni hosil qilamiz. Bu tengsizlik hamma $n \ge 1$ lar uchun o'rinli bo'ladi. (13.9.5) tengsizlikning o'ng tomoni n ga bog'liq emas. Chap tomonidagi yig'indi yuqoridan chegaralangan bo'lganligi uchun n ning o'sishi bilan u kamaymaydi. (13.9.5) tengsizlikda $n \to \infty$ da limitga o'tib Bessel tengsizligi deb ataluvchi

$$\sum_{n=1}^{\infty} a_k^2\|\varphi_k\|^2 \le \int_a^b f^2(x)dx \qquad (13.9.6)$$

tengsizlikni hosil qilamiz.

Xususiy holda, $\{\varphi_k(x)\}$ ortagonal sistemaning o'rniga asosiy trigonametrik funksiyalar sistemasi olinganda (13.9.6) Bessel tengsizligining ko'rinishi ushbu

$$\frac{a_0^2}{2} + \sum_{k=1}^{\infty}(a_k^2 + b_k^2) \le \frac{1}{l}\int_{-l}^{l} f^2(x)dx \qquad (13.9.7)$$

ko'rinishga keladi. (13.9.6) Bessel tengsizligini $[a,b]$ kesmada ixtiyoriy integrallanuvchi $f(x)$ uchun keltirib chiqarildi.

13.9.8-ta'rif. Agar

$$\int_a^b f(x)dx \text{ va } \int_a^b f^2(x)dx$$

integrallar oddiy yoki xosmas ma'nosida mavjud bo'lsa, $f(x)$ funksiya $[a,b]$ oraliqda *kvadrati bo'yicha integrallanuvchi* deyiladi.

(13.9.6) Bessel tengsizligi (13.9.7 ham) ixtiyoriy kvadrati bo'yicha integrallanuvchi $f(x)$ funksiya uchun ham o'rinli bo'ladi. Bundan tashqari (13.9.6) Bessel tengsizligi $\{\varphi_k(x)\}$ ortogonal funksiyalari sistemasi kvdrati bo'yicha integrallanuvchi bo'lgan holda ham o'rinli bo'ladi.

Haqiqatdan ham, $\int_a^b f^2(x)dx$, $\int_a^b \varphi_k^2(x)dx$ integrallarning yaqinlashuvchanligidan va ushbu

$$|f(x)\varphi_k(x)| \le \frac{f^2(x)+\varphi_k^2(x)}{2}, \ |\varphi_i(x)\varphi_k(x)| \le \frac{\varphi_i^2(x)+\varphi_k^2(x)}{2}$$

tengsizliklardan

$$\int_a^b f(x)\varphi_k(x)dx \text{ va } \int_a^b \varphi_i(x)\varphi_k(x)dx$$

integrallarning absalyut yaqinlashuvchanligi kelib chiqadi. Asosiy trigonametrik funksiyalar sistemasi uchun Besseya tengsizligi (13.9.7) dan

$$\frac{a_0^2}{2}+\sum_{k=1}^{\infty}(a_k^2+b_k^2)$$

qatorining yaqinlashuvchiligi va

$$\left.\begin{aligned} \lim_{k\to\infty} a_k &= \frac{1}{l}\lim_{k\to\infty}\int_a^b f(x)\cos\frac{k\pi x}{l}dx = 0, \\ \lim_{k\to\infty} b_k &= \frac{1}{l}\lim_{k\to\infty}\int_a^b f(x)\sin\frac{k\pi x}{l}dx = 0 \end{aligned}\right| \qquad (13.9.8)$$

ekanligi kelib chiqadi. (13.9.8) munosabat 13.6.2-lemma (asosiy) ning xususiy holidir (13.6.7-eslatmaga qarang).

13.10. Funksiyaning silliqlik darajasi bilan uning trigonametrik qatorlarining yaqinlashishi orsidagi bog'lanish. Avvalo funksiya trigonametrik qatorlarining tekis yaqinlashish shartini topishni, so'ngra $f(x)$ funksiya silliqlik darajasi bilan uning Furi koeffitsiyentlari a_k va b_k larning $k\to\infty$ da kamayish tezligi orasidagi bog'lanishni va bu qator yaqinlashish tezligini baholash bilan shug'ullanamiz.

a) Trigonametrik Furye qatorining tekis yaqinlashish sharti.

Trigonametrik Furye qatorining tekis yaqinlashish shartini ba'zi bir qo'shimcha zaruriy shartni qanoatlantiruvchi uzluksiz va bo'lakli silliq funksiyalar uchun isbotlaymiz.

13.10.1-teorema. Agar $[-l,l]$ kesmada uzluksiz va bo'lakli silliq $f(x)$ funksiya kesmaning uchlarida teng qiymatlarni (ya'ni $f(-l)=f(l)$) qabul qilsa, u holda uning trigonametrik Furye qatori

$$S(x)=\frac{a_0}{2}+\sum_{k=1}^{\infty}(a_k\cos\frac{k\pi x}{l}+b_k\sin\frac{k\pi x}{l}), \qquad (13.10.2)$$

$$a_0=\frac{1}{l}\int_{-l}^{l}f(\xi)d\xi,\ a_n=\frac{1}{l}\int_{-l}^{l}f(\xi)\cos\frac{n\pi\xi}{l}d\xi,\ b_n=\frac{1}{l}\int_{-l}^{l}f(\xi)\sin\frac{n\pi\xi}{l}d\xi$$

$[-l,l]$ kesmada tekis yaqinlashuvchi va kesmaning har bir nuqtasida $S(x)=f(x)$ bo'ladi.

13.10.3-eslatma. $f(x)$ funksiyaning $[-l,l]$ oraliqning uchlaridagi qiymatlarining tengligi $f(x)$ funksiya (13.10.2) trigonametrik Furye qatorining oraliqning chetlarida $f(x)$ ga yaqinlashish uchun zaruriy shart bo'ladi.

Haqiqatdan ham, agar bu qator yig'indisi $S(x)$ uchun

$$S(-l)=f(-l),\ S(l)=f(l)$$

shart bajarilsa, u holda $S(-l)=S(l)$ shartga asosan,

$$f(-l)=f(l) \qquad (13.10.4)$$

tenglik bajriladi.

Demak, (13.10.4) tenglik (13.10.2) Furye qatorining $[-l,l]$ da $f(x)$ ga tekis yaqinlashishi uchun zaruriy shart bo'lib hisoblanadi.

b) Funksiyaning silliqlik darajasi bilan uning trigonametrik Furye qatorining yaqinlashish tezligi orasidagi bog'lanish.

13.10.5-teorima. Agar $f(x),f'(x),...,f^{(m)}(x)$ $(m\geq 0)$ funksiyalar $[-l,l]$ oraliqda uzluksiz va orliqning chetki nuqtalrida teng, ya'ni

$$f(-l)=f(l),\ f'(-l)=f'(l),\ f''(-l)=f''(l),...,f^{(m)}(-l)=f^{(m)}(l)$$

qiymatlarni qabul qilsa, hamda $f^{(m-1)}(x)$ funksiya $[-l,l]$ da bo'lakli uzluksiz bo'lsa, u holda $f(x)$ funksiya Furye koeffitsiyentlari uchun

$$a_k = o(\frac{1}{k^{m+1}}), \quad b_k = o(\frac{1}{k^{m+1}})$$

munosabatlar bajariladi va

$$\sum_{k=1}^{\infty} k^{\nu}(|a_k| + |b_k|) \qquad (13.10.6)$$

$\nu = 0,1,2,...,m$ qator yaqinlashuvchi bo'ladi.

13.11. Ortogonal sistemalarning to'liqligi va yopiqligi. $[a,b]$ kesmada hamma bo'lakli-uzluksiz bo'lgan funksiyalar sinfini $Q[a,b]$ deb belgilaymiz. $[a,b]$ kesmada

$$\varphi_1(x), \varphi_2(x), ...\varphi_n(x), ... \qquad (13.11.1)$$

ortogonal funksiyalar sistemsi berilgan bo'lsin.

13.11.2-ta'rif. Agar $\forall f(x) \in Q[a,b]$ funksiyalarning (13.11.1) ortogonal sistemalar bo'yicha

$$\sum_{k=1}^{\infty} c_k \varphi_k(x) \qquad (13.11.3)$$

bunda $\left(c_k = \frac{1}{\|\varphi_k\|^2} \int_a^b f(x)\varphi_k(x)dx, \quad \|\varphi_k\| = \left(\int_a^b \varphi_k^2(x)dx \right)^{1/2} \right)$.

Furye qatori $[a.b]$ kesmada $f(x)$ funksiyaga o'rtacha yaqinlashsa, ya'ni

$$\rho^2(f, \sum_{k=1}^{\infty} c_k \varphi_k) = \int_a^b \left[f(x) - \sum_{k=1}^{\infty} c_k \varphi_k(x) \right]^2 dx \xrightarrow[n \to \infty]{} 0$$

bo'lsa, u holda (13.11.1) ortogonal sistema $Q[a,b]$ *fazoda to'liq* deyiladi. Bu holda (13.11.1) sistema $Q[a,b]$ fazoning bazisini tashkil qiladi deb yuritiladi, chunki (13.11.1) sistema to'liq bo'lganda $\forall f(x) \in Q[a,b]$ uchun

$$f(x) = \sum_{k=1}^{\infty} c_k \varphi_k(x), \quad c_k = \frac{1}{\|\varphi_k\|^2} \int_a^b f(x)\varphi_k(x)dx$$

tenglik o'rinli. Bessel ayniyatiga asosan,

$$\rho^2(f,\sum_{k=1}^{n}c_k\varphi_k)=\int_a^b\left[f(x)-\sum_{k=1}^{n}c_k\varphi_k(x)\right]^2dx=\int_a^b f^2(x)dx-\sum_{k=1}^{n}c_k{}^2\|\varphi_k\|^2.$$

Bunda $n\to\infty$ limitga o'tish orqali

$$\lim_{n\to\infty}\rho^2(f,\sum_{k=1}^{n}c_k\varphi_k)=\int_a^b f^2(x)dx-\sum_{k=1}^{\infty}c_k{}^2\|\varphi_k\|^2,$$

bundan $\lim_{n\to\infty}\rho^2(f,\sum_{k=1}^{n}c_k\varphi_k)=0$ bu esa,

$$\sum_{k=1}^{\infty}c_k{}^2\|\varphi_k\|^2=\int_a^b f^2(x)dx \tag{13.11.4}$$

ga teng kuchli. (13.11.4) tenglikga *Parseval tengligi* deyiladi.

Shunday qilib, (13.11.1) ortogonal sistema $Q[a,b]$ fazoda to'liq bo'lishi uchun $\forall f(x)\in Q[a,b]$ funksiyalar uchun (13.11.4) Parseval tengligining bajarilishi zarur va yetarlidir.

To'liq ortogonal sistemlarning xossalari.

Agar $[a,b]$ da ortogonal bo'lgan, (13.11.1) sistemaning hamma elementlariga $[a,b]$ da ortogonal bo'lgan $\forall f(x)\in Q[a,b]$ funksiy $Q[a,b]$ fazoning nolini tashkil etsa, ya'ni $f(x)$ funksiyaning uzluksizlik nuqtalarida nolga teng bo'lib, qolgan chekli sondagi nuqtlarida noldan farqli bo'lsa, u holda $[a,b]$ da ortogonal bo'lgan, (13.11.1) ortogonal sistema $Q[a,b]$ *fazoda yopiq* deyiladi.

13.11.5-teorema. Agar $[a,b]$ da ortogonal bo'lgan (13.11.1) sistema $Q[a,b]$ da to'liq bo'lsa, u holda u $Q[a,b]$ da yopiq bo'ladi.

13.11.6-teorema. Agar ikkita $f(x)$, $g(x)\in Q[a,b]$ funksiyalar $[a,b]$ da to'liq ortogonal sistemalar bo'yicha bir xil Furye qatoriga ega bo'lsa, u holda ular $Q[a,b]$ ning elementi sifatida bir-biriga teng bo'ladi, ya'ni ular $[a,b]$ ning chikli sondagi nuqtalaridagina bir-biridan farq qiladi.

13.11.7-teorem. Agar (13.11.1) ortogonal sistema $Q[a,b]$ da to'liq bo'lsa, u holda $Q[a,b]$ dagi ixtiyoriy ikkita $f(x)$ va $g(x)$ ($f(x)\in Q[a,b]$, $g(x)\in Q[a,b]$) funksiyalar uchun umumlashgan Parseval tengligi o'rinli bo'ladi, ya'ni

$$\int_a^b f(x)g(x)dx = \sum_{k=1}^{\infty} c_k^f c_k^g \|\varphi_k\|^2, \qquad (13.11.8)$$

bunda c_k^f va c_k^g - mos ravishda $f(x)$ va $g(x)$ funksiyalarning ortogonal sistema bo'yicha Furye koeffitsiyentlari.

13.11.9-teorema. Agar $f(x) \in Q[a,b]$ bo'lib, (13.11.1) ortogonal sistema $Q[a,b]$ da to'liq bo'lsa, u holda $f(x)$ funksiyning (13.11.1) ortogonal sistemalar bo'yicha Furye qatorini hadma-had integrallash mumkin:

$$\int_{x_0}^{x} f(\xi)d\xi = \sum_{k=1}^{\infty} c_k \int_{x_0}^{x} \varphi(\xi)d\xi \qquad (13.11.10)$$

(x_0, x-lar, $[a,b]$ dan olingn ixtiyoriy nuqtalar). (13.11.8) qator x bo'yicha $[a,b]$ tekis yaqinlashuvchi bo'ladi.

13.11.11-eslatma. Xususiy holda $[a,b]$ da (13.11.1) ortogonal sistema, asosiy trigonametrik sisemadan iborat, ya'ni

$$\frac{1}{2}, \cos\frac{\pi x}{l}, \sin\frac{\pi x}{l}, ..., \cos\frac{k\pi x}{l}, \sin\frac{k\pi x}{l}, ...$$

bo'lsa, bu asosiy trigonmetrik sistema $Q[-l,l]$ da to'liq bo'ladi.

13.11.12-misol. Ushbu $f(x) = \mathrm{sgn}(\sin x)$ funksiyani Furye qatoriga yoying va yoyilmadan foydalanib $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)^2}$ qatorning yig'indisini toping; Parseval formulasini yozing; birinchi ikkita qismiy yig'indisining grafigini chizing.

Yechilishi. Berilgan funksiya juft emas, 2π davrga ega. Berilgan funksiya $[-\pi,\pi]$ oraliqning $x=-\pi, x=0, x=\pi$ nuqtalaridan tashqari, hamma joyida uzluksiz, ko'rsatilgan nuqtalarda 1- tur uzilishga ega. $x=-\pi, x=0, x=\pi$ nuqtalardan tashqari $[-\pi,\pi]$ oraliqning hamma nuqtalarida hosilaga ega va bu hosila chegaralangan. Shuning uchun, berilgan funksiyaning Furye qatori hamma x larda yaqinlashuvchi (funksiyaning uzluksizlik nuqtalarida qatorning yig'indisi funksiyaning bu nuqtalardagi qiymatiga teng bo'ladi, $x=-\pi, x=0, x=\pi$ nuqtalarda esa, funksiyaning o'ng va chap limitik qiymatlarining o'rta arifmetigiga teng).

Endi Furye koeffitsiyentlarini topamiz:

$$a_n = 0\ (n = 1, 2, \ldots),\ b_n = \frac{2}{\pi}\int_0^\pi 1 \sin nx dx = -\frac{2}{n\pi}\cos nx\Big|_0^\pi,\ \cos n\pi = (-1)^n$$

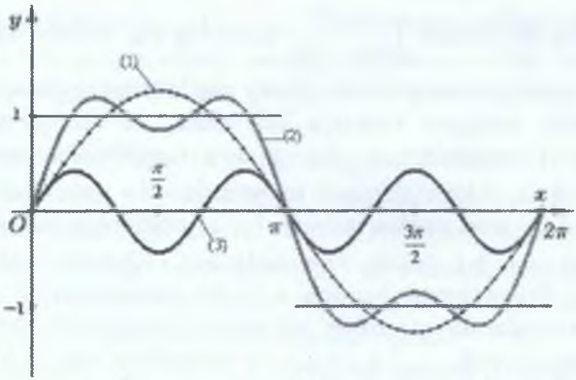
bo'lgani uchun, $b_n = \frac{2}{\pi}\frac{1-(-1)^n}{n}$ $(n = 1, 2, \ldots)$, bundan n- juft bo'lganda $b_n = 0$, n -toq bo'lganda esa, $b_n = \frac{4}{\pi n}$ ga teng bo'ladi. Shunday qilib $\forall x$ lar uchun $\operatorname{sgn}(\sin x) = \frac{\pi}{4}\sum_{n=1}^{\infty}\frac{\sin(2n-1)x}{2n-1}$.Bunda, $x = \frac{\pi}{2}$ bo'lganda

$$1 = \frac{\pi}{4}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2n-1},\ \frac{4}{\pi} = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2n-1}.$$

$x \neq 0$ larda $(\operatorname{sgn}(\sin x))^2 = 1$ bo'lgani uchun, Parseval formulasiga asosan,

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f^2(x)dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2),\ \frac{1}{\pi}\int_{-\pi}^{\pi} 1\, dx = \frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2},$$

bundan $\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$. Birinchi ikkita qismiy yig'indisining grafigi 13.7- chizmada tasvirlangan.



13.7-chizma. $(1) - y = \frac{4}{\pi}\sin x$, $(2) - y = \frac{4}{3\pi}\sin 3x$, $(3) - y = \frac{4}{\pi}(\sin x + \frac{1}{3}\sin 3x)$.

13.3-§. Furye integrali

13.12. Furye integrali. $f(x)$ funksiya butun Ox o'qida aniqlangan bo'lib va o'qning chekli $[-l,l]$qismida bo'lakli-silliq funksiya bo'lsin. U holda trigonometrik Furye qatorining yaqinlashi to'g'risidagi teoremaga asosan istalgan $l>0$ uchun

$$f(x)=\frac{a_0}{2}+\sum_{k=1}^{\infty}(a_k\cos\frac{k\pi}{l}x+b_k\sin\frac{k\pi}{l}x), \qquad (13.12.1)$$

bunda

$$a_0=\frac{1}{l}\int_{-l}^{l}f(t)dt, a_k=\frac{1}{l}\int_{-l}^{l}f(t)\cos\frac{k\pi t}{l}dt, \quad b_k=\frac{1}{l}\int_{-l}^{l}f(t)\sin\frac{k\pi t}{l}dt \quad (13.12.2)$$

tengliklar o'rinli bo'ladi.

Agar $x\in(-l,l)$ bo'lib, $f(x)$ funksiyaning uzluksizlik nuqtasi bo'lsa, (13.12.1) tenglik o'rinli bo'ladi.

Agar $x\in(-l.l)$ bo'lib, $f(x)$ funksiyaning uzilish nuqtasi bo'lsa, (13.12.1) tenglikning chap tomonidagi $f(x)$ funksiyaning o'rniga $\frac{f(x+0)+f(x-0)}{2}$ quyiladi.

(13.12.2) ni (13.12.1) ga olib borib qo'yish natijamida qo'yidagi tenglikka ega bo'lamiz:

$$f(x)=\frac{1}{2l}\int_{-l}^{l}f(t)dt+\frac{1}{l}\sum_{k=1}^{\infty}\int_{-l}^{l}f(t)\cos\frac{k\pi(t-x)}{l}dt. \qquad (13.12.3)$$

Agar $f(x)$ funksiya butun Ox o'qida aniqlangan va unda absalyut integrallanuvchi, ya'ni

$$\int_{-\infty}^{\infty}|f(x)|dx=Q<\infty \qquad (13.12.4)$$

integral mavjud bo'lsin, u holda (13.12.3) tenglikning birinchi qo'shiluvchisi (13.12.4) ga asosan, $l\to\infty$ da nolga intiladi. Haqiqatan, ham

$$\left|\frac{1}{2l}\int_{-l}^{l} f(t)dt\right| \le \frac{1}{2l}\int_{-l}^{l}|f(t)|dt < \frac{1}{2l}\int_{-\infty}^{\infty}|f(t)|dt = \frac{1}{2l}Q \to 0.$$

Demak, (13.12.3) dan

$$f(x) = \lim_{l\to\infty}\frac{1}{l}\sum_{k=1}^{\infty}\int_{-l}^{l} f(t)\cos\frac{k\pi(t-x)}{l}dt. \qquad (13.12.5)$$

Agar qo'yidagi

$$\lambda_k = \frac{k\pi}{l} \text{ va } \Delta\lambda_k = \frac{\pi}{l},\ k = 1,2,....$$

belgilashlarni kiritamiz, u holda (13.12.5) ni qo'yidagicha yozish mumkin:

$$f(x) = \frac{1}{\pi}\lim_{\substack{l\to\infty \\ \Delta\lambda_k\to 0}}\sum_{k=1}^{\infty}\left(\int_{-l}^{l} f(t)\cos\lambda_k(t-x)dt\right)\Delta\lambda_k. \qquad (13.12.6)$$

Endi qat'iy bo'lmagan qo'yidagi mulohazalarni yuritsak, ya'ni:

1) Agar l ning katta qiymatlarida $\int_{-l}^{l} f(t)\cos\lambda_k(t-x)dt$ integralni $\int_{-\infty}^{\infty} f(t)\cos\lambda_k(t-x)dt$ integralga almashtrsak;

2) $\sum_{k=1}^{\infty}\left(\int_{-\infty}^{\infty} f(t)\cos\lambda_k(t-x)dt\right)\Delta\lambda_k$ yig'indi, $\int_{0}^{\infty}\left(\int_{-\infty}^{\infty} f(t)\cos\lambda(t-x)dt\right)d\lambda$ integral uchun integral yig'indi ekanligi e'tborga olsak, u holda ushbu

$$f(x) = \frac{1}{\pi}\int_{0}^{\infty}\left(\int_{-\infty}^{\infty} f(t)\cos\lambda(t-x)dt\right)d\lambda \qquad (13.12.7)$$

ega bo'lamiz.

Agar x nuqta $f(x)$ funksiyaning uzilish nuqtasi bo'lsa, (13.12.7) tenglikning chap tomonidagi $f(x)$ funksiyaning o'riniga $\frac{f(x+0)+f(x-0)}{2}$ yozish kerak.

Odatda (13.12.7) formulaga Furye integral formulasi, uning o'ng tomonidagi integralga *Furye integrali* deyiladi. (13.12.7) formula formal ravishda limitga o'tish natijasida hosil qilindi. Endi yuqoridagi formal ravishda yuritgan mulohazalarimizni asoslaymiz. Buning uchun qo'yidagi teoremani isbotsiz keltiramiz.

13.12.8-teorema. Agar $f(x)$ funksiya Ox o'qining chekli qismida bo'lakli-silliq bo'lib, butun Ox o'qida absolyut interallanuvchi bo'lsa, u holda

$$\lim_{l\to\infty}\frac{1}{\pi}\int_0^l\left(\int_{-\infty}^{\infty}f(t)\cos\lambda(t-x)dt\right)d\lambda=\frac{f(x+0)+f(x-0)}{2} \qquad (13.12.9)$$

tenglik o'rinli bo'ladi.

13.13. Furye integralining kompleks formada tasvirlanishi. Ushbu

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}f(t)e^{-i\lambda(t-x)}dt\right)d\lambda. \qquad (13.13.1)$$

integral Furye interalining *kompleks ko'rishnishini* ifodalaydi va u Furye integralining haqiqiy ko'rinishi (13.12.7) ga ekvivalent.

Haqiqatan ham, $\int_{-\infty}^{\infty}f(t)\cos\lambda(t-x)dt$ integral λ ning juft funksiyasini ifoda qiladi, $\int_{-\infty}^{\infty}f(t)\sin\lambda(t-x)dt$ integral esa, λ ning toq funksiyasini ifoda qiladiyu. Demak,

$$f(x)=\frac{1}{\pi}\int_0^{\infty}\left(\int_{-\infty}^{\infty}f(t)\cos\lambda(t-x)dt\right)d\lambda=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}f(t)\cos\lambda(t-x)dt\right)d\lambda,$$

$$0=-\frac{i}{2\pi}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}f(t)\sin\lambda(t-x))dt\right)d\lambda.$$

Shunday qilib, ushbu $e^{i\lambda(x-t)}=\cos\lambda(x-t)+i\sin\lambda(x-t)$ Eyler formulasini e'tiborga olgan holda

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}f(t)e^{i\lambda(x-t)}dt\right)d\lambda=$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}f(t)(\cos\lambda(t-x)-i\sin\lambda(t-x))dt\right)d\lambda=\frac{1}{\pi}\int_0^{\infty}\left(\int_{-\infty}^{\infty}f(t)\cos\lambda(t-x)dt\right)d\lambda$$

Demak, Furye integralning (13.12.7) ko‘rinishi bilan, (13.13.1) ko‘rinishi ekvivalent ekan, bunda $0=-\frac{i}{2\pi}\int\limits_{-\infty}^{\infty}\left(\int\limits_{-\infty}^{\infty}f(t)\sin\lambda(t-x))dt\right)d\lambda$ integral bosh qiymat ma’nosida tushuniladi:

$$-\frac{i}{2\pi}\int\limits_{-\infty}^{\infty}\left(\int\limits_{-\infty}^{\infty}f(t)\sin\lambda(t-x))dt\right)d\lambda=\lim_{l\to+\infty}\frac{-i}{2\pi}\int\limits_{-l}^{l}\left(\int\limits_{-\infty}^{\infty}f(t)\sin\lambda(t-x))dt\right)d\lambda .$$

13.13. Furyening almashtirishlari. (13.13.1) tenglikni qo‘yidagi ko‘rinishda yozib olamiz:

$$f(x)=\frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty}\left(\frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty}f(t)e^{-i\lambda t}dt\right)e^{i\lambda x}d\lambda \qquad (13.13.1)$$

Agar

$$\tilde{f}(\lambda)=\frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty}f(t)e^{-i\lambda t}dt \qquad (13.13.2)$$

belgilashni olsak, u holda (13.13.1) dan

$$f(x)=\frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty}\tilde{f}(x)e^{i\lambda x}d\lambda \qquad (13.13.3)$$

ga ega bo‘lamiz. Bunda $\tilde{f}(\lambda)$ funksiyaga Furye obrazi yoki $f(x)$ funksiyaning *spektral xarakteristik funksiyasi* deyiladi. (13.13.2) formulaga *Furye almashtrtshi* deyiladi. (13.13.3) formulasiga esa, *teskari Furye almashtirishi* deyiladi. Furye almashtirishlarini e’tiborga olgan holda **13.12.8**- teoremani quyidagicha bayon etish mumukin.

13.13.4-teorema. Agar $f(x)$ funksiya butun son o‘qida absolyut integrallanuvchi, Ox o‘qining chekli qismida bo‘lakli-silliq bo‘lsa, u holda: 1) (13.13.2) formula bilan aniqlanadigan $\tilde{f}(\lambda)$ Furye obrazi mavjud bo‘ladi; 2) (13.13.3) formula bilan aniqlanadigan teskari formulasi o‘rinli bo‘ladi, bunda $f(x)=\frac{1}{\sqrt{2\pi}}\lim\limits_{l\to+\infty}\int\limits_{-l}^{l}\tilde{f}(\lambda)e^{i\lambda x}d\lambda$ deb tushunish kerak.

13.14. Furyening kosinus va sinus almashtirishlari. Butun sonlar o‘qida $(-\infty<x<+\infty)$ aniqlangan funksiyaga qo‘llanlanadigan Furye almashtirishdan tashqari yarim o'qda $(0 \le x < +\infty)$ aniqlangan funksiyalarga qo'llanlanadigan sinus va kosinus Furye almashtirishlari ko'p qo'llanladi.

(13.12.7) – Furye integralini ayirmaning kosinusi formulasidan foydalanib, quyidagicha yozib olamiz:

$$f(x)=\frac{1}{\pi}\int_0^{\infty} d\lambda \int_{-\infty}^{\infty} f(t)\cos\lambda t\cos\lambda x dt+\frac{1}{\pi}\int_0^{\infty} d\lambda \int_{-\infty}^{\infty} f(t)\sin\lambda t\sin\lambda x dt, \qquad (13.15.1)$$

bunda $f(x)$ funksiya butun son o'qida absolyut integrallanuvchi bo'lgani uchun (13.15.1) dagi ikkiala integral ham yaqinlashuvchi.

Agar $f(x)$ juft funksiya bo'lsa, u holda $f(t)\sin\lambda t$ toq, $f(t)\cos\lambda t$ esa juft funksiya bo'ladi. Shuning uchun (13.15.1) ning o'ng tomonidagi ikkinchi qo'shiluvchi nolga aylanadi.

Demak,

$$f(x)=\frac{1}{\pi}\int_0^{\infty} \cos\lambda x d\lambda \int_0^{\infty} f(t)\cos\lambda t dt. \qquad (13.15.2)$$

Xuddi shunday, agar $f(x)$- toq funksiya bo'lsa,

$$f(x)=\frac{1}{\pi}\int_0^{\infty} \sin\lambda x d\lambda \int_0^{\infty} f(t)\sin\lambda t dt \qquad (13.15.3)$$

ekanligini topamiz.

Agar x nuqta $f(x)$ funksiyaning uzilish nuqtasi bo'lsa, u holda (13.15.1) (13.15.2), (13.15.3) formulalarda tenglikning chap tomonidagi $f(x)$ ni $\dfrac{f(x+0)+f(x-0)}{2}$ ga aimshtirish kerak bo'ladi.

Agar $f(x)$ funksiya faqat $0 \le x < +\infty$ da aniqlangan bo'lsa, u holda uni $-\infty < x \le 0$ oraliqqa juft yoki toq davom etirish mumkin, u holda $f(x)$ funksiyani quyidagi ikkita turli xil ko'rinishda tasvirlash mumkin:

$$f(x)=\sqrt{\frac{2}{\pi}}\int_0^{\infty} \cos\lambda x\left(\sqrt{\frac{2}{\pi}}\int_0^{\infty} f(t)\cos\lambda t dt\right)d\lambda, \qquad (13.15.4)$$

$$f(x)=\sqrt{\frac{2}{\pi}}\int_0^{\infty} \sin\lambda x\left(\sqrt{\frac{2}{\pi}}\int_0^{\infty} f(t)\sin\lambda t dt\right)d\lambda. \qquad (13.15.5)$$

Agar $0 \le x < +\infty$ da aniqlangan $f(x)$ funksiya $x=0$nuqtada uzluksiz bo'lsa, u holda juft davom ettirilganda funksiya $x=0$ nuqtada uzluksiz bo'ladi va u butun o'qda aniqlangan funksiya bo'lib, (13.15.4) tenglik $x=0$ da ham o'rinli bo'ladi. (13.15.5) tenglikda esa, $f(0)=0$ bo'lganda, $x=0$da bajariladi, chunki toq davom ettirilganda hamma vaqt funksiya $\frac{f(x+0)+f(x-0)}{2}=0$ bo'ladi.

Agar

$$f_c(\lambda)=\sqrt{\frac{2}{\pi}}\int_0^{\infty} f(t)\cos\lambda t dt \tag{13.15.6}$$

deb belgilasak, u holda (13.15.4) dan

$$f(x)=\sqrt{\frac{2}{\pi}}\int_0^{\infty} f_c(\lambda)\cos\lambda x d\lambda, \tag{13.15.7}$$

bunda $f_c(\lambda)$ funksiyaga $0 \le x < +\infty$ da aniqlangan $f(x)$ funksiyaning *kosinus Furye obrazi* deyiladi. (13.15.6) formulaga kosinus *Furye almasht irish* deyiladi. (13.15.7) formulaga teskari kosinus almashtirish deyiladi.

Xuddi shunday

$$f_s(\lambda)=\sqrt{\frac{2}{\pi}}\int_0^{\infty} f(t)\sin\lambda t dt \tag{13.15.8}$$

deb belgilask, (13.15.8) dan

$$f(x)=\sqrt{\frac{2}{\pi}}\int_0^{\infty} f_s(\lambda)\sin\lambda x d\lambda, \tag{13.15.9}$$

bunda $f_s(\lambda)$ ga $0 \le x < +\infty$aniqlangan $f(x)$ funksiyaning *sinus Furye obrazi* deyiladi. (13.15.9) formulaga esa, *teskari sinus Furye almashtirish* deyiladi.

13-bob bo'yicha nazariy materiallarni mustahkamlash uchun topshiriqlar

13.1. Trigonometrik funksiyalar sistemasining $[-l;l]$ oraliqda ortogonalligi ([17], 474-476 betlar; [9], 2-t., 3- bo'lim; [30], 11- bo'lim).

13.2. 2π davrli $f(x)$ funksiya uchun trigonometrik Furye qatori va uning koeffisiyentlarini hisoblash formulalari ([3], 2-q.,182-185 betlar; [17], 453-456 betlar; [9], 2-t., 3- bo'lim; [30], 11- bo'lim).

13.3. Juft va toq funksiyalarning Furye qatori ([3], 2-q.,185-187 betlar; [17], 456-459 betlar; [9], 2-t., 3- bo'lim; [30], 11- bo'lim).

13.4. Bo'lakli - uzluksiz va bo'lakli - silliq funksiyalar tushunchalari. Misollar ([12], 2-q., 388-391betlar, [17], 459-461 betlar; [9], 2-t., 3- bo'lim; [30], 11- bo'lim).

13.5. Trigonometrik Furye qatorining tekis yaqinlashishi to'g'risidagi Dirixle teoremasi ([3], 2-q.,192-195 betlar; [12], 2-q., 399-405 betlar, [17], 481-484 betlar).

13.6. $f(x)$ funksiya $[-l;l]$ oraliqda bo'lakli - silliq funksiya bo'lsa, uning Furye qatori yig'indisi $S(x)$ uchun qanday tengliklar o'rinli bo'ladi? ([17], 461-462 betlar).

13.7. Trigonometrik Furye qatorining absolyut va tekis yaqinlashishiga oid teorema ([17], 481-484 betlar; [9], 2-t., 2- bo'lim).

13.8. Trigonometrik funksiyalar sistemasi uchun Bessel tengsizligi ([12], 2-q., 412-416 betlar, [17], 477-480 betlar).

13.9. Parseval tengligi ([17], 496-497 betlar).

13.10. Yaqinlashuvchi trigonometrik Furye qatori yig'indisining funksional xossalari ([12], 2-q., 416-419 betlar).

13.11. Funksiyalarni davriy davom ettirish ([12], 2-q., 383-385 betlar, [17], 450-451 betlar).

13.12. Furye integrali ([17], 510-521 betlar, [19], 338-353 betlar).

13.1-amaliy mashg'ulot.
Funksiyalarni Furye (trigonometrik) qatoriga yoyish

1-misol. $[0,2\pi]$ kesmada aniqlangan $f(x)=x^2$ funksiyaning trigonometrik Furye qatori tuzilsin.

Yechilishi. (13.2.11), (13.2.12) va (13.2.13) formulalardan foydalanib tirgonometrik qatorning a_0, a_n, b_n koefitsiyentlarini aniqlaymiz:

$$a_0=\frac{1}{\pi}\int_0^{2\pi}x^2dx=\frac{1}{\pi}\frac{x^3}{3}\Big|_0^{2\pi}=\frac{8\pi^2}{3};$$

$$a_n=\frac{1}{\pi}\int_0^{2\pi}x^2\cos n\pi x dx=\frac{1}{\pi}\left[\frac{x^2\sin nx}{n}\Big|_0^{2\pi}-\frac{1}{n}\int_0^{2\pi}\cos nxdx\right]=$$

$$=-\frac{2}{\pi n}\int_0^{2\pi}x\sin nxdx=\frac{2}{n\pi}\left[\frac{x\cos nx}{n}\Big|_0^{2\pi}-\frac{1}{n}\int_0^{2\pi}\cos nxdx\right]=$$

$$=\frac{2}{\pi n}\left[\frac{2\pi}{n}-\frac{\sin nx}{n^2}\Big|_0^{2\pi}\right]=\frac{4}{n^2}\quad(n=1,2,3,....);$$

$$b_n=\frac{1}{\pi}\int_0^{2\pi}x^2\sin nxdx=\frac{1}{\pi}\left[-\frac{x^2\cos nx}{n}\Big|_0^{2\pi}+\frac{2}{n}\int_0^{2\pi}x\cos nxdx\right]=$$

$$=\frac{1}{\pi}\left[-\frac{4\pi^2}{n}+\frac{2}{n}\left(\frac{x\sin nx}{n}\Big|_0^{2\pi}-\frac{1}{n}\int_0^{2\pi}\sin nxdx\right)\right]=$$

$$=\frac{1}{\pi}\left[-\frac{4\pi^2}{n}+\frac{2}{n^2}\cos nx\Big|_0^{2\pi}\right]=-\frac{4\pi}{n},\quad(n=1,2,3,....)$$

topilgan koeffetsiyentlarning qiymatlarini (13.2.5) ga qo‘yib, ushbu

$$\frac{4\pi^2}{3}+4\sum_{n=1}^{\infty}\left(\frac{\cos nx}{n^2}-\pi\frac{\sin nx}{n}\right)$$

qatorni hosil qilamiz. Bunda $x=0$ desak, ushbu

$$\frac{4\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{1}{n^2}$$

sonli qator hosil bo‘ladi. Ravshanki bu qator yaqinlashuvchi bo‘lib, noldan farqli yig‘indiga ega. Ikkinchi tomondan $f(0)=0$ Bundan

$$f(0)\neq\frac{4\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{1}{n^2}.$$

Agar $x=\pi$ desak, $f(\pi)=\pi^2$ bo‘lib, qator

$$\frac{4\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$$

ko‘rinishga keladi. $x=2\pi$ desak, $f(2\pi)=4\pi^2$ bo‘lib, qator esa, ya’na

$$\frac{4\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{1}{n^2}$$

ko‘rinishda bo‘ladi.

2-misol. $[0,2\pi]$ kesmada aniqlangan $f(x)=\frac{\pi-x}{2}$ funksiyaning Furye qatori tuzilsin.

Yechilishi. Bu funksiya yordamida qatorning koeffetsiyentlarini topamiz.

$$a_0=\frac{1}{\pi}\int_0^{2\pi}\frac{\pi-x}{2}dx=-\frac{(\pi-x)^2}{4\pi}\bigg|_0^{2\pi}=-\frac{\pi}{4}+\frac{\pi}{4}=0;$$

$$a_n=\frac{1}{\pi}\int_0^{2\pi}\frac{\pi-x}{2}\cos nxdx=\frac{1}{\pi}\left[\frac{\pi-x}{2}\cdot\frac{\sin nx}{n}\bigg|_0^{2\pi}+\frac{1}{2\pi}\int_0^{2\pi}\sin nxdx\right]=$$

$$=-\frac{1}{2\pi n^2}\cos nx\Big|_0^{2\pi}=\frac{1}{n},$$

$$b_n=\frac{1}{\pi}\int_0^{2\pi}\frac{\pi-x}{2}\sin nxdx=-\frac{1}{\pi}\left[-\frac{\pi-x}{2}\cdot\frac{\sin nx}{n}\bigg|_0^{2\pi}-\frac{1}{2\pi}\int_0^{2\pi}\cos nxdx\right]=$$

$$=\frac{1}{n}-\frac{1}{2\pi n}\sin nx\Big|_0^{2\pi}=\frac{1}{n}.$$

Shunday qilib, qator $\sum_{n=1}^{\infty}\frac{\sin nx}{n}$ ko‘rinishda bo‘ladi. Bundan

a) $x=0$ deb olsak, $f(0)=\frac{\pi}{2}$ bo‘lib, qator nolga aylanadi.

Demak, $f(0)=\sum_{n=1}^{\infty}\frac{\sin 0}{n}=0$, ya’ni $x=0$ da qator $f(x)$ ga yaqinlashadi;

b) $x=\pi$ bo‘lsa, $f(x)=0$ va qator nolga aylanadi,ya’ni $f(\pi)=0$ va $\sum_{n=1}^{\infty}\frac{\sin n\pi}{n}=0$ ya’ni $f(\pi)=0$.

Demak, $f(\pi)=\sum_{n=1}^{\infty}\frac{\sin n\pi}{n}$, ya'ni $x=\pi$ da qator $f(x)$ ga yaqinlashadi;

c) $x=2\pi$ da $f(2\pi)=-\frac{\pi}{2}$; $\sum_{n=1}^{\infty}\frac{\sin 2n\pi}{n}=0$; $f(2\pi)\neq\sum_{n=1}^{\infty}\frac{\sin 2\pi n}{n}$. Demak, tuzilgan qator $x=2\pi$ da $f(x)$ ga yaqinlashmaydi.

Biz yuqorida ikki misolda ko'rdikki, x ning funksiyaning aniqlanish oralig'iga tegishli istalgan qiymatida qator yig'indisi bilan $f(x)$ funksiya qiymati bir-birga teng bo'lavermaydi. Qatorning berilgan funksiyaga yaqinlashadigan oralig'iga tegishli x lar uchungina qator yig'indisi bilan $f(x)$ ning qiymati o'zaro teng bo'ladi. Shuning uchun qatorning $f(x)$ ga yaqinlashish oralig'ini aniqlamasdan turib, funksiya bilan qator orasiga tenglik belgisini qo'yish mumkin emas. Shunga ko'ra hozircha biz qator bilan funksiya orasiga $(\approx)$ belgisini qo'yamiz:

$$f(x)\approx\frac{a_0}{2}+\sum_{k=1}^{\infty}(a_k\cos\frac{k\pi}{l}x+b_k\sin\frac{k\pi}{l}x)$$

Yuqoridagi ko'rilgan misollarni quyidagicha yozish mumkin: $x\in[0,2\pi]$ uchun

1) $x^2\approx\frac{4\pi^2}{3}+4\sum_{n=1}^{\infty}\left(\frac{\cos nx}{n^2}-\pi\frac{\sin nx}{n}\right)$, 2) $\frac{\pi-x}{2}\approx\sum_{n=1}^{\infty}\frac{\sin nx}{n}$.

3-misol. Ushbu $f(x)=x^2$ ni $[-l,l]$ da Furye qatoriga yoyilsin.

Yechilishi. $f(x)=x^2$ funksiya juft funksiya bo'lgani uchun

$$b_n=0, a_0=\frac{2}{l}\int_0^l x^2dx=\frac{2}{3}l^2,, a_n=\frac{2}{l}\int_0^l x^2\cos\frac{n\pi x}{l}dx, n=1,2,....$$

Asosiy teoremaga asosan,

$$S(x)=\frac{l^2}{3}+\frac{4l^2}{\pi^2}\sum_{k=1}^{\infty}\frac{(-1)^k}{k^2}\cos\frac{k\pi x}{l}=x^2, -l\leq x\leq l, \qquad (1)$$

$S(x)$ funksiya $x=(2k+1)l$ $(k=0,\pm1,\pm2,...)$ nuqtalarda ham uzluksiz. $f(l-0)=f(-l+0)=l^2$.

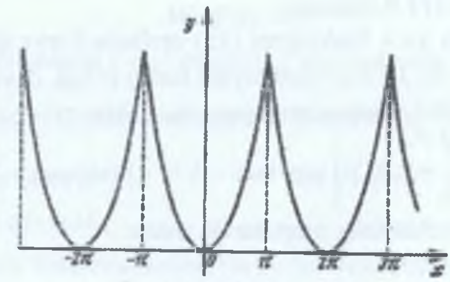
Agar yoyilma $[-\pi,\pi]$ kesmada olib borilgan bo‘lsa, ya’ni $l=\pi$, u holda (1) tenglik

$$x^2=\frac{\pi^2}{3}+4\sum_{k=1}^{\infty}\frac{(-1)^k}{k^2}\cos kx \quad (2)$$

ko‘rinishda bo‘ladi (13.8-chizma). $x=0$ bo‘lganda (2) dan

$$\frac{\pi^2}{12}+\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^2}$$

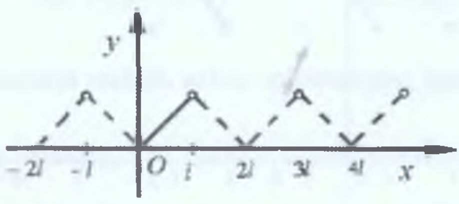
ni hosil qilamiz.



13.8-chizma.

4-misol. Ushbu $f(x)=x$ funksiyani $[0,l]$ oraliqda kosinuslar buyicha Furye qatoriga yoying.

Yechilishi. Masalaning shartiga ko‘ra, $[0,l]$ da berilgan $f(x)=x$ funksiyani, juftlik qonuniga ko‘ra, $[-l,l]$ ga davriy davom ettiramiz, ya’ni $[0,l]$ oraliqda $f(x)=x$ ga teng bo‘lgan $2l$ davrli juft davriy funksiyani Furye qatoriga yoyamiz. Bu funksiya uchun



13.9-chizma.

$$b_n = 0 \quad (n=1,2,\dots), \quad a_0 = \frac{2}{l}\int_0^l x dx = l,$$

$$a_n = \frac{2}{l}\int_0^l x\cos\frac{k\pi}{l}x dx = \frac{2l}{\pi^2 n^2}[(-1)^n - 1] = \begin{cases} 0, & n\text{-juft bo'lganda} \\ -\dfrac{4l}{\pi^2 n^2}, & n\text{-toq bo'lganda} \end{cases}$$

Shunday qilib,

$$x = \frac{l}{2} - \frac{4l}{\pi^2}\left\{\cos\frac{\pi x}{l} + \frac{1}{3^2}\cos\frac{k\pi x}{l} + \frac{1}{5^2}\cos\frac{5\pi x}{l} + \dots\right\}.$$

ko'rinishda bo'ladi (13.9-chizma).

5-misol. Ushbu $y = x$ funksiyani $(3;5)$ oraliqda Furye qatoriga yoying.

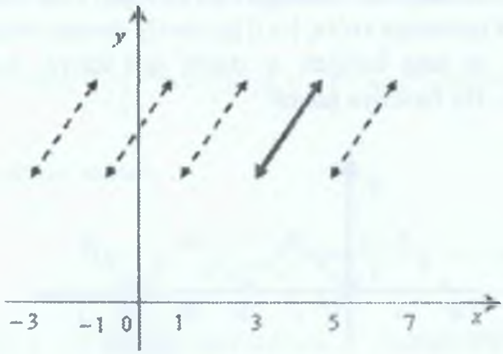
Yechilishi. $y = x$, $3 < x < 5$ funksiyani butun o'qqa, davri 2 ga teng davriy funksiya sifatida davriy davom ettiramiz. Bu holda, $2l = 5 - 3 = 2$, $l = 1$

$$a_0 = \int_3^5 x dx = \left.\frac{x^2}{2}\right|_3^5 = 8, \quad a_n = \int_3^5 x\cos n x dx = 0, \quad b_n = \int_3^5 x\sin n x dx = \frac{2(-1)^{n+1}}{\pi n}, \quad (n \in N).$$

Funksiyaning uzluksizlik nuqtalarida ushbu

$$f(x) = 4 + \sum_{n=1}^{\infty}\frac{2}{\pi n}(-1)^{n+1}\sin n\pi x$$

yoyilma o'rinli bo'ladi (13.10-chizma).



13.10-chizma.

Mustaqil yechish uchun misollar

Quyidagi funksiyalarni $(-\pi;\pi)$ da Furye qatoriga yoying.

1. $f(x)=\sin^4 x$. **2.** $f(x)=\cos^4 x$.

3. $f(x)=\begin{cases}-1, & -\pi<x<0,\\ 1, & 0<x<\pi\end{cases}$ **4.** $f(x)=\begin{cases}2, & -\pi<x<0,\\ -2, & 0<x<\pi\end{cases}$

5. $f(x)=1-2x$. **6** $f(x)=\frac{1}{2}x-3$.

7. $f(x)=\begin{cases}5, & -\pi<x<0,\\ -3, & 0<x<\pi\end{cases}$ **8.** $f(x)=\begin{cases}-7, & -\pi<x<0,\\ 2, & 0<x<\pi\end{cases}$

Quyidagi funksiyalarni $[-\pi;\pi]$ da Furye qatoriga yoying.

9. $f(x)=x^2$. **10.** $f(x)=|x|$.

Quyidagi funksiyalarni $(-\pi;\pi)$ oraliqda Furye qatoriga yoying:

11. $f(x)=\sin ax$ (a – butun son emas). **12.** $f(x)=\begin{cases}-x, & -\pi<x<0,\\ 0, & 0<x<\pi.\end{cases}$

13. $f(x)=\cos ax$ $(a\notin z)$. **13.** $f(x)=\pi^2-x^2$

15. $f(x)=e^{ax}$ $x\in(-h,h)$.

Quyidagi davriy funksiyalarning Furye qatoriga yoying:

16. $f(x)=\operatorname{sgn}(\cos x)$. **17.** $f(x)=\arcsin(\sin x)$.**18.** $f(x)=\arcsin(\cos x)$

Quyidagi funksiyalarni $(0;\pi)$ oraliqda sinus bo'yicha Furye qatoriga yoying.

19. $f(x)=x$. **20.** $f(x)=\frac{\pi}{4}-\frac{x}{2}$. **21.** $f(x)=\begin{cases}x, & 0\le x<\frac{\pi}{2},\\ \pi-x, & \frac{\pi}{2}\le x\le\pi.\end{cases}$

Quyidagi funksiyalarni $[0;\pi]$ da kosinun bo'yicha Furye qatoriga yoying.

22. $f(x)=x$. **23.** $f(x)=\frac{1}{2}x-1$. **24.** $f(x)=-x^2$.

Mustaqil yechish uchun misollarning javoblari

1. $\frac{3}{8}-\frac{1}{2}\cos 2x+\frac{1}{8}\cos 4x$. **2.** $\frac{3}{8}+\frac{1}{2}\cos 2x+\frac{1}{8}\cos 4x$. **3.** $f(x)=\frac{4}{\pi}\sum_{\kappa=1}^{\infty}\frac{\sin(2\kappa-1)x}{2\kappa-1}$.

4. $f(x)=\frac{-8}{\pi}\sum_{\kappa=1}^{\infty}\frac{\sin(2\kappa-1)x}{2\kappa-1}$. **5.** $f(x)=1+4\sum_{n=1}^{\infty}\frac{(-1)^n\sin nx}{n}$.

6. $f(x)=-3+\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\sin nx}{n}$.

7. $f(x)=1-\frac{16}{\pi}\sum_{\kappa=1}^{\infty}\frac{\sin(2\kappa-1)x}{2\kappa-1}$.

8. $\frac{-5}{2}+\frac{18}{\pi}\sum_{\kappa=1}^{\infty}\frac{\sin(2\kappa-1)x}{2\kappa-1}$.

9. $f(x)=\frac{\pi^2}{3}+4\sum_{n=1}^{\infty}(-1)^n\frac{\cos nx}{n^2}$.

10. $f(x)=\frac{\pi}{2}-\frac{4}{\pi}\sum_{\kappa=1}^{\infty}\frac{\cos(2\kappa-1)x}{(2\kappa-1)^2}$.

11. $\frac{2\sin \pi a}{\pi}\left(\frac{\sin x}{1a^2}-\frac{\sin 2x}{2^2-a^2}+\frac{3\sin 3x}{3^2-a^2}-.....\right)$.

12. $-\frac{2}{\pi}\sum_{\kappa=1}^{\infty}\frac{\cos(2\kappa-1)x}{(2\kappa-1)^2}+\sum_{n=1}^{\infty}\frac{1}{n}(-1)^n\sin nx$.

13. $\frac{2\sin \pi a}{\pi}\left[\frac{1}{2a}+\sum_{n=1}^{\infty}(-1)^{n+1}\frac{a\cos nx}{n^2-a^2}\right]$.

13. $\frac{2}{3}\pi^2+4\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2}\cos nx$.

15. $2Shah\left[\frac{1}{2ah}+\sum_{n=1}^{\infty}(-1)^n\frac{ah\cos\frac{n\pi x}{h}-\pi n\sin\frac{n\pi x}{h}}{(ah)^2+(\pi n)^2}\right]$.

16. $\frac{4}{\pi}\sum_{\kappa=0}^{\infty}\left\{(-1)^{\kappa}\frac{\cos(2k+1)x}{2\kappa+1}\right\}$.

17. $\frac{4}{\pi}\sum_{\kappa=0}^{\infty}(-1)^k\frac{\sin(2k+1)x}{(2k+1)^2}$.

18. $\frac{4}{\pi}\sum_{\kappa=0}^{\infty}\frac{\cos(2k+1)x}{(2k+1)^2}$.

19. $2\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin x}{n}$.

20. $\sum_{n=1}^{\infty}\frac{\sin 2nx}{2n}$.

21. $\frac{4}{\pi}\sum_{\kappa=1}^{\infty}\frac{(-1)^{\kappa+1}\sin(2\kappa-1)x}{(2\kappa-1)x}$.

22. $\frac{\pi}{2}-\frac{4}{\pi}\sum_{k=1}^{\infty}\frac{\cos(2\kappa-1)x}{(2\kappa-1)^2}$.

23. $-1+\frac{\pi}{4}-\frac{2}{\pi}\sum_{\kappa=1}^{\infty}\frac{\cos(2\kappa-1)x}{(2\kappa-1)^2}$.

24. $\frac{-\pi^2}{3}-4\sum_{n=1}^{\infty}(-1)^n\frac{\cos nx}{n^2}$.

13.2-amaliy mashg'ulot.
Furye integrali

1-misol. $f(x)=e^{-ax^2}$ $(a>0)$ funksiyaning Furye integral almashtirishlarini toping.

Yechilishi. Ushbu

$$F(\lambda)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-ax^2}e^{-i\lambda x}dx. \quad (1)$$

integralni λ bo'yicha integral ostida differensiyalash mumkin:

$$F'(\lambda)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-ax^2}e^{-i\lambda x}(-ix)dx.$$

λ chekli oraliqda o'zgarganda bu integral tekis yaqinlashadi. Oxirgi integralni bo'laklab integrallash natijasida.

$$F'(\lambda)=\frac{1}{\sqrt{2\pi}}\frac{i}{2a}\int_{-\infty}^{\infty}e^{-i\lambda x}d\left(e^{-ax^2}\right)=\frac{1}{\sqrt{2\pi}}\left\{\frac{i}{2a}[e^{-i\lambda x}e^{-ax^2}]\Big|_{x=-\infty}^{x=\infty}-\right.$$

$$\left.-\frac{i}{2a}\int_{-\infty}^{\infty}e^{-ax^2}e^{-i\lambda x}(-i\lambda)dx\right\}.$$

munosabatga ega bo'lamiz. Bunda integral qo'shiluvchi nolga aylanadi. Natijada,

$$F'(\lambda)=-\frac{\lambda}{2a\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-ax^2}e^{-i\lambda x}dx=-\frac{\lambda}{2a}F(\lambda),$$

bo'ladi. $F'(\lambda)=-\frac{\lambda}{2a}F(\lambda)$ differensial tenglamani yechamiz:

$$F(\lambda)=C\,e^{-\frac{\lambda^2}{4a}} \quad (2)$$

(C-o'zgarmas). (2) da $\lambda=0$ deb olib, $C=F(0)$ ni topamiz. (1) ga asosan, $F(0)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-ax^2}dx$ bo'ladi. Ma'lumki, $\int_{-\infty}^{\infty}e^{-x^2}dx=\sqrt{\pi}$. Buni e'tiborga olsak, $\int_{-\infty}^{\infty}e^{-ax^2}dx=\sqrt{\frac{\pi}{a}}$, bundan $F(0)=\frac{1}{\sqrt{2a}}$.

Shunday qilib, $F(\lambda)=\frac{1}{\sqrt{2a}}e^{-\frac{\lambda^2}{4a}}$. Xususiy holda, $f(x)=e^{-\frac{x^2}{2}}\left(a=\frac{1}{2}\right)$ uchun $F(\lambda)=e^{-\frac{\lambda^2}{2}}$ ekanligini olamiz. ■

Mustaqil yechish uchun misollar

Quyidagi funksiyalarni Furye integralini tasvirlang

1. $f(x)=\begin{cases}1, & |x|<1,\\ 0, & |x|>1.\end{cases}$

2. $f(x)=\begin{cases}signx, & |x|<1,\\ 0, & |x|>1.\end{cases}$

3. $f(x)=sign(x-a)-sign(x-b)\ (b>a)$.

4. $f(x)=\frac{1}{a^2+x^2}\ (a>0)$.

5. $f(x)=\frac{x}{a^2+x^2}\ (a>0)$.

6. $f(x)=e^{-a|x|}\ (a>0)$.

7. $f(x)=\begin{cases}\sin x, & |x|\le\pi,\\ 0, & |x|>\pi.\end{cases}$

8. $f(x)=\begin{cases}\cos x, & |x|\le\pi/2,\\ 0, & |x|>\pi/2.\end{cases}$

9. $f(x)=e^{-a|x|}\cos\beta x\ (a>0)$.

10. $f(x)=e^{-a|x|}\sin\beta x\ (a>0)$.

Mustaqil yechish uchun misollarning javoblari

1. $f(x)=\frac{2}{\pi}\int\limits_0^{\infty}\frac{\sin\lambda}{\lambda}\cos\lambda x\,d\lambda$.

2. $f(x)=\frac{2}{\pi}\int\limits_0^{\infty}\frac{1-\cos\lambda}{\lambda}\sin\lambda x\,d\lambda$.

3. $f(x)=\frac{2}{\pi}\int\limits_0^{\infty}\frac{\sin\lambda(x-a)-\sin\lambda(x-b)}{\lambda}d\lambda$.

4. $\frac{1}{a^2+x^2}=\frac{1}{a}\int\limits_0^{+\infty}e^{-a\lambda}\cos\lambda x\,d\lambda$.

5. $\frac{x}{a^2+x^2}=\frac{1}{a}\int\limits_0^{+\infty}e^{-a\lambda}\sin\lambda x\,d\lambda$.

6. $f(x)=\frac{2a}{\pi}\int\limits_0^{\infty}\frac{\cos\lambda x}{\lambda^2+a^2}d\lambda$.

7. $f(x)=\frac{2}{\pi}\int\limits_0^{\infty}\frac{\sin\lambda\pi}{1-\lambda^2}\sin\lambda x d\lambda$.

8. $f(x)=\frac{2}{\pi}\int\limits_0^{\infty}\frac{\cos 0.5\lambda\pi}{1-\lambda^2}\cos\lambda x d\lambda$.

9. $f(x)=\frac{a}{\pi}\int\limits_0^{\infty}[\frac{1}{(\lambda-\beta)^2+a^2}+\frac{1}{(\lambda+\beta)^2+a^2}]\cos\lambda x d\lambda$.

10. $f(x)=\frac{a}{\pi}\int_0^{\infty}\frac{\lambda\sin\lambda x}{[(\lambda-\beta)^2+a^2][(\lambda+\beta)^2+a^2]}d\lambda$.

13-bob bo'yicha nazariy materiallarni mustahkamlash uchun topshiriqlar

13.1 – masala. $[-\pi;\pi]$ kesmada berilgan ($T=2\pi$ davrga ega bo'lgan) $f(x)$ funksiyani Furye qatoriga yoying.

13.1.1. $f(x)=\begin{cases}0, -\pi\le x<0,\\ x-1, 0\le x\le\pi.\end{cases}$

13.1.2. $f(x)=\begin{cases}0, -\pi\le x<0,\\ x-1, 0\le x\le\pi.\end{cases}$

13.1.3. $f(x)=\begin{cases}2x-1, -\pi\le x\le 0,\\ 0, 0<x\le\pi.\end{cases}$

13.1.4. $f(x)=\begin{cases}-x+1/2, -\pi\le x\le 0,\\ 0, 0<x\le\pi.\end{cases}$

13.1.5. $f(x)=\begin{cases}0, -\pi\le x<0,\\ x/2+1, 0\le x\le\pi.\end{cases}$

13.1.6. $f(x)=\begin{cases}2x+3, -\pi\le x\le 0,\\ 0, 0<x\le\pi.\end{cases}$

13.1.7. $f(x)=\begin{cases}0, -\pi\le x<0,\\ 3-x, 0\le x\le\pi.\end{cases}$

13.1.8. $f(x)=\begin{cases}x-2, -\pi\le x\le 0,\\ 0, 0<x\le\pi.\end{cases}$

13.1.9. $f(x)=\begin{cases}0, -\pi\le x<0,\\ 4x-3, 0\le x\le\pi.\end{cases}$

13.1.10. $f(x)=\begin{cases}5-x, & -\pi\le x\le 0,\\ 0, & 0<x\le\pi.\end{cases}$

13.1.11. $f(x)=\begin{cases}0, -\pi\le x<0,\\ 3x-1, 0\le x<\pi.\end{cases}$

13.1.12. $f(x)=\begin{cases}3-2x, -\pi\le x\le 0,\\ 0, 0<x\le\pi.\end{cases}$

13.1.13. $f(x)=\begin{cases}0, -\pi\le x<0,\\ (\pi-x)/2, 0\le x\le\pi.\end{cases}$

13.1.14. $f(x)=\begin{cases}5x+1, -\pi\le x\le 0,\\ 0, 0<x\le\pi.\end{cases}$

13.1.15. $f(x)=\begin{cases}0, -\pi\le x\le 0,\\ 1-4x, 0\le x\le\pi.\end{cases}$

13.1.16. $f(x)=\begin{cases}3x+2, -\pi\le x\le 0,\\ 0, 0<x\le\pi.\end{cases}$

13.1.17. $f(x)=\begin{cases}0, -\pi\le x<0,\\ 4-2x, 0\le x\le\pi.\end{cases}$

13.1.18. $f(x)=\begin{cases}x+\frac{\pi}{2}, -\pi\le x\le 0,\\ 0, 0<x\le\pi.\end{cases}$

13.1.19. $f(x)=\begin{cases}0, -\pi\le x<0,\\ 6x-5, 0\le x\le\pi.\end{cases}$

13.1.20. $f(x)=\begin{cases}7-3x, -\pi\le x\le 0,\\ 0, 0<x\le\pi.\end{cases}$

13.1.21. $f(x)=\begin{cases}0, -\pi\le x<0,\\ 2x-3, 0\le x\le\pi.\end{cases}$ **13.1.22.** $f(x)=\begin{cases}1, -\pi\le x<0,\\ x+1, 0\le x\le\pi.\end{cases}$

13.1.23. $f(x)=\begin{cases}x-2, -\pi\le x\le 0,\\ 0, 0<x\le\pi.\end{cases}$ **13.1.24.** $f(x)=\begin{cases}-x+1, -\pi\le x\le 0,\\ 0, 0<x\le\pi.\end{cases}$

13.1.25. $f(x)=\begin{cases}0, -\pi\le x<0,\\ 3x-5, 0\le x\le\pi.\end{cases}$ **13.1.26.** $f(x)=\begin{cases}-x, -\pi\le x\le 0,\\ \dfrac{x^2}{\pi}, 0<x\le\pi.\end{cases}$

funksiyani $[-\pi;\pi]$ kesmada Furye qatoriga yoying.

Yechilishi ([9], 2-t., 3.7-bo'lim, [30], 11.11-bo'lim). Berilgan funksiya $[-\pi;\pi]$ da uzluksiz. Uning hosilasi x ning $x_n=\pi n$, $(n=0,\pm1,\pm2,.....)$ nuqtalardan tashqari hamma qiymatlarida uzluksiz va o'zining aniqlanish sohasida chegaralangan.

Demak, berilgan fuknsiyaning Furye qatori x ning hamma qiymatlarida $f(x)$ ga yaqinlashadi. Furye koeffisiyentlarini topamiz:

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)dx=\frac{1}{\pi}\int_{-\pi}^{0}(-x)dx+\frac{1}{\pi}\int_0^{\pi}\frac{x^2}{\pi}dx=-\frac{x^2}{2\pi}\bigg|_{-\pi}^{0}+\frac{x^3}{3\pi^2}\bigg|_0^{\pi}=\frac{\pi}{2}+\frac{\pi}{3}=\frac{5}{6}\pi.$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx\,dx=\frac{1}{\pi}\left(-\int_{-\pi}^{0}x\cos nx\,dx+\int_0^{\pi}\frac{x^2}{\pi}\cos nx\,dx\right).$$

Tenglikning o'ng tomonidagi integrallarni bo'laklab integrallash natijasida

$$a_n=\frac{3(-1)^n-1}{\pi n^2}$$

bo'lishni topamiz. Endi b_n larni topamiz

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nx\,dx=\frac{1}{\pi}\left(-\int_{-\pi}^{0}x\sin nx\,dx+\int_0^{\pi}\frac{x^2}{\pi}\sin nx\,dx\right)=\frac{2[(-1)^n-1]}{\pi^2n^3},$$

Bundan

$$b_n = \begin{cases} 0, & n = 2m,\ m \in N, \\ -\dfrac{4}{\pi^2 (2m-1)^3}, & n = 2m-1,\ m \in N. \end{cases}$$

Shunday qilib, berilgan funksiyaning Furye qatori, quyidagi ko'rinishda bo'ladi:

$$f(x) = \frac{5}{12}\pi + \sum_{m=1}^{\infty}\left[\frac{3(-1)^m - 1}{\pi m^2}\cos mx - \frac{4}{\pi^2(2m-1)^3}\sin(2m-1)x\right].$$

Bu yoyilmada $x = \pi$ deyilsa, u holda

$$\pi = \frac{5\pi}{12} + \sum_{m=1}^{\infty}\frac{3(-1)^m - 1}{\pi m^2}(-1)^m.$$

13.2-masala. $(0;\pi)$ oraliqda berilgan $f(x)$ funksiyani juft va toq davom ettirib (qayta aniqlab) Furye qatoriga yoying.

13.2.1. $f(x) = e^x$. **13.2.2.** $f(x) = x^2$.

13.2.3. $f(x) = 2^x$. **13.2.4.** $f(x) = e^{-x}$.

13.2.5. $f(x) = (x-1)^2$. **13.2.6.** $f(x) = 3^{-x/2}$.

13.2.7. $f(x) = e^{2x}$. **13.2.8.** $f(x) = (x-2)^2$.

13.2.9. $f(x) = 4^{x/3}$. **13.2.10.** $f(x) = e^{4x}$.

13.2.11. $f(x) = (x+1)^2$. **13.2.12.** $f(x) = 5^{-x}$.

13.2.13. $f(x) = e^{-x/4}$. **13.2.14.** $f(x) = (2x-1)^2$.

13.2.15. $f(x) = 6^{x/4}$. **13.2.16.** $f(x) = e^{-3x}$.

13.2.17. $f(x) = x^2 + 1$. **13.2.18.** $f(x) = 7^{-x/7}$.

13.2.19. $f(x) = e^{-2x/3}$. **13.2.20.** $f(x) = 10^{-x}$

13.2.21. $f(x) = e^{3x}$. **13.2.22.** $f(x) = 2x+1$.

13.2.23. $f(x)=3^x$. **13.2.24.** $f(x)=e^{-2x}$.

13.2.25. $f(x)=2x^2+3$.

13.2.26. $f(x)=x$ funksiyani $(0;\pi)$ oraliqda: *a*) juft (kosinuslar); *b*) toq (sinuslar) davom ettirib (qayta aniqlab) Furye qatoriga yoying.

Yechilishi ([9], 2-t., 3.7-bo'lim, [30], 11.11-bo'lim). *a*) Bu holda, berilgan funksiyani juft davom ettirib, son o'qida aniqlangan, ushbu

$$f^*(x)=|x-2k\pi|,\ |x-2k\pi|\le\pi,\quad k=0,\pm1,\pm2,\ldots$$

Yordamchi funksiya tuzamiz. Tuzilgan yordamchi funksiya, x funksiyani Furye qatoriga yoyish to'g'risidagi Furye teoremasining barcha shartlarini qanoatlantiradi. Ma'lumki, $f^*(x)$ juft funksiya bo'lganligi uchun barcha n larda $b_n=0$. Endi a_0 va a_n larni topamiz:

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f^*(x)dx=\frac{2}{\pi}\int_0^{\pi}xdx=\frac{2}{\pi}\frac{x^2}{2}\bigg|_0^{\pi}=\pi;$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f^*(x)\cos nx\,dx=\frac{2}{\pi}\int_0^{\pi}x\cos nx\,dx=$$

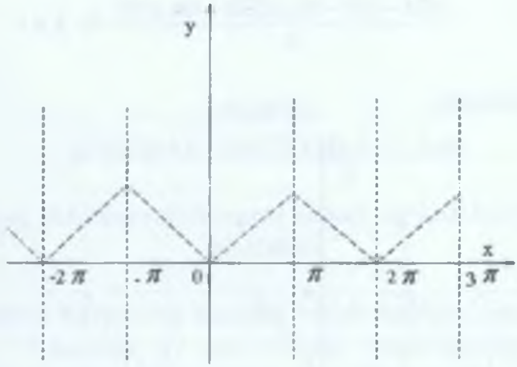
$$=\frac{2}{\pi n^2}(\cos n\pi-1)=\frac{2}{\pi n^2}\left((-1)^n-1\right)=\begin{cases}0, & n=2m,\quad m\in N,\\ -\dfrac{4}{\pi(2m-1)^2}, & n=2m-1, m\in N.\end{cases}$$

Shunday qilib,

$$f^*(x)=\frac{\pi}{2}-\frac{4}{\pi}\sum_{m=0}^{\infty}\frac{\cos(2m+1)x}{(2m+1)^2}\qquad(-\infty<x<\infty),$$

$$f(x)=\frac{\pi}{2}-\frac{4}{\pi}\sum_{m=0}^{\infty}\frac{\cos(2m+1)x}{(2m+1)^2}\qquad(0<x<\pi).$$

Bu qator hamma nuqtalarda yaqinlashada va uning yig'indisi berilgan funksiyaga teng (13.11-chizma).



13.11-chizma.

b) Bu holda, berilgan funksiyani toq (sinuslar bo‘yicha) davom ettirib, butun son o‘qida aniqlangan ushbu

$$f^*(x)=\begin{cases} x-2\pi k, & (2k1-1)/\pi < x < \pi(2k+1), \quad k\in Z, \\ 0, & x=\pi k, \qquad k\in Z \end{cases}$$

funksiyani tuzamiz. $f^*(x)$ funksiya toq bo‘lganligi uchun, $a_0=0,\ a_n=0$ bo‘ladi. Endi b_n larni topamiz:

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f^*(x)\sin nx\,dx=\frac{2}{\pi}\int_0^{\pi} x\sin nx\,dx=(-1)^{n+1}\frac{2}{n}.$$

Shunday qilib,

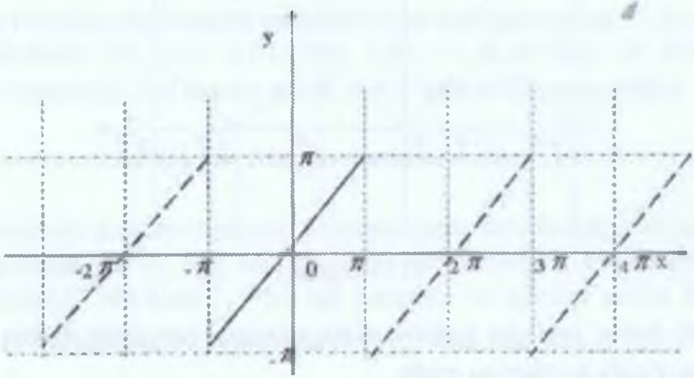
$$f^*(x)=2\sum_{k=1}^{\infty}(-1)^{k-1}\frac{\sin kx}{k} \quad (-\infty<x<\infty),$$

$$f(x)=2\cdot\sum_{k=1}^{\infty}(-1)^{k-1}\frac{\sin\cdot kx}{x} \quad (0<x<\pi)$$

Bu tenglik uzilish nuqtalaridan boshqa barcha nuqtalarda o‘rinli bo‘lib, har bir uzilish nuqtasi, ya’ni $x=(2k-1)\pi,\ k\in Z$ nuqtalardagi qatorning yig‘indisi

$$\frac{f[(2k-1)\pi-0]+f[(2k-1)\pi+0]}{2}=0, \quad k\in Z$$

bo'ladi (13.12-chizma).



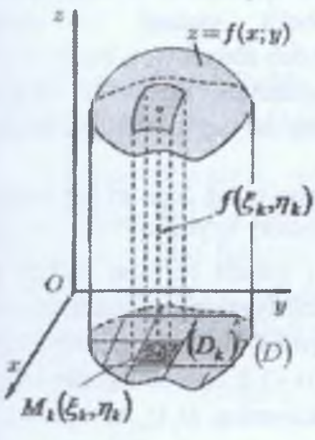
13.12-chizma.

14-BOB.
KARRALI INTEGRALLAR

14.1-§. Ikki karrali integral va uni yig'indi bo'yicha hisoblash

14.1. Silindrik burusning hajmini topish haqidagi masala. Ma'lumki ([15], 7–bob, 7.1-bandga q.) egri chiziqli trapetsiyaning yuzini topish haqidagi masala aniq integral tushinchasiga olib kelgan edi. Xuddi shunday silindrik brusning hajmini topish haqidagi masala ham ikki karrali integral tushinchasiga olib keladi. Biz bu bandda silindrik brusning hajmini topish haqidagi masalani qaraymiz.

Yuqoridan $z=f(x,y)$ sirt bilan, yon tomondan, yasovchilari Oz o'qiga parallel bo'lgan silindrik sirt, pastdan xOy tekisligidagi chegaralangan soha (D) bilan chegaralangan (V) jism berilgan bo'lib, shu jismning hajmi V ni topish talab qilingan bo'lsin. Buning uchun (D) sohani unda butunlay yotuvchi nol yuzli egri chiziqlar yordamida ixtiyoriy ravishda n ta bo'lakka bo'lamiz: $(D_1),(D_2),...,(D_n)$.



14.1-chizma

Bo'luvchi chiziqlarni yo'naltiruvchi sifatida olib, OZ o'qiga parallel silindrik sirtlar o'tkazamiz. Natijada (V) jism, n ta $(V_k)(k=1,2,...,n)$ bo'laqlarga bo'linadi (14.1-chizma).

(D_k) dan ixtiyoriy $M(\xi_k,\eta_k)$ nuqtani olamiz. Agar har bir silindrik sirtlarni balandligi $f(\xi_k,\eta_k)$ ga teng bo'lgan haqiqiy silindr deb faraz qilsak, u holda (V_k) silindrik brusning hajmi taxminan:

$f(\xi_k,\eta_k)D_k$ ga teng bo'ladi, bunda $D_k-(D_k)$ shaklning yuzi. U holda (V) jismning hajmi taxminan

$$V \approx \sum_{k=1}^{n} f(\xi_k,\eta_k)D_k, \tag{14.1.1}$$

chunki, $f(x,y)=f(\xi_k,\eta_k)$, $(x,y)\in(D_k)$ bo'lsin deb hisobladik.

(V) jismning hajmini ifodalovchi (14.1.1) taqribiy formulaning aniqlik darajasini oshirish uchun (D) sohani bo'laklarga bo'lish sonini shunday orttirib boraylikki, natijada har bir (D_k) $(k=1,...,n)$ bo'lakning diametri nolga intilsin. U holda (D_i) sohalar diametrlarining eng kattasi $\lambda\to 0$ da $\sum_{k=1}^{n} f(\xi_k,\eta_k)D_k$ yig'indining limiti izlanuvchi (V) jismning aniq hajmini ifodalaydi, ya'ni

$$V=\lim_{\lambda\to 0}\sum_{k=1}^{n} f(\xi_k,\eta_k)D_k.$$

14.2. Ikki karrali integralning ta'rifi. Biror chegaralangan (D) ($(D)\subset R^2$) soha berilgan bo'lsin. (D) sohaning chegarasidagi ixtiyoriy ikki nuqtani birlashtiruvchi va to'laligicha shu sohada yotuvchi chiziqni (egri chiziqni) Γ chiziq deb belgilaymiz. Bunday Γ chiziqlar (D) sohani bo'laklarga ajratadi. ((D):sohada to'laligicha yotuvchi yopiq chiziqni ham Γ chiziq deb qaraymiz. Bunday chiziqlar ham (D) sohani bo'laklarga ajratadi). (D) sohani bo'laklarga ajratuvchi chekli sondagi Γ chiziqlar sistemasi $\{\Gamma:\Gamma\subset(D)\}$ (D) sohaning *bo'linishi* deb ataladi va u $P=\{\Gamma:\Gamma\subset(D)\}$ kabi belgilanadi. (D) sohani bo'laklarga ajratuvchi har bir Γ chiziq, P bo'linishning bo'luvchi chizig'i, (D) sohaning bo'lagi esa, P bo'linishning *bo'lagi* deyiladi.

P bo'linish bo'laklari diametrining eng kattasi uning diametri deb ataladi va u λ_p kabi belgilanadi. (D) sohaning bo'linishlari to'plamini $\Phi=\{P\}$ orqali belgilaymiz. $f(x,y)$ funksiya (D) ($(D)\subset R^2$) sohada berilgan bo'lsin (1.1-chizma). Bu erda va kelgusida hamma vaqt funksiyaning aniqlanish sohasi (D) ni yuzaga ega bo'lgan soha deb hisoblaymiz. Bu (D) sohaning $P\in\Phi$ bo'linishini va bu bo'linishning har bir (D_k) $(k=1,2,...,n)$ bo'lagidan ixtiyoriy $M_k(\xi_k,\eta_k)$ $(k=1,2,...,n)$ nuqta olib, berilgan funksiyaning $M_k(\xi_k,\eta_k)$ $(k=1,2,...,n)$ nuqtadagi $f(\xi_k,\eta_k)$ qiymatini (D_k) $(k=1,2,...,n)$ sohaning D_k yuziga ko'paytirib, quyidagi

$$\sigma = \sum_{k=1}^{n} f(\xi_k, \eta_k) D_k$$

yig'indini tuzamiz va bu yig'indini $f(x,y)$ funksiyaning *integral yig'indisi yoki Riman yig'indisi* deb ataymiz.

Ravshanki, integral yig'indi berilgan $f(x,y)$ funksiyaga, (D) sohani bo'laklash usuliga, hamda tanlangan $M_k(\xi_k,\eta_k)\in(D_k)$ nuqtaga bog'liq bo'ladi $\sigma_P(f;\xi_k,\eta_k)$. Uni qisqacha σ_P deb belgilaymiz.

14.2.1-ta'rif. Agar (D) sohaning har qanday $P_1, P_2, ..., P_n, ...$ bo'linishlar ketma-ketligi $\{P_n\}$ olinganda ham, unga mos kelgan $\sigma_1, \sigma_2, ..., \sigma_n, ...$ integral yig'indilar ketma-ketligi $\{\sigma_n\}$, $\lambda_P \to 0$ da $M_k(\xi_k,\eta_k)$ $(k = 1,2,...,n)$ nuqtalarni tanlab olishga bog'liq bo'lmasdan, hamma vaqt bitta I songa intilsa, bu I *son σ integral yig'indining limiti* deb ataladi va u

$$\lim_{\lambda_P \to 0} \sigma_P = \lim_{\lambda_P \to 0} \sum_{k=1}^{n} f(\xi_k, \eta_k) D_k = I$$

kabi belgilanadi.

Bu ta'rifni quyidagicha ham ta'riflash mumkin.

14.2.2-ta'rif. Agar $\forall \varepsilon > 0$ olinganda ham, shunday $\delta > 0$ topilsaki, (D) sohaning diametri $\lambda_P < \delta$ bo'lganda, har qanday P bo'linish hamda har bir (D_k) bo'lakdagi ixtiyoriy $M_k(\xi_k,\eta_k)$ nuqtalari uchun,

$$|\sigma - I| < \varepsilon$$

tengsizlik bajarilsa, u holda I *songa σ_P integral yig'indining limiti* deb ataladi va u $\lim\limits_{\lambda_P \to 0} \sigma_P = I$ kabi belgilanadi.

14.2.3-ta'rif. Agar $\lambda_P \to 0$ da $f(x,y)$ funksiyaning σ_P integral yig'indisi chekli limitga ega bo'lsa, $f(x,y)$ funksiya (D) sohada integrallanuvchi (Riman ma'nosida) funksiya deyiladi. Bu σ_P integral yig'indining chekli limiti I son esa, $f(x,y)$ funksiyaning (D) soha bo'yicha *ikki karrali integrali (Riman integrali)* deb ataladi va u

$$I = \lim_{\lambda_P \to 0} \sigma_P = \lim_{\lambda_P \to 0} \sum_{k=1}^{n} f(\xi_k, \eta_k) D_k = \iint_{(D)} f(x,y)\,dD$$

kabi belgilanadi.

Shunday qilib, 14.1 bandda keltirilgan (V) jismning hajmi $f(x,y)$ funksiyaning (D) soha bo'yicha ikki karrli integralini ifoda qilar ekan.

14.3. Ikki karrali integrallar uchun Darbu yig'indilari. $f(x,y)$ funksiya chegaralangan (D) ($(D)\subset R^2$) sohada berilgan va chegaralangan bo'lsin. (D) sohaning biror P bo'linishini qaraylik. Bu bo'linishning har bir (D_k) $(k=1,2,...,n)$ bo'lagida $f(x,y)$ funksiya chegaralangan bo'lib, uning aniq quyi va aniq yuqori chegaralari

$$m_k = \inf_{(x,y)\in(D_k)} \{f(x,y)\},\quad M_k = \sup_{(x,y)\in(D_k)} \{f(x,y)\}$$

mavjud bo'ladi. $f(x,y)$ funksiya (D) ($(D)\subset R^2$) sohada chegaralangan bo'lgani uchun u har bir (D_k) $(k=1,2,...,n)$ da ham chegaralangan bo'ladi:

$\forall(x,y)\in(D_k)$ $(k=1,2,...,n)$ uchun

$$m_k \le f(x,y) \le M_k \tag{14.3.1}$$

tengsizlik o'rinli bo'ladi.

14.3.2-ta'rif. Ushbu

$$s_P = \sum_{k=1}^{n} m_k D_k,\quad S_P = \sum_{k=1}^{n} M_k D_k$$

yig'indilar, mos ravishda, *Darbuning quyi hamda yuqori yig'indilari* deb ataladi.

Darbu yig'indilari berilgan $f(x,y)$ funsiyaga va (D) to'plamning bo'laknishiga bo'g'liq: $s_P = s_P(f)$, $S_P = S_P(f)$. Har doim $s_P(f) \le S_P(f)$ tengsizlik bajariladi. (14.3.1) ga asosan,

$$\sum_{k=1}^{n} m_k D_k \le \sum_{k=1}^{n} f(\xi_k,\eta_k) D_k \le \sum_{k=1}^{n} M_k D_k$$

tengsizlik hosil qilamiz. Bundan (D) sohaning berilgan bo'lishida (ξ_k,η_k) nuqtaning tanlanishiga bog'liq bo'lmagan holda

$$s_P(f) \le \sigma_P \le S_P(f)$$

tengsizlik o'rinli bo'ladi. (ξ_k,η_k) nuqtani tanlash hisobidan $f(\xi_k,\eta_k)$ qiymatini m_k va M_k larga yaqinlashtirish mumkin, shu bilan birga σ_P ni $s_P(f)$ va $S_P(f)$ yig'indilarga istalgancha yaqinlashtirish mumkin.

Shunday qilib, Darbuning quyi va yuqori yigindilari integral yig'indi uchun mos ravishda aniq quyi va aniq yuqori chegara bo'ladi.

(D) sohaning bo'linishlar to'plami $\Phi=\{P\}$ ning har bir $P\in\Phi$ bo'linishiga nisbatan $f(x,y)$ funksiyaning Darbu yig'indilari $s_P(f)$, $S_P(f)$ ni tuzib $\{s_P(f)\}$, $\{S_P(f)\}$ to'plamlarni qaraymiz. Ravshanki, bu to'plamlar chegaralangan bo'ladi.

14.3.3-ta'rif. $\{s_P\}$ $(\{S_P\})$ to'plamlarning aniq yuqori (aniq quyi) chegarasi, ya'ni $\sup\{s_P\}=\underline{J}$ $\left(\inf\{S_P\}=\overline{J}\right)$ miqdorlar, mos ravishda, $f(x,y)$ funksiyaning (D) sohadagi *quyi ikki karrali (yuqori ikki karrali) integrali* deb ataladi va u

$$\underline{I}=\underline{\iint_{(D)}} f(x,y)dD=\sup\{s_P\}\quad\left(\overline{I}=\overline{\iint_{(D)}} f(x,y)dD=\inf\{S_P\}\right)$$

kabi belgilanadi.

14.3.4-ta'rif. Agar $f(x,y)$ funksiyaning (D) soha bo'yicha olingan quyi hamda yuqori ikki karrali integrallari bir-biriga teng, ya'ni $\underline{J}=\overline{J}$ bo'lsa, $f(x,y)$ funksiya (D) sohada *integrallanuvchi* deb ataladi, ularning umumiy qiymati $\underline{J}=\overline{J}=J$ $f(x,y)$ funksiyaning (D) soha bo'yicha olingan ikki karrali deyiladi va u

$$J=\iint_{(D)} f(x,y)dD$$

kabi belgilanadi.

Agar $\underline{J}=\overline{J}$ bo'lsa, u holda $f(x,y)$ funksiya (D) sohada *integrallanmaydi* deyiladi.

Shunday qilib ikki karrali integralning ikki xil ta'rifi berildi. Bu ta'riflar ekvivalent ta'riflardir.

Aniq integralda $f(x,y)$ funksiyaning Darbu yig'indilarinig xossalari keltirilgan bo'lib, ular isbotlangan edi. Xuddi shunday xossalar $f(x,y)$ funksiya Darbu yig'indilari uchun ham o'rinli bo'ladi. Shuning uchun bu holda Darbu yig'indilarining xossalarini isbotsiz keltirish bilan kifoyalanamiz.

Darbu yig'indilari quyidagi xossalarga ega:

1^0. (D) sohaning bo'lish chiziqlariga yangi bo'lish chizig'ini qo'shish bilan Darbuning quyi yig'indisi kamaymaydi, yuqori yig'indisi esa ortmaydi.

Bu xossa (D) sohaning bo'linishidagi bo'laklar soni orta borganda ularga mos Darbuning quyi yig'indisi kamaymasligi, yuqori yig'indining esa, oshmasligini anglatadi.

2^0. (D) sohaning istalgan bo'linishiga mos kelgan Darbuning har bir quyi yig'indisi, har bir yuqori yig'indisidan ortmaydi.

Bu xossa (D) sohaning bo'linishlariga nisbatan tuzilgan quyi yig'indilar to'plami $\{s_P(f)\}$ ning har bir elementi yuqori yig'indilar to'plami $\{S_P(f)\}$ ning istalgan elementidan katta emasligini, yuqori yig'indilar to'plami $\{S_P(f)\}$ ning har bir elementi, quyi yig'indilar to'plami $\{s_P(f)\}$ ning har bir elementi kichik emasligini bildiradi.

3^0. Agar $f(x,y)$ funksiya (D) sohada chegaralangan bo'lsa, u holda $\sup\{s_P(f)\} \le \inf\{S_P(f)\}$ bo'ladi.

Bu xossa, $f(x,y)$ funksiyaning quyi ikki karrali integrali, uning yuqori ikki karrali integralidan katta emasligini bildiradi.

4^0. Agar $f(x,y)$ funksiya (D) sohada chegaralangan bo'lsa, u holda $\forall\varepsilon > 0$ son olinganda ham, shunday $\delta > 0$ son topiladiki, uning (D) sohaning diametri $\lambda_p < \delta$ bo'lgan barcha bo'linishlari uchun

$$S_P(f) < \overline{J} + \varepsilon \quad \left(0 \le S_P(f) - \overline{J} < \varepsilon\right),$$
$$s_P(f) > \underline{J} - \varepsilon \quad \left(0 \le \underline{J} - s_P(f) < \varepsilon\right)$$

bo'ladi.

Bu xossa $f(x,y)$ funksiyaning yuqori hamda quyi integrallari $\lambda_P \to 0$ da mos ravishda Darbuning yuqori hamda quyi yig'indilarning limiti ekanligini anglatadi:

$$\overline{J} = \lim_{\lambda_P \to 0} S_P(f), \ \underline{J} = \lim_{\lambda_P \to 0} s_P(f) \quad .$$

Yuqoridagi mulohazalardan

$$s_P \le \underline{J} \le \overline{J} \le S_P(f)$$

ekanligi kelib chiqadi.

14.4. Ikki karrali integralning mavjudlik sharti. $f(x,y)$ funksiya (D) sohada aniqlangan va chegaralangan bo'lsin.

14.4.1. Teorema (mavjudlik sharti). $f(x,y)$ funksiyaning (D) sohada integrallanuvchi bo'lishi uchun, $\forall\varepsilon>0$ olinganda ham, shunday $\delta>0$ topilib, (D) sohaning diametri $\lambda_p<\delta$ bo'lgan har qanday P bo'linishga nisbatan tuzilgan Darbu yig'indilari

$$S_P(f)-s_P(f)<\varepsilon$$

tengsizlikni qanoatlantirishi zarur va yetarli.

14.5. Integrallanuvchi funksiyalarning sinflari.

14.5.1-teorema. Agar $f(x,y)$ funksiya chegaralangan yopiq (D) ($(D)\subset R^2$) sohada berilgan va uzluksiz bo'lsa, u shu sohada integrallanuvchi bo'ladi.

14.5.2-lemma. (D)sohada nol yuzga ega bo'lgan (L) chiziq berilgan bo'lsin. U holda $\forall\varepsilon>0$ son olinganda ham, shunday $\delta>0$ son topiladiki, (D) sohaning diametri $\lambda_p<\delta$ bo'lgan P bo'linishi olinganda, bu bo'linishning (L) chiziq bilan umumiy nuqtaga ega bo'lgan bo'laklari yuzlarining yig'indisi ε dan kichik bo'ladi.

14.5.3-teorema. Agar $f(x,y)$ funksiya (D) sohada chegaralangan va bu sohaning chekli sondagi nol yuzali chiziqlarida uzilishga ega bo'lib, (D) yopiq sohaning qolgan barcha nuqtalarida uzluksiz bo'lsa, bu funksiya (D) yopiq sohada integrallanuvchi bo'ladi.

14.6. Ikki karrali integralning xossalari. Ikki karrali integral ham aniq integral singari qator xossalarga ega. Bu xossalar ham, aniq integral xossalari singari isbot qilinishini e'tborga olgan holda ularning ba'zilarini isbot qilish bilan chegaralanib, qolganlarini isbot qilishni o'quvchiga havola etamiz.

1°. $f(x,y)$ funksiya $(D)((D)\subset R^2)$ sohada integrallanuvchi bo'lsin. Bu funksiyaning (D) sohada to'lasincha yotuvchi nol yuzaga ega bo'lgan Γ chiziqdagi $(\Gamma\subset(D))$ qiymatlarinigina (chegaralan-ganligini saqlagan holda) o'zgartirishdan hosil bo'lgan $F(x,y)$ funksiya ham (D) sohada integrallanuvchi bo'ladi va

$$\iint\limits_{(D)} f(x,y)dD=\iint\limits_{(D)} F(x,y)dD$$

tenglik o'rinli.

2°. $f(x,y)$ funksiya (D) sohada berilgan bo'lib, (D) soha nol yuzali Γ chiziq yordamida (D_1) va (D_2) sohalarga ajralgan bo'lsin. Agar $f(x,y)$ funksiya (D) sohada integrallanuvchi bo'lsa, u (D_1) va (D_2) sohalarda ham integrallanuvchi bo'ladi va aksinch (D_1) va (D_2) sohalarda integrallanuvchi bo'lsa, u holda (D) sohada ham integrallanuvchi bo'ladi hamda

$$\iint_{(D)} f(x,y)dD = \iint_{(D_1)} f(x,y)dD + \iint_{(D_2)} f(x,y)dD$$

munosabat o'rinli.

3°. Agar $f(x,y)$ funksiya (D) sohada integrallanuvchi bo'lsa, u holda $C \cdot f(x,y)$ $(C = const)$ funksiya ham shu sohada integrallanuvchi bo'ladi va

$$\iint_{(D)} C \cdot f(x,y)dD = C\iint_{(D)} f(x,y)dD$$

formula o'rinli.

4°. Agar $f(x,y)$ va $g(x,y)$ funksiyalar (D) sohada integrallanuvchi bo'lsa, u holda $f(x,y) \pm g(x,y)$ funksiya ham shu sohada integrallanuvchi bo'ladi va

$$\iint_{(D)} [f(x,y) \pm g(x,y)]dD = \iint_{(D)} f(x,y)dD \pm \iint_{(D)} g(x,y)dD$$

formula o'rinli.

14.6.1-natija. Agar $f_1(x,y), f_2(x,y),...,f_n(x,y)$ funksiyalarning har biri (D) sohada integrallanuvchi bo'lsa, u holda

$$\sum_{k=1}^{n} C_k \cdot f_k(x,y) (C_k = const,\ k = 1,2,...,n)$$

funksiya ham shu sohada integrallanuvchi bo'ladi va

$$\iint_{(D)} \sum_{k=1}^{n} C_k \cdot f_k(x,y)dD = \sum_{k=1}^{n} C_k \cdot \iint_{(D)} f_k(x,y)dD$$

tenglik o'rinli.

5°. Agar $f(x,y)$ funksiya (D) sohada integrallanuvchi bo'lib, $\forall(x,y)\in(D)$ uchun $f(x,y)\geq 0$ bo'lsa, u holda

$$\iint_{(D)} f(x,y)dD \geq 0$$

tengsizlik bajariladi.

14.6.2-natija. Agar $f(x,y)$ va $g(x,y)$ funksiyalar (D) sohada integrallanuvchi bo'lib, $\forall(x,y)\in(D)$ uchun $f(x,y)\leq g(x,y)$ bo'lsa, u holda

$$\iint_{(D)} f(x,y)dD \leq \iint_{(D)} g(x,y)dD$$

tengsizlik o'rinli.

6°.Agar $f(x,y)$ funksiya (D) sohada integrallanuvchi bo'lsa, u holda $|f(x,y)|$ funksiya ham shu sohada integrallanuvchi bo'ladi va

$$\left|\iint_{(D)} f(x,y)dD\right| \leq \iint_{(D)} |f(x,y)|dD$$

tengsizlik o'rinli.

7°. *O'rta qiymat haqidagi teorema.* $f(x,y)$ funksiya yuzaga ega bo'lgan (D) sohada aniqlangan va chegralangan bo'lsin, ya'ni $\exists M, m\in R, \forall(x,y)\in(D)$ uchun: $m\leq f(x,y)\leq M$. Agar $f(x,y)$ funksiya (D) sohada integrallanuvchi bo'lsa, u holda shunday o'zgarmas μ son $m\leq\mu\leq M$ mavjudki,

$$\iint_{(D)} f(x,y)dD = D$$

formula o'rinli, bu yerda D -(D) sohaning yuzi.

14.6.3-natija. Agar $f(x,y)$ funksiya yopiq (D) sohada uzluksiz bo'lsa, u holda shunday $(a,b)\in(D)$ nuqta topiladiki,

$$\iint_{(D)} f(x,y)dD = f(a,b)D$$

tenglik o'rinli bo'ladi.

8^0.*O'rta qiymat haqidagi umumlashgan teorema.* Agar $g(x,y)$ funksiya (D) sohada integrallanuvchi bo'lib, u shu sohada o'z ishorasini o'zgartirmasa hamda $f(x,y)$ funksiya (D) sohada uzluksiz bo'lsa, u holda shunday $(a,b)\in(D)$ nuqta topiladiki,

$$\iint\limits_{(D)} f(x,y)g(x,y)dD = f(a,b)\iint\limits_{(D)} g(x,y)dD$$

tenglik o'rinli bo'ladi.

Biz kelgusida kerak bo'ladigan ba'zi tengliklarni keltiramiz:

$$\sum_{i=1}^{n} i = \frac{1}{2}n(n+1),\quad \sum_{i=1}^{n}(i+a) = \frac{1}{2}n(n+1)+an, \tag{14.6.4}$$

$$\sum_{i=1}^{n}(2i-1) = n^2\ ,$$

$$\sum_{i=1}^{n} i^2 = \frac{1}{6}n(n+1)(2n+1),$$

$$\sum_{i=1}^{n} i^3 = \frac{1}{4}n^2(n+1)^2,$$

$$\sum_{k=1}^{n}\sin k\alpha = \frac{\sin\frac{n+1}{2}\alpha\sin\frac{n}{2}\alpha}{\sin\frac{\alpha}{2}}, \forall\alpha\neq 2m\pi, m\in Z,$$

$$\sum_{k=1}^{n}\cos k\alpha = \frac{\sin\frac{n}{2}\alpha\cos\frac{n+1}{2}\alpha}{\sin\frac{\alpha}{2}}, \forall\alpha\neq 2m\pi, m\in Z,$$

$$\sum_{k=1}^{n}\sin(x+k\alpha) = \frac{\sin\frac{n}{2}\alpha\sin(x+\frac{n+1}{2}\alpha)}{\sin\frac{\alpha}{2}}, \forall\alpha\neq 2m\pi, m\in Z$$

$$\sum_{k=1}^{n}\cos(x+k\alpha) = \frac{\sin\frac{n}{2}\alpha\cos(x+\frac{n+1}{2}\alpha)}{\sin\frac{\alpha}{2}}, \forall\alpha\neq 2m\pi, m\in Z;$$

14.2-§. Ikki karrali integrallarni hisoblash va o'zgaruvchilarni almashtirish

Yuqorida $f(x,y)$ funksiyaning (D) sohada ikki karrali integrali integral yig'indining limiti sifatida ta'riflandi. Bu limit murakkab xarakterga ega bo'lib, uni shu ta'rif bo'yicha hisoblash sodda hollarda ham ancha qiyinchilik tug'diradi.

14.7. Ikki karrali integrallarni takroriy integrallarga keltirib hisoblash.

Soha to'g'ri to'rt to'rtburchak bo'lgan holda ikki karrali integralni hisoblash

14.7.1-teorema. $f(x,y)$ funksiya $(D)=\{(x,y)\in R^2 : a\le x\le b,\ c\le y\le d\}$ sohada berilgan va integrallanuvchi bo'lsin. Agar x $(x\in[a,b])$ o'zgaruvchining har bir tayin qiymatida $J(x)=\int\limits_c^d f(x,y)dy$ integral mavjud bo'lsa, u holda

$$\int\limits_a^b J(x)dx=\int\limits_a^b\left[\int\limits_c^d f(x,y)dy\right]dx$$

integral ham mavjud bo'ladi va

$$\iint\limits_{(D)} f(x,y)dD=\int\limits_a^b\left[\int\limits_c^d f(x,y)dy\right]dx \qquad (14.7.2)$$

formula o'rinli.

14.7.3-teorema. $f(x,y)$ funksiya (D) sohada berilgan va integrallanuvchi bo'lsin. Agar $y\in[c,d]$ o'zgaruvchining har bir tayin qiymatida $I(y)=\int\limits_a^b f(x,y)dx$ integral mavjud bo'lsa, u holda $\int\limits_c^d\left[\int\limits_a^b f(x,y)dx\right]dy$ integral ham mavjud bo'ladi va

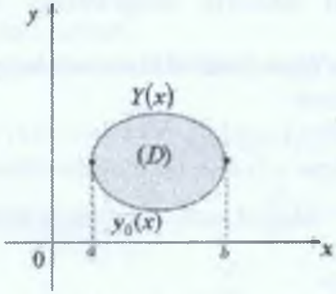
$$\iint\limits_{(D)} f(x,y)dD=\int\limits_c^d\left[\int\limits_a^b f(x,y)dx\right]dy$$

formula o'rinli.

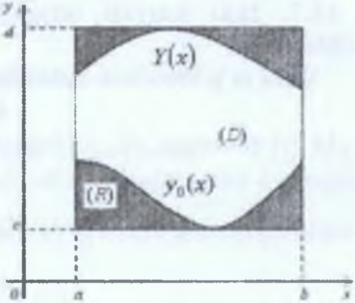
14.7.4-natija. Agar $f(x,y)$ funksiya chegaralangan yopiq $(D)((D)\subset R^2)$ sohada berilgan va uzluksiz bo‘lsa,

$$\iint\limits_{(D)} f(x,y)dD, \quad \int\limits_a^b\left[\int\limits_c^d f(x,y)dy\right]dx, \quad \int\limits_c^d\left[\int\limits_a^b f(x,y)dx\right]dy$$

integrallarning har biri mavjud va ular o‘zaro teng bo‘ladi.



14.2-chizma.



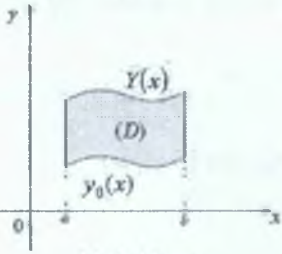
14.3-chizma.

Soha egri chiziq bo‘lgan holda ikki karrali integralni hisoblash

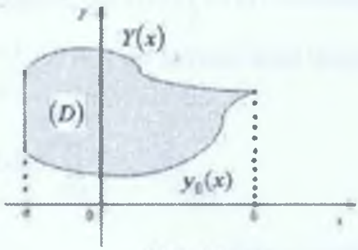
(D) soha, yuqoridan $y=Y(x)$, pastdan $y=y_0(x)$ egri chiziqlar bilan, yon tomonlardan esa $x=a$ va $x=b$ to‘g‘ri chiziqlar bilan chegaralangan soha, ya’ni

$$(D)=\{(x,y)\in R^2 : a\le x\le b, \quad y_0(x)\le y\le Y(x)\}$$

bo‘lsin, bunda $y_0(x)$ va $Y(x)$ funksiyalar $[a,b]$ da uzluksiz deb faraz qilinadi (14.2-,14.3-,14.4-chizma).



14.4-chizma.



14.5-chizma.

14.7.5-teorema. $f(x,y)$ funksiya

$$(D)=\{(x,y)\in R^2 : a\le x\le b,\ y_0(x)\le y\le Y(x)\},\ y_0(x), Y(x)\in C([a,b])$$

sohada berilgan va integrallanuvchi bo'lsin (14.5-chizma). Agar $x\in[a,b]$ o'zgaruvchining har bir tayin qiymatida $J(x)=\int\limits_{y_0(x)}^{Y(x)} f(x,y)dy$ integral mavjud bo'lsa, u holda $\int\limits_a^b dx \int\limits_{y_0(x)}^{Y(x)} f(x,y)dy$ integral ham mavjud bo'ladi va

$$\iint\limits_{(D)} f(x,y)dxdy=\int\limits_a^b dx \int\limits_{y_0(x)}^{Y(x)} f(x,y)dy \tag{14.7.6}$$

tenglik o'rinli.

Agar (D) soha ushbu $(D)=\{(x,y)\in R^2 : c\le y\le d,\ x_0(y)\le x\le X(y)\}$ (14.13-chizma) ko'rinishda bo'lsa, u holda quyidagi teorema o'rinli bo'ladi.

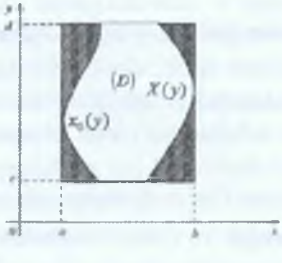
14.7.7-teorema. $f(x,y)$ funksiya

$$(D)=\{(x,y)\in R^2 :\ c\le y\le d,\ x_0(y)\le x\le X(y)\},(x_0(y), X(y)\in C[c;d])$$

sohada berilgan va integrallanuvchi bo'lsin. Agar $y\in[c,d]$ o'zgaruvchining har bir tayin qiymatida $J(y)=\int\limits_{x_0(y)}^{X(y)} f(x,y)dx$ integral mavjud bo'lsa, u holda ushbu $\int\limits_c^d dy \int\limits_{x_0(y)}^{X(y)} f(x,y)dx$ integral ham mavjud bo'ladi va

$$\iint\limits_{(D)} f(x,y)dxdy=\int\limits_c^d dy \int\limits_{x_0(y)}^{X(y)} f(x,y)dx \tag{14.7.8}$$

tenglik o'rinli.

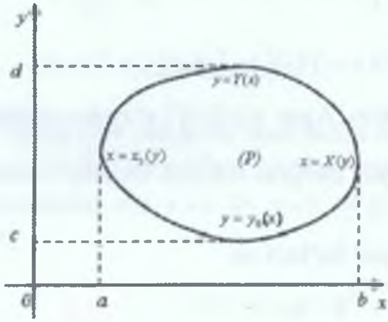


14.6-chizma

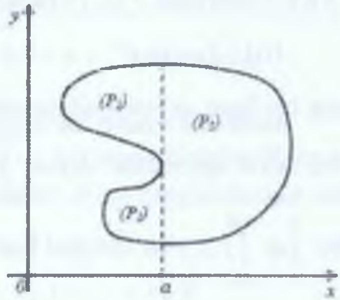
14.7.9-eslatma. Agar (D) sohaning chegarasi OY o'qiga hamda OX o'qiga parallel kesishsa (misol uchun 14.6-chizmadagidek) u holda 14.7.5 va 14.7.7- teoremalardagi shartlar bajarilganda (14.7.6) va (14.7.8) formulalar o'rinli bo'ladi. Ularni solishtirish natijasida

$$\int_a^b dx \int_{y_0(x)}^{Y(x)} f(x,y)dy = \int_c^d dy \int_{x_0(y)}^{X(y)} f(x,y)dx.$$

Agar (D) sohada $f(x,y)$ funksiya uzluksiz bo'lsa, u holda $\iint\limits_{(D)} f(x,y)dD$, $\int_{y_0(x)}^{Y(x)} f(x,y)dy$, $\int_{x_0(y)}^{X(y)} f(x,y)dx$ integrallar mavjud bo'ladi va ikki karrali integral (14.7.6) va (14.7.8) formulalar bilan hisoblanadi (14.7-chizma).



14.7-chizma.



14.8-chizma.

Agar (D) murakkab soha bo'lsa, u holda bu soha yuqorida o'rganilgan sohalar ko'rinishiga keladigan qilib, chekli sondagi bo'laklarga ajratiladi, misol uchun 14.8-chizmadagidek. Natijada hisoblash talab qilingan ikki karrali integral, ajratilgan sohalar bo'yicha olingan ikki karrali integrallar yig'indisiga teng bo'ladi.

14.7.9-eslatma. 1^0. Agar $f(x,y)$ funksiya Oy o'qqa nisbatan simmetrik bo'lgan (D) sohada aniqlangan va integrallanuvchi bo'lib, x o'zgaruvchiga nisbatan $f(x,y)$ juft funksiya, ya'ni $f(x,y)=f(-x,y)$ bo'lsa, u holda quyidagi

$$\iint\limits_{(D)} f(x,y)dxdy = 2\iint\limits_{(D_1)} f(x,y)dxdy$$

formula o'rinli bo'ladi, bunda $(D_1)=(D)\cap\{(x,y): x\geq 0\}$.

2^0. Agar $f(x,y)$ funksiya Oy o'qqa nisbatan simmetrik bo'lgan (D) sohada aniqlangan va integrallanuvchi bo'lib, x o'zgaruvchiga nisbatan $f(x,y)$ toq funksiya, ya'ni $f(x,y)=-f(-x,y)$ bo'lsa, u holda quyidagi

$$\iint\limits_{(D)} f(x,y)dxdy = 0$$

formula o'rinli bo'ladi.

3^0. Agar $f(x,y)$ funksiya Ox o'qqa nisbatan simmetrik bo'lgan (D) sohada aniqlangan va integrallanuvchi bo'lib, y o'zgaruvchiga nisbatan $f(x,y)$ juft funksiya, ya'ni $f(x,y)=f(x,-y)$ bo'lsa, u holda quyidagi

$$\iint\limits_{(D)} f(x,y)dxdy = 2\iint\limits_{(D_1)} f(x,y)dxdy$$

formula o'rinli bo'ladi, bunda $(D_1)=(D)\cap\{(x,y): y\geq 0\}$.

4^0. Agar $f(x,y)$ funksiya Ox o'qqa nisbatan simmetrik bo'lgan (D) sohada aniqlangan va integrallanuvchi bo'lib, y o'zgaruvchiga nisbatan $f(x,y)$ toq funksiya, ya'ni $f(x,y)=-f(x,-y)$ bo'lsa, u holda quyidagi

$$\iint\limits_{(D)} f(x,y)dxdy = 0$$

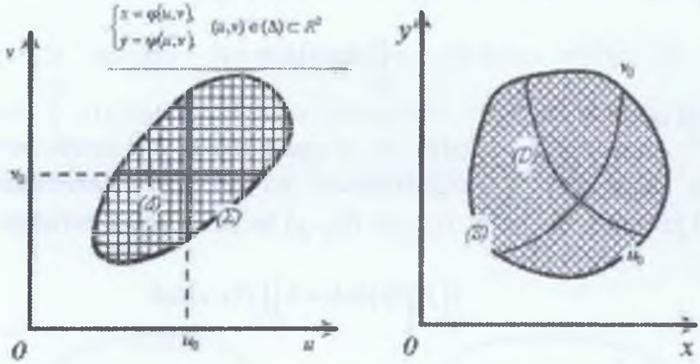
formula o'rinli bo'ladi.

14.8. Ikki karrali integrallarda o'zgaruvchilarni almashtirish. $f(x,y)$ funksiya (D) sohada berilgan va uning chekli

$$\iint\limits_{(D)} f(x,y)dx\,dy \tag{14.8.1}$$

ikki karrali integrali mavjud va uni hisoblash talab qilingan bo'lsin. Agar $f(x,y)$ funksiya va (D) soha murakkab bo'lsa, (14.8.1) integralni hisoblash qiyinlashadi. Bunday hollarda x va y o'zgaruvchilarni ma'lum *qoidaga* ko'ra, boshqa o'zgaruvchilarga almashtirish natijasida integral ostidagi funksiya ham, integrallash sohasi ham soddalashib, ikki karrali integralni hisoblash osonlashadi.

xOy hamda uOv koordinatalar sistemasida (D) va (Δ) chegaralangan sohalar berilgan bo'lsin. Bu sohalarning chegaralari, mos ravishda, $(S)=\partial(D)$ va $(\Sigma)=\partial(\Delta)$ lar, sodda, bo'lakli silliq chiziqlardan iborat bo'lsin. (Δ) sohada



14.9- chizma.

$$\begin{cases} x=\varphi(u,v), \\ y=\psi(u,v), \end{cases} \quad (u,v)\in(\Delta)\subset R^2 \qquad (14.8.2)$$

uzluksiz funksiyalar sistemasi berilgan bo'lsin. Bu funksiyalar shunday funksiyalar bo'lsinki, ulardan tuzilgan (14.8.2) sistema (Δ) dagi (u,v) nuqtani (D) sohadagi (x,y) nuqtaga akslantirsin va bu akslantirishning akslaridan iborat $\{(x,y)\}$ to'plam (D) ga qarashli bo'lsin.

Demak, (14.8.2) sistema (Δ) sohani (D) sohaga akslantiradi (14.9-chizma).

(14.8.2) akslantirish quyidagi shartlarni qanoatlantirsin:

1°. (14.8.2) akslantirish o'zaro bir qiymatli bo'lsin, ya'ni (Δ) sohaning turli nuqtalarini (D) sohaning turli nuqtalariga akslantirsin. (D) sohaning har bir nuqtasi uchun (Δ) sohada unga bitta nuqta mos kelsin. Bu holda (14.8.2) sistema u va v larga nisbatan bir qiymatli yechiladi:

$$\begin{cases} u=\varphi_1(x,y), \\ v=\psi_1(x,y), \end{cases} \quad (x,y)\in(D)\subset R^2 . \qquad (14.8.3)$$

(14.8.3) akslantirish (14.8.2) akslantirishga nisbatan teskari akslantirish bo'lib, (D) sohani (Δ) sohaga akslantiradi, ya'ni

$$\begin{cases} x\equiv\varphi(\varphi_1(x,y),\psi_1(x,y)), \\ y\equiv\psi(\varphi_1(x,y),\psi_1(x,y)), \end{cases} \quad (x,y)\in(D)\subset R^2. \qquad (14.8.4)$$

2°. $\varphi(u,v)$, $\psi(u,v)$ funksiyalar (Δ) sohada, $\varphi_1(x,y)$, $\psi_1(x,y)$ funksiyalar esa, (D) sohada uzluksiz va barcha xususiy hosilalarga ega bo'lib, bu xususiy hosilalar ham uzluksiz bo'lsin.

3°. $\forall(u,v)\in(\Delta)$ uchun (14.8.2) sistemadagi funksiyalarning xususiy hosilalaridan tuzilgan ikkinchi tartibli determinant

$$I(u,v)=\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}=\frac{\partial x}{\partial u}\frac{\partial y}{\partial v}-\frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\neq 0 \qquad (14.8.5)$$

shartni qanoatlantirsin. Odatda, (14.8.5) ikkinchi tartibli determinant-(14.8.2) sistemaning *yakobiani* deyiladi va $I(u,v)$ yoki $\frac{D(x,y)}{D(u,v)}$ kabi belgilanadi.

14.8.6-teorema. $f(x,y)$ funksiya (D) sohada berilgan va uzluksiz bo'lib, (14.8.2) akslantirish 1°-, 2°-, 3°- shartlarni qanoatlantirsin. U holda

$$\iint_{(D)} f(x,y)\,dx\,dy=\iint_{(\Delta)} f(\varphi(u,v),\psi(u,v))|I(u,v)|\,du\,dv \qquad (14.8.7)$$

formula o'rinli. (14.8.7) formula ikki karrali integrallarda *o'zgaruvchilarni almashtirish formulasi* deyiladi.

Faraz qilaylik (14.8.2) sistema (Δ) sohani (D) sohaga akslantirib, u yuqoridagi 1) -3) shartlarni qanoalantirsin. U holda, (D) sohaning yuzi

$$D=\iint_{(D)}|J(u,v)|\,dudv=\iint_{(\Delta)}\left|\frac{D(x,y)}{D(u,v)}\right|dudv \qquad (14.8.8)$$

ga teng bo'ladi. $f(x,y)$ funksiya (D) sohada aniqlangan va uzluksiz bo'lsin.

(D) bo‘lakli-silliq chiziq bilan chegaralangan soha bo‘lsin. U holda $f(x,y)$ funksiya (D) da integrallanuvchi bo‘ladi.

(Δ) sohani bo‘lakli-silliq chiziqlar yordamida bo‘laklarga bo‘lamiz, ya’ni P_Δ bo‘linishini qaraylik. (14.8.2) akslantirish natijasida (D) sohaning P_D bo‘linishi hosil bo‘ladi. Bu bo‘linishga nisbatan $f(x,y)$ funksiyaning

$$\sigma_{P_D}(f)=\sum_{i=1}^{n} f(u_i,v_i)D_i$$

integral yig‘indisini tuzamiz. Ma’lumki,

$$\lim_{\lambda_{P_D}\to 0}\sigma_{P_D}(f)=\lim_{\lambda_{P_D}\to 0}\sum_{i=1}^{n} f(u_i,v_i)D_i=\iint_{(D)} f(x,y)dxdy. \qquad (14.8.9)$$

Yuqoridagi (14.8.8) formulaga ko‘ra,

$$D_i=\iint_{(D)}|J(u,v)|dudv$$

bo‘ladi. O‘rta qiymat haqidagi teoremaga asosan,

$$D_i=\left|J(u^*_i,v^*_i)\right|\Delta_i,(u^*_i,v^*_i)\in(\Delta_i).$$

Buni e’tiborga olgan holda (14.8.8) integral yig‘indi

$$\sigma_{P_D}(f)=\sum_{i=1}^{n} f(u_i,v_i)\cdot\left|J(u^*_i,v^*_i)\right|\Delta_i$$

ko‘rinishga keladi. (u_i,v_i) nuqta (D_i) sohadagi ixtiyoriy nuqta ekanligidan foydalanib, uni

$$\begin{aligned}\varphi(u^*_i,v^*_i)&=u_i\\ \psi(u^*_i,v^*_i)&=v_i\end{aligned},$$

deb olish mumkin. U holda

$$\sigma_p(f) = \sum_{i=1}^{n} f(\phi(u_i^*, v_i^*), \psi(u_i^*, v_i^*))|J(u_i^*, v_i^*)|\Delta_i$$

bo'ladi. Bundan $f(\varphi(u,v),\psi(u,v))|J(u,v)|$ funksiya (Δ) da uzluksiz bo'lgani uchun

$$\lim_{\lambda_p \to 0} \sigma_p(f) = \lim_{\lambda_p \to 0} \sum_{i=1}^{n} f \sum_{i=1}^{n} f(\phi(u_i^*, v_i^*), \psi(u_i^*, v_i^*))|J(u_i^*, v_i^*)|\Delta_i =$$
$$= \iint_{(\Delta)} f(x(u,v), y(u,v))|J(u,v)|dudv. \qquad (14.8.10)$$

$\lambda_{p_\Delta} \to 0$ da $\lambda_{p_D} \to 0$ bo'lgani uchun (14.8.9) va (14.8.10) dan (14.8.7) o'rinli bo'lishi kelib chiqdi.■

14.9. Dekart koordinatalar sistemasidan qutb koordinatalar sistemasiga o'tish. (r,θ) - qutb koordinatalar sistemasi berilgan bo'lsin. Bu sistemada $\theta = const$ munosabat 0 qutbdan chiqib qutb o'qining musbat yo'nalishi bilan (soat strelkasi yo'nalishiga qarama-qarshi) θ burchak tashkil etuvchi nurlar oilasini ifodalaydi $(0 \le \theta \le 2\pi)$. $r = const$ munosabat esa, markazi 0 qutbda va radiusi r ga teng bo'lgan konsentrik aylanalar oilasini ifodalaydi, unda $r \ge 0$. Chiziqlarning yuqorida ko'rsatilgan ikki oilasi butun tekislikni yuzachalarga bo'ladi (14.10-chizma).

Har bir yuza ikki aylana $(r = r_1, r = r_2)$ va ikki nur $(\theta = \theta_1, \theta = \theta_2)$ bilan chegaralangan bo'lib, uning yuzi uchun $\Delta S = rd\theta dr$ formula (yuqori tartibli cheksiz kichik miqdorlar aniqligida) o'rinli. Agar ikki karrali integralda integrallash o'zgaruvchilari

14.10-chizma.

dekart koordinatalaridan iborat bo‘lsa, unda (x,y) dekart koordinatalarini (r,θ) qutb koordinatalariga

$$x = r\cos\theta,\ y = r\sin\theta\ (0 \le r < \infty,\ 0 \le \theta \le 2\pi) \tag{14.9.1}$$

formulalar bo‘yicha almashtiriladi. Bu holda funksional determinant

$$J(r,\theta) = \frac{D(x,y)}{D(r,\theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\,.$$

Natijada (14.8.7) formula ushbu

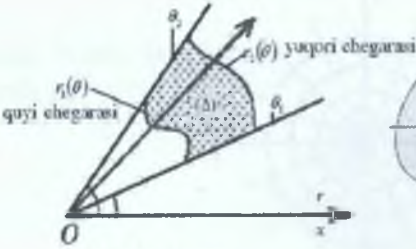
$$\iint_{(D)} f(x,y)dx\,dy = \iint_{(\Delta)} f(r\cos\theta, r\sin\theta)rdrd\theta \tag{14.9.2}$$

ko‘rinishni oladi. Odatda, (14.9.2) munosabat, ikki karrali integralda *qutb koordinatalar sistemasiga o‘tish formulasi* deyiladi.

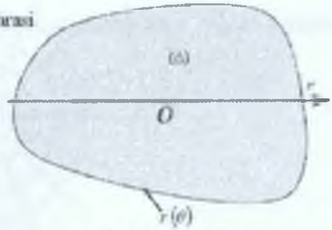
14.9.3-eslatma. 1^0. Agar (Δ) soha $r_1(\theta)$ va $r_2(\theta)$ (bunda $\theta_1 \le \theta \le \theta_2$, $r_1 < r_2$) chiziqlar hamda qutb o‘q bilan θ_1 va θ_2 burchaklar tashkil qiluvchi nurlar bilan chegaralangan bo‘lsa (14.11-chizma), u holda

$$\iint_{(D)} f(x,y)dx\,dy = \iint_{(\Delta)} f(r\cos\theta, r\sin\theta)rdrd\theta = \int_{\theta_1}^{\theta_2} d\theta \int_{r_1(\theta)}^{r_2(\theta)} f(r\cos\theta, r\sin\theta)rdrd\theta$$

formula o‘rinli bo‘ladi.



14.11-chizma.



14.12-chizma.

2^0. Agar (Δ) soha $r(\theta)$ egri chiziq bilan chegaralangan va qutb boshini o'z ichida saqlasa (14.12-chizma), u holda

$$\iint_{(D)} f(x,y)dx\,dy = \iint_{(\Delta)} f(r\cos\theta,\, r\sin\theta)rdrd\theta = \int_0^{2\pi} d\theta \int_0^{r(\theta)} f(r\cos\theta,\, r\sin\theta)rdrd\theta$$

formula o'rinli bo'ladi.

14.9.4-eslatma. Agar ikki karrali integral ostidagi funksiya yoki integrallanuvchi soha chegarasining tenglamalari x^2+y^2 yig'indi orqali ifodalangan bo'lsa, ko'p hollarda (14.9.1) almashtirish yordamida qutb koordinatalar sistemasiga o'tib ikki karrali integralni hisoblash osonlashadi.

14.3-§. Ikki karrali integralning ba'zi bir tatbiqlari

14.10. Ikki karrali integral yordamida jismning hajmini hisoblash. R^3 fazoda yuqoridan $z=f(x,y)$ sirt bilan, yon tomonlaridan yasovchilari Oz o'qqa parallel bo'lgan silindrik sirt hamda pastdan xOy tekislikdagi (D) soha bilan chegaralangan (V) jismning V hajmi ushbu

$$V = \iint_{(D)} f(x,y)dx\,dy \tag{14.10.1}$$

formula yordamida hisoblanadi.

14.11. Ikki karrali integral yordamida silliq sirtning yuzini hisoblash. (S) – sirt tenglamasi $z=f(x,y)$ oshkor ko'rinishda berilgan bo'lsin. Bu sirtning xOy tekislikdagi proyeksiyasi kvadratlanuvchi (D) soha bo'lib, (D) sohaning har bir nuqtasida $z=f(x,y)$ funksiya uzluksiz $p=\frac{\partial z}{\partial x}$, $q=\frac{\partial z}{\partial y}$ xususiy hosilalarga ega bo'lsin. Bu holda (S) – silliq sirtning yuzi

$$S = \iint_{(D)} \sqrt{1+p^2+q^2}\,dxdy \tag{14.11.1}$$

formula orqali hisoblanadi.

14.12. Tekis shaklning yuzi. Ikki karrali integrallar yordamida tekis shaklning yuzasini topish mumkin. Integralning ta'rifidan (D) shaklning yuzi uchun

$$D=\iint\limits_{(D)} dxdy \qquad (14.12.1)$$

formula bevosita kelib chiqadi.

14.13. Ikki karrali integrallar yordamida mexanikaga oid masalalarni yechish. 1. Faraz qilaylik, bir jinsli (D)-xOy tekislikda berilgan zichligi $\rho(x,y)$ ga teng bo'lgan, plastinka berilgan bo'lsin. U holda plastinkaning massasi ushbu

$$m=\iint\limits_{(D)} \rho(x,y)dxdy$$

formula yordamida hisoblanadi.

2. Plastinkaning Ox va Oy o'qlarga nisbatan statik momentlari

$$M_x=\iint\limits_{(D)} y\,\rho(x,y)dxdy,\ M_y=\iint\limits_{(D)} x\,\rho(x,y)dxdy$$

formulalar yordamida hisoblanadi.

3. Plastinka og'irlik markazining koordinatalari esa,

$$x_0=\frac{M_y}{m},\qquad y_0=\frac{M_x}{m}$$

formulalar yordamida hisoblanadi.

4. Plastinkaning Ox va Oy o'qlarga nisbatan inersiya momentlari

$$I_x=\iint\limits_{(D)} y^2\rho(x,y)dxdy,\quad I_y=\iint\limits_{(D)} x^2\rho(x,y)dxdy$$

formulalar yordamida hisoblanadi.

5. Plastinkaning koordinatalar boshiga nisbatan inersiya momenti

$$I_o = I_x + I_y = \iint\limits_{(D)} (y^2 + x^2)\rho(x, y)dxdy$$

formula yordamida hisoblanadi.

14.4-§. Uch karrali integrallar

Biz yuqorida ikki o'zgaruvchili funksiyaning Riman integrali tushunchasi qanday kiritilishini ko'rib o'tdik va uni batafsil o'rgandik. Huddi shunday uch o'zgaruvchili hamda n o'zgaruvchili funksiyalarning Riman integrallari tushunchasi kiritiladi va ikki karrali integralda kiritilgan mulohaza va tasdiqlarning deyarli hammasi o'zgarishsiz qaytariladi. Shuning uchun biz uch karrali integral to'g'risidagi faktlar, tasdiqlarni isbotsiz keltirish bilan chegaralanamiz. Teorema va tasdiqlarning isbotini o'quvchiga havola hilamiz.

14.14. Uch karrali integralning ta'rifi. R^3 fazoda chegaralangan (V) sohada $f(x,y,z)$ funksiuya berilgan bo'lsin. Biz bu erda va kelgusidsida hamma vaqt funksiyaning aniqlanish sohasi (V)ni hajimga ega bo'lgan soha deb qaraymiz. R^3 fazoda biror kublanuvchi (ochiq yoki yopiq) (V) soha berilgan bo'lib, bu sohada chegaralangan $f(x,y,z)$ funksiya berilgan bo'lsin. (V) sohaning ixtiyoriy P bo'linishini olib, bu bo'linishning har bir (V_i), bo'lagidan $\forall M_i(\xi_i,\eta_i,\varsigma_i)$ $(i=1,2,...,n)$, nuqtani olib, berilgan funksiyaning $M_i(\xi_i,\eta_i,\varsigma_i)$ nuqtadagi $f(\xi_i,\eta_i,\varsigma_i)$ qiymatini (V_i) sohaning V_i hajmiga ko'paytirib, quyidagi

$$\sigma_p(f) = \sum_{i=1}^{n} f(\xi_i,\eta_i,\varsigma_i)V_i \qquad (14.14.1)$$

integral yig'indini tuzamiz. Odatda (14.14.1) yig'indi $f(x,y,z)$ funksiyaning Riman *integral yig'indisi* deb ataladi.

(V) sohaning shunday

$$P_1, P_2, ..., P_m, .. \qquad (14.14.2)$$

bo'linishlar ketma-ketligini qaraylikki, ularning diametrlaridan tashkil topgan

$$\lambda_{p_1}, \lambda_{p_2}, \dots, \lambda_{p_m}, \dots$$

ketma - ketlik nolga intilsin. (14.14.2) bo‘linishlar ketma-ketligiga mos ushbu

$$\sigma_{p_1}, \sigma_{p_2}, \dots, \sigma_{p_m}, \dots \qquad (14.14.3)$$

integral yig‘indilar ketma-ketligini tuzamiz.

14.14.4-ta'rif. Agar (V) sohaning har qanday (14.14.2) ko‘rinishdagi bo‘linishlar ketma-ketligi olinganda ham, unga mos (14.14.3) integral yig‘indilar ketma-ketligi $M_i(\xi_i, \eta_i, \varsigma_i)$ nuqtalarni tanlab olishga bog‘liq bo‘lmagan holda hamma vaqt bitta J songa intilsa, u holda bu J son $\sigma_p(f)$ – integral yig‘indining limiti deyiladi va u

$$\lim_{\lambda_p \to 0} \sigma_p(f) = \lim_{\lambda_p \to 0} \sum_{i=1}^{n} f(\xi_i, \eta_i, \varsigma_i) V_i = J$$

kabi belgilanadi.

14.14.5-ta'rif. Agar $\lambda_p \to 0$ da $f(x,y,z)$ funksiyaning integral yig‘indisi $\sigma_p(f)$ chekli limitga intilsa, u holda $f(x,y,z)$ funksiya (V) sohada (Riman ma'nosida) integrallanuvchi deyiladi. Bu $\sigma_p(f)$ yig‘indining chekli limiti J esa, $f(x,y,z)$ funksiyaning (V) soha bo‘yicha *uch karrali integrali* deb ataladi va u

$$J = \iiint\limits_{(V)} f(x,y,z) dV = \iiint\limits_{(V)} f(x,y,z) dx dy dz$$

kabi belgilanadi.

Demak,

$$J = \iiint\limits_{(V)} f(x,y,z) dV = \lim_{\lambda_p \to 0} \sigma_p(f) = \lim_{\lambda_p \to 0} \sum_{i=1}^{n} f(\xi_i, \eta_i, \varsigma_i) V_i.$$

Ravshanki, (14.14.1) integral yig‘indining chekli limiti faqat chegaralangan funksiyalar uchun mavjud bo‘lishi mumkin. Chegaralangan funksiyalarning integral yig‘indisidan tashqari Darbu yig‘indilari tushunchasini ham kiritish mumkin.

$f(x,y,z)$ funksiya (V) $\left((V) \subset R^3\right)$ sohada chegaralangan bo‘lsin:

$$m \le f(x,y,z) \le M, \quad (\forall (x,y,z) \in (V))$$

(V) sohaning bo'linishlar to'plami $\{P\}$ ning P bo'linishiga nisbatan Darbu yig'indilari

$$s_P(f) = \sum_{i=1}^{n} m_i V_i, \; S_P(f) = \sum_{i=1}^{n} M_i \cdot V_i$$

ni $\left(m_i = \inf_{(V_i)} \{f\}, M_i = \sup_{(V_i)} \{f\} \right)$ tuzib $\{s_P(f)\}$, $\{S_P(f)\}$ to'plamlarni qaraymiz. Ma'lumki, bu to'plamlar chegaralangan.

14.14.6-ta'rif. $\{s_P(f)\}$ to'plamning aniq yuqori chegarasiga $f(x,y,z)$ funksiyaning quyi uch karrali integrali deyiladi va u $\underline{J} = \underline{\iiint\limits_{(V)}} f(x,y,z)dV$ kabi belgilanadi.

$\{S_P(f)\}$ to'plamning aniq quyi chegarasiga $f(x,y,z)$ funksiyaning yuqori *uch karrali integrali* deyiladi va u $\overline{J} = \overline{\iiint\limits_{(V)}} f(x,y,z)dV$ kabi belgilanadi.

14.14.7-ta'rif. Agar $f(x,y,z)$ funksiyaning quyi va yuqori uch karrali integrallari o'zaro teng bo'lsa, u holda $f(x,y,z)$ funksiya (V) sohada *integrallanuvchi* deyiladi va ularning umumiy qiymati

$$J = \underline{\iiint\limits_{(V)}} f(x,y,z)dV = \overline{\iiint\limits_{(V)}} f(x,y,z)dV$$

$f(x,y,z)$ funksiyaning *uch karrali integrali* deb ataladi.

Demak,

$$\iiint\limits_{(D)} f(x,y,z)dV = \underline{\iiint\limits_{(V)}} f(x,y,z)dV = \overline{\iiint\limits_{(V)}} f(x,y,z)dV \; .$$

14.15. Uch karrali integralning mavjudlik sharti. $f(x,y,z)$ funksiya (V) $\left((V) \subset R^3\right)$ sohada berilgan bo'lsin.

14.14.1-teorema. $f(x,y,z)$ funksiyaning (V) sohada integrallanuvchi bo'lishi uchun $\forall \varepsilon > 0$ son olinganda ham shunday $\delta > 0$ son topilib, (V)

sohaning diametri $\lambda_p < \delta$ bo'lgan har qanday P bo'linishga nisbatan Darbu yig'indilarining

$$S_p(f) - s_p(f) < \varepsilon \text{ yoki } \sum_{i=1}^{n} \omega_i V_i < \varepsilon, \ \omega_i = M_i - m_i$$

tengsizliklarni qanoatlantirishi zarur va yetarli.

14.16. Integrallanuvchi funksiyalarning sinflari.

14.16.1-teorema. Agar $f(x,y,z)$ funksiya chegaralangan yopiq (V) sohada uzluksiz bo'lsa, u shu sohada integrallanuvchi bo'ladi.

14.16.2-teorema. Agar $f(x,y,z)$ funksiya (V) sohada chegaralangan va bu sohaning chekli sondagi nol hajmli sirtlarida uzilishga ega bo'lib, qolgan barcha nuqtalarida uzluksiz bo'lsa, u shu sohada integrallanuvchi bo'ladi.

14.17. Uch karrali integralning xossalari.

1^0. Uch karrali integralning mavjudligi va uning qiymati funksiyaning chekli sondagi nol hajmli sirtlardagi qiymatiga bog'liq bo'lmaydi.

2^0. Agar (V) soha nol hajimli (S) sirt bilan (V_1) va (V_2) sohalarga ajratilgan bo'lib, $f(x,y,z)$ funksiya (V) sohada integrallanuvchi bo'lsa, funksiya (V_1) va (V_2) cohalarda ham integrallanuvchi bo'ladi va aksincha $f(x,y,z)$ funksiya (V_1) va (V_2) sohalarning har birida integrallanuvchi bo'lsa, funksiya (V) da ham integrallanuvchi va ushbu

$$\iiint\limits_{(V)} f(x,y,z)dV = \iiint\limits_{(V_1)} f(x,y,z)dV + \iiint\limits_{(V_2)} f(x,y,z)dV$$

tenglik o'rinli bo'ladi.

3^0. Agar $f(x,y,z)$ funksiya (V) da integrallanuvchi bo'lsa, u holda $C \cdot f(x,y,z)$ $(C = const)$ funksiya ham shu sohada integrallanuvchi va ushbu

$$\iiint\limits_{(V)} Cf(x,y,z)dV = C\iiint\limits_{(V)} (x,y,z)dV$$

tenglik o'rinli bo'ladi.

4^0. Agar $f(x,y,z)$ va $g(x,y,z)$ funksiyalar (V) sohada integrallanuvchi bo'lsa, u holda $f(x,y,z) \pm g(x,y,z)$ funksiya ham shu sohada integrallanuvchi va ushbu

$$\iiint_{(V)} (f(x,y,z) \pm g(x,y,z))dV = \iiint_{(V)} f(x,y,z)dV \pm \iiint_{(V)} g(x,y,z)dV$$

tenglik o'rinli bo'ladi.

5^0. Agar $f(x,y,z)$ funksiya (V) sohada integrallanuvchi bo'lib, $\forall(x,y,z) \in (V)$ uchun $f(x,y,z) \geq 0$ bo'lsa, u holda

$$\iiint_{(V)} f(x,y,z)dV \geq 0$$

bo'ladi.

6^0. Agar $f(x,y,z)$ funksiya (V) sohada integrallanuvchi bo'lsa. u holda $|f(x,y,z)|$ funksiya ham shu sohada integrallanuvchi va ushbu

$$\left|\iiint_{(V)} f(x,y,z)dV\right| \leq \iiint_{(V)} |f(x,y,z)|dV$$

tengsizlik o'rinli bo'ladi.

7^0. Agar $f(x,y,z)$ funksiya (V) sohada integrallanuvchi bo'lib,

$$m \leq f(x,y,z) \leq M$$

bo'lsa, u holda

$$\iiint_{(V)} f(x,y,z)dv = \mu \cdot V$$

formula o'rinli bo'ladi $(m \leq \mu \leq M)$.

8^0. Agar $f(x,y,z)$ funksiya yopiq (V) sohada uzluksiz bo'lsa, u holda bu sohada shunday $(\bar{x},\bar{y},\bar{z}) \in (V)$ nuqta topiladiki, ushbu

$$\iiint_{(V)} f(x,y,z)dV = f(\bar{x},\bar{y},\bar{z}) \cdot V \qquad (14.17.1)$$

formula o'rinli bo'ladi.

Uch karrali integralning soha bo'yicha funksiya tushunchasini, xususiy holda, sohaning additiv funksiyasi tushunchasini, soha funksiyasining berilgan nuqtadagi hosila tushunchalari ham ikki o'lchovli holdagidek kiritiladi.

Sohaning additiv funksiyasiga misol sifatida ushbu o'zgaruvchan (V) soha bo'yicha olingan

$$\Phi((V))=\iiint_{(V)} f(x,y,z)dV \qquad (14.17.2)$$

integral misol bo'laoladi. Soha funksiyasi $\Phi((V))$ ning berilgan M nuqtadagi hosilasi deb, ushbu nisbatning

$$\lim_{(V)\to M} \frac{\Phi((V))}{V}$$

ga aytiladi.

9^0. Agar integral ostidagi funksiya uzluksiz bo'lsa, u holda (14.17.1) integralning $M(x,y,z)$ nuqtadagi soha bo'yicha hosilasi integral ostidagi funksiyaning shu nuqtadagi qiymatiga teng bo'ladi, ya'ni $f(M)=f(x,y,z)$.

Shunday qilib, yuqoridagi shartlarda (14.17.2) integral qandaydir ma'noda f funksiya uchun boshlang'ich funksiyasi bo'lar ekan.

14.18. Uch karrali integralni takroriy integralga keltirish yo'li bilan hisoblash. $f(x,y,z)$ funksiya

$$(V)=\{(x,y,z)\in R^3 : (x,y)\in(D),\ z_1(x,y)\le z\le z_2(x,y)\ \}$$

(bunda $z_1(x,y), z_2(x,y)$ funksiyalar kvadratlanuvchi sohada uzluksiz) sohada aniqlangan bo'lsin.

14.18.1-teorema. Agar: 1) ushbu

$$\iiint_{(V)} f(x,y,z)dv$$

integral mavjud bo'lsa;

2) $\forall(x,y)\in(D)$ uchun

$$J(x,y)=\int_{z_1(x,y)}^{z_2(x,y)} f(x,y,z)dz$$

aniq integral mavjud bo'lsa, u holda

$$\iint_{(D)} J(x,y)dxdy=\iint_{(D)} dxdy\int_{z_1(x,y)}^{z_2(x,y)} f(x,y,z)dz$$

takroriy integral ham mavjud va ushbu

$$\iiint\limits_{(V)} f(x,y,z)dv = \iint\limits_{(D)} dxdy \int\limits_{z_1(x,y)}^{z_2(x,y)} f(x,y,z)dz \qquad (14.18.2)$$

formula o‘rinli bo‘ladi.

(14.18.2) formulada takroriy integralning birinchisi ikki karrali integral bo‘lib, ichki integral esa, aniq integraldan iborat. Ba’zi hollarda takroriy integralning birinchisi aniq integral bo‘lib, ichki integral esa, ikki karrali integral bo‘lishi mumkin, ya’ni

$(V)=\{(x,y,z)\in R^3 : a\le x\le b; (y,z)\in(G)$ – kvadratlanuvchi soha$\}$ bo‘lsin.

14.18.3-teorema. Agar: 1) $\iiint\limits_{(V)} f(x,y,z)dv$ integral mavjud bo‘lsa;

2) $\forall x\in[a,b]$ uchun $J(x)=\iint\limits_{(G)} f(x,y,z)dydz$ – ikki karrali integral mavjud bo‘lsa, u holda

$$\int\limits_a^b J(x)dx = \int\limits_a^b dx \iint\limits_{(G)} f(x,y,z)dydz$$

takroriy integral ham mavjud va ushbu

$$\iiint\limits_{(V)} f(x,y,z)dv = \int\limits_a^b dx \iint\limits_{(G)} f(x,y,z)dydz$$

tenglik o‘rinli bo‘ladi .

Agar $(D)=\{(x,y)\in R^2 : a\le x\le b,\ y_1(x)\le y\le y_2(x)\}$ (bunda $y_1(x)$ va $y_2(x)$ funksiyalar $[a,b]$ da uzluksiz) soha egri chiziqli trapetsiyadan iborat bo‘lsa, u holda $\iint\limits_{(D)} J(x,y)dxdy$ – ikki karrali integral ushbu ko‘rinishdagi

$$\iint\limits_{(D)} J(x,y)dxdy = \int\limits_a^b dx \int\limits_{y_1(x)}^{y_2(x)} J(x,y)dy = \int\limits_a^b dx \int\limits_{y_1(x)}^{y_2(x)} dy \int\limits_{z_1(x,y)}^{z_2(x,y)} f(x,y,z)dz$$

takroriy integralga keltiriladi. Shunday qilib, bu holda uch karrali integralni hisoblash uchta ketma-ket bir karrali integrallarni hisoblashga olib kelinadi:

$$\iiint\limits_{(V)} f(x,y,z)dv = \int\limits_a^b dx \int\limits_{y_1(x)}^{y_2(x)} dy \int\limits_{z_1(x,y)}^{z_2(x,y)} f(x,y,z)dz. \qquad (14.18.4)$$

Xususiy holda, $(V)=[a,b;c,d;k,l]$ to‘g‘ri burchakli parallelepipeddan iborat bo‘lib, uning yz tekislikdagi proyeksiyasi $(R)=[c,d;k,l]$ to‘g‘ri to‘rtburchakdan iborat bo‘lsa, u holda (14.18.4) formulaning ko‘rinishi

$$\iiint\limits_{(V)} f(x,y,z)dv = \int\limits_a^b dx \iint\limits_{(R)} f(x,y,z)dR$$

shakilda bo‘ladi. Bunda ichkaridagi ikki karrali integralni takroriy integralga keltirsak, natijada

$$\iiint\limits_{(V)} f(x,y,z)dv = \int\limits_a^b dx \int\limits_c^d dy \int\limits_k^l f(x,y,z)dz$$

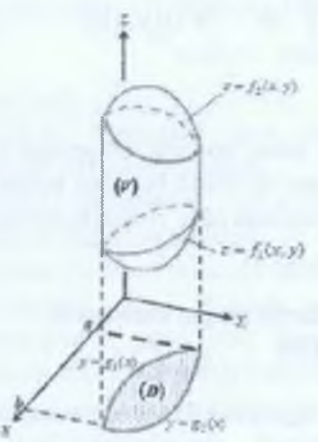
formulaga ega bo‘lamiz, ya’ni bu holda uch karrali integralni hisoblash uchta takroriy aniq integrallarni hisoblashga keltiriladi.

14.19. Uch karrali integralda integrallash chegaralarini topish algoritmi. (V) soha bo‘yicha olingan $\iiint\limits_{(V)} F(x,y,z)dv$ uch karrali integralni, dastlab z ga nisbatan, so‘ngra y ga nisbatan va nihoyat, x ga nisbatan hisoblash quyidagi bosqichlarini o‘z ichiga oladi.

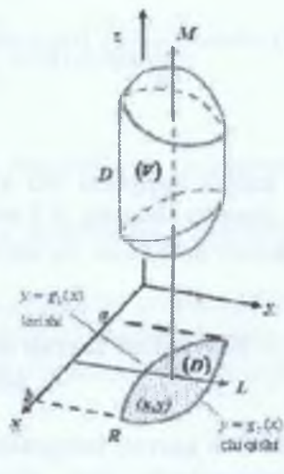
1-qadam. *Chizish.* (V) sohani uning xOy tekislikdagi (vertikal proyeksiyasi) (D) sohasi bilan birga chizish. (V) sohani pastdan va yuqoridan chegaralovchi sirtlarni hamda (D) ni chegaralovchi egri chiziqlarni aniqlash (14.13-chizma).

2-qadam. *Integrallashning z bo‘yicha chegaralari.* (D) dagi biror (x,y) nuqtadan Oz o‘qqa parallel to‘g‘ri chiziqni o‘tkazish. z ortganda M to‘g‘ri chiziq (V) ga $z=f_1(x,y)$ sirt bo‘ylab kiradi va $z=f_2(x,y)$ sirt bo‘yicha undan chiqadi. Ular z bo‘yicha integrallash chegaralari bo‘ladi (14.14-chizma).

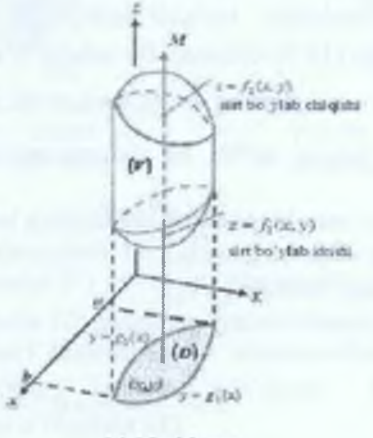
3-qadam. *Integrallashning y bo‘yicha chegaralari.* (x,y) nuqtadan Oy o‘qqa parallel L to‘g‘ri chiziqni o‘tkazish. y ortganda L to‘g‘ri chiziq $y=g_1(x)$ bo‘ylab (D) ga kiradi va $y=g_2(x)$ bo‘ylab (D) dan chiqadi. Ular y bo‘yicha integrallash chegaralari bo‘ladi (14.15-chizma).



14.13-chizma.



14.14-chizma.



14.15-chizma

4-*qadam.* *Integrallashning* x *bo'yicha chegaralari.* x bo'yicha integrallash chegaralari (D) da Oy o'qqa parallel o'tuvchi barcha to'g'ri chiziqlar ichidan tanlanadi (keltirilgan chizmada ular $x=a$ va $x=b$ to'g'ri chiziqlar). Ular x bo'yicha integrallash chegaralari bo'ladi. Bu holda

$$\iiint\limits_{(V)} F(x,y,z)dv = \int\limits_{x=a}^{x=b} dx \int\limits_{y=g_1(x)}^{y=g_2(x)} dy \int\limits_{z=f_1(x,y)}^{z=f_2(x,y)} F(x,y,z)dz$$

bo'ladi.

Uch karrali integralni olti xil usul bilan takroriy integralga keltirish mumkin. Bundan tashqari, (V) sohaga bog'liq holda berilgan integralni bir nechta takroriy integrallar yig'indisi ko'rinishida ham yozish mumkin.

14.5-§. Uch karrali integrallarda o'zgaruvchilarni almashtirish

14.20. Uch karrali integrallarda o'zgaruvchilarni almashtirish. Uch karrali integrallarda ham o'zgaruvchilarni almashtirish ikki karrali integrallarda o'zgaruvchilarni almashtirish tartibida amalga oshiriladi.

Faraz qilaylik, fazoda ($Oxyz$) dekart koordinatalar sistemasida yopiq (V) soha, ($Ouvw$) dekart koordinatalar sistemasida yopiq (G) sohalar berilgan bo'lib, ular mos ravishda, bo'lakli silliq (S) va (σ) sirtlar bilan chegaralangan bo'lsin (14.16-chizma). Bu sohalar o'zaro ushbu

$$x = g(u,v,w), y = h(u,v,w), z = k(u,v,w) \qquad (14.20.1)$$

sistema orqali bog'langan bo'lib, bu sistema quyidagi shartlarni qanoatlantirsin:

1) (14.20.1) – o'zaro bir qiymatli akslantirish bo'lsin;

2) $x = g(u,v,w), y = h(u,v,w), z = k(u,v,w)$ funksiyalar (G) sohada uzluksiz birinchi tartibli xususiy hosilalarga ega;

3) (14.20.1) akslantirishning yakobiani (G) sohada

$$J = \frac{D(x,y,z)}{D(u,v,w)} \neq 0$$

bo'lsin. (V) va (G) sohalar kublanuvchi bo'lsin.

14.20.2-teorema. Agar $f(x,y,z)$ funksiya (V) sohada chegaralangan va bu sohaning chekli sondagi nol hajmli sirtlarida uzilishga ega bo'lib, qolgan barcha nuqtalarida uzluksiz bo'lsa, (14.20.1) akslantirish yuqoridagi 1)-3) shartlarni qanoatlantirsa, u holda ushbu

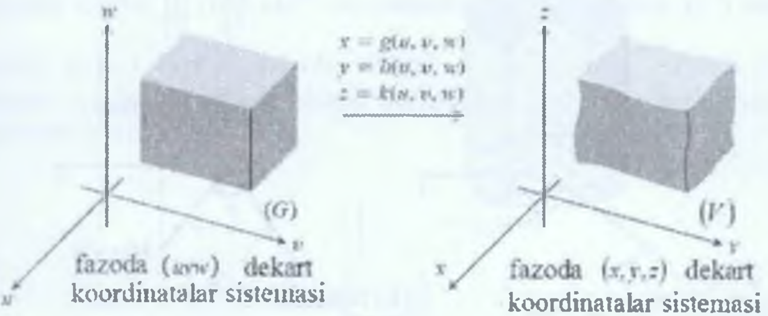
$$\iiint\limits_{(V)} f(x,y,z)dV = \iiint\limits_{(G)} f\big(g(u,v,w),h(u,v,w),k(u,v,w)\big)\left|\frac{D(x,y,z)}{D(u,v,w)}\right| dudvdw \quad (14.20.3)$$

formula o‘rinli.

(14.20.3) formulaga *uch karrali integrallarda o‘zgaruvchilarni almashtirish* formulasi deyiladi.

(14.20.3) formuladagi u,v,w lar x,y,z sistemadagi nuqtalarning *egri chiziqli koordinatalari* deyiladi.

$u = const$, $v = const$, $w = const$ sirtlar (x,y,z) fazoda *egri chiziqli koordinat sirtlari* deyiladi. Egri chiziqda egri chiziqli koordinatalarning ikkitasi o‘zgarmas qiymat qabul qilib, faqat bittasi o‘zgaruvchi bo‘lsa, u holda bu egri chiziqni *koordinat chizig‘i* deyiladi.



14.16-chizma.

Yuqoridagi mulohazalar (V) va (G) sohalar orasidagi qat’iy bir qiymatli moslikka asoslangan.

14.21. (x,y,z) **Dekart koordinatalar sistemasidan** (r,θ,z) **silindrik koordinatalar sistemasiga o‘tish.** Ma’lumki, $(r,\theta,z)-$ silindrik koordinatalar sistemasiga o‘tilganda

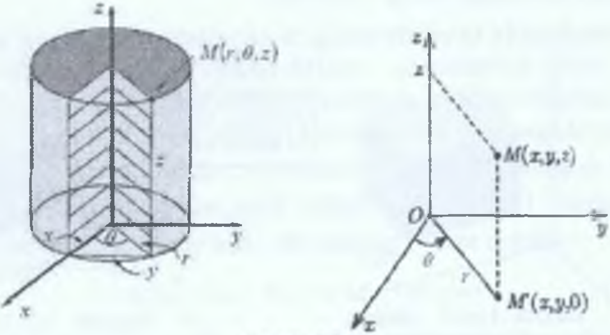
$$x = r\cos\theta,\ y = r\sin\theta,\ z = z, \quad (0 \le r < +\infty, 0 \le \theta \le 2\pi, -\infty < z < \infty) \quad (14.21.1)$$

$$J = \frac{D(x,y,z)}{D(r,\theta,z)} = r \quad (14.21.2)$$

formulalardan foydalaniladi. Bu (14.21.1)-(14.21.2) almashtirish natijasida berilgan uch karrali integral ushbu

$$J=\iiint_{(V)} f(x,y,z)dV=\iiint_{(G)} f(r\cos\theta, r\sin\theta, z)\cdot r\cdot drd\theta dz, \qquad (14.21.3)$$

ko‘rinishga keltiriladi, bunda (G) soha (V) sohaning yangi (r,φ,z) sistemasidagi aksidan iborat. (V) sohadagi $M(x,y,z)$ nuqta uchta (r,φ,z) sonlar juftligi bilan bir qiymatli aniqlanadi. $M(x,y,z)$ nuqtaning tekislikdagi proyeksiyasi $M'(x,y,0)$ bo‘lib hisoblanadi (14.17-chizma), $z-M$ nuqtaning applikatasi.



14.17-chizma.

(r,θ,z) uchlik M nuqtaning *egri chiziqli koordinatasi* deyiladi.

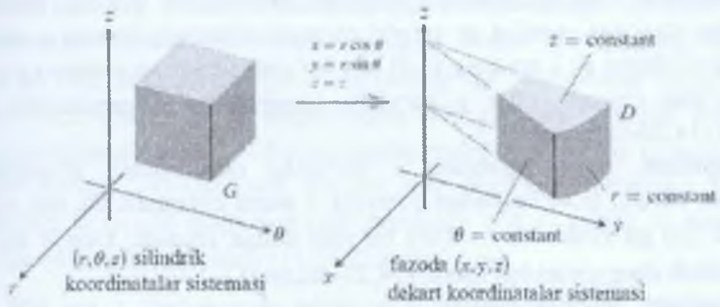
(14.21.1) formula $0\le r<+\infty, 0\le\theta\le 2\pi, -\infty<z<\infty$ sohani x,y,z tekisligiga akslantiradi. $r=0, z=z$ to‘g‘ri chiziqlar bitta $(0;0;z)$ nuqtaga akslantiriladi, bu bilan yuqorida 14.18-chizma aytilgan bir qiymatli moslik buziladi.

Bundan keyingi mulohazalarimizda muhim bo‘lgan ba’zi tushunchalarni keltirib o‘tamiz:

a) $r=const-(r,\theta,z)$ koordinatalar sistemasida o‘qi Oz va radiusi r bo‘lgan doiraviy silindrik sirt tenglamasini ifoda qiladi(14.34-chizma);

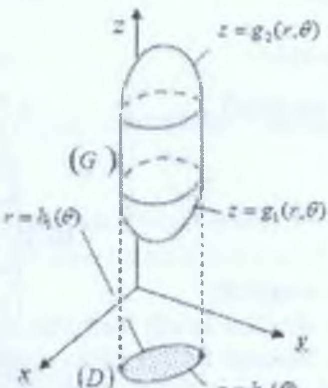
b) $\theta=const-(x,y)$ tekislik bilan φ burchak tashkil etib, $0z$ o‘qidan o‘tuvchi yarim tekislik tenglamasidan iborat (22.2-chizma);

d) $z=const-0z$ o‘qiga perpendi-kulyar $((x,y)$ tekislikka parallel) tekislik tenglamasini ifoda qiladi (14.18-chizma).



14.18-chizma.

14.22. Uch karrali integralni silindrik koordinatalarida integrallash algoritmi. Ushbu $\iiint\limits_{(G)} f(r,\theta,z)dV$ uch karrali integralni fazodagi (G) sohada silindrik (r,φ,z) koordinatalarda, dastlab z ga nisbatan, so'ngra r ga nisbatan va nihoyat, θ ga nisbatan integrallash orqali hisoblash quyidagi qadamlarni o'z ichiga oladi.



14.19-chizma.

1-qadam. *Chizish.* (G) sohani va uning xOy tekislikka proyeksiyasi (D) ni chizish. So'ngra (G)ni va (D) ni chegaralovchi sirtlar va egri chiziqlarni belgilash (14.19-chizma).

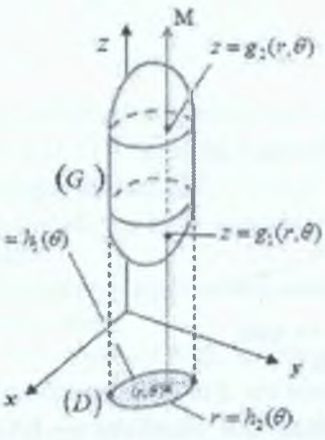
2-qadam. *Integrallashning z bo'yicha chegaralari.* (D) dagi biror (r,θ) nuqtadan Oz o'qqa parallel M to'g'ri chiziqni o'tkazish. Bunda z ortganda M to'g'ri chiziq (G) ga $z=g_1(r,\theta)$ sirt bo'ylab kiradi va undan $z=g_2(r,\theta)$ sirt bo'ylab chiqadi. Ular z bo'yicha integrallash chegaralaridan iborat bo'ladi (14.20-chizma).

3-qadam. *Integrallashning r bo'yicha chegaralari.* Koordinatalar boshidan chiqib, (r,θ) nuqtadan o'tuvchi L nurni chizamiz. Bu nur $r=h_1(\theta)$ bo'ylab (D) ga kiradi va $r=h_2(\theta)$ bo'ylab undan chiqadi. Ular r bo'yicha integrallash chegaralaridan iborat (14.21-chizma).

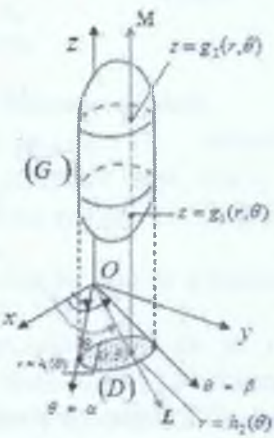
4-qadam. *Integrallashning θ bo'yicha chegaralari.* L nur (D) orqali o'tganda θ burchak Ox o'qning musbat yo'nalishida $\theta=\alpha$ dan $\theta=\beta$ gacha o'zgaradi. Ular θ bo'yicha integrallash chegaralari bo'ladi (14.22-chizma). Unda

$$\iiint\limits_{(G)} f(r,\theta,z)dV=\int\limits_{\alpha}^{\beta}\int\limits_{h_1(\theta)}^{h_2(\theta)}\int\limits_{g_1(r,\theta)}^{g_2(r,\theta)} f(r,\theta,z)\,dz\,r\,dr\,d\theta \qquad (14.22.1)$$

o'rinli bo'ladi.

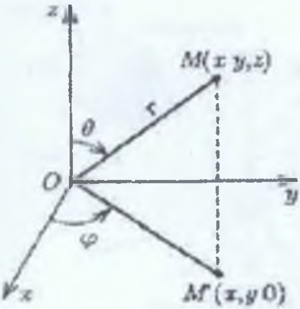


14.21-chizma.

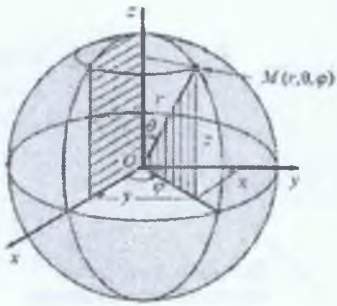


14.22-chizma.

14.23. (x,y,z) **Dekart koordinatalar sistemasidan** (r,θ,φ) **cferik koordinatalar sistemasiga o'tish.** $M(x,y,z)$ nuqta xyz fazoning ixtiyoriy nuqtasi bo'lsin. $M'(x,y,0)$ nuqta M nuqtaning xOy tekislikdagi proyeksiyasi bo'lsin (14.23-chizma). M nuqta uchta (r,θ,φ) sonlar juftligi bilan bir qiymati aniqlanadi, bunda r - koordinatalar boshi O nuqtadan M nuqtagacha bo'lgan masofa, $\theta - OM$ nur va Oz o'q orasidagi burchak, $\varphi - (x,y)$ tekislikdagi $M'(x,y,0)$ nuqtaning qutb burchagi. (r,θ,φ) uchlik M nuqtaning *sferik koordinatalari* deyiladi (14.24-chizma).



14.23-chizma.



14.24-chizma.

To'g'ri burchakli (x,y,z) sistemadan sferik (r,θ,φ) koordinatalar sistemasiga o'tish, ushbu

$$x = r\sin\theta\cos\varphi,\ y = r\cdot\sin\theta\sin\varphi,\ z = r\cos\theta$$
$$(0 \le r < +\infty,\ 0 \le \theta \le \pi,\ 0 \le \varphi \le 2\pi) \tag{14.23.1}$$

formulalar orqali amalga oshiriladi (ba'zi hollarda θ burchak deb OM va OM' nurlar orasidagi burchak olinadi, bu burchak $z>0$ bo'lganda musbat ishora bilan, $z<0$ bo'lganda esa manfiy ishora bilan olinadi, bu holda $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, (14.23.1) formulada $\sin\theta$ ni $\cos\theta$ ga, $\cos\theta$ ni esa, $\sin\theta$ ga almashtirish kerak bo'ladi).

Bu holda ham (V) soha bilan (G) soha o'rtasidagi bir qiymatli moslikning buzilishini ko'ramiz, masalan, (14.23.1) akslantirish yordamida

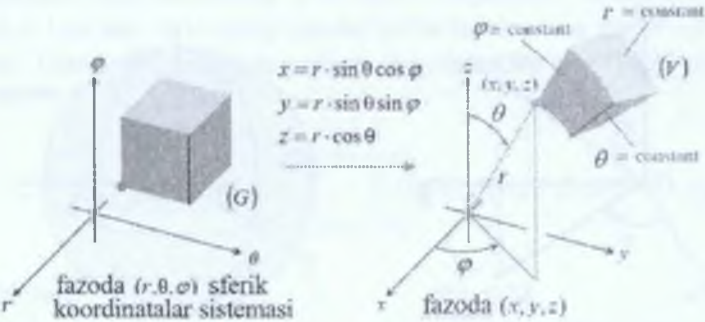
(r,θ,φ) fazodagi $r=0$ tekislik koordinat boshi $x=y=z=0$ ga, $\varphi=0\,(\pi)$, $r=r$ to'g'ri chiziqlar bitta $x=y=0$, $z=r$ nuqtaga akslantiriladi.

Sferik koordinatalar almashtirishda koordinatalar sirtlari quyidagi uchta sirtlar oilasini tashkil qiladi (14.25-chizma):

$a)\, r=const$ – markazi koordinatalar boshida bo'lgan konsentrik sirtlar;

$b)\, \theta=const$ – z o'qidan o'tuvchi yarim tekislik;

$d)\, \varphi=const$ – o'qi z o'qidan iborat bo'lgan aylanma konuslardan tashkil topgan sirtlar oilasidan iborat.



14.25-chizma.

(14.23.1) akslantirishning yakobiani

$$J=\frac{D(x,y,z)}{D(r,\theta,\varphi)}=r^2\sin\theta.$$

(θ burchak ikkinchi usul bilan tanlanganda, yakobian $r^2\cos\theta$ ga teng bo'ladi).

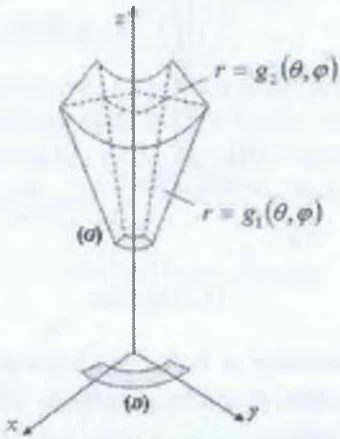
Ba'zi hollarda (x,y,z) to'g'ri burchakli koordinatlar sistemasidan sferik koordinatlar sistemasiga o'tishda umumlashgan sferik koordinatalar deb ataluvchi formulalar orqali amalga oshiriladi:

$$x-x_0=ar^n\sin^\alpha\theta\cos^\beta\varphi,\, y-y_0=br^n\sin^\alpha\theta\sin^\beta\varphi,\; z-z_0=cr^n\cos^\alpha\theta \quad (14.24.2)$$
$$(0\le r<\infty,\, 0\le\theta\le\pi,\; 0\le\varphi\le 2\pi),$$

bunda $x_0, y_0, z_0, a, b, c, n, \alpha, \beta$ – biror sonlar (14.24.2) akslantirishning yakobiani

$$J = \frac{D(x,y,z)}{D(r,\theta,\varphi)} = abcn\alpha\beta r^{3n-1} \sin^{2\alpha-1}\theta \cos^{\alpha-1}\theta \sin^{\beta-1}\varphi \cos^{\beta-1}\varphi.$$

14.24. Uch karrali integralni sferik koordinatalarida integrallash algoritmi. Ushbu $\iiint\limits_{(G)} f(r,\varphi,\theta)dG$ integralni, (G) soha bo'yicha, sferik koordinatalarda, dastlab r bo'yicha, so'ngra θ bo'yicha va nihoyat, φ bo'yicha integrallash yordamida hisoblash quyidagi qadamlarni o'z ichiga oladi.

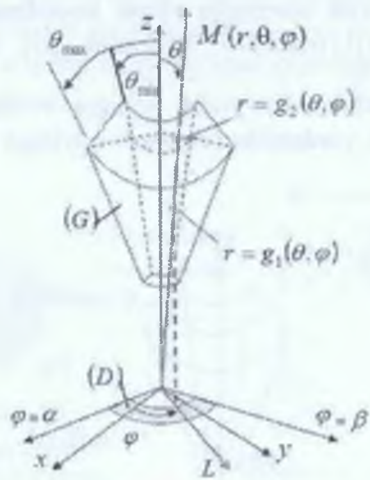


14.26-chizma.

1-qadam. *Chizish.* (G) sohani va uning xOy tekislikdagi (D) proyeksiyasini chizish, (G) ni chegaralovchi sirtlarni belgilash (aniqlash) (14.26-chizma).

2-qadam. *Integrallashning r bo'yicha chegaralari.* Koordinatalar boshidan (G) soha orqali Oz o'qining musbat yo'nalishi bilan θ burchak hosil qiluvchi M nurni o'tkazish. Shuningdek, M nurning xOy tekislikka proyeksiyasini (L proyeksiya deb ataladigan) o'tkazish. L nur Ox o'qning musbat yo'nalishi bilan φ burchak hosil qiladi. r ortganda M nur (G) sohaga $r = g_1(\theta,\varphi)$ bo'ylab kiradi va undan $r = g_2(\theta,\varphi)$ bo'ylab chiqadi. Ular integrallashning r bo'yicha chegaralari bo'ladi (14.27-chizma).

3-qadam. *Integrallashning θ bo'yicha chegaralari.* Ixtiyoriy berilgan φ uchun, M nur Oz o'qning musbat yo'nalishi bilan hosil qiladigan θ burchak $\theta=\theta_{min}$ dan $\theta=\theta_{max}$ gacha o'zgaradi. Shular integrallashning θ bo'yicha chegaralari bo'ladi.



14.27-chizma.

4-qadam. *Integrallashning φ bo'yicha chegaralari.* L nur (D) ustida o'tganda φ burchak α dan β gacha o'zgaradi. Ular integrallashning φ bo'yicha chegaralari bo'ladi.

Demak, uch karrali integral sferik koordinatalar sistemasida quyidagi

$$\iiint\limits_{(G)} f(\rho,\theta,\varphi)dG=\int\limits_{\alpha}^{\beta}\int\limits_{\theta_{min}}^{\theta_{max}}\int\limits_{g_1(\theta,\varphi)}^{g_2(\theta,\varphi)} f\left(r,\theta,\varphi,\right)r^2\sin\theta\, dr\, d\varphi\, d\theta \qquad (14.24.1)$$

ko'rinishda bo'ladi.

14.6-§. Uch karrali integralning ba'zi bir tatbiqlari

14.25. Uch karrali integral yordamida jismning hajmini hisoblash. (x,y,z) fazoda kublanuvchi (V) jismning hajmi

$$V=\iiint\limits_{(V)} dxdydz \qquad (14.25.1)$$

formula orqali ifoda qilinadi.

Agar (V) soha ushbu $(V)=\{(x,y,z)\in R^3:(x,y)\in(D),\ 0\le z\le f(x,y)\}$ (bunda $(D)-(x,y)$ tekislikdagi kvadratlanuvchi soha, $f(x,y)$ funksiya esa, (D) sohada uzluksiz) ko'rinishda bo'lsa, u holda (14.26.1) uch karrali integralni takroriy integrallarga keltirish natijasida:

$$V=\iint\limits_{(D)} dxdy\int\limits_0^{f(x,y)} dz=\iint\limits_{(D)}\left[z\Big|_0^{f(x,y)}\right]dxdy=\iint\limits_{(D)} f(x,y)dxdy \qquad (14.25.2)$$

formula hosil bo'ladi.

Agar (V) soha $x=a$ va $x=b$ tekisliklar orasida bo'lib, (V) sohani $x=const$ tekislik bilan kesganda, har bir kesim yuzi $S(x)$ ga teng bo'lgan kvadratlanuvchi (D) sohadan iborat bo'lsa, u holda (14.25.2) formulaning ko'rinishi

$$V=\int\limits_a^b dx\iint\limits_{(D)} dydz=\int\limits_a^b S(x)dx \qquad (14.25.3)$$

shaklga keladi.

(14.25.1) tenglikda yangi ξ,η,ζ o'zgaruvchilarga (14.25.3) formula orqali o'tganda (V) sohaning hajmi V uchun ushbu

$$V=\iiint\limits_{(V^*)}\left|\frac{D(x,y,z)}{D(\xi,\eta,\zeta)}\right|d\xi d\eta d\zeta$$

formulaga ega bo'lamiz, bunda to'g'ri burchakli koordinatalardagi elementar hajm egri chiziqli koordinatalarda

$$dV=\left|\frac{D(x,y,z)}{D(\xi,\eta,\zeta)}\right|d\xi\, d\eta\, d\zeta$$

ko'rinishga ega bo'ladi.

14.26. Uch karrali integrallar yordamida mexanikaga oid masalalarni yechish. Agar (V) $Oxyz$ fazoda zichligi uzluksiz $\rho(x,y,z)$ funksiyadan iborat bo'lgan (kublanuvchi) jism bo'lsa, u holda ushbu

$$\iiint\limits_{(V)} \rho(x,y,z)dxdydz$$

uch karrali integral (V) jismning massasi m ni ifoda qiladi, ya'ni

$$m = \iiint\limits_{(v)} \rho(x,y,z)dxdydz \ .$$

(V) jismning koordinatlar o'qlariga nisbatan inersiya momentlari:

$$J_z = \iiint\limits_{(V)} \left(x^2 + y^2\right)\rho(M)dv, \quad J_y = \iiint\limits_{(V)} \left(x^2 + z^2\right)\rho(M)dv,$$
$$J_x = \iiint\limits_{(V)} \left(y^2 + z^2\right)\rho(M)dv.$$

Koordinatlar boshiga nisbatan inersiya momenti

$$J_0 = \iiint\limits_{(V)} \left(x^2 + y^2 + z^2\right)dv$$

formulalar orqali topiladi. (V) jism og'irlik markazining koordinatalari

$$x_c = \frac{\iiint\limits_{(V)} x\rho(M)dv}{m}; \quad y_c = \frac{\iiint\limits_{(V)} y\rho(M)dv}{m}; \quad z_c = \frac{\iiint\limits_{(V)} z\rho(M)dv}{m}$$

formulalar orqali ifoda qilinadi.

Agar $\rho(x,y,z) = const$ bo'lsa,

$$x_c = \frac{\iiint\limits_{(V)} xdv}{V}; \quad y_c = \frac{\iiint\limits_{(V)} ydv}{V}; \quad z_c = \frac{\iiint\limits_{(V)} zdv}{V}$$

bunda $V - (V)$ jismning hajmi.

14-bob bo'yicha nazariy materiallarni mustahkamlash uchun topshiriqlar

14.1. Silindrik brusning hajmini topish haqidagi masala ([12], 2-q., 291-219 betlar; [5], 3-t., 122-123 betlar; [17], 15-16 betlar; [30], 15- bo'lim).

14.2. Ikki karrali integralning ta'riflari. ([3], 2-q., 267-270 betlar; [12], 2-q., 292-295 betlar; [5], 3-t., 125-126 betlar; [17], 24-26 betlar; [9], 2-t., 8- bo'lim; [30], 15- bo'lim).

14.3. Darbu yig'indilarining ta'rifi va xossalari. ([3], 2-q., 270-272 betlar; [12], 2-q., 295-296, 298-299 betlar; [5], 3-t., 127-128 betlar; [17], 26-30 betlar; [9], 2-t., 8- bo'lim; [30], 15- bo'lim).

14.4. Yuqori va quyi integrallarning ta'rifi ([3], 2-q., 267-268 betlar; [12], 2-q., 296-297 betlar; [5], 3-t., 128 betlar; [9], 2-t., 8- bo'lim; [30], 15- bo'lim).

14.5. Ikki karrali integralning mavjudlik haqidagi teoremasi ([3], 2-q., 272-273 betlar; [12], 2-q., 299-300 betlar; [5], 3-t., 128 bet; [9], 2-t., 8- bo'lim).

14.6. Ikki karrali integralning mavjudligi shartiga teng kuchli bo'lgan shart ([5], 3-t., 128-130 betlar; [9], 2-t., 8- bo'lim).

14.7. Integrallanuvchi funksiyalarning sinflari. ([3], 2-q., 274-277 betlar; [12], 2-q., 300-303 betlar, [17], 32-33 betlar; [9], 2-t., 8- bo'lim).

14.8. Ikki karrali integralning xossalari ([3], 2-q., 277-279 betlar; [12], 2-q., 303-306 betlar; [5], 3-t., 131-134 betlar; [17], 33-35 betlar; [9], 2-t., 8- bo'lim; [30], 15- bo'lim).

14.9. O'rta qiymat haqidagi teorema. ([3], 2-q., 279-280 betlar; [12], 2-q., 305-306 betlar; [5], 3-t., 133-134 betlar, [17], 35 bet; [9], 2-t., 8- bo'lim; [30], 15- bo'lim).

14.10. Integrallash sohasi to'g'ri to'rt burchakdan iborat bo'lgan holda ikki karrali integralni hisoblash ([3], 2-q., 280-285 betlar; [12], 2-q., 306-311 betlar, [5], 3-t., 137-140 betlar, [17], 48-50 betlar; [9], 2-t., 8- bo'lim; [30], 15- bo'lim).

14.11. Integralash sohasi egri chiziqli bo'lgan holda ikki karrali integralni hisoblash ([3], 2-q., 286-290 betlar; [12], 2-q., 312-315 betlar, [5], 3-t., 149-152 betlar, [17], 50-54 betlar; [9], 2-t., 8- bo'lim; [30], 15- bo'lim).

14.12. Ikki karrali integrallarda o'zgaruvchilarni almashtirish ([3], 2-q.,291-295 betlar; [12], 2-q., 316-321, betlar; [17], 54-56, 66-68 betlar; [9], 2-t., 8- bo'lim; [30], 15- bo'lim).

14.13. Jism hajmining ta'rifi ([3], 2-q., 299-302 betlar; [12], 2-q., 324-326 betlar).

14.14. Hajmni ikki karrali integral orqali hisoblash ([3], 2-q., 299-302 betlar; [12], 2-q., 326-327 betlar, [17], 41-42 betlar).

14.15. Yassi shaklning yuzini ikki karrali integral orqali hisoblash ([3], 2-q., 298-299 betlar; [12], 2-q., 327-328 betlar, [17], 42 bet).

14.16. Uch karrali integralning ta'rifi ([3], 2-q., 305-306 betlar; [12], 2-q., 330-338 betlar; [9], 2-t., 8- bo'lim; [30], 15- bo'lim).

14.17. Uch karrali integralning mavjudligi ([3], 2-q., 307-308 betlar; [12], 2-q., 332-333 betlar; [9], 2-t., 8- bo'lim; [30], 15- bo'lim).

14.18. Uch karrali integrallarda o'zgaruvchilarni almashtirish ([3], 2-q.,311-313 betlar; [12], 2-q., 334-335betlar; [9], 2-t., 8- bo'lim; [30], 15-bo'lim).

14.19. Sirtning yuzi va uni ikki karrali integral orqali ifodalash ([3], 2-q.,302 bet; [12], 2-q., 328-330 betlar).

14.1-amaliy mashg'ulot.
Ikki karrali integrallarni hisoblash

1-misol. Ushbu $\iint\limits_{(D)}(x+y)dxdy$ integralni integral yig'indi yordamida hisoblang, bunda $(D)=\{(x,y)\in R^2 : 0\le x\le 1, 0\le y\le 1\}$.

Yechilishi. Berilgan $f(x,y)=x+y$ funksiya (D) sohada uzluksiz, demak, **14.5.1**- teoremaga ko'ra, u (D) sohada integrallanuvchi bo'ladi. (D) sohani $x=\frac{i}{n}, y=\frac{j}{n}$ $(i,j=\overline{1,n})$ to'g'ri chiziqlar yordamida bo'laklarga ajratamiz va har bir (D_{ij}) sohada $(x_i,y_j)=\left(\frac{i}{n},\frac{j}{n}\right)$ deb qaraymiz. U holda

$$\Delta x_i=x_i-x_{i-1}=\frac{1}{n},\ \Delta y_j=y_j-y_{j-1}=\frac{1}{n},\ D_{ij}=\Delta x_i\,\Delta y_j=\frac{1}{n^2},\quad f(x_i,y_j)=\frac{i}{n}+\frac{j}{n}.$$

(14.6.4) formulani e'tiborga olgan holda, integral yig'indini tuzib, uni hisoblaymiz:

$$\sigma=\sum_{i=1}^{n}\sum_{j=1}^{n}f(x_i,y_j)D_{ij}=\frac{1}{n^3}\sum_{i=1}^{n}\sum_{j=1}^{n}(i+j)=\frac{1}{n^3}\sum_{i=1}^{n}[in+\frac{n(n+1)}{2})]=$$

$$=\frac{1}{n^3}[\frac{n^2(n+1)}{2}+\frac{n^2(n+1)}{2}]=\frac{n+1}{n}.$$

Endi limitni hisoblaymiz: $\lim\limits_{n\to\infty}\sigma=\lim\limits_{n\to\infty}\dfrac{n+1}{n}=1$.

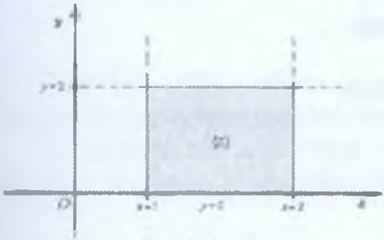
Demak, $\iint\limits_{(D)}(x+y)dxdy=\lim\limits_{n\to\infty}\sigma=1$.

2-misol. $x=0,\ x=1,\ y=0,\ y=1$ to‘g‘ri chiziqlar bilan chegara-langan (D) sohada $f(x,y)=x+y$ funksiyaning integral o‘rta qiymatini hisoblang.

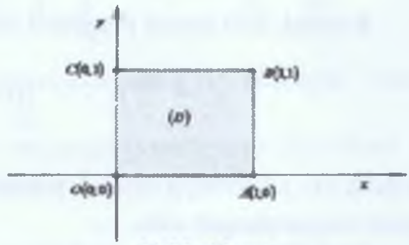
Yechilishi. (D) soha – uchlari $O(0,0)$, $A(1,0)$, $B(1,1)$, $C(0,1)$, nuqtalarda bo‘lgan $OABC$ to‘g‘ri to‘rtburchakdan iborat (14.28- chizma). O‘rta qiymat haqidagi teoremaga asosan, $f(x,y)$ funksiya (D) yopiq sohada uzluksiz, u holda (D) sohada shunday $(\bar{x},\bar{y})\in(D)$ nuqta topiladiki, $\dfrac{1}{D}\iint\limits_{(D)}f(x,y)dD=f(\bar{x},\bar{y})$ formula o‘rinli bo‘ladi, bunda $D-(D)$ sohaning yuzasi.

Ravshanki, $OABC$ to‘g‘ri to‘rtburchakning yuzi $D=1$. U holda 14.6.3-natijaga ko‘ra, $\iint\limits_{(D)}(x+y)dxdy=f(\bar{x},\bar{y})D$ bo‘ladi.

Shunday qilib, integralning o‘rta qiymati $f(\bar{x},\bar{y})=1$.



14.28-chizma.



14.29-chizma

3-misol. Ushbu $\iint\limits_{(D)}\left(10-x^2-y^2\right)dxdy$ integralni hisoblang, bunda $(D)-x=1, x=2, y=0,\ y=2$ to‘g‘ri chiziqlar bilan chegaralangan soha.

Yechilishi. Dastlab funksiyalar grafiklarini chizib olamiz (14.29-chizma). Berilgan ikki karrali integralga $\int\limits_a^b\left[\int\limits_c^d f(x,y)dy\right]dx$ formulani qo‘llab uni

$$\iint\limits_{(D)}\left(10-x^2-y^2\right)dxdy=\int\limits_1^2\left(\int\limits_0^2\left(10-x^2-y^2\right)dy\right)dx$$

takroriy integral ko'rinishda yozamiz.

Ichki integralni x o'zgaruvchini o'zgarmas miqdor deb, hisoblaymiz:

$$\int\limits_0^2\left(10-x^2-y^2\right)dy=\left(10y-x^2y-\frac{y^3}{3}\right)\Bigg|_{y=0}^{y=2}=20-2x^2-\frac{8}{3}=\frac{52}{3}-2x^2$$

Endi x o'zgaruvchi bo'yicha integralni hisoblaymiz:

$$\iint_{(D)}(10-x^2-y^2)dxdy=\int_1^2\left(\frac{52}{3}-2x^2\right)dx=\left(\frac{52}{3}x-\frac{2x^3}{3}\right)\Big|_1^2=\frac{38}{3}$$

Misolni Maple tizimidan foydalanib, yechish:

$$> \mathrm{int}\left(\mathrm{int}\left(10-x\hat{\ }2-y\hat{\ }2,y=0..2\right),x=1...2\right);\quad\frac{38}{3}.$$

4-misol. Ikki karrali integralni hisoblang:

$$\iint\limits_{(D)}(x+y)dxdy,$$

bunda$(D)-y=6-x^2$, $y=2-x^2$ parabolalar, $y=x$ va $x=0$ to'g'ri chiziqlar bilan chegaralangan soha.

Yechilishi. Dastlab funksiyalar grafiklarini chizib olamiz (14.30-chizma), so'ngra

$$\begin{cases}y=6\text{-}x^2\\ y=x\end{cases}\text{ va }\begin{cases}y=2-x^2\\ y=x\end{cases}$$

sistemalarni yechamiz: $x=2,\ y=2,\ x=1,\ y=1$.

Bu funksiyalar $A(1,1)$ va $B(2,2)$ nuqtalar kesishadi. 14.30-chizma.

$$I=\int_0^1\left[\int_{2-x^2}^{6-x^2}(x+y)dy\right]dx+\int_1^2\left[\int_x^{6-x^2}(x+y)dy\right]dx=$$

$$=\int_0^1\left(xy+\frac{y^2}{2}\right)\Big|_{2-x^2}^{6-x^2}dx+\int_1^2\left(xy+\frac{y^2}{2}\right)\Big|_x^{6-x^2}dx=$$

$$=\int_0^1\left(-4x^2+4x+16\right)dx+\int_1^2\left(\frac{x^4}{2}-x^3-\frac{15x^2}{2}+6x+18\right)dx=$$

$$=\left(-\frac{4}{3}x^3++2x^2+16x\right)\Big|_0^1+\left(\frac{x^5}{10}-\frac{x^4}{4}-\frac{5}{2}x^3+3x^2+18x\right)\Big|_1^2=\frac{1537}{60}.$$

5-misol. (D) soha: $x=2y,\ y=2x,\ x+y=6$ to‘g‘ri chiziqlar bilan chegaralangan uchburchakdan iborat.

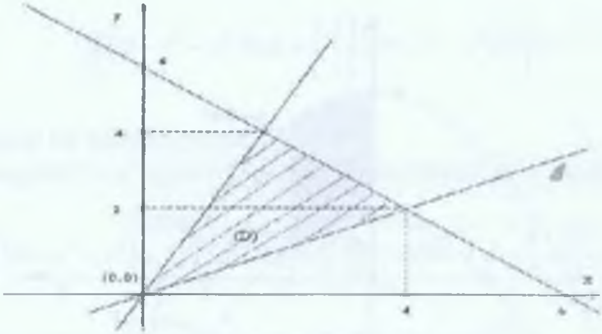
Yechilishi. (D) uchburchak 14.31- chizmada tasvirlangan. (D) sohani $x=2$ to‘g‘ri chiziq yordamida (D_1) va (D_2) sohalarga ajratamiz:

$$(D_1)=\left\{(x,y)\in R^2:\ 0\le x\le 2,\ \ \frac{x}{2}\le y\le 2x\right\},\ (D_2)=\left\{(x,y)\in R^2:2\le x\le 4,\ \ \frac{x}{2}\le y\le 6-x\right\}.$$

(*) formulalarga asosan,

$$\iint\limits_{(D)}f(x,y)dx\,dy=\iint\limits_{(D_1)}f(x,y)dx\,dy+\iint\limits_{(D_2)}f(x,y)dx\,dy=\int_0^2dx\int_{\frac{x}{2}}^{2x}f(x,y)dy+\int_2^4dx\int_{\frac{x}{2}}^{2x}f(x,y)dy$$

yoki $\iint\limits_{(D)}f(x,y)dx\,dy=\int_0^2dy\int_{\frac{y}{2}}^{2y}f(x,y)dx+\int_2^4dy\int_{\frac{y}{2}}^{6-y}f(x,y)dx$ tenglik o‘rinli bo‘ladi.



14.31-chizma.

6-misol. $J=\int_0^1 dx\int_{2x}^1 \sqrt[4]{1-y^2}\,dy$ hisoblang.

Yechilishi. Ichki integral y ga nisbatan murakkab bo‘lgani uchun takroriy integrallarning chegarasi bo‘lgan

$$(D)=\{(x,y)\in R^2 : 0\le x\le 1,\ 2x\le y\le 1\}$$

uchburchakni ushbu

$$(D)=\left\{(x,y)\in R^2 : 0\le y\le 1,\ 0\le x\le \frac{y}{2}\right\}$$

ko‘rinishda ifodalaymiz. Bundan

$$J=\int_0^1 dy\int_0^{\frac{y}{2}} \sqrt[4]{1-y^2}\,dx=-\frac{1}{5}\cdot\left(1-y^2\right)^{\frac{5}{4}}\Big|_0^1=\frac{1}{5}.$$

Mustaqil yechish uchun misollar

Қуйидаги берилган икки каррали интегралларни ҳисобланг:

1. $\iint\limits_{(D)} xydxdy,\ (D)=\{(x,y)\in R^2 : 0\le x\le 1,\ 0\le y\le 1\}$.
2. $\iint\limits_{(D)} (3x-2y)dxdy,\ (D)=\{(x,y)\in R^2 : 1\le x\le 2,\ 1\le y\le 3\}$
3. $\iint\limits_{(D)} (x+y)dxdy,\ (D)=\{(x,y)\in R^2 : 0\le x\le 1, 0\le y\le 1\}$.
4. $\iint\limits_{(D)} (xy-6)dxdy,\ (D)=\{(x,y)\in R^2 : 1\le x\le 3,\ 2\le y\le 5\}$.
5. $\iint\limits_{(D)} (\sin x+\cos y)dxdy:\ (D)=\{(x,y)\in R^2 : 0\le x\le \frac{\pi}{2},\ 0\le y\le \frac{\pi}{2}\}$.

(6)-(9) da ikki karrali integralda integralash tartibini uzgartirib hisoblang.

6. $\int\limits_0^{\pi} dx \int\limits_x^{\pi} \frac{\sin y}{y} dy.$ 7. $\int\limits_0^1 dy \int\limits_y^1 x^2 e^{xy} dx.$

8. $\int\limits_0^{2\sqrt{\ln 3}} dy \int\limits_{y/2}^{\sqrt{\ln 3}} e^{x^2} dx.$ 9. $\int\limits_0^{1/16} dy \int\limits_{\sqrt[4]{y}}^{1/2} \cos\left(16\pi x^5\right) dx.$

10. $\iint\limits_{(D)} (x-y) dxdy$, bu yerda (D)- uchlari $A(1,1)$, $B(4;1)$, $C(4,4)$ nuqtalari bo'lgan uchburchak.

11. $\iint\limits_{(D)} (x+2y) dxdy$, bu yerda $(D)-y=\frac{x^2}{2}$ parabola va $y=3x,\ x=1,\ x=2$ to'g'ri chiziqlar bilan chegaralangan soha.

12. $\iint\limits_{(D)} x dx dy$, bu yerda $(D)-y=\frac{1}{x}$ giperbola va $y=2,\ y=4,\ x+y=6$ to'g'ri chiziqlar bilan chegaralangan soha.

13. $\iint\limits_{(D)} (x+2y) dxdy$, bu yerda $(D)-y=x^2$ parabola va $x+y-2=0,\ y=0$ to'g'ri chiziqlar bilan chegaralangan soha.

14. $\iint\limits_{(D)} x \ln y dxdy$, bu yerda $(D)=\{(x,y)\in R^2 : 0\le x\le 4,\ 1\le y\le e\}$ to'g'ri to'rtburchak.

14. $\iint\limits_{(D)} \left(\cos^2 x+\sin^2 y\right) dxdy$, bu yerda $(D)=\left\{(x,y)\in R^2 : 0\le x\le\frac{\pi}{4},\ 0\le y\le\frac{\pi}{4}\right\}$

16. $\iint\limits_{(D)} e^{x-y} dxdy$, bu yerda $(D)-y=e^x$ egri chiziq va $x=0,\ y=2$ to'g'ri chiziqlar bilan chegaralangan .

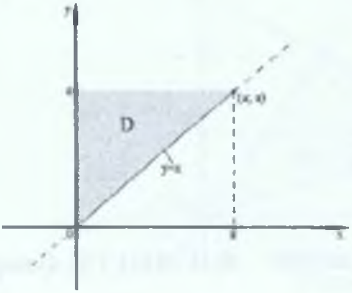
17. $\iint\limits_{(D)} xy dxdy$, bu yerda $(D)-4x^2+y^2=4$ ellips bilan chegaralangan soha.

18. $\iint\limits_{(D)} x.y dxdy$, bu yerda $(D)-y^2=2x$ parabola va $x-y-4=0$ to'g'ri chiziq bilan chegaralangan soha.

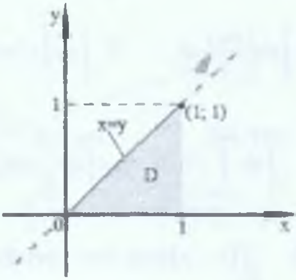
Mustaqil yechish uchun misollarning javoblari

1 $\frac{1}{4}$. 2. 1. 3. 1. 4. 18. 5. π.

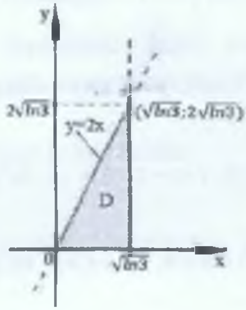
6. 2.



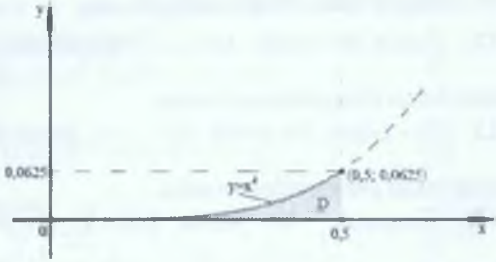
7. $\frac{e-2}{2}$.



8. 2.



9. $1/80\pi$.



10. $4{,}5$. **11.** $\frac{983}{40}$. **12.** $\frac{221}{24}$. **13.** $1{,}45$. **14.** 8. **15** $0{,}9$. **16.** e. **17.** 0. **18.** 90.

14.2-amaliy mashg'ulot.

UCH KARRALI INTEGRALLARNI HISOBLASH

1-misol. Ushbu $\iiint\limits_{(V)}(x+yz)dx\,dy\,dz$ uch karrali integralni integral yig'indi yordamida hisoblang, bunda $(V)=\{(x,y,z):\ 1\le x\le 3,\ 1\le y\le 2,\ 1\le z\le 2\}$– parallelepiped.

Yechilishi. (V) sohani chizamiz (14.32-chizma). (V) sohani P bo'lish yordamida n^3 ta sohalarga (shunday bo'lishni qaraylikki , $n\to\infty$ da $\lambda_p\to 0$

bo'lsin) bo'lamiz. Buning uchun (x,y) yoki $z=0$, (y,z) yoki $x=0$, (z,x) yoki $y=0$ koordinata tekisliklariga parallel bo'lgan

$$z=z_k,\ z_k=1+\frac{1}{n}k,\ \left(k=\overline{1,n},\ \Delta z=z_k-z_{k-1}=1+\frac{1}{n}k-\left(1+\frac{1}{n}(k-1)\right)=\frac{1}{n}\right);$$

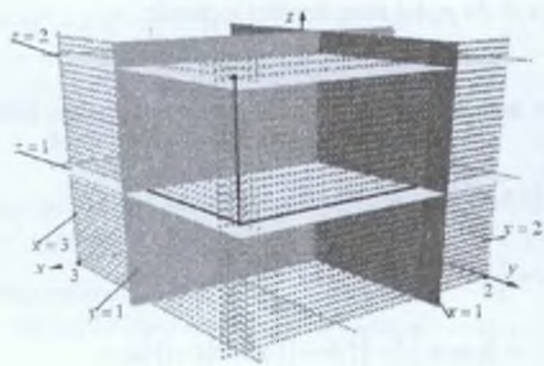
$$x=x_i,\quad x_i=1+\frac{2}{n}i\ \left(i=\overline{1,n},\ \Delta x_i=\frac{2}{n}\right);$$

$$y=y_j,\ y_j=1+\frac{1}{n}j\quad \left(j=\overline{1,n},\ \Delta y_j=\frac{1}{n}\right)$$

tekisliklar o'tkazamiz. Hosil bo'lgan har bir $(V_{i,j,k})$ soha uchun

$$x_{i-1}<x\le x_i,\ y_{j-1}\le y\le y_j,\ z_{k-1}\le z\le z_k\ \left(i=\overline{1,n},\ j=\overline{1,n},\ k=\overline{1,n}\right)$$

shartlar bajariladi va uning hajmi $\Delta V_{i,j,k}=\Delta x_i\cdot\Delta y_j\Delta z_k=\frac{2}{n^3}$ teng bo'ladi.



14.32-chizma.

Integral ostidagi $f(x,y,z)=x+yz$ funksiya qiymatini ixtiyoriy $(x_i,y_j,z_k)\in(V_{i,j,k})$nuqtada hisoblaymiz: $f(x_i,y_j,z_k)=x_i+y_jz_k$, so'ngra natijani $(V_{i,j,k})$ sohachaning $\Delta V_{i,j,k}$ hajmiga ko'paytirib, berilgan uch karrali integral uchun integral yig'indini tuzamiz:

$$\sigma_p(n)=\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{i=1}^{n}\left(x_i+y_jz_k\right)\Delta x_i\Delta y_j\Delta z_k=$$

$$=\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}[1+\frac{2}{n}i+\left(1+\frac{1}{n}j\right)\cdot\left(1+\frac{1}{n}k\right)]\frac{2}{n^3}=$$

$$=\frac{2}{n^3}\sum_{i=1}^{n}\sum_{j=1}^{n}[n+\frac{2}{n}in+\left(1+\frac{1}{n}j\right)\cdot n+\frac{1}{n}\left(1+\frac{1}{n}j\right)\frac{(n+1)n}{2}]=$$

$$=\frac{2}{n^3}\sum_{i=1}^{n}\sum_{j=1}^{n}(2n+2i+j+\frac{n+1}{2}+\frac{1}{n}j\frac{n+1}{2}]=$$

$$=\frac{2}{n^3}\sum_{i=1}^{n}[2n^2+2in+\frac{n(n+1)}{2}+\frac{n(n+1)}{2}+\frac{(n+1)^2}{4}]=$$

$$=\frac{2}{n^3}\sum_{k=1}^{n}[2n^2+2ni+n(n+1)+\frac{(n+1)^2}{4}]=$$

$$=\frac{2}{n^3}[2n^3+2n\frac{n(n+1)}{2}+n^2(n+1)+\frac{n(n+1)^2}{4}]=$$

$$=2\left(2+1+\frac{1}{n}+1+\frac{1}{n}+\frac{1}{4}\left(1+\frac{1}{n}\right)^2\right)=2[4+\frac{1}{4}\left(1+\frac{1}{n}\right)^2+\frac{2}{n}].$$

Endi $n\to\infty$ da $\sigma_p(n)$ ning limitini topamiz:

$$\lim_{n\to\infty}\sigma_P(n)=2[4+\frac{1}{4}]=8\frac{1}{2}.$$

Berilgan uch karrali integralni takroriy integralga keltirish usuli bilan hisoblaymiz:

$$\int_1^3dx\int_1^2dy\int_1^2(x+yz)dz=\int_1^3dx\int_1^2\left(xz+y\frac{z^2}{2}\right)\Big|_1^2dy=$$

$$=\int_1^3dx\int_1^2\left(x+\frac{y}{2}(4-1)\right)dy=\int_1^3dx\int_1^2\left(x+\frac{3}{2}y\right)dy=$$

$$=\int_1^3\left(xy+\frac{3}{2}\frac{y^2}{2}\right)\Big|_1^2dx=\int_1^3\left(x+\frac{3}{4}(4-1)\right)dx=$$

$$=\int_1^3\left(x+\frac{9}{4}\right)dx=\frac{x^2}{2}\Big|_1^3+\frac{9}{4}x\Big|_1^3=\frac{1}{2}(9-1)+\frac{68}{4}=4+4\frac{2}{4}=8\frac{1}{2}.$$

Demak, ikki holda ham, uch karrali integralning natijasi bir xil bo‘ldi.

Misolni Maple tizimidan foydalanib, yechish:

> with(student);

> Tripleint((x+y*z),z=1..2,y=1..2,x=1..3);

$$\int_1^3 \int_1^2 \int_1^2 x + y\, z \, dz\, dy\, dx$$

```
> value(Tripleint((x+y*z),z=1..2,y=1..2,x=1..3));
>
```

$$\frac{17}{2}.$$

2-misol.

$$\iiint_{(V)} (3x+2y)dxdydz, \ (V)=\{(x,y,z): y=0,\ y=x, x=1,\ z=1, z=1+x^2+y^2\}$$

integralni hisoblang.

Yechilishi. (V)sohada $0 \le x \le 1,\ 0 \le y \le x,\ 1 \le 1+x^2+y^2$ tengsizliklar o‘rinli. U holda uch karrali integralni takroriy integrallarga keltirish formulasiga ko‘ra,

$$\iiint_{(V)} (3x+2y)dxdydz = \int_0^1 dx \int_0^x dy \int_1^{1+x^2+y^2} (3x+2y)dz = \int_0^1 dx \int_0^x (3x+2y)z\Big|_1^{1+x^2+y^2} dy =$$

$$= \int_0^1 dx \int_0^x (3x+2y)(x^2+y^2)dy = \int_0^1 dx \int_0^x (3x^3+3xy^2+2yx^2+2y^3)dy =$$

$$= \int_0^1 \left(3x^3y + xy^3 + y^2x^2 + \frac{1}{2}y^4\right)\Big|_0^x dx = \int_0^1 (3x^4+x^4+x^4+x^4+\frac{1}{2}x^4)dx = \frac{11}{10}.$$

Mustaqil yechish uchun misollar

Quyidagi uch karali integrllar ko‘rsatilgan sohada hisoblang:

1. $\iiint_{(v)} (x^2+y^2+z^2)dx\,dy\,dz,$ bunda $(v) - x=0, x=a, y=0,\ y=b, z=0, z=c$ tekisliklar bilanchegaralangan soha.

2. $\iiint_{(v)} y dx\,dy\,dz,$ bu yerda $(v) - x=0, x=2,\ y=0, y=1, z=0,\ z=1-y$.

3. $\iiint_{(v)} (2x+3y-z)dx\,dy\,dz,$ bu yerda (v)–soha $x=0, y=0, x+y=3, z=0,\ z=4$ tekisliklar bilan chegaralangan.

4. $\iiint_{(v)} xyz\ dx\,dy\,dz,$ bu yerda (v)- soha $x=0,\ y=0, z=0,\ x+y+z=1$ tekisliklar bilan chegaralangan.

5. $\iiint_{(v)} xy^2z^3\ dx\,dy\,dz,$ bu yerda (v)- soha $x=1, y=x\ z=0,\ z=xy$ sirtlar bilan chegaralangan.

6. $\iiint\limits_{(v)}(1+x)dx\,dy\,dz$, bu yerda (v)- soha $x=0,\ y=0\ z=0,\ x+y+z=1$ tekisliklar bilan chegaralangan.

7. $\iiint\limits_{(v)}\frac{1}{z}dx\,dy\,dz$ bu yerda (v)- soha $z^2=x^2+y^2$ konus va $z=1,\ z=4$ tekislar bilan chegaralangan.

8. $\iiint\limits_{(v)}z^2dx\,dy\,dz$, bu yerda (v)- soha $z=x^2+y^2$ elliptik paraboloid va $z=2,\ z=6$ tekisliklar bilan chegaralangan.

9. $\iiint\limits_{(v)}z^3dx\,dy\,dz$ bu yerda (v)- soha $z=4-x^2-y^2,\ z=0,\ z=3$ sirtlar bilan chegaralangan.

Mustaqil yechish uchun misollarning javoblari

1. $\frac{abc}{3}\left(a^2+b^2+c^2\right)$ **2.** $\frac{1}{3}$. **3.** 54. **4.** $\frac{1}{720}$. **5.** 2.**6.** $\frac{5}{24}$. **7.** $\frac{15\pi}{2}$. **8.** 50π. **9.** $162\pi/5$.

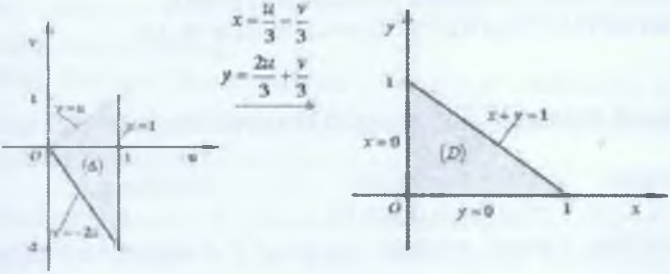
14.3-amaliy mashg'ulot.
Karrali integrallarda o'zgaruvchilarni almashtirish.
Qutb, silindrik va sferik koordinat sistemalariga o'tish usuli

1-misol. $\iint\limits_{(D)}\sqrt{x+y}(y-2x)^2dydx$ ikki karrali integralni hisoblang, bunda (D) soha, $x=0,\ y=0,\ x+y=1$ to'g'ri chiziqlar bilan chegaralangan.

Yechilishi. xOy koordinatalar tekisligida (D) sohani chegar-laydigan sohani chizamiz: $u=x+y$ va $v=y-2x$ almashtirish yordamida integral ostidagi funksiya sodda funksiyaga keltiriladi. Berilgan almashtirishni x va y larga nisbatan yechamiz:

$$x=\frac{u}{3}-\frac{v}{3},\ y=\frac{2u}{3}+\frac{v}{3} \qquad (1)$$

(1) tenglamalar sistemasidan (u,v) koordinatalar tekisligida (Δ) sohani chizib olamiz (14.33- chizma).



14.33 - chizma.

xOy koordinatalar tekisligidagi (D) sohaning chegaralari	(Δ) sohaning chegaralari uchun (u,v) tekisligidagi mosliklar	uOv koordinatalar tekisligidagi (Δ) sohaning chegaralari
$x+y=1$ $x=0$ $y=0$	$\left(\frac{u}{3}-\frac{v}{3}\right)+\left(\frac{2u}{3}+\frac{v}{3}\right)=1$ $\frac{u}{3}-\frac{v}{3}=0$ $\frac{2u}{3}+\frac{v}{3}=0$	$u=1$ $v=u$ $v=-2u$

(1) almashtirishning yakobianini hisoblaymiz:

$$J(u,v)=\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}=\begin{vmatrix} \frac{1}{3} & -\frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} \end{vmatrix}=\frac{1}{3}\neq 0.$$

(14.8.5) formulaga asosan, berilgan ikki karrali integralni yozamiz va hisoblaymiz:

$$\iint\limits_{(D)}\sqrt{x+y}(y-2x)^2\,dxdy=\int\limits_{u=0}^{u=1}\int\limits_{v=-2u}^{v=u}u^{1/2}v^2|J|\,dudv=$$

$$=\int\limits_0^1 du\int\limits_{-2u}^{u}u^{1/2}v^2\frac{1}{3}\cdot dv=\frac{1}{3}\int\limits_0^1 u^{1/2}\left[\frac{1}{3}v^3\right]_{v=-2u}^{v=u}du=$$

$$=\frac{1}{9}\int\limits_0^1 u^{1/2}\left(u^3+8u^3\right)du=\int\limits_0^1 u^{7/2}du=\frac{2}{9}u^{9/2}\Big|_0^1=\frac{2}{9}.$$

Misolni Maple tizimidan foydalanib, yechish:

> int(int((1/3)*(sqrt(u)*v^2),v=-2*u..u),u=0..1);

$$\frac{2}{9}.$$

2-misol. Ushbu $I = \iint\limits_{(D)} e^{x^2-y^2}dxdy$ ikki karrali integralni qutb koordinatalar sistemasidan foydalanib hisoblang, bunda $(D) = \{(x,y): x^2 + y^2 \le 4, x \ge 0, y \ge 0\}$.

Yechilishi. Integral ostidagi $f(x,y) = e^{x^2+y^2}$ funksiya va integrallanuvchi soha chegarasining tenglamasi $x^2 + y^2 = 4$ aylanadan iborat bo'lib, $x^2 + y^2$ bog'liq. **14.9.3**-eslatmaga asosan, (14.9.1) almashtirish yordamida (r,φ) qutb koordinatalar sistemasiga o'tamiz:

$$f(x,y) = e^{x^2+y^2} = e^{r^2\cos^2\varphi + r^2\sin^2\varphi} = e^{r^2},\ r^2\cos^2\varphi + r^2\sin^2\varphi = 4 \text{ yoki } r^2 = 4, r = 2,$$

$$x = 0 \Rightarrow r\cos\varphi = 0 \Rightarrow \varphi = \frac{\pi}{2},\ y = 0 \Rightarrow r\sin\varphi = 0 \Rightarrow \varphi = 0.$$

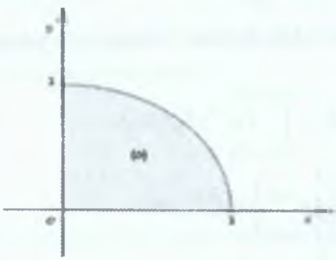
Demak, $(\Delta) = \left\{(r,\varphi):\ 0 \le \varphi \le \frac{\pi}{2},\ 0 \le r \le 2\right\}$ (14.34-chizma). Endi (14.9.2) formulaga asosan, berilgan ikki karrali integralni hisoblaymiz:

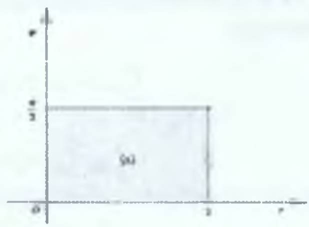
$$I = \iint\limits_{(D)} e^{x^2-y^2}dxdy = \iint\limits_{(\Delta)} e^{r^2} r dr d\varphi = \int\limits_0^{\frac{\pi}{2}} d\varphi \int\limits_0^2 re^{r^2}dr = \frac{\pi}{4}\left(e^4 - 1\right).$$

Misolni Maple tizimidan foydalanib, yechish:

>int(int(y*exp(y^2),y=0..2),x=0..Pi/2);

$$\frac{1}{4}e^4\pi - \frac{1}{4}\pi$$





14.34-chizma.

3-misol. Ushbu $z=4-x^2-y^2$, $x^2+y^2=2y$, $z=0$ sirtlar bilan chegaralangan jismning hajmini toping.

Yechilishi. Berilgan jism yuqoridan $z=4-x^2-y^2$ paraboloid, quyidan esa, $x^2+y^2=2y$ doiraviy silindrik sirt bilan chegaralangan (14.35-chizma). Jismning hajmini hisoblaymiz: $V=\iint\limits_{(D)}\left(4-x^2-y^2\right)dxdy$.

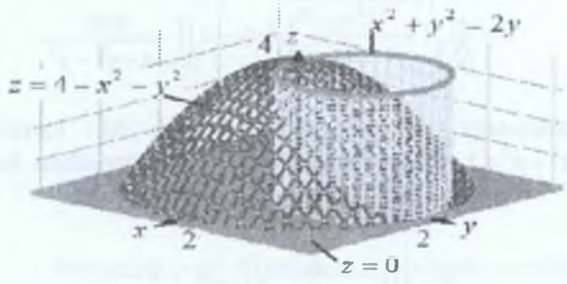
Integrallash chegarasi (D) soha-doira hamda integral ostidagi funksiya x^2+y^2 bog'liq bo'lganligi uchun, qutb koordinatalar sistemasiga o'tib hisoblaymiz. $x^2+y^2=2y$ aylana tenglamasi qutb koordinatalar sistemasida $r^2=2r\sin\varphi$, ya'ni $r=2\sin\varphi$ ko'rinishda bo'lib, (D) sohada φ burchak 0 va π oraliqda o'zgaradi, integral ostidagi funksiya $4-r^2$ bo'ladi. (14.21.3) formulaga asosan, jismning hajmini topamiz:

$$V=\iint\limits_{(D)}\left(4-r^2\right)r\,dr\,d\varphi=\int_0^{\pi}d\varphi\int_0^{2\sin\varphi}\left(4r-r^3\right)dr=$$

$$=\int_0^{\pi}\left(2r^2-\frac{r^4}{4}\right)\Bigg|_0^{2\sin\varphi}d\varphi=\int_0^{\pi}\left(8\sin^2\varphi-4\sin^4\varphi\right)d\varphi=$$

$$=\int_0^{\pi}\left[4(1-\cos2\varphi)-(1-\cos2\varphi)^2\right]d\varphi=$$

$$=\int_0^{\pi}\left(4-4\cos2\varphi-1+2\cos2\varphi+\frac{1+\cos4\varphi}{2}\right)d\varphi=\frac{5\pi}{2}\ \text{(kb. birl).}$$



14.35-chizma.

Misolni Maple tizimidan foydalanib, yechish:

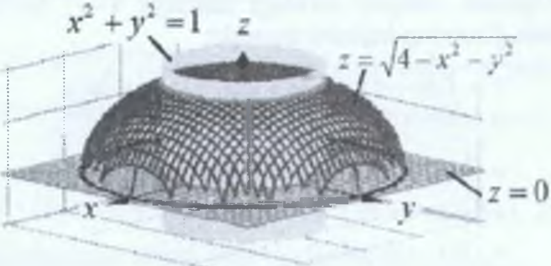
> int(int(4*y-y^3,y=0..2*sin(x)),x=0..Pi);

$\frac{5}{2}\pi$.

4-misol. $x^2+y^2+z^2=4$ $(z\ge 0)$ yarim sferaning $x^2+y^2=1$ silindr bilan ajratilgan qismi yuzini toping.

Yechilishi. (S) sirtning tenglamasi $z=\sqrt{4-x^2-y^2}$ ko'rinishda. $x^2+y^2=1$ aylana bilan chegaralangan (D) soha-doiradan iborat (14.36-chizma).

$$z'_x=p=-\frac{x}{\sqrt{4-x^2-y^2}},\ z'_y=q=-\frac{y}{\sqrt{4-x^2-y^2}}$$



14.36-chizma.

(14.11.1) formuladan foydalanib, izlanayotgan sirt yuzini topamiz:

$$S=\iint\limits_{(D)}\sqrt{1+\frac{x^2+y^2}{4-x^2-y^2}}dxdy=2\iint\limits_{(D)}\frac{dxdy}{\sqrt{4-x^2-y^2}}.$$

Qutb koordinatalar sistemasidan foydalanib ikki karrali integralni hisoblaymiz. $x^2+y^2=1$ aylana tenglamasi $r=1$ ko'rinishda bo'lib, bunda $0\le\varphi\le 2\pi$.

Demak,

$$S=2\iint\limits_{(\Delta)}\frac{r\,dr\,d\varphi}{\sqrt{4-r^2}}=2\int\limits_0^{2\pi}d\varphi\int\limits_0^1\frac{r\,dr}{\sqrt{4-r^2}}=$$

$$=2\int\limits_0^{2\pi}[-\sqrt{4-r^2}]\Big|_0^1 d\varphi=2\cdot 2\pi[-\sqrt{3}+2]=4\pi\left(2-\sqrt{3}\right)\ \text{(kv. birlik)}.$$

Misolni Maple tizimidan foydalanib, yechish:

> **int(int(2*y/sqrt(4-y^2),y=0..1),x=0..2*Pi);**

$$4\frac{\pi}{2+\sqrt{3}}.$$

5-misol. $y=\sqrt{2x}$ parabola va $y=x$ to'g'ri chiziqlar bilan chegaralangan tekis shaklning yuzini hisoblang.

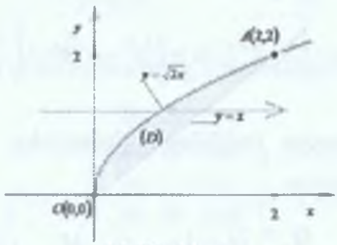
Yechilishi. Tekis shaklning yuzini (14.12.1) formuladan foydalanib hisoblaymiz. Integrallash chegarasini topish uchun, dastlab $y=\sqrt{2x}$ parabola va $y=x$ to'g'ri chiziqlarning kesishish nuqtalarini topamiz:

$$\begin{cases} y^2=2x \\ y=x \end{cases} \Rightarrow x^2=2x \Rightarrow x_1=0,\ y_1=0 \text{ va } x_2=2,\ y_2=2.$$

Demak, berilgan parabola va to'g'ri chiziqning kesishish nuqtalari $O(0,0)$ va $A(2,2)$ bo'la-di. (D) sohaning Oy o'qdagi proyeksiyasi $[0,2]$ kesmadan iborat. Shunday qilib, (14.12.1) formulaga asosan,

$(D)=\left\{(x,y)\in R^2: 0\le y\le 2,\ \dfrac{y^2}{2}\le x\le y\right\}$ yuzani hisoblaymiz (14.37-chizma):

$$D=\int_0^2 dy\int_{y^2/2}^{y}dx=\int_0^2\left(x\Big|_{x=y^2/2}^{x=y}\right)dy=\int_0^2\left(y-\frac{y^2}{2}\right)dy=\left(\frac{y^2}{2}-\frac{y^3}{6}\right)\Big|_0^2=\frac{2}{3}$$



14.37-chizma.

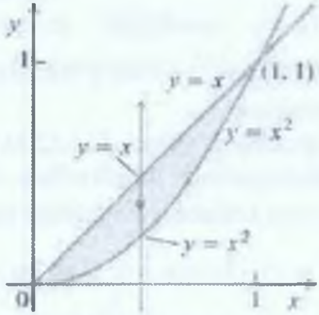
Misolni Maple tizimidan foydalanib, yechish:

> **int(int(1,x=y^2/2..y),y=0..2);**

$$\frac{2}{3}.$$

6-misol. $y=x^2$ parabola va $y=x$ to'g'ri chiziqlar bilan chegaralangan, zichligi $\rho(x,y)=1$ bo'lgan bir jinsli plastinkaning og'irlik markazini toping.

Yechilishi. Dastlab bir jinsli plastinkaning massasini topamiz (14.38-chizma):



14.38- chizma.

$$m=\int_0^1 dx\int_{x^2}^{x} 1\cdot dy=\int_0^1 [y]\Big|_{y=x^2}^{y=x} dx=\int_0^1 \left(x-x^2\right)dx=\left(\frac{x^2}{2}-\frac{x^3}{3}\right)\Big|_0^1=\frac{1}{6}.$$

Ox va *Oy* o‘qlariga nisbatan statistik momentlarini topamiz:

$$M_x=\int_0^1 dx\int_{x^2}^{x} ydy=\int_0^1 [\frac{y^2}{2}]\Big|_{y=x^2}^{y=x} dx=\int_0^1 [\frac{x^2}{2}-\frac{x^4}{2}]dx=[\frac{x^3}{6}-\frac{x^5}{10}]\Big|_0^1=\frac{1}{15},$$

$$M_y=\int_0^1 dx\int_{x^2}^{x} xdy=\int_0^1 (xy)\Big|_{y=x^2}^{y=x} dx=\int_0^1 \left(x^2-x^3\right)dx=\left(\frac{x^3}{3}-\frac{x^4}{4}\right)\Big|_0^1=\frac{1}{12}.$$

m, M_x va M_y lardan foydalanib, plastinka og‘irlik markazining koordinatalarini hisoblaymiz:

$$x_0=\frac{M_y}{m}=\frac{\frac{1}{12}}{\frac{1}{6}}=\frac{1}{2},\qquad y_0=\frac{M_x}{m}=\frac{\frac{1}{15}}{\frac{1}{6}}=\frac{2}{5}.$$

Demak, $C\left(\frac{1}{2};\frac{2}{5}\right)$.

7-misol. Ushbu $\int_0^3 dz\int_0^4 dy\int_{x=y/2}^{x=\frac{y}{2}+1}\left(\frac{2x-y}{2}+\frac{z}{3}\right)dx$ uch karrali integralni

$$u=\frac{2x-y}{2},\ v=\frac{y}{2},\ w=\frac{z}{3} \qquad (2)$$

almashtirish yordamida (V) fazoda integrallash chegaralarini aniqlang va integralni hisoblang.

Yechilishi. (V) sohaning (xyz) fazoda integrallash chegaralari chizmada quyidagicha tasvirlanadi. Uch karrali integralda o'zga-ruvchilarni almashtirish formulasidagi yakobianni hisoblash uchun (2) sistemani x, y va z larga nisbatan yechamiz:

$$x=u+v,\ y=2v,\ z=3w \qquad (3)$$

(x,y,z) koordinatalar tekisligida (V) sohaning chegaralari	(Δ) sohaning chegaralari uchun (u,v,w) tekisligidagi mosliklar	(u,v,w) koordinatalar tekisligidagi (Δ) sohaning chegaralari
$x=\frac{y}{2}$	$u+v=\frac{2v}{2}=v$	$u=0$
$x=\frac{y}{2}+1$	$u+v=\frac{2v}{2}+1=v+1$	$u=1$
$y=0$	$2v=0$	$v=2$
$y=4$	$2v=4$	$w=0$
$z=0$	$3w=0$	$w=1$
$z=3$	$3w=3$	

(3) tenglamalardan almashtirishning yakobianini hisoblaymiz:

$$J(u,v,\omega)=\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}=\begin{vmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{vmatrix}=6.$$

(14.20.3) formulaga asosan, uch karrali integralni hisoblaymiz:

$$\int_0^3 dz\int_0^4 dy\int_{x=y/2}^{x=\frac{y}{2}+1}\left(\frac{2x-y}{2}+\frac{z}{3}\right)dx=\int_0^1 dw\int_0^2 dv\int_0^1 (u+w)\,|J(u,v,w)|\,du=$$

$$= 6\int_0^1 dw \int_0^2 \left(\frac{u^2}{2} + wu\right)\Big|_{u=0}^{u=1} dv = 6\int_0^1 dw \int_0^2 \left(\frac{1}{2} + w\right) dv$$

$$= 6\int_0^1 \left(\frac{v}{2} + wv\right)\Big|_{v=0}^{v=2} dw =$$

$$= 6\int_0^1 (1 + 2w)\, dw = 6[w + w^2]\Big|_{w=0}^{w=1} = 12$$

8-misol. Silindrik koordinatalar sistemasiga o‘tib uch karrali integralni hisoblang: $\iiint\limits_{(V)} \sqrt{x^2 + y^2}\, z dx\, dy\, dz$, bunda (V) soha $0 \le x \le 2,\ 0 \le y \le \sqrt{2x - x^2},\ 0 \le z \le 1$ tengsizliklar bilan berilgan.

Yechilishi. 1-qadam. *Chizish.* (V) sohani va uning xOy tekislikka proyeksiyasi (D) ni chizamiz (14.39-chizma). (V) sohani chegaralovchi sirtlar: $0 \le x \le 2,\ 0 \le y \le \sqrt{2x - x^2},\ 0 \le z \le 1$. (D) sohani chegaralovchi egri chiziqlar: $0 \le x \le 2,\ 0 \le y \le \sqrt{2x - x^2}$.

2-qadam. *Integrallashning z bo‘yicha chegaralari*. (D) dagi biror (r,θ) nuqtadan Oz o‘qqa parallel M to‘g‘ri chiziqni o‘tkazamiz. Bunda z ortganda M to‘g‘ri chiziq (V) ga $z = 0$ sirt bo‘ylab kiradi va undan $z = 1$ sirt bo‘ylab chiqadi. Ular z bo‘yicha integrallash chegaralaridan iborat bo‘ladi: $0 \le z \le 1$.

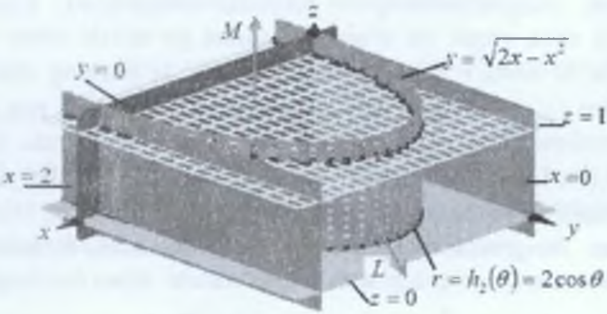
3-qadam. *Integrallashning r bo‘yicha chegaralari*. Koordinatalar boshidan chiqib (r,θ) nuqtadan o‘tuvchi L nurni chizamiz. Bu nur $r = h_1(\theta) = 0$ bo‘ylab (D) ga kiradi va $r = h_2(\theta) = 2\cos\theta$ bo‘ylab undan chiqadi. Ular r bo‘yicha integrallash chegaralari bo‘ladi: $0 \le r \le 2\cos\varphi$.

4-qadam. *Integrallashning θ bo‘yicha chegaralari*. L nur (D) orqali o‘tganda θ burchak Ox o‘qning musbat yo‘nalishida $\theta = 0$ dan $\theta = \frac{\pi}{2}$ gacha o‘zgaradi. Ular θ bo‘yicha integrallash chegaralari bo‘ladi: $0 \le \varphi \le \frac{\pi}{2}$.

(14.22.1) formulaga asosan, uch karrali integralni hisoblaymiz:

$$\iiint\limits_{(V)} \sqrt{x^2 + y^2}\, z dx\, dy\, dz = \iiint\limits_{(T)} r^2 z\, dr\, d\varphi\, dz = \int_0^{\pi/2} d\varphi \int_0^{2\cos\varphi} r^2 d\varphi \int_0^1 z\, dz = \frac{1}{2}\int_0^{\pi/2} d\varphi \int_0^{2\cos\varphi} r^2 dr =$$

$$= \frac{4}{3}\int_0^{\pi/2} \cos^3\varphi\, d\varphi = \frac{4}{3}\int_0^{\pi/2} (1 - \sin^2\varphi)\, d(\sin\varphi) = \frac{4}{3}\left(\sin\varphi\Big|_0^{\pi/2} - \frac{\sin^3\varphi}{3}\Big|_0^{\pi/2}\right) = \frac{4}{3}\left(1 - \frac{1}{3}\right) = \frac{8}{9}.$$



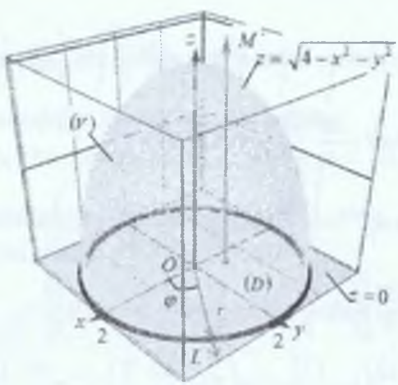
14.39-chizma.

9-misol. Ushbu $\iiint\limits_{(V)}(9-x^2-y^2)dx\,dy\,dz$ uch karrali integralni sferik koordinatalar sistemasiga o‘tib hisoblang, bunda (V) soha $x^2+y^2+z^2\le 4$ yarim shar, $z\ge 0$.

Yechilishi. 1-qadam. *Chizish.* $(x,y,z)\leftrightarrow(r,\varphi,\theta)$ koordinatalar sistemasiga o‘tamiz:

$$f(r\cos\varphi\cos\theta,\ r\sin\varphi\cos\theta,\ r\sin\theta)=9-r^2\cos^2\theta,\quad I=r^2\cos\theta.$$

(G) sohani va uning xOy tekislikdagi (D) proyeksiyasi markazi $O(0,0)$ nuqtada radiusi $R=2$ bo‘lgan doirani $x^2+y^2\le 4$ chizamiz (14.40-chizma). (G)sohani chegaralovchi sirt yarim sfera: $z=\sqrt{4-x^2-y^2}$.



14.40-chizma.

2-qadam. *Integrallashning r bo'yicha chegaralari.* Koordi-natalar boshidan (G) soha orqali Oz o'qining musbat yo'nalishi bilan θ burchak hosil qiluvchi M nurni o'tkazamiz. Shuningdek, M nurning xOy tekislikka proyeksiyasini (L proyeksiya deb ataladigan) o'tkazish. L nur Ox o'qning musbat yo'nalishi bilan φ burchak hosil qiladi. r ortganda M nur (G) sohaga $r = g_1(\theta,\varphi) = 0$ bo'ylab kiradi va undan $r = g_2(\theta,\varphi) = 2$ bo'ylab chiqadi. Ular integrallashning r bo'yicha chegaralari bo'ladi: $0 \le r \le 2$.

3-qadam. *Integrallashning θ bo'yicha chegaralari.* Ixtiyoriy berilgan φ uchun, M nur Oz o'qning musbat yo'nalishi bilan hosil qiladigan θ burchak $\theta = 0$ dan $\theta = \frac{\pi}{2}$ gacha o'zgaradi. Shular integ-rallashning θ bo'yicha chegaralari bo'ladi: $0 \le \theta \le \frac{\pi}{2}$.

4-qadam. *Integrallash-ning φ bo'yicha chegaralari.* L nur (D) ustidan o'tganda φ burchak $\alpha = 0$ dan $\beta = 2\pi$ gacha o'zgaradi. Ular integrallashning φ bo'yicha chegaralari bo'ladi: $0 \le \varphi \le 2\pi$.

Demak, berilgan uch karrali integral sferik koordinatalar sistemasida quyidagi

$$\iiint\limits_{(V)} \left(9 - x^2 - y^2\right) dx\, dy\, dz = \iiint\limits_{(G)} \left(9 - r^2 \cos^2\theta\right) r^2 \cos\theta\, d\varphi\, dr\, d\theta.$$

ko'rinishda bo'ladi. Endi (14.24.1) formulaga asosan, integralni hisoblaymiz:

$$\iiint\limits_{(G)} \left(9r^2 \cos\theta - r^4 \cos^3\theta\right) d\varphi\, dr\, d\theta =$$

$$= \int_0^{2\pi} d\varphi \int_0^{\pi/2} d\theta \int_0^2 \left(9r^2 \cos\theta - r^4 \cos^3\theta\right) dr = \varphi\Big|_0^{2\pi} \int_0^{\pi/2} \left(9\frac{r^3}{3}\Big|_0^2 \cos\theta - \frac{r^5}{5}\Big|_0^2 \cos^3\theta\right) d\theta =$$

$$= 2\pi\left(3\cdot 8\cdot \int_0^{\pi/2} \cos\theta\, d\theta - \frac{32}{5}\int_0^{\pi/2} \left(1 - \sin^2\theta\right) d(\sin\theta)\right) =$$

$$= 2\pi\left(24 \sin\theta\Big|_0^{\pi/2} - \frac{32}{5}\left(\sin\theta\Big|_0^{\pi/2} - \frac{\sin^3\theta}{3}\Big|_0^{\pi/2}\right)\right) =$$

$$= 2\pi\left(24 - \frac{32}{5}\left[\left(\sin\frac{5}{2} - 0\right) - \frac{1}{3}\left(\sin^3\frac{\pi}{2} - \sin 0\right)\right]\right) =$$

$$= 2\pi\left[24 - \frac{32}{5}\left(1 - \frac{1}{3}\right)\right] = 2\pi\left(24 - \frac{32}{5}\cdot\frac{2}{3}\right) = 2\pi\frac{24\cdot 15 - 64}{15} = \frac{292}{15}\pi.$$

Mustaqil yechish uchun misollar

1. Ushbu $u = x + 2y,\ v = x - y$ sistemadan x va y ni u va v lar orqali ifodalang va $J = \dfrac{D(x,y)}{D(u,v)}$ yakobiani toping.

2. (xy) tekislikda $y = 0,\ y = x$ va $x + 2y = 2$ to'g'ri chiziqlar bilan chegaralangan (D) sohada $u = x + 2y,\ \ v = x - y$ almashtirishni tasvirini toping. (uv) tekislikda almashtirish chegaralarini aniqlang.

3. Ushbu $u = 3x + 2y,\ \ v = x + 4y$ sistemasidan x va y ni u va v lar orqali ifodalang, so'ngra $J(u,v) = \dfrac{D(x,y)}{D(u,v)}$ yakobiani toping.

4. $x = 0,\ y = 0$ va $x + y = 1$ to'g'ri chiziqlar bilan chegaralangan sohada $u = 3x + 2y,\ v = x + 4y$ almashtirishning tasvirini toping. (uv) tekislikda almashtirish chegaralarini toping.

5. Ushbu $u = 2x - 3y,\ v = -x - y$ sistemadan x va y ga nisbatan yechib u va v lar orqali ifodalang, so'ngra $J(u,v) = \dfrac{D(x,y)}{D(u,v)}$ yakobianni toping.

6. $x = -3,\ x = 0,\ y = x$ va $y = x + 1$ to'g'ri chiziqlar bilan chegaralangan sohada $u = 2x - 3y,\ v = -x + y$ almashtirishning tasvirini toping. (uv)tekislikda almashtirish chegaralarini toping.

7. $\iint\limits_{(D)} \left(3x^2 + 14xy + 8y^2\right)dxdy,$ bu yerda (D)– soha birinchi chorakda $y = -\dfrac{3}{2}x + 1,\ \ y = -\dfrac{3}{2} + 3,\ y = -\dfrac{1}{4}x$ va $y = -\dfrac{1}{4}x + 1$ to'g'ri chiziqlar bilan chegaralangan.

8. $\iint\limits_{(D)} \left(\sqrt{\dfrac{y}{x}} + \sqrt{xy}\right)dxdy,$ bu yerda (D)– soha $xy = 1,\ xy = 9$ giperbolalar va $y = x,\ y = 4x$ to'g'ri chiziqlar bilan chegaralangan.

9. $\iint\limits_{(D)} (2x - y)dxdy.$ bu yerda $(D) - x + y = 1,\ x + y = 2,\ 2x - y = 1, 2x - y = 3$ to'g'ri chiziqlar bilan chegaralangan parallelogramm.

10. $\iint\limits_{(D)} \sqrt{xy}dxdy.$ bu yerda

$(D) - y^2 = ax,\ y^2 = bx,\ xy = p, xy = q\ \ (0 < a < b,\ 0 < p < q)$ egri chiziqlar bilan chegaralangan.

11. $\iint\limits_{(D)} xydxdy$, bunda $(D) - y = ax^3,\ y = bx^3,\ y^2 = px,\ y^2 = qx\ (0 < a < b,\ 0 < p < q)$ egri chiziqlar bilan chegaralangan soha.

12. $\iint\limits_{(D)} (x+y)^3 (x-y)^2 dxdy$, bu yerda $(D) - x + y = 1, x - y = 1, x + y = 3, x - y = -1$ to‘g‘ri chiziq bilan chegaralan gan kvadrat soha.

Qutb koordinatalar sistemasiga o‘tib ikki ikkarali integralni hisoblang.

13. $\iint\limits_{(D)} \sqrt{x^2 + y^2}\,dxdy;\ D - I\,chorak,\ \ x^2 + y^2 \le 9.$

14. $\iint\limits_{D} y\,dx\,dy;\ D - \quad x^2 + y^2 \le 1, -1 \le x \le 0, 0 \le y \le 1.$

14. $\iint\limits_{D} \frac{dxdy}{\sqrt{x^2 + y^2}};D - x^2 + y^2 \ge 1,\ x^2 + y^2 \le 4\ .$

16. $\iint\limits_{D} \frac{dxdy}{x^2 + y^2 + 1};D - x^2 + y^2 \le 1,\ y \ge 0\ .$

17. $\iint\limits_{D} (x^2 + y^2)dxdy;\ D - \ x^2 + y^2 \le 4x\ .$

Silindrik yoki sferik koordinatalar sistemasiga o‘tib, quyidagi sirtlar bilan chegaralangan jismning hajmini hisoblang:

18. $\iiint\limits_{G} (x^2 + y^2)dx\,dy\,dz;\ G - z = \sqrt{9 - x^2 - y^2}\ .$

19. $\iiint\limits_{G} z\sqrt{x^2 + y^2}dx\,dy\,dz; G - x^2 + y^2 = 2x\ \ z = 0, z = 3$

20. $\iiint\limits_{G} z\,dx\,dy\,dz; G - z^2 = x^2 + y^2 \quad z = 2$

21. $\iiint\limits_{G} (x^2 + y^2)dx\,dy\,dz;\ G - 2z = x^2 + y^2\ \ z = 2$

22. $\iiint\limits_{G} \frac{dx\,dy\,dz}{\sqrt{x^2 + y^2 + z^2}};\ G - x^2 + y^2 + z^2 = 4\ x^2 + y^2 + z^2 = 16\ .$

23. $\iiint\limits_{G} (x^2 + y^2 + z^2)dx\,dy\,dz;\ C - x^2 + y^2 + z^2 = 9\ \ z = \sqrt{x^2 + y^2}\ .$

Mustaqil yechish uchun misollarning javoblari

1. $x = \frac{u + 2v}{3},\ \ y = \frac{u - v}{3} \quad J = -\frac{1}{3}$. **2.** $v = 0,\ u = 2 \quad v = u$

3. $x = \frac{1}{5}(2u - v),\ y = \frac{1}{10}(3v - u);\ J = \frac{1}{10}$. **4.** $3v = u,\ \ v = 2u$ va $3u + v = 10$.

5. $x=-u-3v,\ y=\dfrac{-u-2v}{2},\ J=-\dfrac{1}{2}$. **6.** $u+3v=3,\ u=-3v,\ u=-4v,\ u+4v=2$.

7. $\dfrac{64}{5}$. **8.** $\int\limits_1^2\int\limits_1^3(u+v)\dfrac{2u}{v}\,dudv=8+\dfrac{52}{3}\ln 2$. **9.** $\dfrac{4}{3}$. **10.** $\dfrac{2}{9}\left(q^{3/2}-p^{3/2}\right)\ln\dfrac{b}{a}$

11. $\dfrac{5}{48}\left(a^{-6/5}-b^{-6/5}\right)\left(q^{8/5}-p^{8/5}\right)$. **12.** $\dfrac{20}{3}$. **13.** $9\pi/2$ **14.** $1/2$. **14.** 2π. **16.** $(\pi/2)\ln 2$.

17. 24π. **18.** $324\pi/5$. **19.** 16. **20.** 4π. **21.** $16\pi/3$. **22.** 24π. **23.** $243\left(2-\sqrt{2}\right)\pi/5$.

14-bob bo‘yicha amaliy mashg‘ulotlarni mustahkamlash uchun nazorat topshiriqlari

14.1- masala. Quyidagi intergallarda integrallash tartibini o‘zgartiring.

14.1.1. $\int\limits_0^1 dy\int\limits_0^y f(x,y)\,dx+\int\limits_1^{\sqrt{2}} dy\int\limits_0^{\sqrt{2-y^2}} f(x,y)dx.$

14.1.2. $\int\limits_0^1 dy\int\limits_0^{\sqrt{y}} f(x,y)dx+\int\limits_1^2 dy\int\limits_0^{\sqrt{2-y}} f(x,y)\,dx.$

14.1.3. $\int\limits_{-\sqrt{2}}^{-1} dx\int\limits_{-\sqrt{2-x^2}}^{0} f(x,y)dy+\int\limits_{-1}^{0} dx\int\limits_{x}^{0} f(x,y)dy.$

14.1.4. $\int\limits_{-2}^{-1} dy\int\limits_0^{\sqrt{2+y}} f(x.y)dx+\int\limits_{-1}^{0} dy\int\limits_0^{\sqrt{-y}} f(x,y)dx.$

14.1.5. $\int\limits_{-\sqrt{2}}^{-1} dx\int\limits_0^{\sqrt{2-x^2}} f(x,y)dy+\int\limits_{-1}^{0} dx\int\limits_0^{x^2} f(x,y)dy.$

14.1.6. $\int\limits_0^{\pi/4} dy\int\limits_0^{\sin y} f(x,y)dx+\int\limits_{\pi/4}^{\pi/2} dy\int\limits_0^{\cos y} f(x,y)dx.$

14.1.7. $\int\limits_0^1 dy\int\limits_0^{\sqrt{y}} f(x,y)dx+\int\limits_1^e dy\int\limits_{\ln y}^{1} f(x,y)dx.$

14.1.8. $\int\limits_0^1 dy\int\limits_{-\sqrt{y}}^{0} f(x,y)dx+\int\limits_1^2 dy\int\limits_{-\sqrt{2-y}}^{0} f(x,y)dx.$

14.1.9. $\int\limits_0^1 dy\int\limits_{-y}^{0} f(x,y)dx+\int\limits_1^{\sqrt{2}} dy\int\limits_{-\sqrt{2-y^2}}^{0} f(x,y)dx.$

14.1.10. $\int\limits_0^1 dy \int\limits_0^{y^2} f(x,y)dx + \int\limits_1^2 dy \int\limits_0^{2-y} f(x,y)dx.$

14.1.11. $\int\limits_0^{\pi/4} dx \int\limits_0^{\sin x} f(x,y)dy + \int\limits_{\pi/4}^{\pi/2} dx \int\limits_0^{\cos x} f(x,y)dy.$

14.1.12. $\int\limits_{-\sqrt{2}}^{-1} dy \int\limits_{-\sqrt{2-y^2}}^{0} f(x,y)dx + \int\limits_{-1}^{0} dy \int\limits_y^0 f(x,y)dx$

14.1.13. $\int\limits_0^1 dx \int\limits_0^{x^2} f(x,y)dy + \int\limits_1^2 dx \int\limits_0^{2-x} f(x,y)dy.$

14.1.14. $\int\limits_0^{\sqrt{3}} dx \int\limits_0^{2-\sqrt{4-x^2}} f(x,y)dy + \int\limits_{\sqrt{3}}^{2} dx \int\limits_0^{\sqrt{4-x^2}} f(x,y)dy.$

14.1.5. $\int\limits_0^1 dx \int\limits_{-\sqrt{x}}^{0} f(x,y)dy + \int\limits_1^2 dx \int\limits_{-\sqrt{2-x}}^{0} f(x,y)dy.$

14.1.16. $\int\limits_0^1 dx \int\limits_0^x f(x,y)dy + \int\limits_1^{\sqrt{2}} dx \int\limits_0^{\sqrt{2-x^2}} f(x,y)dy.$

14.1.17. $\int\limits_0^1 dy \int\limits_0^{\sqrt{y}} f(x,y)dx + \int\limits_1^{\sqrt{2}} dy \int\limits_0^{\sqrt{2-y^2}} f(x,y)dx.$

14.1.18. $\int\limits_0^1 dx \int\limits_0^{\sqrt{x}} f(x,y)dy + \int\limits_1^2 dx \int\limits_0^{\sqrt{2-x}} f(x,y)dy.$

14.1.19. $\int\limits_{-2}^{-\sqrt{3}} dx \int\limits_0^{\sqrt{4-x^2}} f(x,y)dy + \int\limits_{-\sqrt{3}}^{0} dx \int\limits_0^{2-\sqrt{4-x^2}} f(x,y)dy.$

14.1.20. $\int\limits_0^1 dx \int\limits_{1-x^2}^{1} f(x,y)dy + \int\limits_1^e dx \int\limits_{\ln x}^{1} f(x,y)dy.$

14.1.21. $\int\limits_0^1 dy \int\limits_{\frac{1}{9}y^2}^{y} f(x,y)dx + \int\limits_1^3 dy \int\limits_{\frac{1}{9}y^2}^{1} f(x,y)dx.$

14.1.22. $\int\limits_3^7 dy \int\limits_{9/y}^{3} f(x,y)dx + \int\limits_7^9 dy \int\limits_{9/y}^{10-y} f(x,y)dx.$

14.1.23. $\int\limits_0^{\sqrt{2}/2} dy \int\limits_y^{\sqrt{1-y^2}} f(x,y)dx + \int\limits_{-\sqrt{2}/2}^{0} dy \int\limits_{-y}^{\sqrt{1-y^2}} f(x,y)dx.$

14.1.24. $\int_1^3 dy \int_0^{\log_3 y} f(x,y)dx + \int_3^4 dy \int_0^{4-y} f(x,y)dx.$

14.1.25. $\int_{-\pi/4}^{3\pi/2} dx \int_0^{\sin x} f(x,y)dy + \int_{\pi/2}^{5\pi/2} dx \int_{\sin x}^{1} f(x,y)dy\,.$

14.1.26. $\int_0^1 dy \int_0^{\sqrt{y}} f(x,y)dx + \int_1^2 dy \int_0^{2-y} f(x,y)dx.$

Yechilishi ([9], 2-t., 8-bo'lim, [30], 14.2-bo'lim). Berilgan masalaning shartiga ko'ra, integrallash sohasi

$$(D)=\{(x,y):0\le y\le 1,\ 0\le x\le \sqrt{y}\}\cup\{(x,y):1\le y\le 2,\ 0\le x\le 2-y\}=$$
$$=(D_1)+(D_2)$$

(D) soha 14.41- chizmada tasvirlangan. Uni quyidagicha ham tasvirlash mumkin:

$$(D)=\{(x,y)\in R^2:0\le x\le 1,\ x^2\le y\le 1\}\cup\{(x,y)\in R^2:0\le x\le 1,\ 1\le y\le 2-x\}$$

Demak, $\int_0^1 dy \int_0^{\sqrt{y}} f(x,y)dx + \int_1^2 dy \int_0^{2-y} f(x,y)dx=$

$$=\int_0^1 dx \int_{x^2}^{1} f(x,y)dy + \int_0^1 dx \int_1^{2-x} f(x,y)dy = \int_0^1 dx \int_{x^2}^{2-x} f(x,y)dy\,.$$

14.2-masala. Ko'rsatilgan (D)- soha uchun $\iint\limits_{(D)} f(x,y)dx\,dy$ integralda qutb koordinatalariga $(x=r\cos\varphi, y=r\sin\varphi)$ o'tib, integrallash chegaralari ikki xil tartibda qo'ying (barcha parametrlar musbat deb qabul qilinadi):

14.2.1. $(D)=\{(x,y)\in R^2:x^2+y^2\le a^2\}$

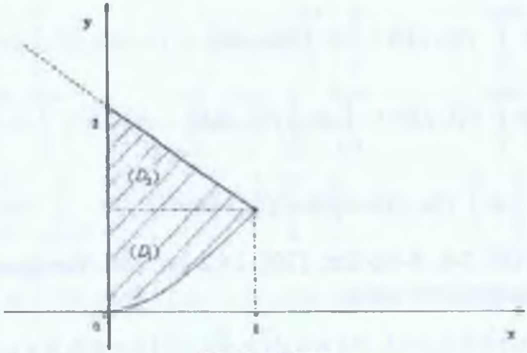
14.2.2. $(D)=\{(x,y)\in R^2:x^2+y^2\le ax\}$

14.2.3. $(D)=\{(x,y)\in R^2:a^2\le x^2+y^2\le b^2\}$

14.2.4. $(D)=\{(x,y)\in R^2:0\le x\le 1,\ 0\le y\le 1-x\}.$

14.2.5. $(D)=\left\{(x,y)\in R^2:-a\le x\le a,\ \dfrac{x^2}{a}\le y\le a\right\}.$

14.2.6. $(D)=\{(x,y)\in R^2:x^2+y^2\le ax,\quad x^2+y^2\le by\}$ doiralarning umumiy qismi.



14.41-chizma.

14.2.7. $(D)=\{(x,y)\in R^2 : y=x,\ \ y=-x, y=1\}$

14.2.8. $(D)=\{(x,y)\in R^2 : a^2\le x^2+y^2\le 4a^2,\ |x|-y\ge 0\}$

14.2.9. $(D)=\{(x,y)\in R^2 : x^2+y^2\le 2ay\}$

14.2.10. $(D)=\{(x,y)\in R^2 :\ (x^2+y^2)^2\le a^2(x^2-y^2),\ x\ge 0\}$

14.2.11. $(D)=\{(x,y)\in R^2 : x^2+y^2\le 2ax,\ \ y\ge x\}$

14.2.12. $(D)=\{(x,y)\in R^2 :\ a^2\le x^2+y^2\le 2ay\}$

14.2.13. $(D)=\{(x,y)\in R^2 :\ (x^2+y^2)^2\le ay(3x^2-y^2),\ x\ge 0.\ \ y\ge 0\}.$

14.2.14. $(D)=\{(x,y)\in R^2 :\ a^2\le x^2+y^2\le 2ax\}$

14.2.15. $(D)=\{(x,y)\in R^2 :\ x^2+y^2\le 2ax,\ \ x^2+y^2\le 2by$ doiralarning umumiy qismi $\}$.

14.2.16. $(D)=\{(x,y)\in R^2 : (x-a)^2+y^2\le 4a^2\}.$

14.2.17. $(D)=\{(x,y)\in R^2 : 0\le x\le a, 0\le y\le x\}.$

14.2.18. $(D)=\{(x,y)\in R^2 : -2\le x\le 0,\ \ x^2\le y\le 2-x\}$

14.2.19. $(D)=\{(x,y)\in R^2 : x\ge y\ge 0,\ \ x+y\le 2a\}.$

14.2.20. $(D)=\{(x,y)\in R^2 :\ 0\le y\le 1,\ y-2\le x\le -\sqrt{y}\}.$

14.2.21. $(D)=\{(x,y)\in R^2 : x^2+y^2\le 16\}$

14.2.22. $(D)=\{(x,y)\in R^2 : x^2+y^2\le 4x\}$

14.2.23. $(D)=\{(x,y)\in R^2 : 4\le x^2+y^2\le 9\}$

14.2.24. $(D)=\{(x,y)\in R^2 : 1\le x\le 2,\ \ 0\le y\le 2-x\}.$

14.2.25. $(D)=\left\{(x,y)\in R^2 : -2\le x\le 2,\ \ \dfrac{x^2}{2}\le y\le 2\right\}.$

14.2.26. $\iint\limits_{(D)} f(x,y)dx\,dy$, $(D)=\{(x,y)\in R^2: x^2+y^2\geq 4,\ x^2+y^2\leq 4y\}$

integralda Dekart koordinatalar sistemasidan qutb koordinatalar sistemasiga o'ting va integrallash chegaralarini ikki xil tartibda qo'ying

Yechilishi ([9], 2-t., 8-bo'lim, [30], 14.3-bo'lim). Qutb koordinatalar sistemasida $(r;\varphi)\in(D)$ nuqtalarning koordinatalariga qo'yilgan shartlarni topamiz: $x=r\cos\varphi, y=r\sin\varphi$ almashtirishni bajarib $r^2\geq 4$, $r^2\leq 4r\sin\varphi$, shartlarga ko'ra, $r\geq 0$ bo'lganligidan, $2\leq r\leq 4\sin\varphi$ munosabatga ega bo'lamiz. $x^2+y^2=2^2$, $x^2+(y-2)^2=2^2$ chiziqlar $(\sqrt{3}:1)$ va $(-\sqrt{3}:1)$ nuqtalarda kesishadi (14.42-chizma). 14.42-chizmadan ko'rinadiki, $\varphi=\varphi_c$ nur (D) soha bilan $\frac{\pi}{6}\leq\varphi_0\leq\frac{5\pi}{6}$ bo'lganda kesishadi. (D) sohaning $(\sqrt{3};1)(-\sqrt{3}:1)$ nuqtalaridan o'tgan $\varphi=\varphi_0$ nurning Ox o'qning musbat yo'nalishi bilan tashkil qilgan burchaklari, mos ravishda, $\varphi_0=\frac{\pi}{6}$ va $\varphi_0=\frac{5\pi}{6}$ bo'ladi. Har bir bunday nur (D) soha bilan $[2;4\sin\varphi_0]$ kesma bo'ylab kesishadi.

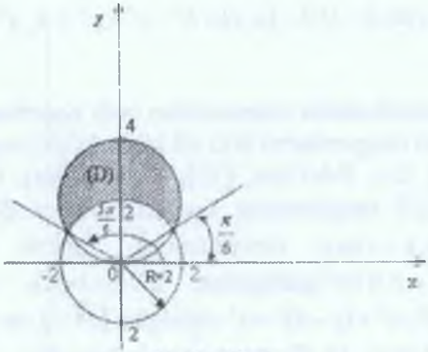
Shunday qilib, (D) sohaning qutb koordinatalar sistemasidagi ko'rinishi $(D)=\left\{(r;\varphi):\ \frac{\pi}{6}\leq\varphi\leq\frac{5\pi}{6},\ 2\leq r\leq 4\sin\varphi\right\}$ shaklda bo'ladi.

Demak, $\iint\limits_{(D)} f(x,y)dx\,dy=\int\limits_{\frac{\pi}{6}}^{\frac{5\pi}{6}} d\varphi \int\limits_{2}^{4\sin\varphi} f(r\cos\varphi, r\sin\varphi)r\,dr.$

Integralning chegarasini boshqacha tartibda joylashtirish uchun yana 14.42-chizmaga murojaat qilamiz. (D) sohaning nuqtalarigacha bo'lgan masofalarning eng kichigi 2 ga, eng kattasi esa, 4 ga teng. Demak, $2\leq r\leq 4$. Markazi $O(0;0)$ nuqtada radiusi $r=C$ ga teng bo'lgan aylana (D) soha bilan $(\alpha,\pi-\alpha)$ yoy bo'ylab kesishadi, bunda $\alpha=arc\sin\frac{r}{4}$.

Shunday qilib, $(D)=\left\{(r;\varphi): 2\leq r\leq 4,\ arc\sin\frac{r}{4}\leq\varphi\leq\pi-\sin\frac{r}{4}\right\}$.

Demak, $\iint\limits_{(D)} f(x,y)dx\,dy=\int\limits_{2}^{4} dr \int\limits_{\arcsin\frac{r}{4}}^{\pi-\arcsin\frac{r}{4}} f(r\cos\varphi, r\sin\varphi)r\,d\varphi.$



14.42-chizma.

14.3 – masala. Ikki karrali va takroriy integrallarni hisoblang.

14.3.1. $\int\limits_1^2 dx\int\limits_3^4 \frac{dy}{(x+y)^2}$.

14.3.2. $\int\limits_3^4 dx\int\limits_0^1 \frac{dy}{(x+y)^3}$.

14.3.3. $\int\limits_0^1 dx\int\limits_0^2 \frac{x^2dy}{1+y^2}$.

14.3.4. $\int\limits_0^2 dx\int\limits_1^2 \frac{x^2dy}{y^2+4}$.

14.3.5. $\int\limits_0^{\pi/2} dx\int\limits_0^{\pi/2}(x\sin x+y\cos y)dy$.

14.3.6. $\iint\limits_{(D)}\frac{x^2}{y^2}dxdy,\ (D)=\{(x,y): x=2,\ y=x,\ xy=1\}$.

14.3.7. $\int\limits_0^a dx\int\limits_{-2\sqrt{ax}}^{2\sqrt{ax}}(x^2+y^2)dy$.

14.3.8. $\int\limits_0^{2\pi} d\varphi\int\limits_{a\sin\varphi}^{a} rdr$.

14.3.9. $\int\limits_0^1 dx\int\limits_0^{\sqrt{1-x^2}}\sqrt{1-x^2-y^2}\,dy$.

14.3.10. $\iint\limits_{(D)}(x-y)dxdy,\ (D)=\{(x,y)\in R^2: y=0,\ y=1,\ x+y=2\}$.

14.3.11. $\iint\limits_{(D)} e^{\frac{x}{y}}dxdy,\ (D)=\{(x,y)\in R^2:\ x=y^2,\ x=0,\ y=1\}$

14.3.12. $\iint\limits_{(D)} \cos(x+y)dxdy, (D)=\{(x,y)\in R^2: x=0,\ y=\pi,\ \ y=x\}.$

14.3.13. $\iint\limits_{(D)} xy^2dxdy, (D)=\{(x,y)\in R^2: x^2+y^2\le a^2, x\ge 0\}$

14.3.14. $\iint\limits_{(D)} (x^3+y^3)dx\,dy,\ (D)=\{(x,y)\in R^2: x^2+y^2\le R^2,\ y\ge 0\}$

14.3.15. $\iint\limits_{(D)} (x+2y)dx\,dy,\ \ (D)=\{(x,y)\in R^2: y=x,\ y=2x,\ x=2,\ x=3\}$

14.3.16. $\iint\limits_{(D)} (x^2+y^2)dx\,dy,\ (D)=\{(x,y)\in R^2: y=x,\ y=x+a, y=a, y=3a\}$

14.3.17. $\iint\limits_{(D)} (x+y)dx\,dy,\ \ (D)=\{(x,y)\in R^2: x^2+y^2\le R^2,\ \ y\ge x\}.$

14.3.18. $\iint\limits_{(D)} e^{x-y}dx\,dy.\ \ (D)=\{(x,y)\in R^2: x=-1,\ x=1, y=x,\ \ y=2x\}$

14.3.19. $\iint\limits_{(D)} \sin\pi(x-y)dx\,dy,\ (D)$- uchlari $(-4;1),\left(-1;-\frac{1}{2}\right),\left(\frac{7}{2};\ \frac{17}{2}\right)$ nuqtalarda bo‘lgan uchburchak.

14.3.20. $\iint\limits_{(D)} xy\,dx\,dy,\ (D)=\{(x,y)\in R^2:\ x^2+y^2\le 25,\ 3x+y\ge 5\}$

14.3.21. $\int\limits_1^2 dx\int\limits_{3x}^{4x}\frac{dy}{(x+y)^2}.$

14.3.22. $\int\limits_3^4 dx\int\limits_0^{2x}\frac{dy}{(x+y)^3}.$

14.3.23. $\int\limits_0^1 dy\int\limits_0^{2y}\frac{x^2dx}{1+y^2}.$

14.3.24. $\int\limits_1^2 dx\int\limits_{x^2}^{2x}\frac{x^2dy}{y^2}.$

14.3.25. $\iint\limits_{(D)} (x+y)dx\,dy,\ \ (D)=\{(x,y)\in R^2: y=x^2,\ y^2=x\}$

14.3.26. $J=\int\limits_0^1 dx\int\limits_{2x}^1\sqrt[4]{1-y^2}\,dy.$

Yechilishi ([9], 2-t., 8-bo‘lim, [30], 14.2-bo‘lim). Ichki integral y ga nisbatan murakkab bo‘lgani uchun takroriy integrallarning chegarasi bo‘lgan

$$(D)=\{(x,y)\in R^2: 0\le x\le 1,\ 2x\le y\le 1\}$$

uchburchakni ushbu

$$(D)=\left\{(x,y)\in R^2: 0\le y\le 1,\ 0\le x\le \frac{y}{2}\right\}$$

ko'rinishda ifodalaymiz. Bundan

$$J=\int_0^1 dx\int_{2x}^1 \sqrt[4]{1-y^2}\,dy=\int_0^1 dy\int_0^{y/2} \sqrt[4]{1-y^2}\,dx=-\frac{1}{5}\left(1-y^2\right)^{5/4}\Big|_0^1=\frac{1}{5}.$$

Maple tizimidan foydalanib, misolning javobini tekshirish:

> int(int((1-y^2)^(1/4),x=0..y/2),y=0..1);

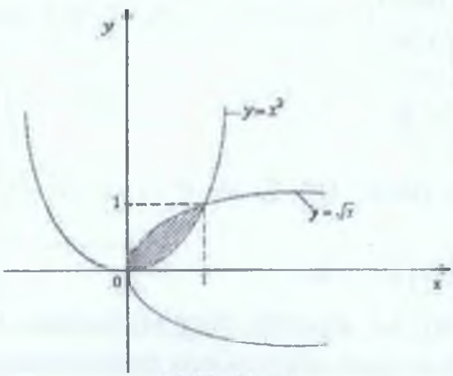
$$\frac{1}{5}$$

14.3.27. $\iint\limits_{(D)}\left(x^2+y\right)dxdy,\ (D)=\{(x,y): y=x^2, y^2=x\}$

Yechilishi. (D) soha $y=x^2,\ y^2=x$ parabolalarning kesishgan qismidan iborat (14.43- chizma). Ikki karrali integralni takroriy integralga keltirish formulasiga asosan,

$$\iint\limits_{(D)}\left(x^2+y\right)dxdy=\int_0^1 dx\int_{x^2}^{\sqrt{x}}\left(x^2+y\right)dy=\int_0^1\left(\frac{yx^2}{3}+\frac{y^2}{2}\right)\Bigg|_{x^2}^{\sqrt{x}}dx=$$

$$=\int_0^1\left(x^{5/2}-\frac{3}{2}x^4+\frac{x}{2}\right)dx=\frac{33}{140}.$$



14.43-chizma.

14.4-masala. Quyidagi uch karrali integrallarni hisoblang.

14.4.1. $\iiint\limits_{(V)} 2y^2 e^{xy}\,dxdydz,\ (V)=\{(x,y,z)\in R^3 : x=0,\ y=1,\ y=x,\ z=0,\ z=1\}$

14.4.2. $\iiint\limits_{(V)} x^2 z\sin(xyz)dxdydz,\ (V)=\{(x,y,z)\in R^3 : x=2, y=\pi,\ z=1, y=0, z=0\}$

14.4.3. $\iiint\limits_{(V)} 8y^2 ze^{2xyz}dxdydz,\ (V)=\{(x,y,z)\in R^3 : x=-1,\ y=2,\ z=1,\ x=0,\ y=0,\ z=0\}$

14.4.4. $\iiint\limits_{(V)} y^2 z\cos(xyz)dxdydz,\ (V)=\{(x,y,z)\in R^3 : x=1, y=\pi,\ z=2,\ x=0, y=0, z=0\}.$

14.4.5. $\iiint\limits_{(V)} xz\sin 0{,}25xyzdxdy,\ (V)=\{(x,y,z)\in R^3 : x=1, y=2\pi, z=4,\ x=0,\ y=0,\ z=0\}$

14.4.6. $\iiint\limits_{(V)} 2y^2 ze^{xyz}dxdydz,\ (V)=\{(x,y,z)\in R^3 : x=1,\ y=1,\ z=1,\ x=0, y=0,\ z=0\}$

14.4.7. $\iiint\limits_{(V)} x^2 zsh(xyz)dxdydz,\ (V)=\{(x,y,z): x=2,\ y=1,\ z=1,\ x=0, y=0,\ z=0\}.$

14.4.8. $\iiint\limits_{(V)} y^2 z\cos(xyz/3)dxdydz,\ (V)=\{(x,y,z): x=3,\ y=1,\ z=2\pi,\ x=0, y=0,\ z=0\}.$

14.4.9. $\iiint\limits_{(V)} 2x^2 z\,sh(xyz)dxdydz,\ (V)=\{(x,y,z): x=1,\ y=-1,\ z=1,\ x=0, y=0,\ z=0\}.$

14.4.10. $\iiint\limits_{(V)} 2x^2 z\,sh(2xyz)dxdydz,\ (V)=\{(x,y,z): x=2,\ y=1,\ z=1,\ x=0, y=0,\ z=0\}.$

14.4.11. $\iiint\limits_{(V)} x^2 z\sin(xyz/2)dxdydz,\ (V)=\{(x,y,z): x=1,\ y=4,\ z=\pi,\ x=0, y=0,\ z=0\}.$

14.4.12. $\iiint\limits_{(V)} y^2 z\,ch(xyz)dxdydz,\ (V)=\{(x,y,z): x=1,\ y=1,\ z=1,\ x=0, y=0,\ z=0\}.$

14.4.13. $\iiint\limits_{(V)} y^2 z\cos(xyz/9)dxdydz,\ (V)=\{(x,y,z): x=9,\ y=1,\ z=2\pi,\ x=0, y=0,\ z=0\}.$

14.4.14. $\iiint\limits_{(V)} y^2 z\,ch(xyz/2)dxdydz,\ (V)=\{(x,y,z): x=2,\ y=-1,\ z=2,\ x=0, y=0,\ z=0\}.$

14.4.15. $\iiint\limits_{(V)} 2y^2 z\,ch(2xyz)dxdydz,\ (V)=\{(x,y,z): x=1/2,\ y=2,\ z=-1,\ x=0, y=0,\ z=0\}.$

14.4.16. $\iiint\limits_{(V)} 8y^2 z e^{-xyz}dxdydz,\ (V)=\{(x,y,z): x=2,\ y=-1,\ z=2,\ x=0, y=0,\ z=0\}.$

14.4.17. $\iiint\limits_{(V)} y^2\,ch(2xy)dxdydz,\ (V)=\{(x,y,z): x=0,\ y=-2, y=4x, z=0,\ z=2\}.$

14.4.18. $\iiint\limits_{(V)} x^2\,sh(3xy)dxdydz,\ (V)=\{(x,y,z): x=1,\ y=2x, y=0, z=0,\ z=36\}.$

14.4.19. $\iiint\limits_{(V)} y^2 ch\left(\frac{\pi}{4}xy\right)dxdydz,\ (V)=\{(x,y,z): x=0,\ y=-1, y=x/2, z=0,\ z=-\pi^2\}.$

14.4.20. $\iiint\limits_{(V)} y^2 e^{-xy}dxdydz,\ (V)=\{(x,y,z): x=0,\ y=-2, y=-4x, z=0,\ z=1\}.$

14.4.21. $\iiint\limits_{(V)} 3x^2e^{xy}\,dxdydz,\ (V)=\{(x,y,z)\in R^3: x=0,\ y=2,\ y=x,\ z=0,\ z=2\}.$

14.4.22. $\iiint\limits_{(V)} x^2 z\cos(xyz)dxdydz,\ (V)=\{(x,y,z)\in R^3: x=0, x=1, y=\pi,\ z=1, y=0, z=0\}$

14.4.23. $\iiint\limits_{(V)} 4x^2ze^{2xyz}dxdydz,\ (V)=\{(x,y,z)\in R^3: x=0,\ y=0,\ z=0,\ x=1, y=2,\ z=1\}$

14.4.24. $\iiint\limits_{(V)} x^2 z\cos(xyz)dxdydz,\ (V)=\left\{(x,y,z)\in R^3: x=1, y=\frac{\pi}{2},\ z=4,\ x=0, y=0, z=0\ \right\}.$

14.4.25. $\iiint\limits_{(V)} xy\sin(xyz)dxdy,\ (V)=\{(x,y,z)\in R^3: x=2, y=2, z=4\pi,\ x=0,\ y=0,\ z=0\}$

14.4.26. $\iiint\limits_{(V)} (3x+2y)dxdydz,\ (V)=\{(x,y,z): y=0,\ y=x, x=1,\ z=1, z=1+x^2+y^2\}$

integralni hisoblang.

Yechilishi ([9], 2-t., 8-bo‘lim, [30], 14.4-bo‘lim). (V)sohada $0\le x\le 1,\ 0\le y\le x,\ 1\le z\le 1+x^2+y^2$ tengsizliklar o‘rinli. U holda uch karrali integralni takroriy integrallarga keltirish formulasiga ko‘ra,

$$\iiint\limits_{(V)}(3x+2y)dxdydz=\int_0^1 dx\int_0^x dy\int_1^{1+x^2+y^2}(3x+2y)dz=\int_0^1 dx\int_0^x (3x+2y)z\Big|_1^{1+x^2+y^2}dy=$$

$$=\int_0^1 dx\int_0^x (3x+2y)(x^2+y^2)dy=\int_0^1 dx\int_0^x\left(3x^3+3xy^2+2yx^2+2y^3\right)dy=$$

$$=\int_0^1\left(3x^3y+xy^3+y^2x^2+\frac{1}{2}y^4\right)\Big|_0^x dx=\int_0^1(3x^4+x^4+x^4+x^4+\frac{1}{2}x^4)dx=\frac{11}{10}.$$

Misolni Maple tizimidan foydalanib, yechish:

```
> int(int(int(3*x+2*y,z=1..1+x^2+y^2),y=0..x),x=0..1);
```

$$\frac{11}{10}.$$

15-BOB.
EGRI CHIZIQLI VA SIRT INTYEGRALLARI

15.1-§. Birinchi tur egri chiziqli integralning ta'rifi

15.1. Birinchi tur egri chiziqli integral va unga olib keladigan mexanik masala. Egri chiziqli integral tushunchasiga olib keladigan ushbu mexanik masalani qaraymiz.

Tekislikda sodda uzluksiz to'g'rilanuvchi (hozircha ochiq) (AB) egri chiziq berilgan bo'lsin (15.1-chizma). Bu egri chiziq bo'ylab biror massa tarqalgan bo'lib, uning $\forall\ M$ nuqtasidagi chiziqli zichligi $\rho(M)$ bo'lsin. (AB) egri chiziq bo'ylab tarqalgan massa m ni topish talab qilingan bo'lsin. Massani topish uchun (AB) egri chiziqni $A_0, A_1, ..., A_{n-1}(A_0 = A, A_{n-1} = B)$ nuqtalar yordamida bo'laklarga bo'lamiz (bo'lishni B dan A ga qarab ham olib borish mumkin edi, bu umumiy mulohazaga halaqit bermaydi). (AB) egri chiziqning $A_i\overline{A}_{i+1}$ yoyidan ixtiyoriy M_i nuqtani olamiz.

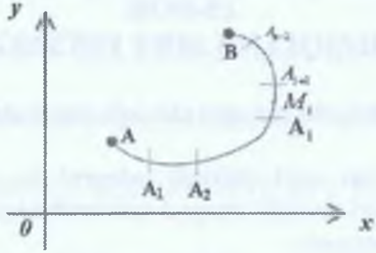
Bu nuqtaning chiziqli zichligi $\rho(M_i)$. Faraz qilaylik, $A_i\overline{A}_{i+1}$ yoyining hamma nuqtasidagi chiziqli zichlik $\rho(M_i)$ bo'lsin. $A_i\overline{A}_{i+1}$ yoyining uzunligini σ_i, $A_i\overline{A}_{i-1}$ yoy bo'ylab tarqalgan massani esa, m_i deb belgilaymiz. U holda $A_i\overline{A}_{i+1}$ dagi massa $m_i \approx \rho(M_i)\sigma_i$ bo'ladi. (AB) egri chiziq bo'ylab tarqalgan butun massa

$$m \approx \sum_{i=o}^{n-1} \rho(M_i)\sigma_i .$$

Agar $\max\limits_{1\le i\le n} \sigma_i = \lambda$ deb belgilab yuqoridagi yig'indida $\lambda \to 0$ da limitga o'tsak, talab qilingan massani topish uchun aniq formulaga ega bo'lamiz, ya'ni

$$m = \lim_{\lambda\to 0} \sum_{i=0}^{n-1} \rho(M_i)\sigma_i . \qquad (15.1.1)$$

15.2. Birinchi tur egri chiziqli integralning ta'rifi. Tekislikda xOy koordinatlar sistemasi tanlagan bo'lsin. Tekislikda biror silliq yoki bo'lakli silliq[1] (AB) chiziq berilgan. Bu egri chiziqda ikki yo'nalishdan birini musbat yo'nalish, ikkinchisini manfiy yo'nalish deb qabul qilaylik.



15.1-chizma.

$f(x,y)$ funksiya (AB) egri chiziqda aniqlangan va chegaralangan bo'lsin. (AB)egri chiziqni yuqoridagi singari $A = A_0, A_1, A_2, .., A_{n-1}, A_n = B$ nuqtalar bilan n ta bo'lakka bo'lamiz. Bu $A_0, A_1, ..., A_n$ nuqtalar sistemasi $\breve{AB}$ yoyning bo'linishi deb ataladi va u

$$P = \{A_0, A_1, A_2, ..., A_n\}$$

kabi belgilanadi.

$\breve{A_i A_{i+1}}$ yoy uzunligini Δs_i $(i = 0,1,2,...,n-1)$ bilan, $\max\{\Delta s_i\} = \lambda_P$ deb belgilaymiz.

(AB) egri chiziqning ixtiyoriy $P = \{A_0, A_1, A_2, ..., A_n\}$ bo'lishini va uning har bir $A_i\breve{A}_{i+1}$ yoyidan ixtiyoriy $M_i(\xi_i, \eta_i)$ $(i = 0,1,2,..., n-1)$ nuqtani olib, berilgan funksiyaning bu nuqtadagi $f(\xi_i, \eta_i)$ qiymatini $A_i\breve{A}_{i+1}$ yoyining uzunligi Δs_i ga ko'paytirib, quyidagi

$$\sigma_P = \sum_{i=0}^{n-1} f(\xi_i, \eta_i)\Delta s_i$$

yig'indini tuzamiz.

$(A\breve{B})$ egri chiziqning shunday

$$P_1, P_2, ..., P_m, ... \qquad (15.2.1)$$

[1] Agar egri chiziq tenglamasi $x = \varphi(t)$, $y = \psi(t), \alpha \le t \le \beta$ ko'rinishda berilgan bo'lib, $\varphi(t)$, $\psi(t)$ funksiyalar uzluksiz va uzluksiz bir vaqtda nolga aylanmaydigan $\varphi'(t)$ ва $\psi'(t)$ hosilalar ega bo'lsa, u holda bu egri chiziq silliq chiziq deyiladi.

Chekli sondagi silliq bo'laklardan tuzilgan uzluksiz egri chiziqqa bo'lakli-silliq deyiladi.

bo'linishlar ketma-ketligini qaraylikki, ularning mos diametrlaridan tuzilgan

$$\lambda_{P_1}, \lambda_{P_2}, \ldots, \lambda_{P_m}, \ldots$$

ketma-ketlik nolga intilsin: $\lambda_{P_m} \to 0$. (15.2.1) bo'linishlarga mos kelgan ushbu

$$\sigma_{P_1}, \sigma_{P_2}, \ldots, \sigma_{P_m}, \ldots \tag{15.2.2}$$

yig'indilar ketma-ketligini tuzamiz. Bu ketma-ketlikning har bir hadi $M_i(\xi_i, \eta_i)$ nuqtalarga bog'liq.

15.2.3-ta'rif. Agar (AB) egri chiziqning (15.2.1) ko'rinishidagi har qanday bo'linishlar ketma-ketligi $\{P_m\}$ olinganda ham, unga mos kelgan (15.2.2) ko'rinishdagi yig'indilar $\{\sigma_m\}$ ketma-ketligi $M_i(\xi_i, \eta_i)$ nuqtalarning tanlab olinishiga bog'liq bo'lmagan holda hamma vaqt bitta chekli J songa intilsa, bu son σ_p *yig'indining limiti* deb ataladi va u

$$\lim_{\lambda_P \to 0} \sigma_P = \lim_{\lambda_P \to 0} \sum_{i=0}^{n-1} f(\xi_i, \eta_i) \Delta s_i = J$$

kabi belgilanadi. Bu holda $f(x, y)$ funksiya (AB) egri chiziq bo'yicha *integrallanuvchi* deyiladi. Bu J limitga $f(x, y)$ funksiyaning (AB) egri chiziq bo'yicha *birinchi tur egri chiziqli integrali* deb ataladi va u

$$J = \int_{(AB)} f(x, y) ds \tag{15.2.4}$$

kabi belgilanadi.

Shunday qilib, kiritilgan egri chiziqli integral tushunchasining o'ziga xosligi qaralayotgan ikki argumentli $f(x, y)$-funksiyaning berilish sohasi tekislikdagi (AB) egri chiziq ekanligidadir. Qolgan boshqa hamma mulohazalar (bo'linishlarning olinishi, bo'linish oraliqlaridan ixtiyoriy nuqtani tanlab olib integral yig'indi tuzish, limitga o'tish) 8-bob, 8.13-bandda kiritilgan aniq integral tushunchasi kabidir.

Demak, yuqorida keltirilgan (15.1.1) ifodani, ya'ni egri chiziq bo'ylab tarqalgan m massani ham quyidagicha

$$m = \int_{(AB)} \rho(M)ds \qquad (15.2.5)$$

yozish mumkin bo‘lar ekan. Ta’kidlaymizki, yuqorida keltirilgan ta'rifda yo‘nalish hech qanday rol o‘ynamaydi. Masalan, egri chiziq yopiq bo‘lmasdan egri chiziq yo‘nalishini A dan B ga qarab olingandagi egri chiziqli integral bilan, yo‘nalish B dan A ga qarab olingandagi egri chiziqli integralning qiymati o‘zaro teng bo‘ladi, ya’ni

$$\int_{(AB)} f(x,y)ds = \int_{(BA)} f(x,y)ds$$

Shunday qilib birinchi tur egri chiziqli integral egri chiziqda olingan yo‘nalishga bog‘liq bo‘lmaydi.

Xuddi shunday, egri chiziq fazoda berilganda, egri chiziqli integral tushunchasi kiritiladi:

$$\int_{(AB)} f(M)ds = \int_{(AB)} f(x,y,z)ds.$$

15.2-§. Birinchi tur egri chiziqli integralning mavjudlik sharti va uni aniq integralga keltirish

15.3. Birinchi tur egri chiziqli integralning mavjudlik sharti. Faraz qilaylik (AB) sodda to’g’rilanuvchi egri chiziqda ixtiyoriy ravishda yo‘nalish (ikki yo‘nalishdan biri) aniqlangan bo‘lsin. Egri chiziqdagi M nuqtaning holati A boshlang‘ich nuqtadan hisoblanuvchi $s = \breve{AB}$ uzunligi bilan aniqlansin. U holda (AB) egri chiziq parametrik tenglamasi ushbu

$$x = x(s), \ \ y = y(s) \ \ (0 \le s \le S),$$

ko‘rinishda bo‘lsin, bunda S, $\breve{AB}$ ning uzunligi ko‘rinishda ifodalanadi, egri chiziqda berilgan $f(x,y)$ funksiya esa, s ning murakkab $f(x(s), y(s))$ funksiyasiga keltiriladi. (AB) egri chiziqning ixtiyoriy $P = \{A_0, A_1, ..., A_n\}$ bo‘linishni va har bir $A_i A_{i+1}$ dan $\forall M_i(\xi_i, \eta_i)$ nuqtani olaylik. Har bir A_i nuqtaga mos kelgan $\breve{AA_i}$ yoyining uzunligi s_i, har bir M_i nuqtaga mos keladigan $\breve{AA_i}$

yoyning uzunligini s'_i deb belgilaylik. U holda, A_iA_{i+1} yoyining uzunligi $s_{i+1} - s_i = \Delta s_i$ bo'ladi. Natijada P bo'linishga nisbatan tuzilgan integral yig'indi

$$\sigma_P = \sum_{i=0}^{n-1} f(\xi_i, \eta_i) \cdot \Delta s_i ,$$

ushbu

$\sigma_P(f) = \sum_{i=0}^{n-1} f(x(s'_i), y(s'_i))\Delta s_i$ ko'rinishga keladi, bunda $\xi_i = x(s'_i)$, $\eta_i = y(s'_i)$.

Bu yig'indining $[0, S]$ oraliqda $f(x(s), y(s))$ funksiyaning Riman integral yig'indisi ekanligiga ishonch hosil qilish qiyin emas.

Agar $f(x,y)$ funksiya (AB) egri chiziqda uzluksiz bo'lsa, $f(x(s), y(s))$ funksiya ham $[0, S]$ da uzluksiz va integrallanuvchi bo'ladi, ya'ni

$$\lim_{\lambda_P \to 0} \sigma_P = \lim_{\lambda_P \to 0} \sum_{i=0}^{n-1} f(x(s_i'), y(s'_i)) \cdot \Delta s_i = \int_0^S f(x(s), y(s))ds$$

mavjud bo'ladi.

Shunday qilib quyidagi teoremaga kelamiz.

15.3.1-teorema. Agar $f(x,y)$ funksiya (AB) egri chiziqda uzluksiz bo'lsa, u holda bu funksiyaning (AB) egri chiziq bo'yicha birinchi tur egri chiziqli integrali mavjud va ushbu

$$\int_{(AB)} f(x,y)ds = \int_0^S f(x(s), y(s))ds) \qquad (15.3.2)$$

tenglik o'rinli bo'ladi.

15.4. Birinchi tur egri chiziqli integralning aniq integralga keltirish.

1. (AB) egri chiziq o'zining parametrik $x = \varphi(t), y = \psi(t)$ $(t_0 \le t \le T)$ tenglamasi bilan berilgan bo'lsin, bunda $\varphi(t)$ va $\psi(t)$ funksiyalar uzluksiz va uzluksiz $\varphi'(t)$, $\psi'(t)$ hosilalarga ega. Bundan tashqari egri chiziqni karrali nuqtalariga ega emas deb faraz qilamiz. Bu holda egri chiziq to'g'rilanuvchi bo'ladi. $S = \overset{\smile}{AM} = s(t)$ yoyning o'sishi parametr t ning o'sishiga mos kelsa, u holda, ma'lumki, o'zgaruvchan yoy differensiali $ds = \sqrt{[\varphi'(t)]^2 + [\psi'(t)]^2}dt$

bo'ladi. (15.3.2) da s o'zgaruvchini t o'zgaruvchiga almashtirish bilan ushbu egri chiziqli integralni

$$\int_{(AB)} f(x,y)ds=\int_{t_0}^{T} f(\varphi(t),\psi(t))\sqrt{[\varphi'(t)]^2+[\psi'(t)]^2}\,dt \qquad (15.4.1)$$

hisoblash formulasiga keltiramiz.

2. Agar xususiy holda (AB) egri chiziq tenglamasi $y=y(x)$ $(a\le x\le b)$ ko'rinishda bo'lib, $y(x)$ – uzluksiz va uzluksiz $y'(x)$ hosilaga ega bo'lsa, u holda, o'zgaruvchan yoy differensiali $ds=\sqrt{1+(y')^2}\,dx$ bo'ladi va birinchi tur egri chiziq integral mavjud va u ushbu

$$\int_{(AB)} f(x,y)ds=\int_{a}^{b} f(x,y(x))\sqrt{1+y_x'^2}\,dx \qquad (15.4.2)$$

formula orqali hisoblanadi.

Xuddi shunday, (AB) egri chiziq tenglamasi $x=x(y)$ $(c\le y\le d)$ ko'rinishda bo'lib, $x(y)$ – uzluksiz va uzluksiz $x'(y)$ hosilaga ega bo'lsa, u holda, o'zgaruvchan yoy differensiali $ds=\sqrt{1+(x_y')^2}\,dy$ bo'ladi va birinchi tur egri chiziq integral mavjud bo'ladi va u ushbu

$$\int_{(AB)} f(x,y)ds=\int_{c}^{d} f(x(y),y)\sqrt{1+x_y'^2}\,dy$$

formula orqali hisoblanadi.

3. (AB) egri chiziq qutb koordinatalar sistemasida ushbu $r=r(\theta)$ $(\theta_1\le\theta\le\theta_2)$ tenglama berilgan bo'lib, $r(\theta)$ funksiya $[\theta_1,\theta_2]$ da uzluksiz va uzluksiz hosilaga ega bo'lsin. Agar $f(x,y)$ funksiya shu (AB) egri chiziqda berilgan va uzluksiz bo'lsa, u holda egri chiziqli integral mavjud bo'ladi va u ushbu

$$\int_{(AB)} f(x,y)ds=\int_{\theta_1}^{\theta_2} f(r(\theta)\cos\theta,r(\theta)\sin\theta)\sqrt{r^2+r'^2}\,d\theta \qquad (15.4.3)$$

formula bilan hisoblanadi.

4. Agar $(K)=(AB)$ egrichiziq $\varphi=\varphi(\rho)(\rho_1\le\rho\le\rho_2)$ tenglama bilan berilgan bo'lsa, u holda egrichiziqli integral mavjud bo'ladi va u ushbu

$$\int_{(K)} f(x,y)ds = \int_{\rho_1}^{\rho_2} f(\rho\cos\varphi, \rho\sin\varphi)\sqrt{1+\rho^2(\varphi'_\rho)^2}\,d\rho$$

formula bilan hisoblanadi.

15.5. Birinchi tur egri chiziqli integralning xossalari. Ushbu $x=\varphi(t), y=\psi(t)$ $(t_0 \le t \le T)$ sistema bilan aniqlangan (AB) egri chiziqda $f(x,y)$ uzluksiz funksiya berilgan bo'lsin.

Biz yuqorida ko'rdikki birinchi tur egrichiziqli integrallari Riman integraliga keltirilar ekan, shu tufayli egrichiziqli integrallar ham Riman integrali kabi hossalariga ega bo'ladi. Shuning uchun egrichiziqli integrallarning asosiy xossalarini isbotsiz keltiramiz.

1^0. Agar $AB = AC + CB$ bo'lsa u holda

$$\int_{(AB)} f(x,y)ds = \int_{(AC)} (x,y)ds + \int_{(CB)} f(x,y)ds$$

bo'ladi.

2^0.Ushbu

$$\int_{(AB)} Cf(x,y)ds = C\int_{(AB)} f(x,y)ds (C = const)$$

tenglik o'rinli.

3^0. (AB) egri chiziqda uzluksiz $f(x,y)$ bilan birgalikda uzluksiz $g(x,y)$ ham berilgan bo'lsin, u holda

$$\int_{(AB)} [f(x,y) \pm g(x,y)]ds = \int_{(AB)} f(x,y)ds \pm \int_{(AB)} g(x,y)ds$$

formula o'rinli bo'ladi.

4^0. Agar $\forall (x,y) \in (AB)$ da $f(x,y) \ge 0$ bo'lsa, u holda

$$\int_{(AB)} f(x,y)dx \ge 0$$

bo'ladi.

5^0. Agar $f(x,y)$ funksiya (AB) egrichiziqda integrallanuvchi bo'lsa, $|f(x,y)|$ funksiya ham (AB)da integrallanuvchi va

$$\left|\int\limits_{(AB)} f(x,y)ds\right| \le \int\limits_{(AB)} |f(x,y)|ds$$

tengsizlik o'rinli.

6⁰ . Agar $f(x,y)$ funksiya (AB) egrichiziqda uzluksiz bo'lsa, u holda shunday $(\bar{x},\bar{y}) \in (AB)$ nuqta topiladiki,

$$\int\limits_{(AB)} f(x,y)ds = f(\bar{x},\bar{y})S$$

bo'ladi, bunda $S - (\breve{AB})$ yoy ning uzundigi.

15.3-§. Birinchi tur egri chiziqli integrallarning ba'zi tatbiqlari

Birinchi tur egri chiziqli integral yordamida yoy uzunligini, jismning massasini, og'irlik markazi koordinatalarini, statik momentlarini topish mumkin:

15.6. Yoy uzunligini va jismning massasini topish. Tekislikda sodda to'g'irlanuvchi (AB) egri chiziq berilgan bo'lib, unda $f(x,y)$ uzluksiz funksiya aniqlangan bo'lsin. Agar xususiy holda $f(x,y)=1$ deb faraz qilsak, u holda 15.6.3- ta'rifga asosan,

$$\int\limits_{(AB)} f(x,y)ds = \lim_{\lambda_P \to 0} \sum_{i=0}^{n-1} f(\xi_i,\eta_i)\Delta s_i = \lim_{\lambda_P \to 0} \sum_{i=0}^{n-1} \Delta s_i = S.$$

Demak,

$$S = \int\limits_{(AB)} ds. \qquad (15.6.1)$$

15.6.2-misol. Ushbu

$$x = a(\cos t + t\sin t), y = a(\sin t - t\cos t) \ (0 \le t \le 2\pi)$$

sistema bilan berilgan, (K) chiziqning yoy uzunligi topilsin.

Yechilishi. Berilgan egri chiziq yoy uzunligini (15.6.1) formula yordamida topamiz:

$$S=\int_{(K)} ds=\int_0^{2\pi}\sqrt{x_t'^2+y_t'^2}\,dt;\ x_t'=a(-\sin t+\sin t+t\cos t)=at\cos t,$$

$$y_t'=a(\cos t-\cos t+t\sin t)=a\ t\cdot\sin t,$$

$$S=\int_0^{2\pi}a\sqrt{t^2\cos^2 t+t^2\sin^2 t}\,dt=a\int_0^{2\pi}t\,dt=a\frac{t^2}{2}\Bigg|_0^{2\pi}=2a\pi^2.\blacksquare$$

Tekislikda uzluksiz sodda to‘g‘rilanuvchi egri chiziq berilgan bo‘lib, bu chiziq bo‘ylab biror massa tarqalgan bo‘lsin. Uning chiziqli zichligi $\rho(M)$ $(\forall M\in(AB))$. Bu m massa

$$m=\int_{(AB)}\rho(M)ds=\int_{(AB)}\rho(x,y)ds \tag{15.6.3}$$

formula orqali topiladi.

15.6.5-misol. Ushbu $x=a\cos^3 t,\ y=a\sin^3 t$ parametrik tenglamasi bilan berilgan astroida yoyi bo‘ylab biror massa tarqalgan. Astroida yoyidagi ixtiyoriy $M(x,y)$ nuqtadagi massaning zichligi $\rho(M)=|xy|$ formula bilan aniqlanadi. Astroida yoyi bo‘ylab tarqalgan massa topilsin.

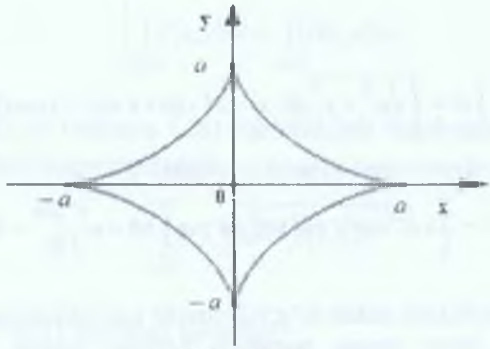
Yechilishi. Ma’lumki, massa ushbu (15.6.3) formula bilan hisoblanadi. Astroidaning parametrik tenglamasi (15.2-chizma): $x=a\cos^3 t,\ y=a\sin^3 t\ (0\le t\le 2\pi)$. Bundan,

$$x'_t=-3a\sin t\cos^2 t,\ y_t'=3a\cos t\cdot\sin^2 t.$$

$$ds=3a\sqrt{\cos^2 t\cdot\sin^2 t}\,dt=3a\cdot|\cos t\cdot\sin t|dt.\qquad \rho(M)=|xy|=a^2|\cos^3 t\cdot\sin^3 t|$$

$$m=\int_{(k)}|xy|ds=3a^3\int_0^{2\pi}\cos^4 t\sin^4 t\,dt=\frac{3a^3}{16}\int_0^{2\pi}\sin^4 2t\,dt=\frac{3a^3}{64}\int_0^{2\pi}(1-\cos 4t)^2dt=$$

$$=\frac{3a^3}{64}\left(\frac{3}{2}t-\frac{1}{2}\sin 4t+\frac{1}{16}\sin 8t\right)\Bigg|_0^{2\pi}=\frac{9\pi a^3}{64}.\blacksquare$$



15.2-chizma.

15.7. Egri chiziqning statik momenti va og'irlik markazini topish. Tekislikda sodda to'g'rilanuvchi uzluksiz (AB) egri chiziqning koordinatalar o'qlariga nisbatan statik va inersiya momentlari hamda og'irlik markazining koordinatalari, mos ravishda, ushbu

$$K_y = \int\limits_{(K)} \rho\, x ds,\ K_x = \int\limits_{(K)} \rho\, y ds,$$

$$x_c = \frac{K_y}{m} = \frac{\int\limits_{(K)} \rho\, x ds}{\int\limits_{(K)} \rho\, ds},\quad y_c = \frac{K_x}{m} = \frac{\int\limits_{(K)} \rho\, y ds}{\int\limits_{(K)} \rho\, ds}, \tag{15.7.1}$$

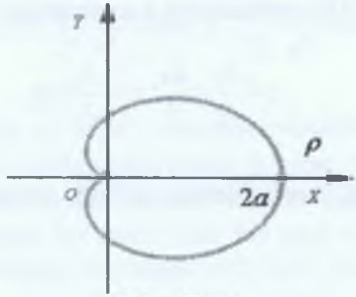
$$J_x = \int\limits_{(AB)} y^2 ds,\ J_y = \int\limits_{(AB)} x^2 ds.$$

formulalar orqali topiladi.

15.7.2-misol. $r = a(1+\cos\varphi)$ kardioida yoyi bo'ylab bir jinsli massa joylashgan. Bu kardioida yoyining $(0 \le \varphi \le 2\pi)$ og'irlik markazi topilsin.

Yechilishi. Bir jinsli massa zichligi $\rho = const$ (soddalik uchun $\rho = 1$ deb olamiz). Kardioida yoyining og'irlik markazini $M(\xi, \eta)$ deb belgilaylik. U holda, og'irlik markazining koordinatalari (15.7.1) formulalar orqali hisoblanadi.

Ma'lumki, kardioidaning parametrik tenglamasi $x = a(1+\cos\varphi)\cdot\cos\varphi$, $y = a(1+\cos\varphi)\sin\varphi$, kardioida uchun yoy differensiali $ds = 2a\left|\cos\frac{\varphi}{2}\right| d\varphi.$



15.3 - chizma.

Endi ushbu $J_1 = \int\limits_{(k)} xds, \quad J_2 = \int\limits_{(k)} yds$ egri chiziqli integrallarni hisoblaymiz (15.3 - chizma) :

$$J_1 = \int\limits_{(K)} x \cdot ds = \int\limits_0^{2\pi} a(1+\cos\varphi)\cos\varphi \cdot 2a\left|\cos\frac{\varphi}{2}\right| d\varphi =$$

$$= 2a^2\left[\int\limits_0^{\pi}(1+\cos\varphi)\cos\varphi\cdot\cos\frac{\varphi}{2}d\varphi + \int\limits_{\pi}^{2\pi}(1+\cos\varphi)\left(-\cos\frac{\varphi}{2}\right)d\varphi\right] =$$

($\varphi = 2\pi - t$ almashtirish olinadi) $= 4a^2\int\limits_0^{\pi}(1+\cos\varphi)\cos\varphi\cdot\cos\frac{\varphi}{2}d\varphi.$

Bunda, $u = \sin\frac{\varphi}{2}$ almashtirishni bajaramiz:

$$2du = \cos\frac{\varphi}{2}du, \ \cos^2\frac{\varphi}{2} = 1-u^2.$$

$$J_1 = 8a^2\int\limits_0^{\pi}\cos^2\frac{\varphi}{2}\cdot\left(\cos^2\frac{\varphi}{2}-\sin^2\frac{\varphi}{2}\right)\cdot\cos\frac{\varphi}{2}d\varphi = 16a^2\int\limits_0^1\left(1-3u^2+2u^4\right)du = \frac{3^2}{5}a^2;$$

$$J_2 = \int\limits_{(k)} yds = 2a^2\int\limits_0^{2\pi}(1+\cos\varphi)\sin\varphi\left|\cos\frac{\varphi}{2}\right|d\varphi =$$

$$2a^2\left[\int\limits_0^{\pi}4\cos^3\frac{\varphi}{2}\cdot\sin\frac{\varphi}{2}\cdot\cos\frac{\varphi}{2}d\varphi + \int\limits_{\pi}^{2\pi}4\cos^3\frac{\varphi}{2}\sin\frac{\varphi}{2}\left(-\cos\frac{\varphi}{2}\right)d\varphi = \right.$$

$$= \frac{16a^2}{5}\left[-\cos^5\frac{\varphi}{2}\Big|_0^{\pi} + \cos^5 - \cos^5\frac{\varphi}{2}\Big|_{\pi}^{2\pi} = -\frac{16a^2}{5}(1-1) = 0;\right.$$

$$J_3 = \int\limits_{(k)} ds = 2a\int\limits_0^{2\pi}\left|\cos\frac{\varphi}{2}\right|d\varphi = 2a\cdot\left[\int\limits_0^{\pi}\cos\frac{\varphi}{2}d\varphi - \int\limits_{\pi}^{2\pi}\cos\frac{\varphi}{2}d\varphi\right] = 8a.$$

Shunday qilib, og'irlik markazining koordinatalari:

$$\xi = \frac{J_1}{J_3} = \frac{4a}{5}; \ \eta = \frac{J_2}{J_3} = 0.$$

Demak, tekislikdagi kardioidaning og'irlik markazi $M\left(\frac{4a}{5};0\right)$ nuqtada bo'lar ekan. ■

15.8. Material nuqtalarning tortilish kuchini topish

m_0 massaga ega bo'lgan M_0 moddiy nuqta massasi m ga teng bo'lgan M moddiy nuqta bilan M_0 nuqtadan M moddiy nuqtaga qarab yo'nalgan biror kuch bilan tortilayotgan bo'lsin (15.4 - chizma).

15.4 - chizma.

Bu kuchning son qiymatini Nyutonning butun olam tortishish qonuni

$$F = G\frac{m_0 m}{r^2}$$

dan foydalanib hisoblash mumkin. Bu yerda r- M_0 va M nuqtalar orasidagi masofa, G– gravitatsiya, ya'ni tortishish doimiysi bo'lib, uning son qiymati quyidagiga teng: $G = 6{,}67 \cdot 10^{-11} \frac{н \cdot м^2}{кг^2}$.

M_0 moddiy nuqta - massalari, mos ravishda, $m_1\ m_2,....,m_n$ bo'lgan moddiy nuqtalar bilan tortishayotgan bo'lsin. U holda teng ta'sir qiluvchi kuch har bir nuqta bilan tortilish kuchlarining geometrik yig'indisiga teng bo'ladi. O'z vaqtida tortilish kuchining koordinatalar o'qlaridagi proyeksiyasi har bir nuqtadagi tortilish kuchlari proyeksiyalarining algebraik yig'indisiga teng bo'ladi. Teng ta'sir qiluvchi kuchning koordinatalar o'qlardagi proyeksiyalarini X va Y orqali, $\vec{r_i} = \overrightarrow{M_0 M_i}$ vektorning Ox o'q bilan tashkil qilgan burchagini φ_i deb belgilasak, u holda

$$X = \sum_{i=1}^{n} \frac{m_0 m_i}{r_i^2} \cos\varphi, \quad Y = \sum_{i=1}^{n} \frac{m_0 m_i}{r_i^2} \sin\varphi_i$$

bo'ladi, bunda r_i miqdor- $\vec{r_i} = \overrightarrow{M_0 M_i}$ vektorning uzunligi.

Endi, faraz qilaylik, tortilayotgan massa (K) egri chiziq bo'ylab uzluksiz tarqalgan bo'lsin. Bu holda tortilish kuchini topish uchun, (K) egri chiziqni n ta bo'laklarga bo'lamiz. Har bir egri chiziq bo'lakchasidagi massani ixtiyoriy M_i nuqtaga to'plangan deb faraz qilib, teng ta'sir qiluvchi kuchning o'qlardagi proyeksiyalarni taqribiy ravishda topamiz:

$$X \approx \sum_{i=1}^{n} \frac{m_0 \rho(M_i)\delta_i}{r_i^2} \cos\varphi_i, \quad Y \approx \sum_{i=1}^{n} \frac{m_0 \rho(M_i)\delta_i}{r_i^2} \sin\varphi_i,$$

bunda har bir uchastkadagi m_i massa $\rho(M_i)\delta_i$ ga teng, $\delta_i = M_i M_{i+1}$. Agar $\max\delta_i \to 0$ deb olsak, u holda ta'sir qiluvchi kuchning o'qlardagi proyeksiyalarining aniq qiymatlarini hosil qilamiz:

$$X = m_0 \int_{(K)} \frac{\rho(m)\cos\varphi}{r^2} ds, \quad Y = m_0 \int_{(K)} \frac{\rho(M)\sin\varphi}{r^2} ds,$$

bu yerda, r – $\vec{r} = \overrightarrow{M_0 M}$ vektorning uzunligi, φ esa, vektorning Ox o'q bilan tashkil etgan burchagi.

15.4-§. Ikkinchi tur egri chiziqli integrallar

15.9. Ikkinchi tur egri chiziqli integralning ta'rifi. xOy koordinatalar sistemasida to'g'rilanuvchi sodda (AB) egri chiziq (hozircha ochiq) berilgan bo'lib, unda biror $f(x,y)$ funksiya aniqlangan bo'lsin. Egri chiziqni $A_i(i = 0,1,2,...,n)$ nuqtalar yordamida bo'laklarga bo'lamiz, ya'ni uning ixtiyoriy $P = \{A_0, A_1,..., A_n\}$ bo'linishini qaraymiz. Bu bo'linishning har bir $A_i A_{i+1}$ $(i = 0,...,n-1$ yoyidan ixtiyoriy $M_i(\xi_i, \eta_i)$ $(i = 0,...,n-1)$ nuqtani olib, berilgan funksiyaning shu nuqtadagi qiymati $f(\xi_i, \eta_i)$ ga $A_i A_{i+1}$ yoyining $Ox(Oy)$ o'qidagi proyeksiyasiga ko'paytirib, ushbu

$$\sigma' = \sum_{i=0}^{n-1} f(\xi_i,\eta_i)\Delta x_i \left(\sigma'' = \sum_{i=0}^{n-1} f(\xi_i,\eta_i)\Delta y_i\right)$$

yig'indini tuzamiz. (AB) egri chiziqning shunday

$$P_1, P_2, \ldots, P_m, \ldots \qquad (15.9.1)$$

bo'linishlar ketma-ketligini qaraylikki, ularning diametrlaridan tashkil topgan mos $\lambda_{P_1}, \lambda_{P_2}, \ldots, \lambda_{P_m}, \ldots$ ketma-ketlik nolga intilsin: $\lambda_{P_m} \to 0$, bu bo'linishlarga mos ushbu

$$\sigma'_1, \sigma'_2, \ldots, \sigma'_m \, (\sigma''_1, \sigma''_2, \ldots, \sigma''_m \ldots)$$

yig'indilar ketma-ketligini tuzamiz.

15.9.2-ta'rif. (AB) egri chiziqning har qanday (15.9.1) ko'rinishdagi bo'linishlar ketma-ketligi $\{P_m\}$ olinganda ham, unga mos yig'indilar ketma-ketligi $\{\sigma'_m\}(\{\sigma''_m\})$ $M_i(\xi_i,\eta_i)$ nuqtalarning $(M_i(\xi_i,\eta_i) \in A_i\breve{A}_{i+1})$ tanlab olinishiga bog'liq bo'lmagan holda hamma vaqt bitta J' songa (J'' songa) intilsa, bu son $\sigma'(\sigma'')$ yig'indining limiti deb ataladi va

$$\lim_{\lambda_p \to 0} \sigma' = \lim_{\lambda_p \to 0} \sum_{i=0}^{n-1} f(\xi_i,\eta_i) \cdot \Delta x_i = J' \qquad \left(\lim_{\lambda_p \to 0} \sigma'' = \lim_{\lambda_p \to 0} \sum_{i=0}^{n-1} f(\xi_i,\eta_i)\Delta y_i = J''\right)$$

kabi belgilanadi. Bu ta'rifni quyidagicha keltirish ham mumkin.

15.9.3-ta'rif. Agar $\forall\varepsilon > 0$ son olinganda ham, shunday $\delta > 0$ son topilsaki, (AB) egri chiziqning diametri $\lambda_p < \delta$ bo'lgan har qanday P bo'linish uchun tuzilgan $\sigma'(\sigma'')$ yig'indi uchun ixtiyoriy $M_i(\xi_i,\eta_i)$ nuqtalarda $\left(M_i \in A_i\breve{A}_{i+1}, i = 0,1,\ldots,n-1\right)$ $|\sigma' - J'| < \varepsilon \left(|\sigma'' - J''| < \varepsilon\right)$ tengsizlik bajarilsa, J' son (J'' son) $\sigma'(\sigma'')$ yig'indining $\lambda_p \to 0$ dagi limiti deb ataladi.

Yuqoridagi keltirilgan ta'riflar o'zaro ekvivalent ta'riflardir.

15.9.4-ta'rif. Agar $\lambda_p \to 0$ da $\sigma'(\sigma'')$ yig'indi chekli limitga ega bo'lsa, u holda $f(x,y)$ funksiya (AB) egri chiziq bo'yicha integrallanuvchi deyiladi. Bu limitga $f(x,y)$ funksiyaning (AB) egri chiziq bo'yicha ikkinchi tur egri chiziqli integrali deb ataladi va u

$$\iint\limits_{(AB)} f(x,y)dx \left(\int\limits_{(AB)} f(x,y)dy \right)$$

kabi belgilanadi.

Demak,

$$\int\limits_{(AB)} f(x,y)dx = \lim_{\lambda_p \to 0} \sigma' = \lim_{\lambda_p \to 0} \sum_{i=0}^{n-1} f(x,y)\Delta x_i$$

$$\left(\int\limits_{(AB)} f(x,y)dy = \lim_{\lambda_p \to 0} \sigma'' = \lim_{\lambda_p \to 0} \sum_{i=0}^{n-1} f(x,y)\Delta y_i \right).$$

Umumiy holda, agar (AB) egri chiziqda ikkita $P(x,y)$ va $Q(x,y)$ funksiyalar berilgan bo'lib, $\int\limits_{(AB)} P(x,y)dx$, $\int\limits_{(AB)} Q(x,y)dy$ integrallar, ularning ikkinchi tur egri chiziqli integrali bo'lsa, u holda ushbu

$$\int\limits_{(AB)} P(x,y)dx + \int\limits_{(AB)} Q(x,y)dy$$

yig'indi umumiy holda *ikkinchi tur egri chiziqli integral* deb ataladi va

$$\int\limits_{(AB)} P(x,y)dx + Q(x,y)dy$$

kabi belgilanadi.

Demak,

$$\int\limits_{(AB)} P(x,y)dx + Q(x,y)dy = \int\limits_{(AB)} P(x,y)dx + \int\limits_{(AB)} Q(x,y)dy.$$

Birinchi va ikkinchi tur egri chiziqli integrallarning ta'riflarini solishtirib ko'rganimizda, ular orasida muhim farq borligini ko'ramiz, ya'ni birinchi tur integral yig'indini tuzishda funksiyaning $f(M_i)$ qiymatini $A_i\bar{A}_{i+1}$ yoyning uzluksizligi $\sigma_i = \Delta s_i$ ga ko'paytiriladi, ikkinchi turda esa, funksiyaning $f(M_i)$ qiymatini $A_i\bar{A}_{i+1}$ yoyning $Ox\,(Oy)$ o'qdagi proyeksiyasi $\Delta x_i(\Delta y_i)$ ga ko'paytiriladi.

Ravshanki, birinchi tur egri chiziqli integralda $A_i\bar{A}_{i+1}$ yoyning uzunligi (AB) egri chiziqning yo'nalishiga bog'liq bo'lmaydi, ikkinchi tur egri chiziqli integralda yoyning proyeksiyasi egri chiziqning yo'nalishiga bog'liq bo'ladi, ya'ni (AB) egri chiziqning yo'nalishi teskariga o'zgarganda $A_i A_{i+1}$ yoyning o'qdagi proyeksiyasi teskari ishoraga o'zgaradi. Shunday qilib ikkinchi tur egri chiziqli inegrallarda

$$\int_{(BA)} f(x,y)dx = -\int_{(AB)} f(x,y)dx \left(\int_{(BA)} f(x,y)dy = -\int_{(AB)} f(x,y)dy \right)$$

bo'ladi, bunda o'ng tomondagi integralning mavjudligidan chap tomonidagi integralining mavjudligi va aksincha chap tomondagi integralning mavjudligidan o'ng tomondagi integralning mavjudligi kelib chiqadi.

Agar (AB) fazoviy egri chiziq bo'lib, bu chiziqda $f(x,y,z)$ funksiya berilganda ikkinchi tur egri chiziqli integralning ta'rifi xuddi yuqoridagi singari ta'riflanadi va ular

$$\int_{(AB)} f(x,y,z)dx, \ \int_{(AB)} f(x,y,z)dy, \ \int_{(AB)} f(x,y,z)dz$$

kabi belgilanadi.

Umumiy holda (AB) egri chiziqda $P(x,y,z), Q(x,y,z), R(x,y,z)$ funksiyalar berilgan bo'lib,

$$\int_{(AB)} P(x,y,z)dx, \ \int_{(AB)} Q(x,y,z)dx, \ \int_{(AB)} R(x,y,z)dz$$

integrallar mavjud bo'lsa, u holda ushbu

$$\int_{(AB)} P(x,y,z)dx + \int_{(AB)} Q(x,y,z)dy + \int_{(AB)} R(x,y,z)dz$$

yig'indi ikkinchi tur egri chiziqli integralning *umumiy ko'rinishi* deb ataladi va u

$$\int_{(AB)} P(x,y,z)dx + Q(x,y,z)dy + R(x,y,z)dz$$

kabi belgilanadi. Demak,

$$\int\limits_{(AB)} P(x,y,z)dx + \int\limits_{(AB)} Q(x,y,z)dy + \int\limits_{(AB)} R(x,y,z)dz =$$
$$= \int\limits_{(AB)} P(x,y,z)dx + Q(x,y,z)dy + R(x,y,z)dz. \qquad (15.9.5)$$

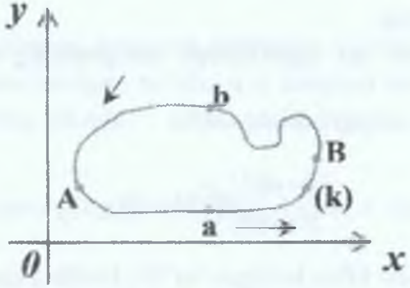
15.9.6-eslatma. Agar (AB) egri chiziq Ox o'qiga (Oy o'qiga) perpendikulyar to'g'ri chiziq kesmasidan iborat bo'lib, $f(x,y)$ funksiya shu chiziqda berilgan bo'lsa, u holda

$$\int\limits_{(AB)} f(x,y)dx = 0 \left(\int\limits_{(AB)} f(x,y)dy = 0 \right)$$

bo'ladi. Bu tenglik bevosita ikkinchi tur egri chiziqli integralning ta'rifidan kelib chiqadi.

Faraz qilaylik (AB)- sodda yopiq chiziq bo'lsin, ya'ni A va B nuqtalar ustma-ust tushsin. Bundan keyin yopiq chiziqni (k) deb belgilaymiz. Bu sodda yopiq chiziqda ham ikki yo'nalish bo'ladi. Bularning birini musbat, ikkinchisini manfiy yo'nalish deb qabul qilamiz. Kuzatuvchi yopiq chiziq bo'ylab harakat qilganda, yopiq chiziq bilan chegaralangan soha kuzatuvchiga nisbatan chapda qolgan holga musbat yo'nalish, o'ngda qolgan holga esa, manfiy yo'nalish deb qabul qilamiz.

(k)-sodda yopiq chiziqda $f(x,y)$ funksiya berilgan bo'lsin. Bu chiziqda ixtiyoriy ikkita turli A va B nuqtalarni olamiz. Natijada yopiq chiziq (AaB) va (BbA) chiziqlarga ajraladi (15.5-chizma).



15.5-chizma

Agar $\int\limits_{(AaB)} f(x,y)dx$ va $\int\limits_{(BbA)} f(x,y)dx$ integrallar mavjud bo'lsa, u holda

$$\int\limits_{(AaB)} f(x,y)dx + \int\limits_{(BbA)} f(x,y)dx$$

integral $f(x,y)$ funksiyaning (k) yopiq chiziq bo'yicha ikkinchi tur egri chiziqli integral deyiladi va u $\int\limits_{(k)} f(x,y)dx$ kabi belgilanadi.

Demak,

$$\int\limits_{(k)} f(x,y)dx = \int\limits_{(AaB)} f(x,y)dx + \int\limits_{(BbA)} f(x,y)dx.$$

huddi shunday $\int\limits_{(k)} f(x,y)dy$ integral ham ta'riflanadi , umumiy holda

$$\int\limits_{(k)} P(x,y)dx + Q(x,y)dy$$

integralning ham ta'rifi yuqoridagilar singari beriladi.

Agar (AB) egri chiziq o'zining $x = x(y)(c \le y \le d)$ oshkor tenglamasi bilan berilgan bo'lib, $x(y)$ funksiya uzluksiz va uzluksiz $x'(y)$ hosilaga ega bo'lsa, u holda

$$\int\limits_{(AB)} P(x,y)dx + Q(x,y)dy = (R)\int\limits_{c}^{d}\left[P(x(y),y)\cdot x_y' + Q(x(y),y)\right]dy$$

formula o'rinli bo'ladi.

15.10. Ikkinchi tur egrichiziqli integralning mavjudligi va uni hisoblash.

Fazar qilaylik, (AB)egri chiziq ushbu

$$\left.\begin{array}{l} x = \varphi(t) \\ y = \psi(t) \end{array}\right\}(\alpha \le t \le \beta) \qquad (15.10.1)$$

parametrik tenglamasi bilan berilgan bo'lib, bunda $\varphi(t),\psi(t)$ funksiyalar $[\alpha,\beta]$ da uzluksiz va uzluksiz $\varphi'(t),\psi'(t)$ hosilalarga ega, hamda $(\varphi(\alpha),\psi(\alpha)) = A$ va $(\varphi(\beta),\psi(\beta)) = B$ bo'lsin.

t parametr α dan β ga qarab o'zgarganda $(x,y)=(\varphi(t),\psi(t))$ nuqta A dan B ga qarab (AB) egri chiziqni chizsin. $f(x,y)$ funksiya (AB) egri chiziqda berilgan va uzluksiz bo'lsin. Shu shartlar bajarilganda ushbu

$$\int_{(AB)} f(x,y)dx=(R)\int_{\alpha}^{\beta} f(\varphi(t),\psi(t))\varphi'(t)dt, \tag{15.10.2}$$

$$\int_{(AB)} f(x,y)dy=(R)\int_{\alpha}^{\beta} f(\varphi(t),\psi(t))\psi'(t)dt \tag{15.10.3}$$

tengliklar o'rinli bo'ladi.

Umumiy holda $P(x,y), Q(x,y)$ funksiyalar (AB) egri chiziqda berilgan va uzluksiz bo'lsa hamda (15.10.1) dagi $\varphi(t),\psi(t)$ funksiyalar yuqoridagi shartlarni qanoatlantirsa, u holda

$$\int_{(AB)} P(x,y)dx+Q(x,y)dy=(R)\int_{\alpha}^{\beta}\left[P(\varphi(t),\psi(t)\varphi'(t)+Q(\varphi(t),\psi(t))\psi'(t)\right]dt \tag{15.10.3}$$

tenglik o'rinli bo'ladi.

Agar (AB) egri chiziq o'zining $y=y(x)(a\le x\le b)$ tenglamasi bilan berilgan bo'lib, $y(x)$ funksiya uzluksiz va uzluksiz $y'(x)$ ga ega bo'lsa, u holda x ni parametr deb qarasak, u holda

$$\int_{(AB)} P(x,y)dx+Qdy=(R)\int_{a}^{b}\left[P(x,y(x))+Q(x,y(x))y'(x)\right]dx. \tag{15.10.4}$$

Formulani hosil qilamiz. Agar (AB) egrichiziq o'zining $x=x(y)(c\le y\le d)$ oshkor tenglamasi bilan berilgan bo'lib, $x(y)$ funksiya uzluksiz va uzluksiz $x'(y)$ hosilaga ega bo'lsa, u holda

$$\int_{(AB)} P(x,y)dx+Q(x,y)dy=(R)\int_{c}^{d}\left[P(x(y),y)\cdot x'_y+Q(x(y),y)\right]dy.$$

15.11. Birinchi va ikkinchi tur egri chiziqlar orasidagi bog'lanish. (AB) sodda silliq egri chiziq, o'zining ushbu $x=x(s), y=y(s)$ $(0\le s\le S)$

tenglamasi bilan berilgan bo'lib, bunda $x(s)$ va $y(s)$ funksiyalar uzluksiz. $x'(s)$, $y'(s)$ hosilalarga ega bo'lsin. Ravshanki, bu egri chiziq, uzining har bir nuqtasida urinmaga ega.

Urinmaning Ox va Oy o'qlar bilan yoy o'sishi tomoniga qarab yo'nalishi orasidagi burchaklarni mos raavishda α va β deyilsa, unda $x'(s)=\cos\alpha, y'(s)=\cos\beta$ bo'ladi.

$f(x,y)$ funksiya (AB) egri chiziqda berilgan va uzluksiz bo'lsin. U holda $\int\limits_{(AB)} f(x,y)dx$ integral mavjud bo'ladi va u (15.10.2) formula orqali hisoblanadi:

$$\int\limits_{(AB)} f(x.y)dx = (R)\int\limits_0^S f(x(s),y(s))x'(s)ds = \int\limits_0^S f(x(s),y(s))\cos\alpha ds =$$

$$= \int\limits_{(AB)} f(x,y)\cos\alpha\, ds. \qquad (15.11.1)$$

Xuddi shunday

$$\int\limits_{(AB)} f(x,y)dy = (R)\int\limits_0^S f(x(s),y(s))y'(s)ds = \int\limits_0^S f(x(s),y(s))\cos\beta ds =$$

$$= \int\limits_{(AB)} f(x,y)\cos\beta ds. \qquad (15.11.2)$$

Umumiy holda, agar (AB) egri chiziqda $P(x,y), Q(x,y)$ uzluksiz funksiyalar berilganda

$$\int\limits_{(AB)} P(x,y)dx + Q(x,y)dy = \int\limits_{(AB)} (P(x,y)\cos\alpha + Q(x,y)\cos\beta)ds \qquad (15.11.3)$$

formula o'rinli bo'ladi. (15.11.1), (15.11.2) va (15.11.3) formulalar birinchi va ikkinchi tur egri chiziqli integrallar orasidagi bog'lanishlarni ifodalovchi formulalar bo'lib hisoblanadi.

Agar (AB) egri chiziq fazoda berilganda (15.11.3) ga o'xshash formulaning ko'rinishi ushbu

$$\int\limits_{(AB)} Pdx + Qdy + Rdz = \int\limits_{(AB)} (P\cos\alpha + Q\cos\beta + R\cos\gamma)ds$$

ko'rinishda bo'ladi, bunda P,Q,R funksiyalar (AB) da uzluksiz, $\cos\alpha,\cos\beta$ va $\cos\gamma$ lar urinmaning yo'naltiruvchi kosinuslari bo'lib, uning yo'nalishi egri chiziq yo'nalishiga mos deb faraz qilinadi.

15.5-§. Grin formulasi

15.12. Grin formulasi. a) Yuqoridan $y=Y(x)$, quyidan $y=y_0(x)\,(a\le x\le b)$ egri chiziqlar bilan yon tomonlardan esa, OY o'qqa parallel (PS) va (RQ) to'g'ri chiziq kesmalari bilan chegaralangan (D)– egri chiziqli trapesiyani qaraylik (15.6-chizma). Shu (D) sohada $P(x,y)$ uzluksiz va uzluksiz $\frac{\partial P}{\partial y}$ xususiy hosilaga ega bo'lgan funksiya berilgan bo'lsin. Shu shartlarda ushbu

$$\iint\limits_{(D)}\frac{\partial P}{\partial y}dxdy=-\int\limits_{(K)}P(x,y)dx \qquad (15.12.1)$$

formula o'rinli bo'ladi, bunda (K)– (D) sohaning chegarasi.

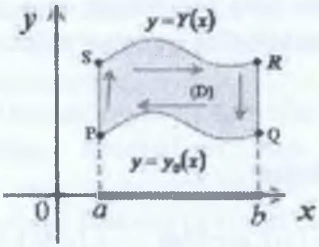
b) Xuddi shunday, (D) yuqoridan (SR) quyidan (PQ) to'g'ri chiziq kesmalari bilan, yon tomonlaridan esa, $x=x_0(y)$, $x=x(y)\,(c\le y\le d)$ egri chiziqlar bilan chegaralangan soha bo'lib (15.7- chizma), unda $Q(x,y)$ uzluksiz va uzluksiz $\frac{\partial Q}{\partial x}$ xususiy hosilaga ega bo'lgan funksiya bo'lganda, ushbu

$$\iint\limits_{(D)}\frac{\partial Q}{\partial x}dxdy=\int\limits_{(K)}Q(x,y)dy \qquad (15.12.2)$$

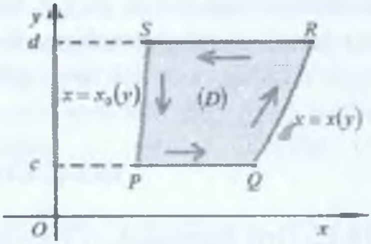
formula o'rinli bo'ladi. Endi (D) soha yuqorida qaralgan ikkala holning ham shartlarini qanoatlantirsa, bu sohada $P(x,y)$ va $Q(x,y)$ funksiyalar yuqoridagi shartlarni qanoatlantirsa, u holda (15.12.1), (15.12.2) formulalar o'rinli bo'ladi. (15.12.2) formuladan (15.12.1) formulani ayirsak, quyidagi

$$\iint\limits_{(D)}\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dxdy=\int\limits_{(K)}P\,dx+Qdy \qquad (15.12.3)$$

formulaga ega bo'lamiz. (15.12.3) formulaga *Grin formulasi* deyiladi.



15.6-chizma.



15.7-chizma.

Ushbu $\iint\limits_{(D)} \frac{\partial P}{\partial y} dxdy$ integralni hisoblash talab qilingan bo‘lsin. Buning uchun soha egri chiziqli soha bo‘lganda ikki karrali integralni hisoblash formulasidan foydalanamiz, ya’ni

$$\iint\limits_{(D)} \frac{\partial P}{\partial y} dxdy = \int\limits_a^b dx \int\limits_{Y_0}^{Y(x)} \frac{\partial P}{\partial y} dy = \int\limits_a^b P(x, Y(x))dx - \int\limits_a^b P(x, y_0(x))dx .$$

Keyingi tenglikning o‘ng tomonidagi integrallarning ikkalasini ham (15.10.4) formulaga asosan, egri chiziqli integralga aylantiramiz:

$$\int\limits_a^b P(x, Y(x)dx = \int\limits_{(SR)} P(x, y)dx, \quad \int\limits_{(SR)} P(x, y_0(x))dx = \int\limits_{(PQ)} P(x, y)dx.$$

Natijada,

$$\iint\limits_{(D)} \frac{\partial P}{\partial y} dxdy = \int\limits_{(SR)} P(x, y)dx + \int\limits_{(PQ)} P(x, y)dx,$$

bu tenglikning o‘ng tomoniga

$$\int\limits_{(PS)} P(x, y)dx = 0, \quad \int\limits_{(RQ)} P(x, y)dx = 0$$

(PS va RQ kesmalar Ox o’qiga perpendikulyar bo’lgani uchun yuqoridagi integrallar nolga teng) integrallarni qo‘shamiz:

$$\iint\limits_{(D)}\frac{\partial P}{\partial y}dxdy=\int\limits_{(SR)}P(x,y)dx+\int\limits_{(QP)}P(x,y)dx+\int\limits_{(PS)}P(x,y)dx+\int\limits_{(RQ)}P(x,y)dx.$$

Natijada keyingi tenglamaning o‘ng tomoni (D) sohaning chegarasi (k) yopiq chiziqni ifoda qiladi, lekin yo‘nalish manfiy yo‘nalishda olingan. Shuning uchun

$$\iint\limits_{(D)}\frac{\partial P}{\partial y}dxdy=-\int\limits_{(k)}P(x,y)dx.$$

Xudi shunday

$$\iint\limits_{(D)}\frac{\partial Q}{\partial x}dxdy=\int\limits_{(k)}Q(x,y)dy$$

formula ham isbot qilinadi, bunda $Q(x,y)$ funksiya (D) sohada uzluksiz va uzluksiz $\frac{\partial Q}{\partial x}$ xususiy xosilaga ega, (D) soha esa, yuqoridan (SR), qo‘yidan (PQ) Ox o'qga parallel bo’lgan to'g‘ri chiziq kesmalari bilan, yon tomonlardan esa, $x=x_0(y), x=x(y)$ $(c\le y\le d)$ egri chiziqlar bilan chegralangan soha.

15.12.5-eslatma. Grin formulasi (D) soha bo‘yicha olingan ikki karrali integral bilan, sohaning chegarasi bo‘yicha olingan ikkinchi tur egri chiziqli integrallarni bog‘lovchi formula ekan. Biz yuqorida Grin formulasini maxsus ko‘rinishdagi (D) sohalar (egri chiziqli trapetsiyalar) uchun isbotladik.

Aslida bu formula ancha keng sinfdagi sohalar uchun ham o‘rinli bo'ladi, ya’ni (D) sohani chekli sondagi egri chiziqli trapesiya yig’indisi sifatida tasverlash bilan isbot qilinadi.

15.13. Grin formulasining ba'zi bir qo‘llanishlari.

1.Grin formulasi yordamida yassi shaklning yuzini egri chiziqli integrallar orqali hisoblash. Haqiqatdan ham, agar (15.12.3) formuladagi $P(x,y)$ va $Q(x,y)$ funksiyalarni $\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}$ ayirma 1 ga teng bo‘ladigan qilib tanlasak, u holda ikki karrali integral (D) sohaning yuzini ifodalaydi, natijada (D) sohaning yuzi D ushbu

$$D = \int_{(k)} Pdx + Qdy$$

formula orqali, ya'ni egri chiziqli integral orqali ifoda qilinadi. Misol uchun

1) $Q = x,\ P = 0$ deb olinsa, $D = \int_{(k)} xdy$.

2) $Q = 0,\ P = -y$ deb olinsa, $D = -\int_{(k)} ydx$.

3) $Q = \frac{1}{2}x,\ P = -\frac{1}{2}y$ deb olinsa,

$$D = \frac{1}{2}\int_{(k)} xdy - ydx \tag{15.13.1}$$

formulaga ega bo'lamiz.

2. Egri chiziqli integralning integrallash yo'liga bog'liq bo'lmaslik sharti. Chegaralangan bog'lamli $(D)\ \left((D) \subset R^2\right)$ sohada $P(x,y)$ va $Q(x,y)$ funksiyalar berilgan bo'lib, ular uzluksiz va uzluksiz $\frac{\partial P}{\partial y}, \frac{\partial Q}{\partial x}$ xususiy hosilalarga ega bo'lsin.

15.13.2-teorema. (D) sohaga qarashli har qanday (k) yopiq egri chiziq bo'yicha olingan integral

$$\int_{(k)} Pdx + Qdy = 0 \tag{15.13.3}$$

bo'lishi uchun

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \tag{15.13.4}$$

shartning bajarilishi zarur va yetarlidir.

15.13.5-teorema. $P(x,y)$ va $Q(x,y)$ funksiyalar bir bog'lamli yopiq (D) sohada yuqoridagi shartlarni qanoatlantirganda

$$\int_{(AB)} Pdx + Qdy \qquad \left((AB) \subset (D)\right) \tag{15.13.6}$$

integralning integrallash yo'liga (A va B nuqtalarni birlashtiruvchi egri chiziqqa) bog'liq bo'lmasligi uchun (15.13.4) shartining bajarilishi zarur va yetarli.

3. $Pdx + Qdy$ **ifodaning biror** $F(x,y)$ **funksiyaning to'liq differensiali bo'lishlik sharti.** Ushbu

$$Pdx + Qdy \tag{15.13.7}$$

ifoda bizga biror $F(x,y)$ funksiyaning to‘liq differensiali bo‘lishligini eslatadi, ya’ni

$$dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy,$$

bunda $\frac{\partial F}{\partial x} = P, \frac{\partial F}{\partial y} = Q$ desak, (15.13.7) ifoda $F(x,y)$ funksiyaning to‘liq differensialini ifoda qiladi, lekin (15.13.7) ifoda har doim ham biror $F(x,y)$ funksiyaning to‘liq differensialini ifoda qilavermaydi. Bu ifoda biror $F(x,y)$ funksiyaning to‘liq differensialini ifoda qilish uchun qanday shart bajarilishi kerak degan savolning qo‘yilishi tabiiy. Bu savolga quyidagi teorema javob beradi.

15.13.10-teorema. $P(x,y)$ va $Q(x,y)$ funksiyalar yuqoridagi shartlarni qanoatlantirganda (15.13.7) ifoda biror $F(x,y)$ funksiyaning to‘liq differensialini ifodalashligi uchun (15.13.4) shartning bajarilishi zarur va yetarli.

15.13.11-natija. $P(x,y)$ va $Q(x,y)$ funksiyalar bir bog‘lamli yoyiq (D) sohada yuqoridagi shartlarni qanoatlantirganda

$$\int\limits_{(AB)} Pdx + Qdy$$

integralning integrallash yo‘liga bog‘liq bo‘lmasligi, $\int\limits_{(k)} Pdx + Qdy$ integralning nolga teng bo‘lishi ((k)-yopiq chiziq), $Pdx + Qdy$ ifodaning biror ikki o‘zgaruvchili funksiyaning to‘liq differensiali bo‘lishi uchun (15.13.4) shartning bajarilishi zarur va yetarli.

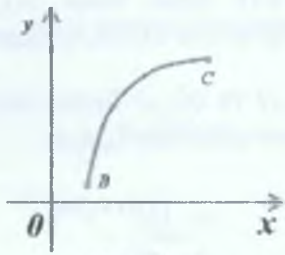
15.14. Egri chiziqli integrallarning fizikaga qo‘llanishi. Biror material nuqta $\vec{F}$ kuch ta’siri ostida uzluksiz tekis BC chizig‘i bo‘ylab B dan C ga qarab harakatlanayotgan bo‘lsin. $\vec{F}$ kuchni o‘zgaruvchan (BC chiziqdagi nuqtaning holatiga bog‘liq) deb faraz qilamiz (15.8-chizma). Materal nuqtani B holatdan C holatga o‘tkazishda $\vec{F}$ kuchni bajargan ishini hisoblaymiz. Buning uchun (BC) chiziqning ixtiyoriy

$$P = \{B = M_0, M_1, M_2, ..., M_n = C\}$$

bo'lishini qaraymiz. Bu bo'linishning M_iM_{i+1} bo'lagida $\vec{F}$ kuchni taqribiy ravishda o'zgarmas $\vec{F}_i \approx \vec{F}(M_i)$ qiymatga teng deb faraz qilib, nuqtaning $\breve{M_iM}_{i+1}$ yoy bo'ylab harakatini esa, $\overline{M_iM_{i+1}}$ kesma bo'ylab harakatga almashtirsak, u holda o'zgarmas $\overline{F}(M_i)$ kuchning M_iM_{i+1} kesma bo'ylab bajargan ishini taqribiy ravishda $\vec{F}$ o'zgaruvchan kuchning M_iM_{i+1} yoy bo'ylab bajargan ishi deb qabul qilish mumkin:

$$A_i \approx \bar{F}(M_i) \cdot \overline{M_iM_{i+1}}$$

Bu taqribiy tenglikning o'ng tomoni $\vec{F}(M_i)$ va $\overrightarrow{M_iM_{i+1}}$ vektorning skalyar ko'paytmasini ifodalaydi. Ravshanki, skalyar ko'paytma $\bar{F}(M_i) = \{P(M_i), Q(M_i)\}$, $\overline{M_iM_{i+1}} = \{\Delta x_i, \Delta y_i\}$ vektorlarning mos ravishda koordinatalari ko'paytmasining yig'indisiga teng:



15.8-chizma.

$$A_i \approx P(M_i)\Delta x_i + Q(M_i)\Delta y_i.$$

Bundan, (BC) egri chiziq bo'ylab bajarilgan ish

$$A \approx \sum_{i=1}^{n}[P(M_i)\Delta x_i + Q(M_i)\Delta y_i] = \sum_{i=1}^{n}\left[P(M_i)\Delta x_i + \sum_{i=1}^{n} Q(M_i)\Delta y_i\right] \qquad (15.14.1)$$

Bajarilgan ishning aniq qiymati (15.14.1) tenglikda $\lambda_p = \max\left|\breve{M_iM}_{i+1}\right| \to 0$ da limitga o'tish bilan topiladi. Ikkinchi tomondan (15.14.1) yig'indi (BC) egri chiziqda berilgan $P(x,y)$ va $Q(x,y)$ funksiyalarning integral yig'indilarining yig'indisini ifoda qiladi. Bu yig'indilarning $\lambda_p \to 0$ dagi limiti ikkinchi tur egri chiziqli integraldan iborat.

Demak, bajarilgan ish

$$A = \int_{(BC)} P(x,y)dx + Q(x,y)dy$$

formula bilan topilar ekan, bunda P va Q lar $\bar{F}$ kuchning (koordinata o'qlardagi proyeksiyalari) koordinatalari. Agar berilgan masalani fazoda qarasak, u holda ish

$$A = \int_{(BC)} Pdx + Qdy + Rdz$$

formula orqali topiladi.

15.6-§. Sirtning tomoni va uning yuzini ikki karrali integral orqali hisoblash

15.15. Sirtning parametrik shaklida tasvirlanishi. Ma'lumki, egri chiziqning fazodagi parametrik tenglamasi ushbu

$$x = \varphi(t),\ y = \psi(t),\ z = \chi(t)\ \ (\alpha \le t \le \beta)$$

ko'rinishda bo'ladi, bunda egri chiziqdagi nuqtalarning o'rni parametr t ning qiymatlari bilan aniqlanadi. Tenglamasi oshkor

$$z = f(x,y) \qquad (15.15.1)$$

ko'rinishda berilgan (S) sirtdagi nuqtalarning o'rni esa, ikkita x va y parametrlarning (x,y) juft qiymatlar bilan aniqlanadi. Umumiy holda (S) sirtning ixtiyoriy u va v parametrlar orqali tenglamisi ushbu

$$x = \varphi(u,v),\ \ y = \psi(u,v),\ z = g(u,v),\ \ (u,v) \in (\Delta) \qquad (15.15.2)$$

ko'rinishda bo'lib, bunda φ, ψ, g – lar (Δ) sohada uzluksiz va uzluksiz birinchi tartibli xususiy hosilalarga ega, (Δ) - chegaralangan, yopiq soha, uning chegarasi o'zaro kesishmaydigan bo'lakli siliq chiziqdan iborat, hamda (Δ) sohaning har xil (u,v) ichki nuqtasiga (S) sirtning har xil (x,y,z) nuqtasi mos keladi, (Δ) ning har bir chegara nuqtasiga (S) sirtning bitta chegara nuqtasi mos keladi deb faraz qilinadi. (Δ) sohaning chegara nuqtalariga mos keladigan (S) sirtning nuqtalar to'plami sirtning chegarasini tashkil qiladi. Shunday qilib, (15.15.2) sistema yordamida (S) sirt nuqtalari bilan (Δ) soha nuqtalari o'rasida o'zaro bir qiymatli moslik o'rnatiladi. Bu holda sirtni sodda sirt deyiladi. u va v parametrlar (S) sirt nuqtalarining egrichiziqli kordinalari deyiladi. (S) sirtning (15.15.2) ko'rinishdagi parametrik tenglamasi uning oshkor (15.15.1) ko'rinishdagi tenglamasiga qandaydir ma'noda, ya'ni biror nuqtaning kichik atrofida teng kuchli ekanligini ko'rsatamiz. Buning uchun (S) sirtning biror $M_0(x_0,y_0,z_0)$ nuqtasini olib, u nuqta parametrlarning bitta $u=u_0$, $v=v_0$ juft qiymatlariga mos kelsin deb faraz qilamiz.

(15.15.2) sitemaning ushbu

$$\begin{pmatrix} \varphi_u' & \psi_u' & g_u' \\ \varphi_v' & \psi_v' & g_v' \end{pmatrix} \qquad (15.15.3)$$

funksiyaonal matritsasini qaraymiz.

Parametrlarning (u_0,v_0) qiymati bilan aniqlangan $M_0(x_0,y_0,z_0)$ nuqtada (15.15.3) matritsaning hech bo'lmaganda bitta determenanti noldan farqli bo'lsin.

Masalan,

$$\begin{vmatrix} \varphi_u' & \psi_u' \\ \varphi_v' & \psi_v' \end{vmatrix} \neq 0.$$

Bu holda M_0 nuqtani oddiy (maxsus emas) nuqta deyiladi. Agarda (15.15.3) matritsaning hamma determinanti M_0 nuqtada birdaniga nolga teng bo'lsa, M_0 nuqta *maxsus nuqta* deyiladi. Maxsus nuqtaga ega bo'lmagan sodda sirtni silliq sirt deb ataymiz.

(15.15.2) ning birinchi ikkitasini ushbu

$$\varphi(u,v)-x=0, \quad \psi(u,v)-y=0$$

ko'rininshda yozib olamiz. Oshkormas funksiyaning mavdjudligi haqidagi – teoremaga asosan, u,v,x,y o'zgaruvchili ikkita tenglamalar sistemasidan u,v lar x va y larning bir qiymatli funksiyasi sifatida aniqlanadi:

$$u = u(x,y),\ v = v(x,y),$$

bu funksiyalar o'zlarining xususiy xosilalari bilan birgalikda uzluksiz. u va v larning topilgan $u(x,y)$, $v(x,y)$ ifodalarini (15.15.2) ning uchunchisiga olib borib qo'ysak, natijada sirtning $M_0(x_0,y_0,z_0)$ nuqta atrofdagi oshkor $z = g(v(x,y), v(x,y)) = f(x,y)$ tenglamasi hosil bo'ladi,bunda $f(x,y)$ ham uzluksiz, uzluksiz xususiy hosilalarga ega bo'ladi. M_0 nuqta maxsus nuqta bo'lsa, sirt tenglamasini, bu nuqta atrofida oshkor shaklda ifodalab bo'lmaydi. $M(x,y,z)$ nuqta (15.15.2) sirtning oddiy ichki nuqtasi bo'lsin. Yuqorida ko'rdikki, bu nuqtaning biror kichik atrofida sirt tenglamasi oshkor shaklda ifoda qilinadi va sirt $M(x,y,z)$ nuqtada o'rinma tekislikga ega bo'ladi. Ma'lumki bu o'rinma tekislik tenglamasi ushbu

$$A(X-x) + B(Y-y) + C(Z-z) = 0 \qquad (15.15.4)$$

ko'rinishda bo'ladi, bunda A,B,C hozircha no'malum koeffitsiyentlar. Agar (15.15.2) sirt tenglamasida v ning belgilangan M nuqtaga mos keladigan qiymatini belgilasak, natijada M nuqtadan o'tuvchi va sirtda butunlay yotuvchi $x = \varphi(u,v_0)$, $y = \psi(u,v_0)$, $z = g(u,v_0)$ chiziq hosil bo'ladi. v_0 ning qiymatlarini o'zgartirish natijasida M nuqtadan o'tuvchi "v" koordinat chiziqlari oilasi hosil bo'ladi. Bu chiziqqa M nuqtada o'tkazilgan o'rinma chiziq tenglamasi

$$\frac{X-x}{x'_u} = \frac{Y-y}{y'_u} = \frac{Z-z}{z'_u}$$

ko'rinishda bo'ladi. Xuddi shunday u ni belgilasak M nuqtadan o'tuvchi boshqa "u" koordinat chiziqlari oilasi hosil bo'ladi. Bu chiziqqa M nuqtada o'tkazilgan o'rinma chiziq tenglamasi ushbu

$$\frac{X-x}{x'_v} = \frac{Y-y}{y'_v} = \frac{Z-z}{z'_v}$$

ko'rinishda bo'ladi. Bu o'rinma chiziqlarning ikkalasi ham (15.15.4) tekislikda yotadi. U holda,

$$Ax'_u + By'_u + Cz'_u = 0,$$
$$Ax'_v + By'_v + Cz'_v = 0$$

shartlar o'rinli bo'ladi. Bu holda A, B, C koeffitsiyentlar, (15.15.3) matritsaning determinantlariga (qaralayotgan nuqtada hamma determinantlar birdaniga nolga teng emasligini takidalaymiz) proporsional bo'lishi kerak, ya'ni ularni bundan keyin bu determinantlarga teng deb olish mumkin, ya'ni

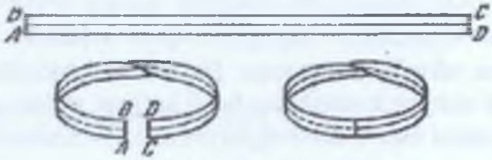
$$A=\begin{vmatrix} y'_u & z'_u \\ y'_v & z'_v \end{vmatrix},\ B=\begin{vmatrix} z'_u & x'_u \\ z'_v & x'_v \end{vmatrix},\ C=\begin{vmatrix} x'_u & y'_u \\ x'_u & y'_v \end{vmatrix} \qquad (15.15.5)$$

Ma'lumki, $M(x,y,z)$ nuqtaga o'tkazilgan normalning yo'naltiruvchi kosinuslari:

$$\cos\alpha=\frac{A}{\pm\sqrt{A^2+B^2+C^2}},\ \cos\beta=\frac{B}{\pm\sqrt{A^2+B^2+C^2}},\ \cos\gamma=\frac{C}{\pm\sqrt{A^2+B^2+C^2}}.$$

15.16. Sirtning tomoni. Ko'p hollarda sirt tomoni tushunchasini ichki (intuitiv) hissiyatlarimiz asosida aniqlashimiz mumkin. Masalan, agar sirt biror jismni qoplab turgan bo'lsa, u holda sirtning jismga qaragan tomonini, sirtning ichki tomoni, jismning fazoga qaragan tomonini tashqi tomon deb qarash mumkin. Agar sirt tenglamasi $z=f(x,y)$ oshkor shaklda tasvirlangan bo'lsa, sirtning yuqori va pastki tomoni to'g'risida gapirish mumkin (bunda z o'qi vertikal yuqoriga qaralgan deb tushuniladi). Bu ichki hissiyatlarimiz yordamida sirtning tomoni ta'rifini berishga harakat qilamiz. (S)- siliq sirtning ixtiyoriy M_0 nuqtasini olib, undan (S)- sirtga $\vec{n}$ normal o'tkazamiz va uning biror yo'nalishini belgilaymiz. (S) sirtning M_0 nuqtasidan o'tuvchi va unga qaytib keluvchi, sirtning chegarasini kesib o'tmaydigan biror (K) yopiq egri chiziqni olib, M nuqtani bu yopiq (K) chiziq bo'ylab uzluksiz haraktlantiramiz va uning har bir holatiga normal o'tkazib, uning yo'nalashishini belgilab boramiz, natijada, M nuqta yana M_0 nuqtaga qaytib kelganda, normalning yo'nalishi ikki yo'nalishdan biri, ya'ni oldingi belgilangan yo'nalishda yoki unga qarama-qarshi yo'nalishda bo'lishi mumkin.

Agar belgilangan M_0 nuqtadan o'tuvchi va (S) sirtning chegasini kesib o'tmaydigan ixtiyoriy yopiq (K) egri chiziq bo'ylab xarakatrlanayotgan M nuqta yana oldingi belgilangan M_0 ga qaytib kelganda normal o'zining yo'nalishini o'zgartirmasa, ya'ni yo'nalishi oldingi belgilangan yunalishga mos tushsa, u holda (S) sirt ikki tomonli sirt deyiladi. Ikki tomonli sirtga, tenglamasi $z = f(x,y)$ (bunda $f(x,y)$, $f'_x(x,y)$, $f'_y(x,y)$ – biror D sohada uzluksiz funksiyalar) shaklda berilgan sirtlar misol bo'laoladi.

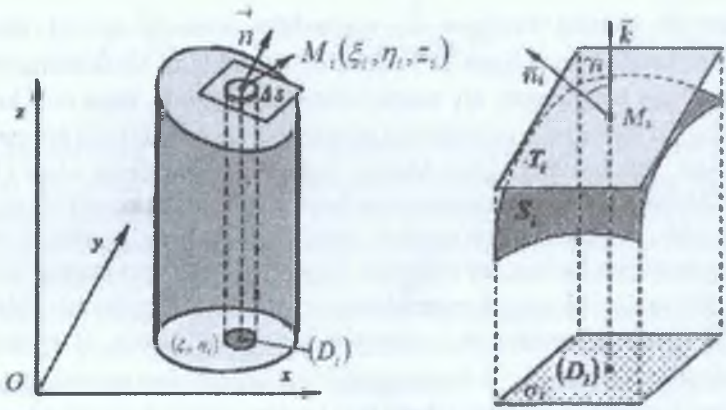


15.9-chizma.

Agar (S) sirtning berilgan M_0 nuqtasidan o'tuvchi va (S) sirtning chegarasini kesib o'tmaydigan (K) yopiq chiziq bo'ylab xarakatlanayotgan M nuqta oldingi belgilangan M_0 nuqta holatiga ke0lganda, unga mos kelgan normalning yo'nalishi o'z yo'nalishini o'zgartirsa, u holda (S) sirt *bir tomonli sirt* deyiladi. Bir tomonli sirtga Mebius yaprog'i misol bo'la oladi (15.9-chizma). Mebius yaprog'ini quyidagicha hosil qilish mumkin: ABSD qog'oz bo'lagini olib, uni shunday bo'raylikki, natijada A nuqta C nuqta, B nuqta esa D nuqta ustiga tushsin, so'ngra uni kleylaymiz. Mebius yaprog'ida M_0 nuqtani olib unga o'tkazilgan normalning yo'nalishini belgilaymiz. Mebius yaprog'i bo'ylab M nuqtani va $\vec{n}$ normalni harakatlantiramiz. M nuqta M_0 nuqta holatiga kelganda $\vec{n}$ normalning yo'nalishi oldingi belgilangan yo'nalishga qarama – qarshi bo'ladi. Biz bundan keyin faqat ikki tomonli sirtni qaraymiz. Ikki tomonli sirtda bitta belgilangan nuqtadagi normalning tanlangan yo'nalishi, sirtning qolgan nuqtalardagi normallarning yo'nalishini bir qiymatli aniqlaydi. Sirtning, unda ko'rsatilgan qoida bo'yicha normallarning yo'nalishlari aniqlangan barcha nuqtalari to'plami sirtning *belgilangan tomoni* (*oriyentirlanuvchi sirt) deyiladi*, sirtning biror tomonini tanlash esa, oriyentatsiya deyiladi.

Bir tomonli sirt – *oriyentirlanmaydigan sirt* deb ataladi.

15.17. Sirt yuz i va uni ikki karrali integral orqali hisoblash. Avvalo sirtning yuzi tushunchasini kiritamiz. (S) sirt o'zining oshkor $z = z(x,y)$ tenglamasi bilan berilgan bo'lib, uning xOy tekislikdagi proyeksiyasi (D) soha bo'lsin (15.2-chizma). (D) sohada $z(x,y)$ – uzluksiz va uzluksiz $z'_x(x,y)$, $z'_y(x,y)$) xususiy hosilalarga ega bo'lgan funksiya aniqlangan bo'lsin. (S) sirtning yuzini topish uchun (D) sohaning ixtiyoriy P bo'linishini qaraymiz: $(D_1), (D_2), ..., (D_n)$. Bu bo'linishning bo'luvchi chiziqlarini yo'naltiruvchilar sifatida qarab, ular orqali yasovchilari Oz o'qiga parallel bo'lgan silindrik sirtlarni o'tkazamiz. Bu silindrik sirtlar (S) sirtni $(S_1), (S_2), ..., (S_n)$ bo'laklarga ajratadi. Har bir $(D_i)(i=1,2,...,n)$ dan ixtiyoriy (ξ_i, η_i) nuqtani olib, (S) sirtda unga mos kelgan $(\xi_i, \eta_i, z_i)(z_i = z(\xi_i, \eta_i))$ nuqtani topamiz. (S) sirtning shu nuqtasiga o'rinma tekislik o'tkazamiz. Bu o'rinma tekislik bilan yuqorida aytilgan silindirik sirtning keshishidan hosil bo'lgan o'rinma tekislik qismini (T_i) bilan, uning yuzini esa, T_i deb belgilaymiz (15.9-chizma). Ma'lumki (D_i) soha (T_i) ning ortogonal preksiyasi bo'lib, uning yuzi



15.9-chizma.

$$D_i = T_i|\cos\gamma_i|, \quad T_i = \frac{D_i}{|\cos\gamma_i|} \tag{15.17.1}$$

formula bilan topiladi, bunda $\gamma_i - (s)$ sirtga $(\xi_i, \eta_i, z_i)\,(z_i = z(z_i, \eta_i))$ nuqtada o'tkazilgan o'rinma tekislik normalining Oz o'qi bilan takshil etgan burchagi. Ravshanki, $\lambda_p \to 0$ da $(S_i)(i=1,2,...,n)$ ning diametri ham nolga intiladi.

Agar $\lambda_p \to 0$ da $\sum_{i=1}^{n} T_i$ yig'indi chekli limitga ega bo'lsa, bu limit (S) *sirtning yuzi* deyiladi, ya'ni (S) sirtning yuzi

$$S=\lim_{\lambda_p\to 0}\sum_{i=1}^{n}T_i.$$

$z=z(x,y)$ funksiyaga yuqoridagi shartlar qo‘yilganda

$$\cos\gamma_i=\frac{1}{\sqrt{1+\left(z_x'(\xi_i,\eta_i)\right)^2+\left(z_y'(\xi_i,\eta_i)\right)^2}}$$

bo‘ladi. Unda (15.17.1) munosabatdan $T_i=\frac{D_i}{\cos\gamma_i}$.

Shunday qilib,

$$\sum_{i=1}^{n}T_i=\sum_{i=1}^{n}\frac{D_i}{\cos\gamma_i}=\sum_{i=1}^{n}\sqrt{1+\left(z_x'(\xi_i,\eta_i)\right)^2+\left(z_y'(\xi_i,\eta_i)\right)^2}\,D_i. \qquad (15.17.2)$$

Ravshanki, (15.17.2) ning o‘ng tomonidagi yig‘indi

$$\sqrt{1+\left(z_x'(x,y)\right)^2+\left(z_y'(x,y)\right)^2}$$

funksiya uchun integral yig‘indi bo‘lib hisoblanadi.

Bu funksiya (D) sohada uzluksiz, demak, u integrallanuvchi. Shuning uchun

$$\lim_{\lambda_p\to 0}\sum_{i=1}^{n}\sqrt{1+\left(z_x'(\xi_i,\eta_i)\right)^2+\left(z_y'(\xi_i,\eta_i)\right)^2}\cdot D_i=\iint_{(D)}\sqrt{1+\left(z_x'(x,y)\right)^2+\left(z_y'(x,y)\right)^2}\,dxdy.$$

Demak, (S) sirtning yuzi ushbu

$$S=\iint_{(D)}\sqrt{1+\left(z_x'\right)^2+\left(z_y'\right)^2}\,dxdy=\iint_{(D)}\frac{1}{|\cos\gamma|}dxdy \qquad (15.17.3)$$

formula bilan hisoblanar ekan.

2. Umumiy holda sirt yuzi. (S) sirt tenglamasi parametrik shaklida:

$$x=x(u,v),\ y=y(u,v),\ z=z(u,v),\ \ (u,v)\in(\Delta).$$

berilgan bo'lib, $x(u,v)$, $y(u,v)$, $z(u,v)$ lar (Δ) sohada uzliksiz va uzliksiz hususiy hosilalarga ega hamda karrali va maxsus nuqtalarga ega emas deb faraz qilamiz.

Ixtiyoriy $M \in (S)$ nuqtani olamiz va shu nuqtada $C \neq 0$ bo'lsin deb faraz qilamiz. 15.1-bandda aytilganlarga va yuqoridagi (S) sirtga qo'yilgan shartlarda quyidagi tasdiqlar o'rinli bo'ladi:

(S) sirtning M nuqtani o'rab turuvchi shunday (S_1) qismi mavjud bo'lib, u quyidagi xossalarga ega bo'ladi:

1^0. (S_1) sirt tenglamasi oshkor $z = z(x,y)$ shalida ifoda qilinadi.

2^0. (S_1) ga sirt mos keluvchi uOv tekislikdagi (Δ) sohaning qismini (δ) deb belgilasak, u holda (δ) da $C \neq 0$ bo'ladi.

Agar A yoki B determinatlar noldan farqli bo'lsa, u holda $z = z(x,y)$ tenglama, $x = x(y,z)$ tenglamaga yoki $y = y(z,x)$ tenglamaga almashiladi. Bundan (S) sirt chekli sondagi (S_1) sirt bo'laklariga yoyiladi degan xulosani chiqarish mumkin. Endi yuqoridagi ta'rifga ko'ra (S_1) sirt bo'lagining yuzini hisoblaymiz. Agar (S_1) sirt bo'lagining xOy tekislikdagi proyeksiyasini (d) deb belgilasak, u holda uning yuzi ((15.17.3) qarang).

$$S_1 = \iint\limits_{(d)} \frac{dxdy}{|\cos\gamma|} \qquad (15.17.4)$$

formula bilan hisoblanadi.

(d) sohaning nuqtalari (δ) sohaning nuqtalari bilan bir qiymatli moslikda ekanligini e'tiborga olgan holda va bundan tashqari tekis figuralarni almashtirish mavzusidagi talablar bajarilgan bo'lsin degan farazda, u holda (15.17.4) da u va v o'zgaruvchilarga o'tsak hamda

$$\frac{D(x,y)}{D(u,v)} = C, \quad |\cos\gamma| = \frac{|C|}{\sqrt{A^2+B^2+C^2}}$$

ekanligini iborga olgan holda

$$S_1 = \iint\limits_{(\delta)} \sqrt{A^2+B^2+C^2}\,dudv$$

ni topamiz. Endi (S) sirtning yuzi S ni uning (S_1) bo'laklari yuzlarining yig'indisi sifatida topamiz:

$$S=\iint\limits_{(\Delta)}\sqrt{A^2+B^2+C^2}\,dudv. \qquad (15.17.5)$$

(15.17.5) formulani boshqacha ko'rinishda ham yozish mumkin:

$$x_u'^2+y_u'^2+z_u'^2=E,\ x_v'^2+y_v'^2+z_v'^2=G,\ x_u'x_v'+y_u'y_v'+z_u'z_v'=F$$

belgilashlarni kiritsak, natijada $A^2+B^2+C^2=EG-F^2$ ekanligiga ishonch hosil qilamiz. U holda (15.17.5) ko'rinishi

$$S=\iint\limits_{(\Delta)}\sqrt{EG-F^2}\,dudv \qquad (15.17.6)$$

ko'rinishga keladi. E,G,F koeffitsiyentlarni Gauss koeffitsiyentlari deb yuritiladi.

15.7-§. Birinchi tur sirt integrali va uning fizikaga qo'llanishi

15.18. Birinchi tur sirt integralining ta'rifi. $f(x,y,z)$ funksiya biror ikki tomonli silliq yoki bo'lakli siliq $(S)\left((S)\subset R^3\right)$ sirtda berilgan bo'lsin. Bu sirtni bo'lakli – silliq chiziqlar to'ri yordamida $(S_1),(S_2),...,(S_n)$ bo'laklarga ajratamiz. Bu bo'linishni P bo'lininsh deb ataymiz. Bu bo'linishning $(S_i)\,(i=1,2,...,n)$ bo'lagidan ixtiyoriy $M_i(x_i,y_i,z_i)$ nuqtani olib, bu nuqtadagi berilgan funksiyaning $f(x_i,y_i,z_i)$ qiymatiga (S_i) sirtning yuzi S_i ni ko'paytirib, quyidagi yig'indini tuzamiz:

$$\sigma=\sum_{i=1}^{n}f(x_i,y_i,z_i)S_i=\sum_{i=1}^{n}f(M_i)S_i$$

Bu yig'indiga $f(x,y,z)$ funksiyaning Riman integral yig'indisi deb ataladi. (S) sirtning shundan

$$P_1,\ P_2,...,\ P_n,.... \qquad (15.18.1)$$

bo'linishlar ketma ketligini qaraylikki, ularning mos diametrlaridan tashkil topgan $\lambda_{P_1},\lambda_{P_2},...,\lambda_{P_n},...$ ketma – ketlik nolga intilsin: $\lambda_{P_n}\to 0$.

$\{P_m\}$ bo'linishlar ketma – ketligiga mos kelgan $f(x,y,z)$ funksiyaning integral yig'indilar ketma – ketligini tuzamiz:

$$\sigma_1, \sigma_2, \ldots, \sigma_n, \ldots$$

15.18.2. Ta'rif. Agar (S) sirtning har qanday (15.4.1) ko'rinishdagi bo'linishlar ketma – ketligi $\{P_m\}$ olingan ham unga mos kelgan integral yig'indilar ketma – ketligi $\{\sigma_m\}$ $M_i(x_i, y_i, z_i)$ nuqtalarni tanlash qonuniga bog'liq bo'lmagan holda, hamma vaqt bitta J limitga intilsa, u holda bu limit σ integral *yig'indining limiti* deb ataladi va u

$$\lim_{\lambda_P \to 0} \sigma = \lim_{\lambda_P \to 0} \sum_{i=1}^{n} f(x_i, y_i, z_i) s_i = J$$

kabi belgilanadi.

15.18.3. Ta'rif. Agar $\lambda_P \to 0$ da $f(x,y,z)$ funksiyaning integral yig'indisi σ chekli limitga ega bo'lsa, u holda $f(x,y,z)$ funksiya (S) sirt bo'yicha integrallanuvchi funksiya deb ataladi. Bu yig'indining chekli limiti J ga $f(x,y,z)$ funksiyaning *birinchi tur sirt integrali* deyiladi va u

$$\iint_{(s)} f(M)ds = \iint_{(s)} f(x,y,z)ds \qquad (15.18.4)$$

kabi belgilanadi.

15.19.Birinchi sirt integralining mavjudlik sharti va uni hisoblash. (S) sirt o'zining (15.15.2) ko'rinishdagi parametrik tenglamasi bilan brilgan bo'lib, (S)-sodda siliq sirt bo'lsin, ya'ni φ, φ, g funksiyalar (Δ) sohada $((u,v) \in (\Delta))$ uzluksiz va uzluksiz xususiy hosilalarga ega bo'lib maxsus nuqtalarga ega bo'lmasin.

15.19.1-teorema. Agar $f(x,y,z)$ funksiya tenglamasi (15.15.2) ko'rishda berilgan (S) silliq sirtda uzluksiz bo'lsa, u holda (15.18.4) integral mavjud bo'ladi va ushbu

$$\iint_{(s)} f(x,y,z)ds = (R)\iint_{(\Delta)} f(\varphi(u,v), \psi(u,v), g(u,v))\sqrt{EG - F^2}\,dudv \qquad (15.19.2)$$

tenglik o'rinli bo'ladi.

Agar (S) sirt tenglamasi oshkor $z = z(x,y)$ shaklda berilgan bo'lsa, u holda (15.19.2) furmulaning ko'rinishi

$$\iint_{(s)} f(x,y,z)\,ds = (R)\iint_{(D)} f(x,y,z(x,y))\sqrt{1+(z_x^{'})^2+(z_y^{'})^2}\,dxdy \qquad (15.19.3)$$

shaklida bo'ladi, bunda $(D)-(S)$ sirtning xOy tekislikdagi proyeksiya

$$\sqrt{1+(z_x^{'})^2+(z_y^{'})^2}=\frac{1}{|\cos\gamma|}.$$

($\gamma-(S)$ sirtga o'tkazilgan normalning Oz o'qi bilan tashkil qilgan burchagi) bo'lishini e'tiborga olib, (15.19.3) formulani

$$\iint_{(s)} f(x,y,z)ds=(R)\iint_{(D)} f(x,y,z(x,y))\frac{dxdy}{|\cos\gamma|}$$

ko'rinishda yozish mumkin. Shunga o'xshash, (S) siliq sirt $x=x(y,z)\in(D_1)$ tenglama bilan berilganda (bu yerda $(D_1)-(S)$sirtning (y,z) tekislikdagi proyeksiyasi), sirt differensiali $ds=\sqrt{1+x_y^{'2}+x_z^{'2}}\,dydz$ ko'rinishda yoziladi. Shuning uchun sirt integralini quyidagi ko'rinishda yozish mumkin:

$$\iint_{(s)} f(x,y,z)ds=(R)\iint_{(D_1)} f(x(y,z),y,z)\sqrt{1+x_y^{'2}+x_z^{'2}}\,dzdy \qquad (15.19.4)$$

yoki

$$\iint_{(s)} f(x,y,z)ds=(R)\iint_{(D_1)} f((x,y),y,z)\cdot\frac{dydz}{|\cos\alpha|}$$

(bu yerda $\alpha-(S)$ sirtga o'tkazilgan $\vec{n}$ normal vektorning Ox o'qi bilan tashkil burchagi, ya'ni $\alpha=\left(\widehat{Ox,\vec{n}}\right)$). (S) sirt tenglamasi $y=y(x,z)$ $((x,z)\in(D_2))$ tenglama bilan berilganda, sirt differensiali $ds=\sqrt{1+y_x^{'2}+y_z^{'2}}\,dxdz$ ko'rinishda yoziladi va

$$\iint_{(s)} f(x,y,z)ds=(R)\iint_{(D_2)} f(x,y(x,z),z)\sqrt{1+y_x^{'2}+y_z^{'2}}\,dxdz \qquad (15.19.5)$$

yoki

$$\iint_{(S)} f(x,y,z)ds = (R)\iint_{(D_2)} f(x,y(x,z),z)\frac{dxdz}{|\cos\beta|}$$

formula o'rinli bo'ladi (bu yerda $(D_2)-(S)$sirtining (x,z) tekislikdagi proyeksiyasi, $\beta = \left(\widehat{Oy,\vec{n}}\right)$).

15.19.6-eslatma. (15.19.2), (15.19.3), (15.19.4) va (15.19.5) formulalar bo'lakli sirtlar uchun ham o'rinli.

15.20. Birinchi tur sirt integralining fizikaga qo'llanishi. $\forall M(x,y,z)$ nuqtadagi sirt zichligi $\rho(x,y,z)$ bo'lgan material (S) sirt berilgan bo'lsin $(M \in (S))$. U holda, quyidagi formulalar o'rinli bo'ladi:

$a)$ $m = \iint_{(S)} \rho(x,y,z)ds$ sirtning massasi;

$b)$ $M_{xy} = \iint_{(S)} z\rho(x,y,z)ds,\ M_{xz} = \iint_{(S)} y\rho(x,y,z)ds,\ M_{yz} = \iint_{(S)} x\rho(x,y,z)ds$

xOy, xOz, yOz koordinatlar tekisliklariga nisbatan statik momentlar;

$v)$ $x_0 = \frac{M_{yz}}{m},\ y_0 = \frac{M_{xz}}{m},\ z_0 = \frac{M_{xy}}{m}$

(S)sirtning og'irlik markasining koordinatalari;

$g)$ $J_x = \iint_{(s)} (y^2+z^2)\rho(x,y,z)ds$

(S) sirtning Ox o'qiga nisbatan infersiya momenti ;

$d)$ $J_{yz} = \iint_{(s)} x^2\rho(x,y,z)ds$

(S)sirtning yOz tekisligiga nisbatan infersiya momenti;

$e)$ $J_0 = \iint_{(s)} (x^2+y^2+z^2)\rho(x,y,z)ds$

(S) sirtning koordinatlar boshiga nisbatan momenti;

$j)$ $\vec{F} = \{F_x, F_y, F_z\}$–massasi m_0 ga teng bo'lgan material $M_0(x_0,y_0,z_0)$ nuqtadagi material sirtning tortish kuchi, bunda

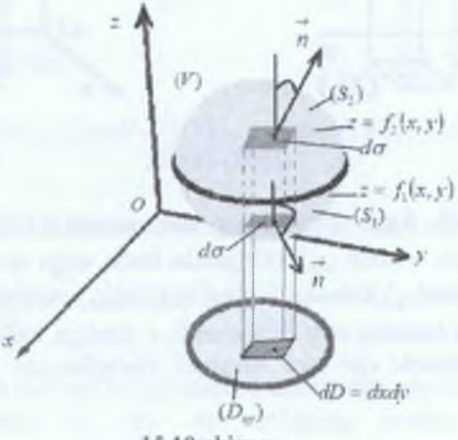
$$F_x = vm_0 \iint_{(s)} \frac{x-x_0}{r^3}\rho(x,y,z)ds, F_y = vm_0 \iint_{(s)} \frac{y-y_0}{r^3}\rho(x,y,z)ds,$$

$$F_z = vm_0 \iint_{(s)} \frac{z-z_0}{r^3}\rho(x,y,z)ds,$$

$\vec{r} = \{x-x_0, y-y_0, z-z_0\}$, $r = |\vec{r}|$, v – gravitatsion o'zgarmasi.

15.8-§. Ikkkinchi tur sirt integrali va uning fizikaga qo'llanishi

15.21. Ikkinchi tur sirt integral iva uning ta'rifi. Ikki tomonli siliq yoki bo'lakli silliq (S) sirt berilgan bo'lib, uning ikki tomonidan birini belgilaymiz. Faraz qilaylik (S) sirt oshkor tenglamasi $z = z(x, y)$ bilan berilgan bo'lsin. Bu sirtda $f(x,y,z)$ funksiya aniqlangan. Sirtning ixtiyoriy P bo'linishini va bo'lishning har bir (S_i) bo'lagidan $(i=1,2,...,n)$ ixtiyoriy $M_i(\xi_i,\eta_i,\zeta_i)$ nuqtani olib, bu nuqtadagi berilgan funksiyaning $f(\xi_i,\eta_i,\zeta_i)$ qiymatni (S_i) ning xOy tekislikdagi preksiyasi (D_i) ning yuzi D_i ga ko'paytirib ushbu (5.10-,15.11-chizmalar)



15.10-chizma.

$$\sigma = \sum_{i=1}^{n} f(\xi_i,\eta_i,\zeta_i)D_i$$

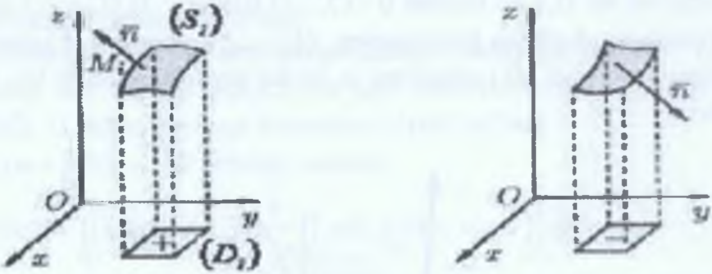
yig'indini tuzamiz. (S) sirtning shunday

$$P_1, P_2, ..., P_m, ... \qquad (15.21.\ 1)$$

bo'linishlar ketma – ketligini qaraylikki, ularning mos diametrlaridan tuzilgan $\lambda_{p1}, \lambda_{p_2}, ..., \lambda_{p_m}, ...$ ketma–ketlik nolga intilsin, ya'ni $\lambda_{p_m} \to 0$.(15.21.1) ko'rinishdagi bo'linishlarga mos kelgan $f(x,y,z)$ funksiyaning integral yig'indilar ketma- ketligi

$$\sigma_1, \sigma_2, ..., \sigma_m, ... \qquad (15.21.2)$$

ni tuzamiz.



15.11-chizma.

15.21.3-ta'rif. Agar (S) sirtning har qanday (15.21.1) ko'rinishdagi bo'linishlar ketma – ketli $\{p_m\}$ olinganda ham, unga mos kelgan (15.21.2) integral yig'indilar $\{\sigma_m\}$ ketma – ketligi $M_k(\xi_k, \eta_k, \zeta_k)$ nuqtani tanlashga bog'liq bo'lmagan holda hamma vaqt bita chekli J limitga intilsa, u holda bu limt $f(x,y,z)$ ning *ikkinchi tur sirt integrali* (belgilangan tomonni bo'yicha) deyiladi va u

$$J = \iint\limits_{(S)} f(x,y,z)dxdy$$

kabi belgilanadi. Bu holda $f(x,y,z)$ funksiya (S) sirtda integrallanuvchi deb ataladi.

Demak,

$$\lim_{\lambda_p \to 0} \sigma = \lim_{\lambda_p \to 0} \sum_{i=1}^{n} f(\xi_i, \eta_i, \zeta_i) D_i = \iint_{(S)} f(x,y,z)\, dxdy.$$

Ravshanki, $f(x,y,z)$ funksiyaning (S) sirtning bir tomoni bo'yicha olingan ikkinchi tur sirt integrali, sirtning ikkinchi tomoni bo'yicha olingan ikkinchi tur sirt integralidan faqat ishoraga farq qiladi. (S) sirtda aniqlangan $P(x,y,z)$ $[Q(x,y,z)]$ funksiyalarning sirtning belgilangan tomoni bo'yicha y va z (z va x) o'zgaruvchilar bo'yicha sirt integralining ta'rifi ham xuddi yuqoridagi singari beriladi:

$$\iint\limits_{(S)} P(x,y,z)dydz \quad (\iint\limits_{(S)} Q(x,y,z)dzdx).$$

Umumiy holda, (S) sirtda $P(x,y,z)$, $Q(x,y,z)$, $R(x,y,z)$ funksiyalar berilgan bo'lib,

$$\iint\limits_{(S)} P(x,y,z)dydz, \iint\limits_{(S)} Q(x,y,z)dzdx, \iint\limits_{(S)} R(x,y,z)dxdy$$

integrallar mavjud bo'lsa,

$$\iint\limits_{(S)} P(x,y,z)dydz + \iint\limits_{(S)} Q(x,y,z)dzdx + \iint\limits_{(S)} R(x,y,z)dxdy$$

yig'indi umumiy holda ikkinchi tur sirt integrali deyiladi va u

$$\iint\limits_{(S)} P(x,y,z)dydz + Q(x,y,z)dzdx + R(x,y,z)dxdy$$

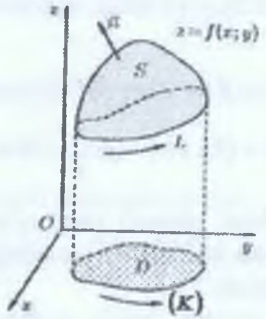
kabi belgilanadi.

15.22. Ikkinchi tur sirt integralining mavjudlik sharti va uni karrali integralga keltirish. 1. (S) sirt o'zining oshkor ko'rinishdagi $z = z(x,y)$, $(x,y) \in (D)$ tenglamasi bilan berilgan bo'lib, $z(x,y)$ funksiya chegaralangan yopiq (D) sohada uzluksiz va uzluksiz $z_x'(x,y)$, $z_y'(x,y)$ xususiy hosilalarga ega bo'lsin.

15.22.1-teorema. Agar $f(x,y,z)$ funksiya (S) sirtda berilgan va uzluksiz bo'lsa, u holda bu funksiyadan (S) sirt bo'yicha olingan ikkinchi tur sirt integrali mavjud bo'ladi va u ushbu

$$\iint_{(S)} f(x,y,z)\,dxdy = \iint_{(D_3)} f\big(x,y,z(x,y)\big)\,dxdy \qquad (15.22.2)$$

formula bilan hisoblanadi.



15.12-chizma.

$$\int\limits_{(L)} P(x,y,z)dx = \iint\limits_{(S)} \frac{\partial P}{\partial z} dxdz - \frac{\partial P}{\partial y} dxdy\,. \qquad (15.24.2)$$

Isboit. Shartga ko'ra $\int\limits_{(L)} P(x,y,z)dx$ – egri chiziqli integral mavjud. Avvalo egri chiziq bo'yicha olingan egri chiziqli integralni (L) ning xOy tekislikdagi proyeksiyasi (K) egri chiziq bo'yicha olingan egri chiziqli integralga aylantiramiz:

$$\int\limits_{(L)} Pdx = \int\limits_{(K)} P\cdot\left(\frac{\partial x}{\partial u}du + \frac{\partial x}{\partial v}\partial v\right). \qquad (15.24.\ 3)$$

(15.24.3) tenglikning to'g'riligini ko'rsatish uchun (K) egri chiziqning parametrik tenglamasi:

$$u = u(t), v = v(t)\ (\alpha \le t \le \beta)$$

dan foydalanamiz: (K) egri chiziq tenglamasiga ko'ra (L) egri chiziq tenglamasi $x = x(u(t),v(t)),\ \ y = y(u(t),v(t)),\ \ z = z(u(t),v(t))$ ko'rinishda bo'ladi. U holda ikala integral ham bir xil t bo'yicha olingan oddiy

$$\int\limits_{\alpha}^{\beta} P\cdot\left(\frac{\partial x}{\partial u}\cdot\frac{du}{dt} + \frac{\partial x}{\partial v}\cdot\frac{dv}{dt}\right)dt$$

integralga keltiriladi. Endi (15.24.3) integralning o'ng tomoniga Grin formulasini qo'llaymiz:

$$\int_{(\kappa)} P\cdot\left(\frac{\partial x}{\partial u}du+\frac{\partial x}{\partial v}dv\right)=\iint_{(\Delta)}\left\{\frac{\partial}{\partial u}\left(P\frac{\partial x}{\partial v}\right)-\frac{\partial}{\partial v}\left(P\frac{\partial x}{\partial u}\right)\right\}dudv. \qquad (15.24.4)$$

Keyingi tenglikning o'ng tomonidagi integral ostidagi {..........} ifodaning to'liq kengaytirilgan ko'rinishini yozamiz:

$$\{........\}=\left(\frac{\partial P}{\partial x}\frac{\partial x}{\partial u}+\frac{\partial P}{\partial y}\frac{\partial y}{\partial u}+\frac{\partial P}{\partial z}\frac{\partial z}{\partial u}\right)\cdot\frac{\partial x}{\partial v}+P\frac{\partial^2 x}{\partial u\partial v}-$$

$$-\left(\frac{\partial P}{\partial x}\frac{\partial x}{\partial v}+\frac{\partial P}{\partial y}\cdot\frac{\partial y}{\partial v}+\frac{\partial P}{\partial z}\frac{\partial z}{\partial v}\right)\cdot\frac{\partial x}{\partial u}-P\frac{\partial^2 x}{\partial u\partial v}=$$

$$=\frac{\partial P}{\partial z}\left(\frac{\partial z}{\partial u}\frac{\partial x}{\partial v}-\frac{\partial z}{\partial v}\frac{\partial x}{\partial u}\right)-\frac{\partial P}{\partial y}\left(\frac{\partial x}{\partial u}\frac{\partial u}{\partial v}-\frac{\partial x}{\partial v}\cdot\frac{\partial y}{\partial u}\right)$$

(15.15.5) formulani e'tiborga olgan holda

$$\{..........\}=\frac{\partial P}{\partial z}B-\frac{\partial P}{\partial y}C$$

ni hosil qilamiz. U holda (15.24.4) ning o'ng tomonidagi integral

$$\iint_{(\Delta)}\left\{\frac{\partial P}{\partial z}B-\frac{\partial P}{\partial y}C\right\}dudv$$

ko'rinishga keladi. Bu integralni (15.23.2) formula yordamida sirtning belgilangan tomoni bo'yicha sirt integraliga keltiramiz:

$$\iint_{(\Delta)}\left\{\frac{\partial P}{\partial z}B-\frac{\partial P}{\partial y}C\right\}dudv=\iint_{(S)}\frac{\partial P}{\partial z}dzdx-\frac{\partial P}{\partial y}dxdy$$

shu bilan teorema isbot bo'ladi.

Xuddi shunday (S) sirtda berilgan $Q(x,y,z)$, $R(x,y,z)$ funksiyalar tegishli shartlarni qanoatlantirganda ushbu

$$\int_{(L)} Q(x,y,z)dy = \iint_{(s)} \frac{\partial Q(x,y,z)}{\partial x}dxdy - \frac{\partial Q(x,y,z)}{\partial z}dydz, \qquad (15.24.5)$$

$$\int_{(L)} R(x,y,z)dz = \iint_{(s)} \frac{\partial R(x,y,z)}{\partial y}dydz - \frac{\partial R(x,y,z)}{dx}dzdx \qquad (15.24.6)$$

formulalarning o‘rinli ekanligini ko‘rsatish mumkin. (15.24.2), (15.24.5) va (15.24.5) formulalar hadlab qo‘shish natijasida

$$\int_{(L)} Pdx + Qdy + Rdz = \iint_{(S)} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dxdy +$$
$$+\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)dydz + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)dzdx \qquad (15.24.7)$$

formulani hosil qilamiz. Odatda (15.24.7) formula *Stoks formulasi* deb ataladi.

Shunday qilib, Stoks formulasi fazoviy yopiq (L) egri chiziq bo‘yicha olingan egri chiziqli integralni shu yopiq chiziqdan bilan chegaralangan (S) sirt bo‘yicha olingan sirt integralini bog‘lovchi formula bo‘lib hisoblanar ekan. (15.24.7) formulani birinchi tur sirt integrali orqali ham yozish mumkin:

$$\int_{(L)} Pdx + Qdy + Rdz =$$
$$= \iint_{(S)} \left[\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\cos\alpha + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\cos\beta + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\cos\gamma\right]ds \qquad (15.24.8)$$

(15.24.8) formulani ham *Stoks formulasi* deyiladi.

15.24.9-eslatma. Grik formulasi, Stoks formulasining xususiy holidir. Haqiqatan ham (15.24.7) formulada (S) sirt sifatida Oxy tekisligidagi (D) soha olinsa, unda $z = 0$ bo‘lib, (15.24.7) formuladan

$$\int_{(K)} Pdx + Qdy = \iint_{(D)} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dxdy$$

Grin formulasi kelib chiqadi.

15.25. Stoks formulasining fazodagi egri chiziqli integrallarni hisoblashga qo'llanishi. Ochiq (V) sohada P, Q, R uzluksiz funksiyalar berilgan bo'lib, ular uzluksiz $\frac{\partial Q}{\partial x}, \frac{\partial P}{\partial y}, \frac{\partial R}{\partial y}, \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z}, \frac{\partial R}{\partial x}$ xususiy hosilalarga ega bo'lsin. Stoks formulasi yordamida fazodagi egri chiziqli $\int\limits_{(L)} Pdx + Qdy + Rdz$ integralning nolga teng bo'lishi uchun yetarli va zaruriy shartni keltirib chiqarish mumkin (bunda (L), (V) sohada yotuvchi sodda chiziq):

Agar

$$\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}; \frac{\partial R}{\partial y} = \frac{\partial Q}{\partial z}; \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x} \qquad (15.25.1)$$

shartlar bajarilganda, Stoks formulasidan fazoviy yopiq (L) egri chiziq bo'yicha olingan egri chiziqli integralning nolga teng bo'lishi kelib chiqadi:

$$\int\limits_{(L)} Pdx + Qdy + Rdz = 0 \qquad (15.25.2)$$

Bu holda, egri chiziqli integral integrallash yo'liga bog'liq bo'lmasligi kelib chiqadi. Tekislikdagi singari (15.25.1) shart, (15.25.2) tenglikning bajarilishi uchun zaruriy va yetarli bo'lib hisoblanadi. (15.25.1) yoki (15.25.2) shartlarining bajarilishi integral ostidagi $Pdx + Qdy + Rdz$ ifodaning biror $u(x, y, z)$ funksiyaning to'liq differensiali: $du = Pdx + Qdy + Rdz$ bo'lishi uchun zarur va yetarli shart bo'lib hisoblanadi

15.26. Ostroradskiy formulasi. Yuqoridan $(S_2): z_2 = Z_2(x, y)$, pastdan $(S_1): z_1 = Z_1(x, y)$ - silliq sirtlar bilan, yon tomondan esa, yasovchilari Oz o'qiga parallel bo'lgan silindrik (S_t) sirt bilan chegaralangan (V) sohani qaraymiz. Uning xOy tekislikdagi proyeksiyasi (D) ning chegarasi bo'lakli – silliq (K) yopiq chiziq (nol yuzga ega) bo'lib, u silindrik sirtning yo'naltiruvchisi sifatida olinadi.

(15.14-chizma) $(z_1 \leq Z_2)(V)$ sohada biror $R(x, y, z)$ funksiya aniqlangan bo'lib u (V) sohada va uning chegarasida uzluksiz va uzluksiz $\frac{\partial R}{\partial z}$ xususiy hosilaga ega bo'lsin. Shu shartlarda

$$\iiint\limits_{(V)} \frac{\partial R}{\partial z} dxdydz = \iint\limits_{(S)} Rdxdy \qquad (15.26.1)$$

$$\int_{(L)} Q(x,y,z)dy = \iint_{(s)} \frac{\partial Q(x,y,z)}{\partial x}dxdy - \frac{\partial Q(x,y,z)}{\partial z}dydz, \qquad (15.24.5)$$

$$\int_{(L)} R(x,y,z)dz = \iint_{(s)} \frac{\partial R(x,y,z)}{\partial y}dydz - \frac{\partial R(x,y,z)}{dx}dzdx \qquad (15.24.6)$$

formulalarning o‘rinli ekanligini ko‘rsatish mumkin. (15.24.2), (15.24.5) va (15.24.5) formulalar hadlab qo‘shish natijasida

$$\int_{(L)} Pdx + Qdy + Rdz = \iint_{(S)}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dxdy +$$

$$+\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)dydz + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)dzdx \qquad (15.24.7)$$

formulani hosil qilamiz. Odatda (15.24.7) formula *Stoks formulasi* deb ataladi.

Shunday qilib, Stoks formulasi fazoviy yopiq (L) egri chiziq bo‘yicha olingan egri chiziqli integralni shu yopiq chiziqdan bilan chegaralangan (S) sirt bo‘yicha olingan sirt integralini bog‘lovchi formula bo‘lib hisoblanar ekan. (15.24.7) formulani birinchi tur sirt integrali orqali ham yozish mumkin:

$$\int_{(L)} Pdx + Qdy + Rdz =$$

$$= \iint_{(S)}\left[\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\cos\alpha + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\cos\beta + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\cos\gamma\right]ds \qquad (15.24.8)$$

(15.24.8) formulani ham *Stoks formulasi* deyiladi.

15.24.9-eslatma. Grik formulasi, Stoks formulasining xususiy holidir. Haqiqatan ham (15.24.7) formulada (S) sirt sifatida Oxy tekisligidagi (D) soha olinsa, unda $z = 0$ bo‘lib, (15.24.7) formuladan

$$\int_{(K)} Pdx + Qdy = \iint_{(D)}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dxdy$$

Grin formulasi kelib chiqadi.

15.25. Stoks formulasining fazodagi egri chiziqli integrallarni hisoblashga qo'llanishi. Ochiq (V) sohada P, Q, R uzluksiz funksiyalar berilgan bo'lib, ular uzluksiz $\frac{\partial Q}{\partial x}, \frac{\partial P}{\partial y}, \frac{\partial R}{\partial y}, \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z}, \frac{\partial R}{\partial x}$ xususiy hosilalarga ega bo'lsin. Stoks formulasi yordamida fazodagi egri chiziqli $\int\limits_{(L)} Pdx + Qdy + Rdz$ integralning nolga teng bo'lishi uchun yetarli va zaruriy shartni keltirib chiqarish mumkin (bunda (L), (V) sohada yotuvchi sodda chiziq):

Agar

$$\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}; \frac{\partial R}{\partial y} = \frac{\partial Q}{\partial z}; \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x} \tag{15.25.1}$$

shartlar bajarilganda, Stoks formulasidan fazoviy yopiq (L) egri chiziq bo'yicha olingan egri chiziqli integralning nolga teng bo'lishi kelib chiqadi:

$$\int\limits_{(L)} Pdx + Qdy + Rdz = 0 \tag{15.25.2}$$

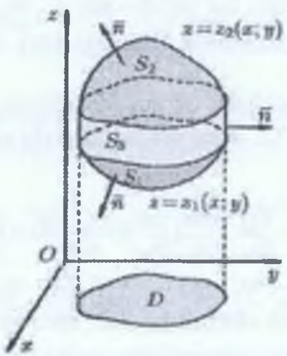
Bu holda, egri chiziqli integral integrallash yo'liga bog'liq bo'lmasligi kelib chiqadi. Tekislikdagi singari (15.25.1) shart, (15.25.2) tenglikning bajarilishi uchun zaruriy va yetarli bo'lib hisoblanadi. (15.25.1) yoki (15.25.2) shartlarining bajarilishi integral ostidagi $Pdx + Qdy + Rdz$ ifodaning biror $u(x, y, z)$ funksiyaning to'liq differensiali: $du = Pdx + Qdy + Rdz$ bo'lishi uchun zarur va yetarli shart bo'lib hisoblanadi

15.26. Ostroradskiy formulasi. Yuqoridan $(S_2): z_2 = Z_2(x, y)$, pastdan $(S_1): z_1 = Z_1(x, y)$ - silliq sirtlar bilan, yon tomondan esa, yasovchilari Oz o'qiga parallel bo'lgan silindrik (S_3) sirt bilan chegaralangan (V) sohani qaraymiz. Uning xOy tekislikdagi proyeksiyasi (D) ning chegarasi bo'lakli – silliq (K) yopiq chiziq (nol yuzga ega) bo'lib, u silindrik sirtning yo'naltiruvchisi sifatida olinadi.

(15.14-chizma) $(z_1 \le Z_2)(V)$ sohada biror $R(x, y, z)$ funksiya aniqlangan bo'lib u (V) sohada va uning chegarasida uzluksiz va uzluksiz $\frac{\partial R}{\partial z}$ xususiy hosilaga ega bo'lsin. Shu shartlarda

$$\iiint\limits_{(V)} \frac{\partial R}{\partial z} dxdydz = \iint\limits_{(S)} Rdxdy \tag{15.26.1}$$

formula o‘rinli bo‘ladi, bunda $(S)-(V)$ sohani chegarasi, o‘ng tomonidagi integral (S) integralning tashqi tomoni bo‘yicha olingan.



15.14-chizma.

Isboti. Bu holda $\iiint\limits_{(V)} \frac{\partial R}{\partial z} dxdydz$ - integral mavjud bo‘ladi va u (14.18.2) formulaga ko‘ra quyidagicha hisoblanadi:

$$\iiint_{(V)} \frac{\partial R}{\partial z} dxdydz = \iint_{(D)} dxdy \int_{z_1(x,y)}^{z_2(x,y)} \frac{\partial R}{\partial z} = \iint_{(D)} R(x,y,Z_2(x,y))\,dxdy -$$

$$-\iint_{(D)} R(x,y,z_1(x,y))\,dxdy.$$

Tenglikning o‘ng tomonidagi ikki karali integrallarni (15.22.2) formuladan foydalanib ikkinchi tur sirt integraliga keltiramiz: (15.26.1) ning o‘ng tomonidagi integral (S) ning tashqi tomoni bo‘yicha olinganligi e’tiborga olib,

$$\iiint_{(V)} \frac{\partial R}{\partial z} dxdydz = \iint_{(S_2)} R(x,y,z)\,dxdy - \iint_{(S_1)} R(x,y,z)\,dxdy =$$

$$= \iint_{(S_2)} R(x,y,z)\,dxdy + \iint_{(S_1)} R(x,y,z)\,dxdy. \qquad (15.26.2)$$

(S_3) sirt yasovchilari Oz o‘qiga parallel bo‘lgan silindrik, sirt bo‘lganligi uchun

$$\iint_{(S_1)} R(x,y,z)dxdy = 0 \tag{15.26.3}$$

bo'ladi. (15.26.2) ning o'ng tomoniga (15.26.3) ni qo'shib

$$\iiint_{(V)} \frac{\partial R}{\partial z} dxdydz = \iint_{(S)} R(x,y,z)dxdy$$

ni hosil qilamiz, bunda $(S)-(V)$ jismni o'rab turuvchi sirt: $(S)=(S_1)+(S_2)+(S_3)$

Xuddi shunday yo'l bilan (V) sohada aniqlangan $P(x,y,z), Q(x,y,z)$ funksiyalar $R(x,y,z)$ funksiya singari shartlarni qanoatlantirganda quyidagi

$$\iiint_{(V)} \frac{\partial P}{\partial x} dxdydz = \iint_{(S)} R\,dydz \tag{15.26.4}$$

$$\iiint_{(V)} \frac{\partial Q}{\partial y} dxdydz = \iint_{(S)} Qdzdx \tag{15.26.5}$$

formulaning o'rinli ekanligini ko'rsatish mumin. (15.26.1), (15.26.4) (15.26.5) formulalarni hadlab qo'shish natijasida Ostrogradskiy formulasi deb ataluvchi ushbu

$$\iiint_{(V)} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \right) dxdydz = \iint_{(S)} Pdydz + Qdzdx + Rdxdy \tag{15.26.6}$$

formulani hosil qilamiz. (15.26.6) ning o'ng tomonidagi ikkinchi tur sirt integralini birinchi tur sirt integraliga aylantirish natijasida uning ko'rinishini

$$\iiint_{(V)} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \right) dxdydz = \iint_{(S)} (P\cos\alpha + Q\cos\beta + R\cos\gamma)ds \tag{15.26.7}$$

shaklida ham yozish mumkin. (15.26.7) formulaga ham Ostrogradskiy formulasi deyiladi, bunda $\cos\alpha, \cos\beta, \cos\gamma$ lar (S) sirtga o'tkazilgan tashqi normalning mos ravishda koordinatalar o'qlari bilan tashkil qilgan burchaklari. Ostrogradskiy formulasi (V) fazoviy soha bo'yicha olingan uchkarrali integralni (V) sohaning chegarasi (S) sirt bo'yicha olingan sirt integrali bilan bog'laydigan formuladir.

15.27. Ostrogradskiy formulasining qo'llanishi. (15.26.6) formulada P, Q, R funksiyalarni, shunday tanlash kerakki, natijada uch karali integral ostidagi ifoda birga teng bo'lsin, ya'ni (V) jismning hajmi V, (V) jismni o'rab turuvchi (S) sirt bo'yicha olingan sirt integrali bilan ifodalansin:

$$V = \iint_{(S)} Pdydz + Qdzdx + Rdxdy,$$

bunda

1) $P = x, Q = 0, R = 0$; 2) $P = 0, Q = y, R = 0$; 3) $P = 0, Q = 0, R = z$

deb tanlasak, natijada

$$V = \iint_{(S)} x\,dydz = \iint_{(S)} ydzdx = \iint_{(S)} zdxdy.$$

4) $P = \frac{1}{3}x,\ Q\frac{1}{3}y,\ R = \frac{1}{3}z$

deb tanlasak

$$V = \frac{1}{3}\iint_{(S)} xdydz + ydzdx + zdxdy$$

yoki

$$V = \frac{1}{3}\iint_{(S)} (x\cos\alpha + y\cos\beta + z\cos\gamma)ds$$

(bunda $\cos\alpha, \cos\beta, \cos\gamma$ lar (S) sirtga o'tkazilgan tashqi $\vec{n}$ normalning yo'naltiruvchi kosinuslari)

15-bob bo'yicha nazariy materiallarni mustahkamlash uchun topshiriqlar

15.1. Birinchi tur egri chiziqli integralining ta'rifi ([3], 2-q., 323-324 betlar; [12], 2-q., 335-337 betlar; [5], 3-t., 11-13 betlar; [17], 150-152 betlar; [9], 2-t., 9- bo'lim; [30], 16- bo'lim).

15.2. Egri chiziqli integralni oddiy integralga keltirish ([3], 2-q., 324-327 betlar; [12], 2-q., 340-343 betlar, [5], 3-t., 13-15 betlar, [17], 153-154 betlar; [9], 2-t., 9- bo'lim; [30], 16- bo'lim).

15.3. Birinchi tur egri chiziqli integrallarning xossalari ([12], 2-q., 339-340 betlar, [5], 3-t., 12-13 betlar, [17], 154-155 betlar; [9], 2-t., 9- bo'lim; [30], 16- bo'lim).

15.4. Birinchi tur egri chiziqli integralning ba'zi tadbiqlari ([3], 2-q.,328-331 betlar; [12], 2-q., 343-344 betlar, [5], 3-t., 15-19 betlar, [17], 155-157 betlar; [9], 2-t., 9- bo'lim; [30], 16- bo'lim).

a) Egri chiziqning statik momenti va og'irlik markazini topish.

b) Tekis figuraning statik momenti va og'irlik markazini topish.

15.5. Ikkinchi tur egri chiziqli integralning ta'rifi ([3], 2-q., 331-335 betlar;

[12], 2-q., 344-349 betlar, [5], 3-t., 20-22 betlar, [17], 158-160 betlar; [9], 2-t., 9- bo'lim; [30], 16- bo'lim).

15.6. Ikkinchi tur egri chiziqli integralning mavjudligi haqidagi teorema ([3], 2-q., 335-337 betlar; [10], 2-q., 349-350 betlar, [5], 3-t., 22-25 betlar, [17], 152 bet; [9], 2-t., 9- bo'lim; [30], 16- bo'lim).

15.7. Egri chiziq tenglamasi oshkor shaklda berilganda ikkinchi tur egri chiziqli integralni hisoblash ([3], 2-q.,338-339 betlar; [10], 2-q., 352-353 betlar, [17], 162-163 betlar; [30], 16- bo'lim).

15.8. Egri chiziq tenglamasi parametrik shaklda berilganda ikkinchi tur egri chiziqli integralni hisoblash ([3], 2-q., 335-337 betlar; [10], 2-q., 352 bet, [17], 162 bet; [30], 16- bo'lim).

15.9. Birinchi tur va ikkinchi tur egri chiziqli integrallar orasidagi bog'lanish ([12], 2-q., 363 bet, [5], 3-t., 38-40 betlar, [17], 160-162 betlar; [30], 16- bo'lim).

15.10. Grin formulasi ([3], 2-q., 343-347 betlar; [12], 2-q., 354-356 betlar, [5], 3-t., 174-179 betlar, [17], 168-173 betlar; [9], 2-t., 9- bo'lim; [30], 16- bo'lim).

15.11. Egri chiziqli integralning integrallash yo'liga bog'liq bo'lmaslik shartlari ([3], 2-q., 347-348 betlar; [12], 2-q., 359-362 betlar, [5], 3-t.,46-49 betlar, [17], 174-177 betlar; [9], 2-t., 9- bo'lim; [30], 16- bo'lim).

15.12. $P(x,y)dx + Q(x,y)dy$ ifoda biror $F(x,y)$ funksiyaning to'liq differensiali bo'lish sharti ([12], 2-q., 359-362 betlar, [5], 3-t., 45-46 betlar, [17], 178-179 betlar; [9], 2-t., 9- bo'lim; [30], 16- bo'lim).

15.13. Egri chiziqli integral orqali tekis figuraning yuzini hisoblash ([10], 2-q., 356-357 betlar, [5], 3-t., 32-35 betlar, [17], 173-174 betlar; [9], 2-t., 9- bo'lim; [30], 16- bo'lim).

15.14. Sirtning berilish usullari ([10], 2-q., 175-176 betlar; [17], 117-121 betlar; [30], 16- bo'lim).

15.15. Silliq sirt tushunchasi ([10], 2-q., 175-177 betlar; [5], 3-t., 247-248 betlar; [30], 16- bo'lim).

15.16. Sirtning yuzi tushunchasi ([10], 2-q., 181-182 betlar; [5], 3-t., 251-257 betlar; [17], 129-135 betlar; [28], 563-566 betlar; [9], 2-t., 9- bo'lim; [30], 16- bo'lim).

15.17. Birinchi tur sirt integralining ta'rifi, [12], 2-q., 364-365 betlar; ([10], 2-q., 185-186 betlar; [5], 3-t., 274 bet; [28], 567-569 betlar; [30], 16- bo'lim).

15.18. Birinchi tur sirt integralining mavjudlik sharti ([12], 2-q., 365-369 betlar; [10], 2-q., 187-188 betlar; [30], 16- bo'lim).

15.19. Birinchi tur sirt integralining xossalari([12], 2-q., 369 bet; [30], 16- bo'lim).

15.20. Birinchi tur sirt integralini hisoblash ([12], 2-q., 369-370 betlar; [5], 3-t., 275-277 betlar; [17], 184-188 betlar; [30], 16- bo'lim).

15.21. Birinchi tur sirt integralining tadbiqlari: 1) sirtning yuzi; 2) sirtning massasi; 3) sirtning og'irlik markazini topish; 4) sirtning Ox, Oy, Oz koordinatalar o'qlariga nisbatan inersiya momentlari([5], 3-t., 277-285 betlar; [17], 188-189 betlar; [30], 16- bo'lim).

15.22. Bir tomonli va ikki tomonli sirtlar tushunchalari([5], 3-t., 241-246 betlar; [17], 191-195 betlar; [28], 559-562 betlar; [30], 16- bo'lim).

15.23. Ikkinchi tur sirt integralining ta'rifi ([12], 2-q., 372-374 betlar, [10], 2-q., 185-186 betlar;[5], 3-t., 285-286 betlar; [28], 569-572 betlar; [30], 16- bo'lim).

15.24. Ikkinchi tur sirt integralining mavjudlik sharti ([12], 2-q., 374-376 betlar; [17], 199-201 betlar; [30], 16- bo'lim).

15.25. Ikkinchi tur sirt integralining xossalari([12], 2-q., 376 bet; [30], 16- bo'lim).

15.26. Birinchi va ikkinchi tur sirt integrallari orasidagi bog'lanish ([12], 2-q., 377 bet; [10], 2-q., 188-189 betlar; [17], 200-201 betlar; [30], 16-bo'lim).

15.27. Ikkinchi tur sirt integralini hisoblash ([12], 2-q., 376-377 betlar; [17], 199-201 betlar; [30], 16- bo'lim).

15.28. Stoks formulasi ([12], 2-q., 378-380 betlar; [17], 207-210 betlar; [9], 2-t., 9- bo'lim; [30], 16- bo'lim).

15.29. Ostrogradskiy formulasi ([12], 2-q., 380-382 betlar; [17], 201-206 betlar; [9], 2-t., 9- bo'lim; [30], 16- bo'lim).

15.1-amaliy mashg'ulot. Birinchi va ikkinchi tur egri chiziqli integrallar

1-misol. $\int\limits_{(K)}(3x^2+y)dx+(x-2y^2)dy$, bunda (K): uchlar $O(0;0)$, $A(2;0)$ va $B(0;2)$ nuqtalarda uchburchakning chegarasi - $(OABO)$.

Yechilishi. Berilgan ikkinchi tur egri chiziqli integralning integrallash konturi 15.15- chizmada tasvirlangan. $(K)=(OABO)$ chiziqning yo'nalishi 15.15- chizmada ko'rsatilgan. Berilgan egri chiziqli integralni hisoblash uchun uchburchakning harbir tomoni bo'yicha (ko'rsatilgan yo'nalishda) integralni hisoblab, so'ngra egri chiziqli integralning additivlik xossasiga asosan, uchala tomon bo'yicha hisoblangan integrallarning qiymatlarini qo'shamiz.

1) uchburchak OA tomonining tenglamasi $y=0, dy=0$. Unda $\int\limits_{(OA)}3x^2dx=3\int\limits_0^2 x^2dx=8$. 2) uchburchak AB tomonining tenglamasi : $x+y=2$, bunda $y=2-x$, va

$$\int\limits_{(AB)}(3x^2+y)dx+(x-2y^2)dy=\int\limits_2^0\left[3x^2+(2-x)-x+2(2-x)^2\right]dx=$$

$$=\int\limits_2^0(5x^2-10x+10)dx=(\frac{5}{3}x^3-5x^2+10x)\Big|_2^0=-\frac{40}{3}.$$

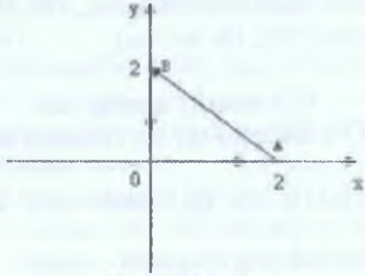
3) uchburchak BO tomonining tenglamasi : $x=0$, $dx=0$, va

$$\int\limits_{(BO)}-2y^2dy=-\int\limits_2^0 2y^2dy=\frac{16}{3}.$$

Shunday qilib, 1), 2) va 3) lardan uchala tomon bo'yicha hisoblangan integrallarning qiymatini qo'shsak:

$$\int\limits_{(OABC)} \left(3x^2 + y\right)dx + \left(x - 2y^2\right)dy = 8 - \frac{40}{3} + \frac{16}{3} = 0.$$

Bu natijani integralni hisoblamasdan ham olish mumkin, chunki integral ostidagi $Pdx + Qdy = \left(3x^2 + y\right)dx + \left(x - 2y^2\right)dy$ ifoda $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ $\left(\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} = 1\right)$ shartni qanoatlantiradi.



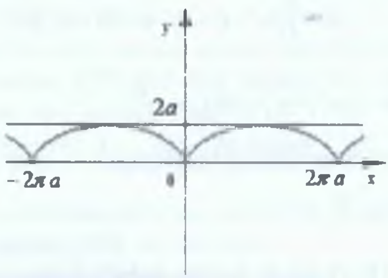
15. 15-chizma.

2-misol. Ushbu $J = \int\limits_{(K)} yds$ birinchi tur egri chiziqli integral hisoblansin, bunda (K): $x = a(t - \sin t)$, $y = a(1 - \cos t)(0 \leq t \leq 2\pi)$ sikloidaning birinchi arkasi.

Yechilishi. Misolning berilishiga ko'ra, $f(x, y) = y$.

$$ds = \sqrt{x_t'^2 + y_t'^2}\,dt = \sqrt{a^2(1 - \cos t)^2 + a^2 \sin^2 t}\,dt =$$
$$= a\sqrt{2(1 - \cos t)}\,dt = \sqrt{2}a\sqrt{1 - \cos t}\,dt = 2a \sin\frac{t}{2}\,dt$$

(15.4.1) formulaga asosan (15.16 - chizma),



15.16-chizma.

$$J=\int_0^{2\pi} a(1-\cos t)2a\sin\frac{t}{2}dt=a^2\int_0^{2\pi}\sin^3\frac{t}{2}dt=8a^2\int_0^{\pi}\sin^3 x dx=$$

$$=8a^2\left(-\cos x+\frac{\cos^3 x}{3}\right)\Big|_0^{\pi}=16a^2\left(1-\frac{1}{3}\right)=\frac{32}{3}a^2.$$

3-misol. Ushbu $\int\limits_{(AB)} y ds$ egri chiziqli integralni hisoblang, bunda $(AB)-y^2=2x$ parabola yoyining $(0;0)$ va $(2;2)$ nuqtalarni birlashtiruvchi qismi.

Yechilishi. Berilgan (AB) egri chiziqning tenglamasi $y=\sqrt{2x},\ y'=\frac{1}{\sqrt{2x}}$.

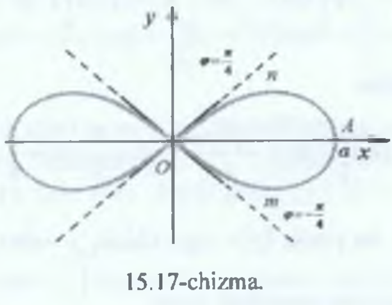
(15.4.2) formulaga asosan, $ds=\sqrt{1+y_x'^2}dx=\sqrt{1+\frac{1}{2x}}dx$,

$$\int\limits_{(AB)} y ds=\int_0^2\sqrt{2x}\cdot\sqrt{1+\frac{1}{2x}}dx=\int_0^2\sqrt{2x+1}dx=\frac{1}{3}[2x+1]^{3/2}\Big|_0^2=\frac{1}{3}\left(5\cdot\sqrt{5}-1\right).$$

4-misol. Ushbu $\int\limits_{(K)}(x+y)ds$ egri chiziqli integralni hisoblang, bunda $(K)-r^2=a^2\cos^2\varphi$ Bernulli lemniskatasining o'ng yaprog'i.

Yechilishi. Lemniskataning o'ng yaprog'i $\varphi=-\frac{\pi}{4}$ va $\varphi=\frac{\pi}{4}$ nurlar orasida joylashgan bo'ladi (15.17-chizma). $(OmAnO)$ egri chiziqda

$$r=a\sqrt{\cos 2\varphi},\quad r'_\varphi=-\frac{a\sin 2\varphi}{\sqrt{\cos 2\varphi}},\quad ds=\sqrt{r^2+r_\varphi'^2}d\varphi=\frac{a}{\sqrt{\cos 2\varphi}}d\varphi.$$



15.17-chizma.

Bernulli lemniskatasining parametrik tenglamasi:
$x = a\sqrt{\cos 2\varphi}\cdot\cos\varphi,\ y = a\sqrt{\cos 2\varphi}\cdot\sin\varphi.$

(15.4.3) formulaga asosan,

$$\int_{(K)}(x+y)ds = a^2\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\left(\sqrt{\cos 2\varphi}\cdot\cos\varphi + \sqrt{\cos 2\varphi}\cdot\sin\varphi\right)\cdot\frac{d\varphi}{\sqrt{\cos 2\varphi}} =$$

$$= a^2\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}(\cos\varphi+\sin\varphi)d\varphi = 2\cdot a^2\int_0^{\frac{\pi}{4}}\cos\varphi d\varphi = 2a^2\sin\varphi\Big|_0^{\frac{\pi}{4}} = 2a^2\cdot\frac{\sqrt{2}}{2} = \sqrt{2}a^2.$$

5-misol. Ushbu $x(t) = a\cos^3 t,\ y(t) = a\sin^3 t,\ (0 \le t \le 2\pi)$ sistema bilan berilgan (AB) egri chiziq (astroida) yoy ning uzunligi topilsin.

Yechilishi. Astroida koordinata o‘qlariga nisbatan simmetrik bo‘lgani uchun (15.6.1) formula bo‘yicha topamiz:

$$S = \int_{(AB)} ds = 4\int_0^{\pi/2}\sqrt{x'^2(t)+y'^2(t)}dt = \int_0^{\pi/2}\sqrt{\left(-3a\cos^2 t\sin t\right)^2 + \left(3a\sin^2 t\ \cos t\right)^2}dt =$$

$$= 4\int_0^{\pi/2}\sqrt{\frac{9a^2}{4}\sin^2 2t}dt = 6a\int_0^{\pi/2}\sin 2t dt = 6a.$$

6-misol. $y^2 = 2px$ parabola, Ox o‘qi bilan hamda x ga mos keluvchi ordinata bilan chegaralangan figuraning statik momentlari K_x, K_y va og‘irlik markazining koordinatalari topilsin.

Yechilishi. $y = \sqrt{2px}$ bo‘lgani uchun

$$K_x = \frac{1}{2}2p\int_0^x x dx = \frac{1}{2}px^2,\ K_y = \sqrt{2p}\int_0^x x^{\frac{3}{2}}dx = \frac{2\sqrt{2p}}{5}x^{\frac{5}{2}}.$$

Ikkinchi tomondan

$$P = \sqrt{2p}\int_0^x x^{\frac{1}{2}}dx = \frac{2\sqrt{2p}}{5}x^{\frac{3}{2}},\ \xi = \frac{3}{5}x,\ \eta = \frac{3}{8}\sqrt{2px} = \frac{3}{8}y.$$

7-misol. $\int_{(\gamma)} xydx$, bu yerda (γ)– egri chiziq $y = \sin x$ sinusoida chiziqning $(0{,}0)$ hamda $(\pi;0)$ nuqtalar orasidagi qismi.

Yechilishi. Ushbu $\int\limits_{(\gamma)} P(x,y)dx = \int\limits_a^b P(x, f(x))dx$ formulaga asosan,

$$\int\limits_{(\gamma)} xydx = \int\limits_0^\pi x\sin xdx = -x\cos x\Big|_0^\pi + \int\limits_0^\pi \cos xdx = \pi.$$

8-misol. Ushbu $\int\limits_{(AB)} x^2dx + xydy$ integralni hisoblang, bunda (AB): $x=\cos t$, $y=\sin t$ $\left(0\le t\le \frac{\pi}{2}\right)$ aylananing to‘rtdan biri, $t=0$ ga A nuqta, $t=\frac{\pi}{2}$ ga B nuqta mos keladi.

Yechilishi. (15.10.3) formulaga asosan,

$$\int\limits_{(AB)} x^2dx + xydy = \int\limits_0^{\frac{\pi}{2}} \left(-\cos^2 t\cdot\sin t + \cos^2 t\sin t\right)dt = 0.$$

9-misol. Grin formulasi yordamida ushbu

$$\int\limits_{(k)} (x-y)dx + (x+y)dy$$

egri chiziqli integralni hisoblang, bunda (k) – aylana: $x^2+y^2=R^2$.

Yechilishi. Bu holda, $P(x,y)=x-y,\ Q(x,y)=x+y,\ \frac{\partial P}{\partial y}=-1,\ \frac{\partial Q}{\partial x}=1$ funksiyalar $x^2+y^2\le R^2$ yopiq doirada uzluksiz bo‘lgani uchun berilgan integralga Grin formulasini qo‘llash mumkin:

$$\int\limits_{(k)} (x-y)dx + (x+y)dy = \iint\limits_{(D)} [1-(-1)]dxdy = 2\iint\limits_{(D)} dxdy = 2\cdot\pi R^2.$$

10-misol. Yarim o‘qlari a va b bo‘lgan ellipsning yuzi topilsin.

Yechilishi. Ellipsning parametrik tenglamasi $x=a\cos t, y=b\sin t(0\le t\le 2\pi)$ dan foydalanib (15.13.1) formula bo‘yicha topamiz:

$$D=\frac{1}{2}\int\limits_{(k)} xdy - ydx = \frac{1}{2}\int\limits_0^{2\pi} (a\cos t\cdot b\cos t + b\sin t\cdot a\sin t)dt = \frac{1}{2}\int\limits_0^{2\pi} abdt = \pi ab.$$

Mustaqil yechish uchun misollar

1. $\int\limits_{\gamma}(x+y)de$, bunda γ - uchlari $D(0,0)$, $A(1,0)$, $B(0,1)$ nuqtalarda bo'lgan uchburchak konturi.

2. $\int\limits_{\gamma}xyde$, bu yerda γ - uchlari $A(-2;2)$, $B(6.1)$, $C(2;5)$ nuqtalarda bo'lgan uchburchak konturi.

3. $\int\limits_{\gamma}\frac{de}{\sqrt{x^2+y^2+1}}$ bu yerda γ - tekislikning $0\,(0,0)$ va $A(1,1)$ nuqtalarni birlashtiruvchi to'g'ri chiziq kesmasi.

Quyidagi egri chiziqli integrallarni ko'rsatilgan egri chiziq bo'ylab hisoblang

4. $\int\limits_{\gamma}yde$, bunda $\gamma=\{(x,y): x=a\cos t,\ y=a\sin t\}$.

5. $\int\limits_{\gamma}xy\,de$, bunda $\gamma: x=t-\sin t,\ y=1-\cos t,\ 0\le t\le 2\pi$

6. $\int\limits_{\gamma}(x+y)dl$, bunda γ-chiziq $r^2=a^2\cos 2\varphi$ leminskataning o'ng yaprog'i.

7. $\int\limits_{\gamma}(2x-3y)dl$, bunda γ-chiziq leminskataning o'ng yoprog'i: $r=a\sqrt{\cos 2\varphi}$.

Quyidagi ikkinchi tur egri chiziqli integrallarni hisoblang.

8. $\int\limits_{\gamma}y\,dx$, bu yerda $\gamma - y=x^2(0\le x\le 1)$ parabola .

9. $\int\limits_{\gamma}xydx$, bu yerda $\gamma-$ egri chiziq $y=\sin x$ sinusoida chiziqning $(0,0)$ hamda $\left(\frac{\pi}{2};0\right)$ nuqtalar orasidagi qismi.

10. $\int\limits_{\gamma}x\,dy$, bu yerda $\gamma-$ egri chiziq $\frac{x}{3}+\frac{y}{4}=1$ to'g'ri chiziqning $(3,0)$ va $(0,4)$ nuqtalari orasidagi qismi.

11. $\int\limits_{\gamma}ydx-(y+x^2)dy$, bu yerda $\gamma-$egri chiziq $y=2x-x^2$ parabola yoyining $A(2,0)(\cdot)$ dan va $B(0,0)$ nuqtagacha bo'lgan qismi.

Quyidagi ikkinchi tur egri chiziqli integralni hisoblang.

12. $\int\limits_{\gamma}(2-y)dx+x\,dy$, bunda $\gamma=\{(x,y): x=t-\sin t,\quad y=1-\cos t,\quad 0\le t\le 2\pi\}$.

13. $\int\limits_{\gamma}y^2dx+x^2dy$, bu yerda γ egri chiziq $A(-a,0)$ dan $B(a,0)$ gacha bo'lgan yarim ellips yoyidir: $x=a\cos t,\quad y=b\sin t$.

14. $\int\limits_{\gamma}\frac{x^2dy-y^2dx}{x^{2/3}+y^{2/3}}$, bu yerda γ-astroidaning $A(R,0)$ nuqtasidan $B(0,R)$ nuqtasigacha bo'lgan yoyi: $x=R\cos^3 t$, $y=R\sin^3 t$.

Integral ostidagi ifoda to'liq differensial ekanligini tekshirib, berilgan egri chiziqli integralni hisoblang.

15. $\int\limits_{(-1,3)}^{(2,3)} x\,dy+y\,dx$. **15.** $\int\limits_{(0;1)}^{(3;-4)} x\,dx+y\,dy$.

17. $\int\limits_{(-1;-2)}^{(1;0)}(2x-y)dx+(3y-x)dy$. **18.** $\int\limits_{(0,1)}^{(1,0)}(3x^2-2xy+y^2)dx-(x^2-2xy)dy$.

Mustaqil yechish uchun misollarning javoblari

1. $1+\sqrt{2}$. **2.** $\frac{31}{16}$. **3.** 5. **4.** $2a^2$. **5.** 0. **6.** $a^2\sqrt{2}$. **7.** $2\sqrt{2}a^2$. **25.8.** $\frac{1}{3}$. **9.** 1. **10.** 6. **11.** -4. **12.** -2π. **13.** $\frac{4}{3}ab^2$. **14.** $\frac{32}{105}R^{7/3}$. **15.** 9. **15.** 12. **17.** -4. **18.** 1.

15.2-amaliy mashg'ulot.
Birincni va ikkinchi tur sirt integrallar

1-misol. $\iint\limits_{(S)}\sqrt{x^2+y^2}ds$, bunda $(S): x^2+y^2=z^2$ konus sirtning $z=0$ va $z=3$ tekisliklar orasidagi qismi.

Yechilishi. Berilgan sirt tenglamasidan $z=\sqrt{x^2+y^2}$ ekanligini olamiz. Bu sirt qaralayotgan qismining Oxy tekislikdagi proyeksiyasi $(D): x^2+y^2\le 9$-doiradan iborat. Berilgan 1- tur sirt integrali (15.19.3) formula bilan hisoblanadi: $z'_x=\frac{x}{\sqrt{x^2+y^2}}$, $z'_y=\frac{y}{\sqrt{x^2+y^2}}$ larni e'tiborga olgan holda ,

$$\iint\limits_{(S)}\sqrt{x^2+y^2}ds=\iint\limits_{(D)}\sqrt{x^2+y^2}\sqrt{1+\frac{x^2+y^2}{x^2+y^2}}dx\,dy=\sqrt{2}\iint\limits_{(D)}\sqrt{x^2+y^2}dx\,dy.$$

bo'lishini olamiz. Keyingi ikki karrali integralda $x=\rho\cos\varphi$, $y=\rho\sin\varphi$ almashtirishni bajarib, quyidagiga ega bo'lamiz:

$$\sqrt{2}\iint\limits_{(D)}\sqrt{x^2+y^2}dx\,dy=\sqrt{2}\int\limits_0^{2\pi}d\varphi\int\limits_0^3\rho^2d\rho=\sqrt{2}\cdot 2\pi\cdot\frac{27}{3}=18\sqrt{2}\cdot\pi.$$

Demak, $\iint\limits_{(S)} \sqrt{x^2+y^2}\,ds = 18 \cdot \sqrt{2} \cdot \pi$.

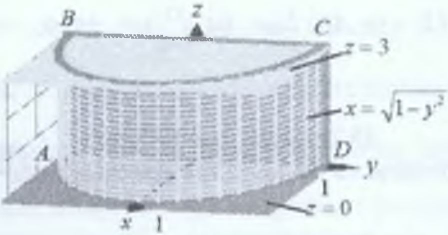
2-misol. Ushbu

$$J = \iint\limits_{(S)} x(y+1)ds$$

birinchi tur sirt integralini hisoblang, bunda (S) - $x=\sqrt{1-y^2}$ silindrik sirtning $z=0$ va $z=3$ tekisliklar orasidagi qismi.

Yechilishi. (S) sirtning yOztekislikdagi proyeksiyasi (D) soha- $ABCD$ to'g'ri to'rtburchakdan iborat, ya'ni

$$(D)=\{(y,z):-1\le y\le 1,\ 0\le z\le 3\}$$



15.18-chizma.

(15.18-chizma). Bu (D) sohada $x=\sqrt{1-y^2}$ funksiya uzluksiz va $x_y'=-\dfrac{y}{\sqrt{1-y^2}}$, $x_z'=0$ uzluksiz xususiy hosilalarga ega. (S) sirtda berilgan $f(x,y,z)=x(y+1)$ funksiya shu sirtda uzluksiz. Berilgan birinchi tur sirt integralini (15.19.5) formuladan foydalanib, hisoblaymiz:

$$J=\iint\limits_{(D)}\sqrt{1-y^2}(y+1)\sqrt{1+\frac{y^2}{1-y^2}}dydz=\iint\limits_{(D)}(y+1)dydz=$$

$$=\int_{-1}^{1}(y+1)dy\int_{0}^{3}dz=\frac{(y+1)^2}{2}\bigg|_{-1}^{1} z\bigg|_{0}^{3}=6.$$

Misolni Maple tizimidan foydalanib, yechish:

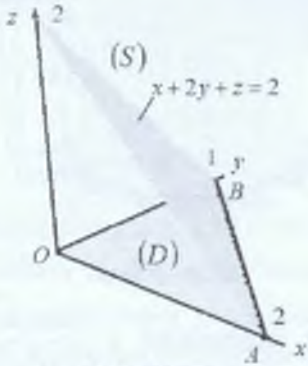
```
> int(int(y+1,z=0..3),y=-1..1);
> 6
```

3-misol. Ushbu

$$J = \iint\limits_{(S)} [2x + y + 3z]ds$$

birinchi tur sirt integralini hisoblang, bunda (S) sirt- $x+2y+z=2$ tekislikning birinchi oktantdagi qismi.

Yechilishi. (S) sirt tenglamasini z ga nisbatan yechamiz: $z = 2 - x - 2y$. (S) sirtning $xOy(z=0)$ tekislikdagi proyeksiyasi (D) soha esa, ABO uchburchakdan iborat (15.19-chizma). Bu (D) sohada z funksiya uzluksiz va



15.19-chizma.

$z_x' = -1$, $z_y' = -2$ uzluksiz xususiy hosilalarga ega. $f(x,y,z) = 2x + y + 3z$ funksiya (S) sirtda uzluksiz. (15.19.3) formuladan foydalanib, integralni hisoblaymiz:

$$J = \iint\limits_{(D)} [2x + y + 6 - 3x - 6y]\sqrt{6}dxdy = \sqrt{6}\iint\limits_{(D)} [6 - x - 5y]\sqrt{6}dxdy =$$

$$= \sqrt{6}\int\limits_0^1 dy \int\limits_0^{2-2y} (6 - x - 5y)dx = \sqrt{6}\int\limits_0^1 [(6-5y)x - \frac{x^2}{2}]\Bigg|_{x=0}^{x=2-2y} dy =$$

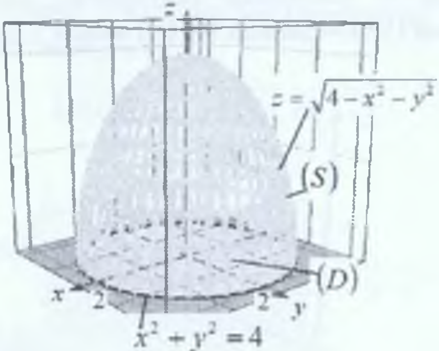
$$= 2\sqrt{6}\int\limits_0^1 (5 - 9y + 4y^2)dy = \frac{11}{3}\sqrt{6}.$$

4-misol. Ushbu

$$I = \iint\limits_{(S)} (x+2)z^2 dxdy$$

sirt integralini hisoblang, bu yerda (S) - $x^2 + y^2 + z^2 = 4$ $(z \geq 0)$ yarim sferaning yuqori tashqi tomoni.

Yechilishi. Berilgan yarim sferaning xOy tekislikdagi proyeksiyasi $x^2 + y^2 = 4$ aylana bilan chegralangan (D) doiradan iborat (15.20- chizma.).



15.20- chizma.

(S) yarim sferaning tenglamasini $z = \sqrt{4-x^2-y^2}$ ko‘rinishda yozish mumkin. Endi ikkinchi tur sirt integralini (15.22.2) formulaga asosan, ikki karrali integralga keltirib hisoblaymiz:

$$I = \iint\limits_{(S)} (x+2)z^2 dxdy = \iint\limits_{(D)} (x+2)(4-x^2-y^2)dxdy =$$

$$= \int_0^2 dr \int_0^{2\pi} (4-r^2)r(r\cos\varphi + 2)d\varphi = 4\pi\int_0^2 (4r - r^3)dr = 16\pi.$$

Misolni Maple tizimidan foydalanib, yechish:

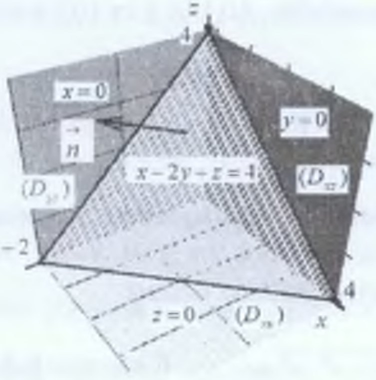
```
> int(int((4-r^2)*r*(r*cos(x)+2),x=0..2*Pi),r=0..2);
```

$$16\pi$$

5-misol. Ushbu

$$I = \iint\limits_{(S)} xdydz - zdzdx + 6dxdy$$

ikkinch tur sirt integralini hisoblang, bu yerda (S) - ushbu $x-2y+z=4,\ x=0,\ y=0,\ z=0,\ z=0$ tekisliklar bilan chegaralangan piramidaning yuqori tashqi sirti.



15.21- chizma.

Yechilishi. Tekislikning joylashishi 15.21- chizmadan ko‘rinib turibdi. Tekislikning $\vec{n}$ normal vektori Oy o‘q bilan o‘tmas burchak, Ox, Oz o‘qlar bilan esa, o‘tkir burchaklar tashkil yetadi. Bu tekislikning $\vec{n}=\{1,-2,1\}$ normal vektorining yo‘naltiruvchi kosinuslari quyidagicha topiladi:

$$\left|\vec{n}\right| = 2 = \sqrt{6},\ \cos\alpha = \frac{1}{\sqrt{6}} > 0,\ \cos\beta = \frac{-2}{\sqrt{6}} < 0,\ \cos\alpha = \frac{1}{\sqrt{6}} > 0\,.$$

Endi ikkinchi tur sirt integralini (15.22.2)- (15.22.4) formulalarga asosan, ikki karrali integralga keltirib hisoblaymiz, bunda birinchi, uchinchi integrallarni “plyus”, ikkinchi integralni “minus” ishora bilan yozamiz:

$$I = \iint\limits_{(D_{yz})} (2y - z + 4)dydz + \iint\limits_{(D_{xz})} zdzdx + \iint\limits_{(D_{xy})} 6dxdy =$$

$$= \int_{-2}^{0} dy \int_{0}^{4+2y} (2y - z + 4)dz + \int_{0}^{4} dx \int_{0}^{4-x} zdz + 6\cdot\frac{1}{2}\cdot 2\cdot 4 =$$

$$= \frac{-8}{3} + \frac{32}{3} + 24 = 32.$$

6-misol. $J=\iint\limits_{(S)} xdydz+dxdz+xz^2dxdy$, bunda $(S): x^2+y^2+z^2=1$ sfera birinchi oktantdagi qismining yuqori tomoni.

Yechilishi. Berilgan (S) sirtning Oyz, Oxz, Oxy tekisliklardagi proyeksiyalarini, mos ravishda, $(D_x), (D_y)$ va (D_z) kabi belgilab, berilgan J integralni uchta:

$$J_1=\iint\limits_{(S)} xdydz,\ J_2=\iint\limits_{(S)} dxdz,\ J_3=\iint\limits_{(S)} xz^2dxdy$$

integrallar yig'indisi shaklida tasvirlaymiz. J_1 integralda $P=x, Q=R=0$; J_2 integralda $Q=1, P=R=0$; J_3 da esa, $P=Q=0,\ R=xz^2$. Har bir integral uchun (15.22.2)-(15.22.4) formulalarni qo'llaymiz:

$$J_1=\iint\limits_{(D_x)} \sqrt{1-y^2-z^2}\,dydz,\ J_2=\iint\limits_{(D_y)} dxdz,\ J_3=\iint\limits_{(D_z)} x\left(1-x^2-y^2\right)dxdy$$

$(D_x), (D_y)$ va (D_z) sohalar mos koordinatalar tekisliklarida joylashgan radiusi 1 ga teng bo'lgan doiraning to'rtdan bir qismiga teng. Shuning uchun, $J_2=\frac{\pi}{4}$. J_1 va J_3 integrallarda qutb koordinatalar sistemasiga o'tib, hisoblash bajaramiz:

$$J_1=\iint\limits_{(D_x)} \sqrt{1-y^2-z^2}\,dydz=\iint\limits_{(D_x)} \sqrt{1-\rho^2}\cdot\rho d\rho d\varphi=-\frac{1}{2}\int_0^{\frac{\pi}{2}} d\varphi\int_0^1\left(1-\rho^2\right)^{1/2} d\left(1-\rho^2\right)=\frac{\pi}{6}.$$

$$J_3=\int_0^{\frac{\pi}{2}} d\varphi\int_0^1 \rho\cos\varphi\left(1-\rho^2\right)\rho d\rho=\sin\varphi\Big|_0^{\frac{\pi}{2}}\cdot\left(\frac{\rho^3}{3}-\frac{\rho^5}{5}\right)\Bigg|_0^1=\frac{2}{15}.$$

Demak, $J=J_1+J_2+J_3=\frac{\pi}{6}+\frac{\pi}{4}+\frac{2}{15}=\frac{5\pi}{12}+\frac{2}{15}$.

Mustaqil yechish uchun misollar

Qo'yidagi birinchi tur sirt integrallarini hisoblang.

1. $\iint\limits_{(S)}(x+y+z)dS$, bunda (S)- $x\ge 0,\ y\ge 0,\ z\ge 0$ ajratilgan shartda $x+2y+4z=4$ tekislik qismi.

2. $\iint\limits_{(S)}(x+y+z)dS$, bunda $(S)-z\geq 0$ ajratilgan shartda $x^2+y^2+z^2=4$ sfera qismi.

3. $\iint\limits_{(S)}(x+y+z)dS$, bunda (S) kubning to'liq sirti: $0\leq x\leq a,\quad 0\leq y\leq a,\quad 0\leq z\leq a$.

4. $\iint\limits_{(S)}(6x+4y+3z)dS$, bunda $(S)-x+2y+3t=6$ tekislikning birinchi oktantdagi qismi.

5. $\iint\limits_{(S)}(x^2+y^2+z^2)dS$, bunda $(S)-x^2+y^2+z^2=R^2$ sfera.

6. $\iint\limits_{(S)}(x^2+y^2+z^2)dS$, bunda $(S)-|x|\leq a,\quad |y|\leq a,\quad |z|\leq a$ kub sirti.

7. $\iint\limits_{(S)}z^2dxdy$, bu erda (S) - ushbu $x^2+y^2+z^2=a^2\ (z\geq 0)$ yarim sferaning tashqi tomoni.

8. $\iint\limits_{(S)}x^2dydz$, bu erda (S) - ushbu $x\geq 0, y\geq 0, 0\leq z\leq 1$ sohadagi $z=x^2+y^2$ paraboloid sirtning tashqi tomoni.

9. $\iint\limits_{(S)}-xdydz+zdzdx+5dxdy$, bu erda (S) - birinchi oktantada joylashgan $2x-3y+z=6$ tekislikning yuqori qismidagi tomoni.

10. $\iint\limits_{(S)}xdydz+ydzdx+zdxdy$, bu erda (S) - ushbu $x^2+y^2+z^2=R^2$ sferaning tashqi tomoni.

11. $\iint\limits_{(S)}x^3dydz+y^3dzdx+z^3dxdy$, bu erda (S) - ushbu $x^2+y^2+z^2=R^2$ sferaning tashqi tomoni.

Mustaqil yechish uchun misollarning javoblari

1. $\frac{7\sqrt{21}}{3}$. **2.** π. **3.** $9a^3$. **4.** $54\sqrt{14}$. **5.** $4\pi R^4$. **6.** $40a^4$. **7.** $0{,}5\pi a^4$. **8.** $\frac{4}{15}$.

9. -9. **10.** $4\pi R^3$. **11** $\frac{12}{5}\pi R^5$.

15-bob bo'yicha amaliy mashg'ulotlarni mustahkamlash uchun nazorat topshiriqlari

15.1– masala. Quyidagi birinchi tur egri chiziqli integralni hisoblang.

15.1.1. $\int\limits_{(K)}\sqrt{x^2+y^2}ds$ bunda $(K): x=a(\cos t+t\sin t),$
$y=a(\sin t-t\cos t)\ (0\leq t\leq 2\pi)$.

15.1.2. $\int\limits_{(K)} \sqrt{2y}ds$, bunda $(K): x = a(t - \sin t), y = a(1 - \cos t)$ $(0 \le t \le 2\pi)$.

15.1.3 $\int\limits_{(K)} y^3 ds$, bunda $(K): x = a(t - \sin t),\ y = a(1 - \cos t),\ (0 \le t \le 2\pi)$ sikloidaning bir arkasidagi qismi.

15.1.4. $\int\limits_{(K)} (x + y)ds$ bunda $(K): x = a\cos t, y = a\sin t \quad 0 \le t \le \frac{\pi}{2}$.

15.1.5. $\int\limits_{(K)} (4x^2 - y^2)ds$, bunda $(K): x = a\cos^3 t,\ y = a\sin^3 t$.

15.1.6. $\int\limits_{(K)} xy\,ds$, bunda $(K): x = t - \sin t,\ y = 1 - \cos t,\ 0 \le t \le 2\pi$

15.1.7. $\int\limits_{(K)} \frac{ds}{x^2 + y^2}$, bunda $(K): x = \cos t + t\sin t, y = \sin t - t\cos t, 0 \le t \le 2\pi$.

15.1.8. $\int\limits_{(K)} (x^2 + y^2)^n ds$, bunda $(K): x = a\cos t, y = a\sin t$ aylana.

15.1.9. $\int\limits_{(K)} \frac{ds}{x^2 + y^2 + z^2}$, bunda $(K): x = a\cos t, y = a\sin t, z = bt$.

15.1.10. $\int\limits_{(K)} \left(4\sqrt[3]{x} - 3\sqrt[3]{y}\right)ds$. bunda $(K): x = \cos^3 t,\ y = \sin^3 t$ – astroidaning $A(1;0)$ va $B(0,1)$ nuqtalar orasidagi qismi.

15.1.11. $\int\limits_{(K)} \sqrt{2 - z^2}\left(2z - \sqrt{x^2 + y^2}\right)ds$, bunda $(K): x = t\cos t,\ y = t\sin t,\ z = t,$ $0 \le t \le 2\pi$.

15.1.12. $\int\limits_{(K)} y^2 ds$, bunda $(K): x = t - \sin t, y = 1 - \cos t$ sikloidning birinchi arkasi.

15.1.13. $\int\limits_{(K)} (x + z)ds$, bunda $(K): x = t,\ y = \frac{3}{\sqrt{2}}t^2,\ z = t^3, 0 \le t \le 1$.

15.1.14. $\int\limits_{(K)} (x + y)ds$, bunda $(K): \rho^2 = a^2\cos 2\varphi, -\frac{\pi}{4} \le \varphi \le \frac{\pi}{4}$.

15.1.15. $\int\limits_{(K)} x\,ds$, bunda $(K): \rho = 1 + \cos\varphi$ $(0 \le \varphi \le 2\pi)$ chiziq yuqori qismining yarmi.

15.1.15. $\int\limits_{(K)} \left(x^{\frac{4}{3}} + y^{\frac{4}{3}}\right)ds$, bunda $(K): x = a\cos^3 t,\ y = a\sin^3 t,\ (0 \le t \le \pi)$.

15.1.17. $\int\limits_{(K)}(x^2+y^2+z^2)ds$, bunda $(K): x=a\cos t,\ y=a\sin t,$ $z=bt\ (0\le t\le 2\pi)$.

15.1.18. $\int\limits_{(K)} zds$, bunda $(K):\ x=t\cos t,\ \ y=t\sin t,\ \ z=t\ (0\le t\le t_0)$.

15.1.19. $\int\limits_{(K)}(2a-y)ds$, bunda $(K):\ x=a(t-\sin t),\ y=a(1-\cos t)$ $(0\le t\le 2\pi)$.

15.1.20. $\int\limits_{(K)}(x^2+y^2)ds$, bunda $(K): x=a(\cos t+t\sin t)\ \ y=a(\sin t-t\cos t),$ $(0\le t\le 2\pi)$.

15.1.21. $\int\limits_{(K)}(x^2+y^2)ds$. bunda $(K):\ \ x=4\cos t,\ y=\sin 2t,\ 0\le t\le\frac{\pi}{2}$.

15.1.22. $\oint\limits_{(K)}(x^2-y^2)dx+(x^2+y^2)dy$, bunda $(K):\ \ x=2\cos t,$ $y=2\sin t,\ 0\le t\le\frac{\pi}{2}$.

15.1.23. $\int\limits_{(K)}(4x^2-y^2)ds$, bunda astroidaning $A(1;0)$ va $B(0,1)$ nuqtalar orasidagi qismi.

15.1.24. $\int\limits_{(K)}(x^2+y^2)^4ds$, bunda $(K): x=4\cos t, y=4\sin t$ aylana.

15.1.25. $\int\limits_{(K)}(x^2+y^2)^2ds$, bunda $(K): x=t-\sin t, y=1-\cos t$ sikloidning birinchi arkasi.

15.1.26. $\int\limits_{(K)}(x+y)ds$, bunda $(K):\ x=a(t-\sin t),\ y=a(1-\cos t)$ $(0\le t\le 2\pi)$ sikloidaning $O(0;0)$ va $B(4a\pi;0)$nuqtalar orasidagi yoyi $(K)=(\breve{OB})$.

Yechilishi ([9], 2-t., 9-bo'lim, [30], 15.1-bo'lim). (K): chiziq siklondaning ikkita silliq yoy bo'laklaridan iborat: $(K)=(\breve{OA})+(\breve{AB})$ bunda $A(2\pi a;\ 0)$(15.22-chizma).

$(\breve{OA})=\{(x,y)\in R^2:\ x=a(t-\sin t),\ \ y=a(1-\cos t),\ 0\le t\le 2\pi\};$

$(\breve{AB})=\{(x,y)\in R^2:\ x=a(t-\sin t),\ y=a(1-\cos t),\ 2\pi\le t\le 4\pi\}$

bo'lganligi sababli ikkala yoy uchun

$$ds=\sqrt{x_t'^2+y_t'^2}\,dt=a\sqrt{2-2\cos t}=2a\left|\sin\frac{t}{2}\right|dt,$$

$$(x+y)ds=2a^2\left(t-2\sin\frac{t}{2}\cos\frac{t}{2}+\sin^2\frac{t}{2}\right)\left|\sin\frac{t}{2}\right|dt.$$

Egri chiziqli integralning addativlik xossasiga binoan,

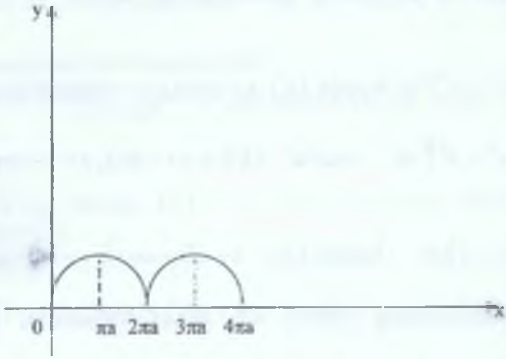
$$\int\limits_{(K)}(x+y)ds=\int\limits_{(\breve{OA})}(x+y)ds+\int\limits_{(\breve{AB})}(x+y)ds=$$

$$=2a^2\int\limits_0^{2\pi}\left(t-2\sin\frac{t}{2}\cos\frac{t}{2}+\sin^2\frac{t}{2}\right)\cdot\sin\frac{t}{2}\,dt-$$

$$-2a^2\int\limits_{2\pi}^{4\pi}\left(t-2\sin\frac{t}{2}\cos\frac{t}{2}+\sin^2\frac{t}{2}\right)\cdot\sin\frac{t}{2}\,dt=$$

$$=8a^2\left[\int\limits_0^{\pi}2z\sin z\,dz-2\int\limits_0^{\pi}\sin^2 z\cos z\,dz+\int\limits_0^{\pi}\left(1-\cos^2 z\right)\sin dz\right]+$$

$$+16\pi a^2=8a^2\left(2\pi+\frac{4}{3}\right)+16\pi a^2=\frac{32}{3}a^2(1+3\pi).$$



15.22-chizma.

15.2–masala. Quyidagi ikkinchi tur egri chiziqli integrallarni hisoblang.

15.2.1. $\int\limits_{(K)}xydx$, bunda (K): $y=\sin x,\ 0\le x\le\pi$.

15.2.2. $\int\limits_{(K)}xdy$, bunda (K): $\dfrac{x}{a}+\dfrac{y}{b}=1$ to'g'ri chiziqning $A(a;0)$ nuqtadan $B(0;b)$ nuqtagacha bo'lgan qismi.

15.2.3. $\int\limits_{(K)}(x^2+y^2)dx$, bunda $(K): y=x^2$ parabolaning $O(0;0)$ va $A(2;4)$ nuqtalar orasidagi yoyi.

15.2.4. $\int\limits_{(K)}(x^2+y^2)dy$, bunda $(K): x=1, x=3, y=1, y=5$ to'g'ri chiziqlar bilan chegaralangan to'g'ri to'rtburchak (yo'nalish soat strelkasiga teskari).

15.2.5. $\int\limits_{(K)}(x^2-y^2)dx+(x^2+y^2)dy$, bunda (K): musbat yo'nalishda olingan $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ ellips.

15.2.6. $\int\limits_{(K)}4x\sin^2 ydx+y\cos^2 2xdy$, bunda $(K): O(0;0)$ va $A(3;6)$ nuqtalardan o'tuvchi to'g'ri chiziq.

15.2.7. $\int\limits_{(K)}xdy$, bunda (K): koordinatalar o'qlari va $\frac{x}{2}+\frac{y}{3}=1$ to'g'ri chiziq bilan chegaralangan uchburchakning chegarasi (yo'nalish musbat yo'nalish bo'yicha olingan).

15.2.8. $\int\limits_{(0;0)}^{(1;1)}2xy\,dx+x^2dy$, bunda $(K): y=x$.

15.2.9. $\int\limits_{(0;0)}^{(1;1)}2xydx+x^2dy$, bunda $(K): y=x^2$.

15.2.10. $\int\limits_{(0;0)}^{(1;1)}2xydx+x^2dy$, bunda $(K): y=x^3$

15.2.11. $\int\limits_{(0;0)}^{(1;1)}2xydx+x^2dy$, bunda $(K): y^2=x$.

15.2.12. $\int\limits_{(0;0)}^{(1;1)}xydx+(y-x)dy$, bunda $(K): y=x$.

15.2.13. $\int\limits_{(0;0)}^{(1;1)}xydx+(y-x)dy$, bunda $(K): y=x^2$.

15.2.14 $\int\limits_{(0;0)}^{(1;1)}xydx+(y-x)dy$, bunda $(K): y^2=x$.

15.2.15. $\int\limits_{(0;0)}^{(1;1)}xydx+(y-x)dy$, bunda$(K): y=x^3$.

15.2.15. $\int\limits_{(K)}\left(x^2+y^2\right)dx+xydy$, bunda $(K): y=e^x$ chiziqning $A(0;1)$ va $B(1;e)$ nuqtalar orasidagi qismi.

15.2.17. $\int\limits_{(K)}(xy-1)dx+x^2ydy$, bunda (K): $4x+y^2=4$ parabolaning birinchi kvadrantdagi qismi.

15.2.18. $\oint\limits_{(K)}\left(x^2-y^2\right)dx+\left(x^2+y^2\right)dy$, bunda (K): $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ ellips.

15.2.19. $\oint\limits_{(K)}2xdx-(x+2y)dy$, bunda (K): uchlari $A(-1;0)$, $B(0;2)$, $C(2;0)$ nuqtalarda bo'lgan uchburchak (chegarasi)

15.2.20. $\oint\limits_{(K)}\frac{(x+y)dx-(x-y)dx}{x^2+y^2}$, bunda (K): $x^2+y^2=a^2$ – aylana.

15.2.21. $\int\limits_{(K)}(x+y)dx$, bunda (K): $y=\ln x$, $1\le x\le e$.

15.2.22. $\int\limits_{(K)}ydx-xdy$, bunda (K): $y=\lg x$, $1\le x\le 10$.

15.2.23. $\int\limits_{(K)}(x^2+y^2)dx+\left(x^2-y^2\right)dy$, bunda (K): $y=|x|$, $-1\le x\le 2$.

15.2.24. $\int\limits_{(K)}(x^2+y^2)dx+\left(x^2-y^2\right)dy$, bunda (K): $y=x^3$, $0\le x\le 1$.

15.2.25. $\int\limits_{(K)}(x^2+y^2)dx+2xydy$, bunda (K): $y=x^4$, $0\le x\le 1$.

15.2.26. $\int\limits_{(K)}\left(3x^2+y\right)dx+\left(x-2y^2\right)dy$, bunda (K): uchlar $O(0;0)$, $A(2;0)$ va $B(0;2)$ nuqtalarda uchburchakning chegarasi - $(OABO)$

Yechilishi ([9], 2-t., 9-bo'lim, [30], 15.2-bo'lim). Berilgan ikkinchi tur egri chiziqli integralning integrallash konturi 15.23- chizmada tasvirlangan.

$(K)=(OABO)$ chiziqning yo'nalishi 15.23- chizmada ko'rsatilgan. Berilgan egri chiziqli integralni hisoblash uchun uchburchakning harbir tomoni bo'yicha (ko'rsatilgan yo'nalishda) integralni hisoblab, so'ngra egri chiziqli integralning additivlik xossasiga asosan, uchala tomon bo'yicha hisoblangan integrallarning qiymatini qo'shamiz.

1) uchburchak OA tomonining tenglamasi $y=0,\ dy=0$.

$$\int\limits_{(OA)}3x^2dx=3\int\limits_0^2x^2dx=8.$$

$$\int\limits_{(AB)} \left(3x^2 + y\right)dx + \left(x - 2y^2\right)dy = \int\limits_2^0 \left[3x^2 + (2-x) - x + 2(2-x)^2\right]dx =$$

$$= \int\limits_2^0 \left(5x^2 - 10x + 10\right)dx = \left(\frac{5}{3}x^3 - 5x^2 + 10x\right)\Bigg|_2^0 = -\frac{40}{3}\cdot$$

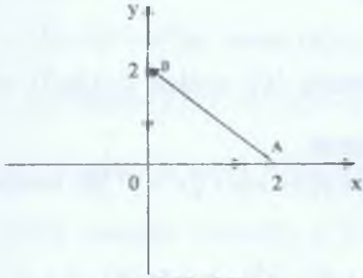
3) uchburchak BO tomonining tenglamasi : $x = 0,\ dx = 0$,

$$\int\limits_{(BO)} -2y^2 dy = -\int\limits_2^0 2y^2 dy = \frac{16}{3}.$$

Shunday qilib, 1), 2) va 3) lardan uchala tomon bo'yicha hisoblangan integrallarning qiymatini qo'shamiz:

$$\int\limits_{(OABO)} \left(3x^2 + y\right)dx + \left(x - 2y^2\right)dy = 8 - \frac{40}{3} + \frac{16}{3} = 0.$$

Bu natijani integralni hisoblanmasdan ham olish mumkin, chunki integral ostidagi $Pdx + Qdy = \left(3x^2 + y\right)dx + \left(x - 2y^2\right)dy$ ifoda $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}\ \left(\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} = 1\right)$ shartni qanoatlantiradi.



15.23-chizma.

15.3–masala. Grin formulasidan foydalanib, quyidagi egri chiziqli integrallarni hisoblang.

15.3.1. $\int\limits_{(K)} (xy + x + y)dx + (xy + x - y)dy$, bunda (K): $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ - ellips

15.3.2. $\int\limits_{(K)} (xy + x + y)dx + (xy + x - y)dy$, bunda (K): $x^2 + y^2 = ax$ – aylana

15.3.3. $\int\limits_{(K)} (2xy - y)dx + x^2 dy$, bunda $(K): \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ - ellips

15.3.4. $\int\limits_{(K)} \frac{xdy + ydx}{x^2 + y^2}$, bunda $(K): (x-1)^2 + (y-1)^2 = 1$ - aylana

15.3.5. $\int\limits_{(K)} (x+y)^2 dx - (x^2 + y^2)dy$, bunda (K): uchlari $(1;1), (3;2), (2,5)$ nuqtalarda bo'lgan uchburchakning chegarasi.

15.3.6. $\int\limits_{(K)} (y - x^2)dx + (x + y^2)dy$, bunda $(K): 0 < r < R,\ 0 < \varphi < \alpha \le \frac{\pi}{2}$ aylana sektorining chegarasi, (r, φ)– qutb koordinatalari.

15.3.7. $\int\limits_{(K)} e^x[(1 - \cos y)dx - (y - \sin y)dy]$, bunda (K): $0 \le x \le \pi,\quad 0 \le y \le \sin x$ sohaning chegarasi.

15.3.8. $\int\limits_{(K)} e^{y^2 - x^2}(\cos 2xydx + \sin 2xydy)$, bunda $(K): x^2 + y^2 = R^2$ – aylana.

15.3.9. $\int\limits_{(K)} (e^x \sin y - y)dx + (e^x \cos y - 1)dy$, bunda (K): $x^2 + y^2 < ax,\ y > 0$ sohaning chegarasi.

15.3.10. $\int\limits_{(K)} \frac{dx - dy}{x + y}$, bunda (K): uchlari $(1;0), (0;1), (-1;0), (0;-1)$ nuqtalarda bo'lgan kvadratning chegarasi.

15.3.11. $\int\limits_{(K)} \sqrt{x^2 + y^2}dx + y\left(xy + \ln\left(x + \sqrt{x^2 + y^2}\right)\right)dy$, bunda $(K): x^2 + y^2 = R^2$ aylana.

15.3.12. $\int\limits_{(K)} (1 - x^2)y\,dx + x(1 + y^2)dy$, bunda $(K): x^2 + y^2 = R^2$ aylana

11.3.5.13. $\int\limits_{(K)} (e^{xy} + 2x\cos y)dx + (e^{xy} - x^2 \sin y)dy$, bunda $(K): x^2 + y^2 = 2x$ aylana.

15.3.14. $\int\limits_{(K)} (xy + x + y)dx + (xy + x - y)dy$, bunda $(K): \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$-ellips.

15.3.15. $\int\limits_{(K)} (xy + x + y)dx + (xy - x - y)dy$, bunda $(K): x^2 + y^2 = ax$ aylana.

15.3.15. $\int\limits_{(K)} xy^2 dy - x^2 y dx$, bunda $(K): x^2 + y^2 = R^2$ aylana.

15.3.17. $\int\limits_{(K)}(x+y)dx-(x-y)dy$. bunda $(K):\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ ellips.

15.3.18. $\int\limits_{(K)}(x-y)dx-(x-y)dy$, bunda $(K):x^2+y^2=3x$ aylana.

15.3.19. $\int\limits_{(K)}(x+y)^2dx-(x-y)^2dy$, bunda $(K):AB$ kesma $(A(1;1), B(2;6))$ va A, B, hamda $O(0;0)$ nuqtalardan o'tuvchi $y=ax^2+bx+c$ parabola yoyi bilan chegaralangan sohaning chegarasi.

15.3.20. $\int\limits_{(K)}\left(e^x\sin y-y\right)dx+\left(e^x\cos y-1\right)dy$, bunda$(K)$: $\{0<x<\pi,\ 0<y<\sin x\}$ sohaning chegarasi.

15.3.21. $\int\limits_{(K)}(x+y)dx-(x-y)dy$, bunda $(K):x^2+y^2=2x$ aylana.

15.3.22. $\int\limits_{(K)}y^2dx-x^2dy$, bunda $(K):x^2+y^2=16$ aylana.

15.3.23. $\int\limits_{(K)}(2x+2y)dx-(3x-3y)dy$, bunda $(K):\dfrac{x^2}{25}+\dfrac{y^2}{9}=1$ ellips.

15.3.24. $\int\limits_{(K)}\left(x^2-y^2\right)dx-\left(x^2-y^2\right)dy$, bunda $(K):x^2+y^2=3x$ aylana.

15.3.25. $\int\limits_{(K)}xdy-ydx$, bunda $(K):(x-1)^2+(y-1)^2=4$ aylana.

15.3.26. $\int\limits_{(AmB)}\left(e^x\sin y-my\right)dx+\left(e^x\cos y-m\right)dy$, bunda$(K)=(AmB)$: $A(a;0)$ nuqtadan chiqib $O(0;0)$ nuqtaga boruvchi $x^2+y^2=ax$ aylananing yuqori qismi.

Yechilishi ([9], 2-t., 9-bo'lim, [30], 15.1-bo'lim). $[0,a]$ kesma tenglamasi $y=0$ bo'lgani uchun bu kesma bo'yicha $\int\limits_{(OA)}\left(e^x\sin y-my\right)dx+\left(e^x\cos y-m\right)dy=0$ bo'ladi. Shuning uchun, (AmO) chiziq bo'yicha olingan integral $(AmOA)$ yopiq chiziq bo'yicha olingan integralga teng bo'ladi. (AmO) chiziq va $[0,a]$ kesma bilan chegaralangan soha $(D)=\left\{(x,y)\in R^2:0\le x\le a, 0\le y\le\sqrt{ax-x^2}\right\}$ ko'rinishda bo'ladi.

Berilgan integral ostidagi ifodada $P(x,y)=e^x\sin y-my$, $Q(x,y)=e^x\cos y-m$ deb belgilasak, ravshanki, $P(x,y)$va $Q(x,y)$ funksiyalar Grin teoremasining shartlarini qanoatlantiradi. Shuning uchun, Grin formulasiga asosan,

$$\int\limits_{(AmOA)} \left(e^x \sin y - my\right)dx + \left(e^x \cos y - m\right)dy =$$

$$= \iint\limits_{(D)} [e^x \cos y - e^x \cos y + m]dx\,dy = m \cdot \iint\limits_{x^2+y^2 \le ax} dxdy = m \cdot D = \frac{m}{8} \cdot \pi a^2.$$

15.4 – masala. Ushbu $\iint\limits_{(S)} f(x,y,z)ds$ integralni hisoblang, bunda (S) sirt - (P) tekislikning $x=0;\ y=0;\ z==0$ koordinatalar tekisliklari bilan kesilgan qismi.

15.4.1. $\iint\limits_{(S)} (2x+3y+2z)ds,\ (P): x+3y+z=3.$

15.4.2. $\iint\limits_{(S)} (1+y-7x+9z)ds,\ (P): 2x-y-2z=-2.$

15.4.3. $\iint\limits_{(S)} (6x+y+4z)ds,\ (P): 3x+3y+z=3.$

15.4.4. $\iint\limits_{(S)} (x+2y+3z)ds,\ (P): x+y+z=2.$

15.4.5. $\iint\limits_{(s)} (3x-2y+6z)ds,\ (P): 2x+y+2z=2.$

15.4.6. $\iint\limits_{(S)} (2x+5y-z)ds,\ (P): x+2y+z=2.$

15.4.7. $\iint\limits_{(S)} (sx-8y-z)ds,\ (P): 2x-3y+z=6.$

15.4.8. $\iint\limits_{(S)} (3y-x-z)ds,\ (P): x-y+z=2.$

15.4.9. $\iint\limits_{(S)} (3y-2x-2z)ds,\ (P): 2x-y-2z=-2.$

15.4.10. $\iint\limits_{(S)} (2x-3y+z)ds,\ (P): x+2y+z=2.$

15.4.11. $\iint\limits_{(S)} (5x+y-z)ds,\ (P): x+2y+2z=2.$

15.4.12. $\iint\limits_{(S)} (3x-2y+2z)ds,\ (P): 3x+2y+2z=6.$

15.4.13. $\iint\limits_{(S)} (2x+3y-z)ds,\ (P): 2x+y+z=2.$

15.4.14. $\iint\limits_{(S)} (9x+2y+z)ds,\ (P): 2x+y+z=4.$

15.4.15. $\iint\limits_{(S)}(3x+8y+8z)ds,\ (P): x+4y+2z=8.$

15.4.15. $\iint\limits_{(S)}(4y-x+4z)ds.\ (P): x-2y+2z=2.$

15.4.17. $\iint\limits_{(S)}(7x+y+2z)ds,\ (P): 3x-2y+2z=6.$

15.4.18. $\iint\limits_{(S)}(2x+3y+z)ds,\ (P): 2x+3y+z=6.$

15.4.19. $\iint\limits_{(S)}(4x-y+z)ds,\ (P): x-y+z=6.$

15.4.20. $\iint\limits_{(S)}(6x-y+8z)ds,\ (P): x+y+2z=2.$

15.4.21. $\iint\limits_{(S)}(x+3y+2z)ds,\ (P): 2x+4y+z=8.$

15.4.22. $\iint\limits_{(S)}(x+y+9z)ds,\ (P): 2x-3y-2z=-6.$

15.4.23. $\iint\limits_{(S)}(x+3y+2z)ds,\ (P): 4x+3y+z=12.$

15.4.24. $\iint\limits_{(S)}(x+3y-2z)ds,\ (P): x+3y+3z=9.$

15.4.25. $\iint\limits_{(S)}(2x-2y-3z)ds,\ (P): 3x+4y+2z=12.$

15.4.26. $\iint\limits_{(S)}(4x-4y-z)ds,\ (P): x+2y+2z=4.$

Yechilishi ([9], 2-t., 9.3-bo'lim, [30], 15.5-bo'lim). Masala shartidan ko'rinadiki, (S) - uchburchakli piramidaning sirti bo'ladi. Integral ostidagi funksiya $f(x,y,z)=4x-4y-z.$ (S) sirt tenglamasini $z=2-\frac{1}{2}x-y$ ko'rinishda yozish mumkin. Bu funksiyadan $z'_x=-\frac{1}{2}$, $z'_y=-1$ xususiy hosilalar olib, birinchi tur sirt integralini (15.19.4) formula yordamida hisoblaymiz:

$$\iint\limits_{(S)}(4x-4yz)ds=\iint\limits_{(D)}\left(4x-4y-2+\frac{1}{2}x+y\right)\sqrt{1+\frac{1}{4}+1}dxdy=$$
$$=\frac{3}{2}\iint\limits_{(D)}\left(\frac{9}{2}x-3y-2\right)dxdy,$$

bunda $(D)=\left\{(x,y)\in R^2: 0\le x\le 4,\ 0\le y\le 2-\frac{1}{2}x\right\}$. Endi ikki karrali integralni hisoblaymiz:

$$\iint_{(S)}(4x-4y-z)ds=\frac{3}{2}\int_0^4 dx\int_0^{2-\frac{1}{2}x}\left(\frac{9}{2}x-3y-2\right)dy=$$

$$=\frac{3}{2}\int_0^4\left(\frac{9}{2}xy-\frac{3}{2}y^2-2y\right)\Bigg|_0^{2-\frac{1}{2}x}dx=\frac{3}{2}\int_0^4\left(-6+12x-\frac{21}{8}x^2\right)dx=24.$$

15.5-masala. Quyidagi 2- tur sirt integrallari sirtning ko'rsatilgan tomoni bo'yicha hisoblansin.

15.5.1. $\iint_{(S)}x^2dydz+y^2dzdx+z^2dxdy$, bunda $(S): x^2+y^2+z^2=a^2$ sferaning tashqi tomoni.

15.5.2. $\iint_{(S)}(2z-x)dydz+(x+2z)dzdx+3zdxdy$, bunda $(S): x+4y+z=4,\ x\ge 0,\ y\ge 0,\ z\ge 0$ uchburchakning tashqi tomoni.

15.5.3. $\iint_{(S)}yz\,dy\,dz+zx\,dz\,dx+xy\,dx\,dy$, bunda $(S): x+y+z\le 1,\ x\ge 0,\ y\ge 0,\ z\ge 0$ – sirtning ichki tomoni.

15.5.4. $\iint_{(S)}z^2dxdy$, bunda $(S): \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ ellipsoidning tashqi tomoni.

15.5.5. $\iint_{(S)}ydzdx$, bunda $(S): x^2+y^2+z^2=R^2$ sferaning tashqi tomoni.

15.5.6. $\iint_{(S)}x^2dydz$, bunda $(S): x^2+y^2+z^2=R^2$ sferaning tashqi tomoni.

15.5.7. $\iint_{(S)}(x^2+z)dydz$, bunda $(S): x^2+y^2+z^2=R^2,\ z\le 0$ yarim sferaning ichki qismi.

15.5.8. $\iint_{(S)}x^2y^2z\,dx\,dy$, bunda $(S): x^2+y^2+z^2=R^2,\ z\le 0$ yarim sferaning ichki qismi.

15.5.9. $\iint_{(S)}xdydz+ydzdx+zdxdy$, bunda $(S): x^2+y^2+z^2=R^2$ sferaning tashqi tomoni.

15.5.10. $\iint\limits_{(S)} z^2 dxdy$, bunda $(S):(x-a)^2+(y-b)^2+z^2=R^2, z\geq 0$ yarim sferaning ichkm qismi.

15.5.11. $\iint\limits_{(S)} dzdx$, bunda $(S):\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ ellipsoidning tashqi tomoni.

15.5.12. $\iint\limits_{(S)} x\,dydz$, bunda $(S):\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ ellipsiodning tashqi tomoni.

15.5.13. $\iint\limits_{(S)} x^2 dydz$, bunda $(S):\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ ellepsiodning tashqi tomoni.

15.5.14. $\iint\limits_{(S)} \frac{dxdy}{z}$, bunda $(S):\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ ellipsiodning tashqi tomoni.

15.5.15. $\iint\limits_{(S)} yzdzdx$, bunda $(S):\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, $z\geq 0$ ellipsiodning tashqi tomoni.

15.5.15. $\iint\limits_{(S)} x^2dydz+y^2dzdx$, bunda $(S):\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ $z\geq 0$ ellipisning tashqi tomoni.

15.5.17. $\iint\limits_{(S)} (y-z)dydz+(z-x)dzdx+(x-y)dxdy$, bunda

$(S):x^2+y^2=z^2$ $(0\leq z\leq h)$ konus sirtning tashqi tomoni .

15.5.18. $\iint\limits_{(S)} x^2yzdydz+xy^2zdzdx+xyz^2dxdy$, bunda

$(S): x^2+y^2+z^2=1, x=0, y=0, z=0$ sirtlar bilan chegaralangan jismning to'liq tashqi tomoni.

15.5.19. $\iint\limits_{(S)} xdydz+ydzdx+zdxdy$, bunda

$(S): x^2+y^2+z^2=a^2$, $0\leq z\leq H$ silindr to'liq sirtning tashqi tomoni.

15.5.20. $\iint\limits_{(S)} x^3dydz+y^3dzdx+z^3dxdy$, bunda

$(S): x+y+z\leq a$, $x\geq 0$, $y\geq 0$, $z\geq 0$ tetraedr sirtning tashqi tomoni.

15.5.21. $\iint\limits_{(S)} (2x^2+y^2+z^2)dzdy$, $(S):x^2+y^2=z^2$ $(0\leq z\leq h)$ konus sirtning tashqi tomoni.

15.5.22. $\iint\limits_{(S)} yzdzdx$, $(S): x^2+y^2+z^2=4, z\ge 0$ sferaning tashqi tomoni.

15.5.23. $\iint\limits_{(S)} x^3dydz+y^3dzdx$, $(S): x^2+y^2+z^2=4, z\ge 0$ sferaning tashqi tomoni.

15.5.24. $\iint\limits_{(S)} (x-2)^3dydz$, $(S): x^2+y^2+z^2=4x, z\ge 0$ sferaning tashqi tomoni.

15.5.25. $\iint\limits_{(S)} xzdydz+xydzdx+yzdxdy$, bunda $(S): x^2+y^2=4, x\le 0,$ $y\ge 0, 0\le z\le 2$ sirtlar bilan chegaralangan jismning to'liq tashqi tomoni.

15.5.26. $J=\iint\limits_{(S)} xdydz+dxdz+xz^2dxdy$, bunda $(S): x^2+y^2+z^2=1$ sfera birinchi oktantdagi qismining yuqori tomoni.

Yechilishi ([9], 2-t., 9.3-bo'lim, [30], 15.5-bo'lim). Berilgan (S) sirtning Oyz, Oxz, Oxy tekisliklardagi proyeksiyasini, mos ravishda, (D_x), (D_y) va (D_z) kabi belgilab, berilgan J integralni uchta:

$$J_1=\iint\limits_{(S)} xdydz,\ J_2=\iint\limits_{(S)} dxdz,\ J_3=\iint\limits_{(S)} xz^2dxdy\ .$$

integrallar yig'indisi shaklida tasvirlaymiz. J_1 integralda $P=x, Q=R=0;$ J_2 integralda $Q=1, P=R=0;$ J_3 da esa, $P=Q=0,\ R=xz^2$. Har bir integral uchun (15.22.2)-(15.22.4) formulani qo'llaymiz:

$$J_1=\iint\limits_{(D_x)}\sqrt{1-y^2-z^2}\,dydz,\ J_2=\iint\limits_{(D_y)} dxdz,\ J_3=\iint\limits_{(D_z)} x\left(1-x^2-y^2\right)dxdy.$$

(D_x), (D_y) va (D_z) sohalar mos koordinatalar tekisliklarida joylashgan radiusi 1 ga teng bo'lgan doiraning to'rtdan bir qismiga teng. Shuning uchun $J_2=\frac{\pi}{4}$. J_1 va J_3 integrallarda qutb koordinatalar sistemasiga o'tib, hisoblash bajaramiz:

$$J_1=\iint\limits_{(D_x)}\sqrt{1-y^2-z^2}\,dydz=\iint\limits_{(D_x)}\sqrt{1-\rho^2}\cdot\rho d\rho d\varphi=-\frac{1}{2}\int\limits_0^{\frac{\pi}{2}}d\varphi\int\limits_0^1\left(1-\rho^2\right)^{1/2}d\left(1-\rho^2\right)=\frac{\pi}{6}.$$

$$J_3=\int\limits_0^{\frac{\pi}{2}}d\varphi\int\limits_0^1\rho\cos\varphi\left(1-\rho^2\right)\rho d\rho=\sin\varphi\Big|_0^{\frac{\pi}{2}}\cdot\left(\frac{\rho^3}{3}-\frac{\rho^5}{5}\right)\Bigg|_0^1=\frac{2}{15}.$$

Demak, $J=J_1+J_2+J_3=\frac{\pi}{6}+\frac{\pi}{4}+\frac{2}{15}=\frac{5\pi}{12}+\frac{2}{15}.$

16-BOB.
MAYDONLAR NAZARIYASI

16.1. Skalyar va vektorli maydonlar

16.1.1-ta'rif. E sohaning ixtiyoriy M nuqtasiga biror qonun bo'yicha $U(M)$ son - kattalik mos qo'yilgan bo'lsa, u holda bu sohada *skalyar maydon* berilgan deyiladi.

Skalyar maydonning berilishi, $U(M)$ skalyar funksiyaning berilishiga ekvivalent. $U(M)$ skalyar maydonning biror Dekart koordinatalar sistemasida berilishi, $U(x,y,z)$ funksiyaning berilishiga ekvivalent. Biz, bu funksiyani, bundan buyon, uzluksiz va uzluksiz xususiy hosilalarga ega, deb faraz qilamiz. Skalyar maydonning misoli sifatida: 1) qizdirilgan jismning ichkarisidagi temperaturalar maydoni; 2) potensial energiyalar maydoni va h.k. larni ko'rsatish mumkin.

Agar skalyar $U(M)$ funksiya t vaqtga bog'liq bo'lsa, maydon *stasionar bo'lmagan maydon* deyiladi.

16.1.2– ta'rif. E sohaning har bir M nuqtasiga biror $\vec{A}(M)$ vektor mos qo'yilgan bo'lsa, u holda E sohada *vektorli maydon* berilgan deyiladi (vektor - funksiya berilgan deyiladi).

Vektorli maydonning berilishi, $\vec{A}(M)$ vektor - funksiyaning berilishiga ekvivalent bo'ladi. Agar fazoda $\vec{A}(M)$ vektorli maydon berilgan bo'lib, bu yerda Dekart koordinatalar sistemasi o'rnatilgan bo'lsa, u holda $\vec{A}(M)$ vektor - funksiya

$$\vec{A}(M)=P(x,y,z)\vec{i}+Q(x,y,z)\vec{j}+R(x,y,z)\vec{k}$$

shaklda tasvirlanadi. Biz, bundan keyin, $P(M)$, $Q(M)$, $R(M)$ funksiyalarni uzluksiz va uzluksiz xususiy hosilalarga ega, deb faraz qilamiz.

16.2. Yo'nalish bo'yicha hosila va skalyar maydonning gradiyenti. Biror E sohada $U(M)$ skalyar maydon berilgan bo'lsin. Skalyar maydondan bir–biriga yaqin M_0 va M nuqtalarni olamiz va $\dfrac{U(M)-U(M_0)}{h}$ ifodani tuzamiz, bunda $h=|M_0M|$. $U(M)$ skalyar maydonda biror $\vec{l}$ yo'nalish berilgan bo'lsin. M_0M chiziqning yo'nalishi $\vec{l}$ ning yo'nalishi bilan mos tushsin.

16.2.1–ta'rif. Agar $\lim\limits_{M \to M_0} \dfrac{U(M)-U(M_0)}{h}$ limit mavjud bo'lsa, bu limit $U(M)$ skalyar maydonning M_0 nuqtadagi $\vec{l}$ *yo'nalish bo'yicha hosilasi* deyiladi va u $\dfrac{\partial U(M)}{\partial \vec{l}}$ kabi belgilanadi.

$\dfrac{\partial U(M)}{\partial \vec{l}}$ hosila $U(M)$ skalyar maydonning $\vec{l}$ yo'nalish bo'yicha o'zgarish tezligini ifodalaydi. $\dfrac{\partial U(M)}{\partial \vec{l}}$ hosilani hisoblash uchun, Dekart koordinatalar sistemasini olamiz. U holda $U(M)$ skalyar maydon $U(x,y,z)$ shaklida ifoda qilinadi. M va M_0 nuqtalarning koordinatalari, mos ravishda, $M_0(x_0,y_0,z_0)$, $M(x,y,z)$ bo'lsin. U holda $M_0M=\{x-x_0,y-y_0,z-z_0\}$. $\vec{l^0}=\{\cos\alpha,\cos\beta,\cos\gamma\}$ birlik vektor bo'lsin. U holda $M(x,y,z)$ nuqtaning koordinatalari, $x=x_0+h\cos\alpha, y=y_0+h\cos\beta, z=z_0+h\cos\gamma$ kabi yoziladi.

$U(x=x_0+h\cos\alpha, y=y_0+\cos\beta, z=z_0+h\cos\gamma)$ skalyar maydonning $\vec{l}$ yo'nalish bo'yicha hosilasi

$$\frac{\partial U(M)}{\partial \vec{l}}=\frac{\partial U}{\partial x}\cos\alpha+\frac{\partial U}{\partial y}\cos\beta+\frac{\partial U}{\partial z}\cos\gamma \qquad (16.2.2)$$

formula orqali topiladi.

$U(x,y,z)$ skalyar maydon berilgan bo'lsin. $U(x,y,z)$ funksiya uzluksiz va uzluksiz xususiy hosilalarga ega, deb faraz qilamiz.

16.2.3-ta'rif. Komponentalari $\left\{\dfrac{\partial U}{\partial x},\dfrac{\partial U}{\partial y},\dfrac{\partial U}{\partial z}\right\}$ bo'lgan vektor $U(M)$ skalyar maydonning *gradiyenti* deyiladi va u

$$gradU=\frac{\partial U}{\partial x}\vec{i}+\frac{\partial U}{\partial y}\vec{j}+\frac{\partial U}{\partial z}\vec{k}$$

kabi belgilanadi.

(16.2.2) tenglikning o‘ng tomoni, $gradU$ va $\vec{l}$ vektorning skalyar ko‘paytmasini ifodalaydi, ya’ni $\frac{\partial U}{\partial \vec{l}} = \left(gradU, \vec{l} \right)$.

Skalyar maydonning gradiyenti quyidagi xossalarga ega:

1^0. Gradiyent sirtga o‘tkazilgan normal bo‘yicha yo‘nalgan bo‘ladi.

2^0. Agar $\vec{l}$ ning yo‘nalishi $gradU$ ning yo‘nalishi bilan mos tushsa, $\frac{\partial U(M)}{\partial \vec{e}}$ yo‘nalish bo‘yicha hosila o‘zining eng katta qiymatiga erishadi.

Haqiqatan, $gradU$ bilan $\vec{l}$ ning orasidagi burchak $\varphi = 0$ bo‘ladi va $\cos\varphi = 1$. Unda

$$\frac{\partial U}{\partial \vec{l}} = \left(gradU, \vec{l} \right) = |gradU| \cdot \left| \vec{l} \right| \cdot \cos\varphi,$$

bu yerdan, $\max \frac{\partial U}{\partial \vec{l}} = |gradU| = \sqrt{\left(\frac{\partial U}{\partial x} \right)^2 + \left(\frac{\partial U}{\partial z} \right)^2 + \left(\frac{\partial U}{\partial z} \right)^2}$ bo‘ladi.

3^0. Qolgan hamma yo‘nalishlar bo‘yicha $\frac{\partial U(M)}{\partial \vec{l}}$ ifoda noldan farqli qiymat qabul qiladi, va u absolyut qiymati bo‘yicha $|gradU|$ dan oshmaydi, ya’ni

$$\frac{\partial U}{\partial \vec{l}} = |gradU||\cos\varphi| < |gradU|$$

Gradiyentning ta’rifidan foydalanib, quyidagi tengliklarni isbotlash qiyin emas:

1) $grad(U + C) = gradU (C = const)$,

2) $grad(CU) = CgradU$,

3) $grad(U + V) = gradU + gradV$,

4) $grad(U,V) = UgradV + VgradU$,

5) $gradf(U) = f'(U) gradU$.

16.3. Vektor chiziqlar va vektorli trubka. E sohada $\vec{A}(M)$ vektorli maydon aniqlangan bo‘lsin.

16.3.1–ta'rif. Agar E da yotgan silliq (K) chiziqning har bir nuqtasidagi urinmaning yo'nalishi, shu nuqtada $\vec{A}(M)$ vektorning yo'nalishiga mos tushsa, u holda (K) chiziq *vektor chizig'i* deyiladi.

Agar vektorli maydon $\vec{A} = P\vec{i} + Q\vec{j} + R\vec{k}$ ko'rinishda bo'lsa, u holda vektor chizig'ining tenglamasi $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$ shaklda bo'ladi. Buni differensiallash natijasida, $\begin{cases} \varphi_1(x,y,z) = C_1, \\ \varphi_2(x,y,z) = C_2 \end{cases}$ ikki parametrli vektor chiziqlar hosil bo'ladi.

E sohada (S) sirt va bu sirtda $\vec{A}(M)$ vektorli maydon berilgan bo'lsin.

16.3.2– ta'rif. Agar (S) sirtning har bir nuqtasiga o'tkazilgan normal, shu nuqtadagi $\vec{A}(M)$ vektorga ortogonal bo'lsa, u holda $\vec{A}(M)$ vektorli maydon *vektorli trubka* deyiladi.

Demak, vektorli trubka, sirtning bir qismi bo'lib, u butun vektor chiziqlardan tashkil topgan bo'ladi. Bu chiziqlarning hammasi vektorli trubkaning ichida yoki uning tashqarisida joylashgan bo'ladi.

16.4. Potensial maydon. $U(M)$ skalyar maydon berilgan bo'lsin. Bu maydonning har bir M nuqtasiga $gradU = \vec{A}$ vektor mos qo'yilgan bo'lsin. Natijada vektorli maydonning – gradiyentlar maydoni hosil bo'ladi.

16.4.1– ta'rif. Agar $\vec{A}(M)$ vektorli maydonni biror skalyar maydonning gradiyenti, ya'ni $\vec{A}(M) = gradU$ shaklida tasvirlash mumkin bo'lsa, u *potensial maydon* deyiladi.

Qanday shartlar bajarilganda, $\vec{A}(M)$ vektorli maydon potensial maydon bo'ladi? Bu savolga javob berish uchun quyidagini eslaymiz, ya'ni $Pdx + Qdy + Rdz$ ifoda biror bir qiymatli $U(x,y,z)$ funksiyaning to'liq differensialini ifodalashi uchun,

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \quad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}, \quad \frac{\partial R}{\partial x} = \frac{\partial P}{\partial z} \qquad (16.4.2)$$

shartlarning bajarilishi zarur va yetarli.

Agar $Pdx + Qdy + Rdz = dU$ bo'lsa, u holda

$$P = \frac{\partial U}{\partial x}, \; Q = \frac{\partial U}{\partial y}, \; R = \frac{\partial U}{\partial z}, \; \vec{A} = (P, Q, R) = (\frac{\partial U}{\partial x}, \frac{\partial U}{\partial y}, \frac{\partial U}{\partial z}).$$

Demak, (16.4.2) shartning bajarilishi, $\vec{A}(M)$ vektorli maydonning potensial maydon bo'lishi uchun zarur va yetarli bo'lar ekan.

16.5. Sirtdan o'tuvchi vektorli maydon oqimi. Vektorli maydon divergensiyasi. Solenoidal maydon. $\vec{A}(M)$ vektorli maydon berilgan bo'lsin, ya'ni $P(x,y,z)$, $Q(x,y,z)$ va $R(x,y,z)$ funksiyalar berilgan. (S) sirtni olamiz va uning tomonini belgilaymiz, ya'ni uni orentrlaymiz. Bu sirtga o'tkazilgan $\vec{n}$ normalning yo'naltiruvchi kosinuslarini, mos ravishda, $\cos\alpha$, $\cos\beta, \cos\gamma$ orqali belgilaymiz: $\vec{A} = \{P; Q; R\}$, $\vec{n} = \{\cos\alpha, \cos\beta, \cos\gamma\}$.

16.5.1-ta'rif. Ushbu

$$\iint_{(S)} (P\cos\alpha + Q\cos\beta + R\cos\gamma)dS \qquad (16.5.2)$$

sirt integraliga, (S) sirtning belgilangan tomonidan o'tgan vektor *oqimi* deyiladi va O orqali belgilanadi.

(16.5.2) ni quyidagicha ham yozish mumkin:

$$O = \iint_{(S)} (\vec{A}, \vec{n})dS$$

(S) sirt bilan chegaralangan (V) jism berilgan bo'lsin. $\vec{n}$ esa (S) sirtga o'tzilgan normal bo'lsin. $P = A_x, Q = A_y, R = A_z$ deb belgilaylik. U holda Gauss-Ostrogradskiy formulasiga asosan,

$$\iint_{(S)} (\vec{A}, \vec{n})ds = \iint_{(S)} (A_x\cos\alpha + A_y\cos\beta + A_z\cos\gamma)ds =$$

$$= \iiint_{(V)} (\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z})dV$$

bo'ladi.

16.5.3-ta'rif. Ushbu $\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$ ifodaga $\vec{A}(M)$ vektorli maydonning *divergensiyasi* deyiladi va u $div\,\vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$ kabi belgilanadi.

Shunday qilib, Ostrogradskiy formulasini, qisqacha,

$$\iint\limits_{(S)} (\vec{A}, \vec{n})ds = \iiint\limits_{(V)} div\,\vec{A}\,dV$$

ko'rinishda yozish mumkin ekan. Divergensiyaning bu ta'rifi koordinatalar sistemasini tanlashga bog'liq. Bu kamchilikdan qutulish uchun, uning koordinatalar sistemasiga bog'liq bo'lgan ta'rifini berish mumkin.

M nuqtani biror kichik (V) jism bilan o'raymiz. Bu jismning sirtini (S) orqali belgilaymiz.

16.5.4-ta'rif. Agar $\lim\limits_{V \to M} \dfrac{\iint\limits_{(S)} (\vec{A}, \vec{n})ds}{V}$ limit mavjud bo'lsa, bu limit, $\vec{A}(M)$ vektorli maydonning M nuqtadagi *divergensiyasi* deyiladi.

Vektorli maydon oqimi quyidagi xossalarga ega:

1^0. $\iint\limits_{(S)} (\vec{A}, \vec{n})ds = -\iint\limits_{(S^-)} (\vec{A}, \vec{n})ds$.

2^0. Agar (S) sirt $(S_1), (S_2), \ldots, (S_n)$ silliq sirtlar yig'indisidan iborat bo'lsa, u holda $O = \sum\limits_{k=1}^{n} \iint\limits_{(S_k)} (\vec{A}, \vec{n})ds$.

3^0. $div(\vec{A} + \vec{B}) = div\,\vec{A} + div\,\vec{B}$. 4^0. $div\left(\varphi \vec{A}\right) = \varphi\, div\vec{A} + \left(grad\varphi, \vec{A}\right)$.

16.5.5-ta'rif. Agar vektorli maydonning divergensiyasi nolga teng, ya'ni $div\,\vec{A} = 0$ bo'lsa, u holda vektorli maydonga *solenoidal (trubkasimon)* maydon deyiladi.

Vektorli maydon solenoidal maydon bo'lishi uchun, jismning sirtidan o'tayotgan vektor oqimi nolga teng bo'lshi kerak, ya'ni

$$O = \iint\limits_{(S)} (\vec{A}, \vec{n})ds = 0.$$

(S_1) va (S_2) sirtlar bilan chegaralangan vektorli trubkani qaraylik. Trubkaning o'zining sirti (S_3) bo'lsin. Bu uchala sirt (S) sirtni tashkil etadi. U holda vektorli maydon oqimining 2-xossasiga asosan,

$$\iint\limits_{(S)} (\vec{A}, \vec{n})ds = \iint\limits_{(S_1)} (\vec{A}, \vec{n})ds + \iint\limits_{(S_2)} (\vec{A}, \vec{n})ds + \iint\limits_{(S_3)} (\vec{A}, \vec{n})ds.$$

Bu holda normal tashqariga yo'nalgan bo'ladi. Shuning uchun,

$$\iint\limits_{(S_1)} (\vec{A}, \vec{n}) ds = 0$$

chunki vektor oqimi normalga perpendikulyar bo'ladi. Agar (S_1) kesimda normalning yo'nalishini (S_2) dagi normalning yo'nalishi singari o'zgartirsak, u holda yuqoridagi tenglamadan

$$\iint\limits_{(S_1)} (\vec{A}, \vec{n}) ds = \iint\limits_{(S_2)} (\vec{A}, \vec{n}) ds$$

bo'ladi.

Demak, solenoidal maydonda vektorli maydon oqimi trubkaning ko'ndalang kesimlarida birdek bo'lar ekan.

16.6. Vektor maydonning sirkulyasiyasi va rotori. $\vec{A}(M)$ vektorli maydon, ya'ni P, Q, R funksiyalar va (K) silliq yoki bo'lakli silliq chiziq berilgan bo'lsin.

16.6.1-ta'rif. Ushbu

$$\int\limits_{(K)} Pdx + Qdy + Rdz \qquad (16.6.2)$$

egri chiziqli integralga $\vec{A}(M)$ vektorli maydonning (K) chiziq bo'yicha *sirkulyasiyasi* deyiladi.

Agar $\vec{A}(M)$ maydon kuchlar maydonini tashkil qilsa, u holda egri chiziqli integral, maydonning (K) egri chiziq bo'ylab bajargan ishini ifodalaydi.

Agar (K) chiziq yopiq bo'lsa, u holda (16.6.2) ni Stoks formulasi yordamida

$$\int\limits_{(K)} Pdx + Qdy + Rdz = = \iint\limits_{(S)} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy + \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) dydz + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) dxdz,$$

ko'rinishda ham yozish mumkin, bunda $(S)-(K)$ chiziq bilan chegaralangan sirt.

16.6.3-ta'rif. Ushbu

$$\iint\limits_{(S)}(\vec{A},\vec{n})ds=\iiint\limits_{(V)}div\,\vec{A}\,dV$$

ko'rinishda yozish mumkin ekan. Divergensiyaning bu ta'rifi koordinatalar sistemasini tanlashga bog'liq. Bu kamchilikdan qutulish uchun, uning koordinatalar sistemasiga bog'liq bo'lgan ta'rifini berish mumkin.

M nuqtani biror kichik (V) jism bilan o'raymiz. Bu jismning sirtini (S) orqali belgilaymiz.

16.5.4-ta'rif. Agar $\lim\limits_{V\to M}\dfrac{\iint\limits_{(S)}(\vec{A},\vec{n})ds}{V}$ limit mavjud bo'lsa, bu limit, $\vec{A}(M)$ vektorli maydonning M nuqtadagi *divergensiyasi* deyiladi.

Vektorli maydon oqimi quyidagi xossalarga ega:

1^0. $\iint\limits_{(S)}(\vec{A},\vec{n})ds=-\iint\limits_{(S^-)}(\vec{A},\vec{n})ds$.

2^0. Agar (S) sirt $(S_1),(S_2),\ldots,(S_n)$ silliq sirtlar yig'indisidan iborat bo'lsa, u holda $O=\sum\limits_{k=1}^{n}\iint\limits_{(S_k)}(\vec{A},\vec{n})ds$.

3^0. $div(\vec{A}+\vec{B})=div\,\vec{A}+div\,\vec{B}$. 4^0. $div\left(\varphi\vec{A}\right)=\varphi\,div\vec{A}+\left(grad\varphi.\ \vec{A}\right)$.

16.5.5-ta'rif. Agar vektorli maydonning divergensiyasi nolga teng, ya'ni $div\,\vec{A}=0$ bo'lsa, u holda vektorli maydonga *solenoidal (trubkasimon)* maydon deyiladi.

Vektorli maydon solenoidal maydon bo'lishi uchun, jismning sirtidan o'tayotgan vektor oqimi nolga teng bo'lshi kerak, ya'ni

$$O=\iint\limits_{(S)}(\vec{A},\vec{n})ds=0.$$

(S_1) va (S_2) sirtlar bilan chegaralangan vektorli trubkani qaraylik. Trubkaning o'zining sirti (S_3) bo'lsin. Bu uchala sirt (S) sirtni tashkil etadi. U holda vektorli maydon oqimining 2-xossasiga asosan,

$$\iint\limits_{(S)}(\vec{A},\vec{n})ds=\iint\limits_{(S_1)}(\vec{A},\vec{n})ds+\iint\limits_{(S_2)}(\vec{A},\vec{n})ds+\iint\limits_{(S_3)}(\vec{A},\vec{n})ds.$$

Bu holda normal tashqariga yo'nalgan bo'ladi. Shuning uchun,

$$\iint_{(S_3)} (\vec{A}, \vec{n})ds = 0$$

chunki vektor oqimi normalga perpendikulyar bo'ladi. Agar (S_1) kesimda normalning yo'nalishini (S_2) dagi normalning yo'nalishi singari o'zgartirsak, u holda yuqoridagi tenglamadan

$$\iint_{(S_1)} (\vec{A}, \vec{n})ds = \iint_{(S_2)} (\vec{A}, \vec{n})ds$$

bo'ladi.

Demak, solenoidal maydonda vektorli maydon oqimi trubkaning ko'ndalang kesimlarida birdek bo'lar ekan.

16.6. Vektor maydonning sirkulyasiyasi va rotori. $\vec{A}(M)$ vektorli maydon, ya'ni P,Q,R funksiyalar va (K) silliq yoki bo'lakli silliq chiziq berilgan bo'lsin.

16.6.1-ta'rif. Ushbu

$$\int_{(K)} Pdx + Qdy + Rdz \tag{16.6.2}$$

egri chiziqli integralga $\vec{A}(M)$ vektorli maydonning (K) chiziq bo'yicha *sirkulyasiyasi* deyiladi.

Agar $\vec{A}(M)$ maydon kuchlar maydonini tashkil qilsa, u holda egri chiziqli integral, maydonning (K) egri chiziq bo'ylab bajargan ishini ifodalaydi.

Agar (K) chiziq yopiq bo'lsa, u holda (16.6.2) ni Stoks formulasi yordamida

$$\int_{(K)} Pdx + Qdy + Rdz = = \iint_{(S)} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dxdy + \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)dydz + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)dxdz,$$

ko'rinishda ham yozish mumkin, bunda $(S)-(K)$ chiziq bilan chegaralangan sirt.

16.6.3-ta'rif. Ushbu

$$\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)\vec{i}+\left(\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z}\right)\vec{j}+\left(\frac{\partial P}{\partial z}-\frac{\partial R}{\partial x}\right)\vec{k}$$

vektorga vektorli maydonning *rotori* deyiladi va u quyidagicha belgilanadi:

$$rot\,\vec{A}=\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)\vec{i}+\left(\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z}\right)\vec{j}+\left(\frac{\partial P}{\partial z}-\frac{\partial R}{\partial x}\right)\vec{k}.$$

Stoks formulasini rotor yordamida quyidagicha yozamiz:

$$\int\limits_{(K)}\vec{A},d\vec{r}=\int\limits_{(S)}\int\left(rot\,\vec{A},\vec{n}\right)ds.$$

Demak, (K) yopiq chiziq bo'yicha vektorli maydonning sirkulyasiyasi, shu maydon rotorining (S) sirtdan o'tayotgan vektor oqimiga teng bo'lar ekan. Rotorning koordinatalar sistemasiga bog'liq bo'lmagan ta'rifini berish uchun, M nuqtadan o'tuvchi $\vec{n}$ vektorga perpendikulyar bo'lgan tekislik o'tkazamiz. M nuqtani o'rab turgan (S) sirtni chegaralovchi egri chiziqni (K) deb belgilaymiz.

16.6.4-ta'rif. Agar $\lim\limits_{(S)\to M}\dfrac{\oint\limits_{(K)}(\vec{A},d\vec{r})}{S}$ chekli limit mavjud bo'lsa, u holda bu limit, $\vec{A}(M)$ vektorli maydonning *rotori deyiladi.*

$\vec{A}(M)=(P,Q,R)$ vektorli maydonning rotorini quyidagicha ham yozish mumkin:

$$rot\,\vec{A}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ P & Q & R\end{vmatrix}.$$

$\vec{A}(M)$ vektorli maydon potensial maydon bo'lishi uchun, $rot\,\vec{A}=0$ bo'lishi zarur va yetarli.

Demak, rotor orqali ifodalangan vektorli maydon solenoidal maydon bo'lar ekan.

Rotor quyidagi xossalarga ega:

1^0 $rot\,\vec{c}=0$ ($\vec{c}=\{c_1,c_2,c_3\}$ – o‘zgarmas vektor); 2^0 $rot(C\vec{A})=C\,rot\,\vec{A}$; 3^0 $rot(\vec{A}+\vec{B})=rot\,\vec{A}+rot\,\vec{B}$; 4^0 $rot\,\vec{r}=0$, $\vec{r}=\{x,\,y,\,z\}$ – radius vektori.

16.7. Ikkinchi tartibli differensial amallar. Maydonlar nazariyasining asosiy tushunchalari quyidagilardir:

1. $u(x,y,z)$ skalyar funksiyaning gradiyenti $\overrightarrow{grad}\,u=\frac{\partial u}{\partial x}\vec{i}+\frac{\partial u}{\partial y}\vec{j}+\frac{\partial u}{\partial z}\vec{k}$.

2. $\vec{a}(x,y,z)$ vektor - funksiyaning divergensiyasi $div\,\vec{a}=\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}$. bu yerda P,Q,R – $\vec{a}$ vektorning koordinatalari.

3. $\vec{a}(x,y,z)$ vektorning rotori (uyurmasi)

$$rot\,\vec{a}=\left(\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z}\right)\vec{i}+\left(\frac{\partial P}{\partial z}-\frac{\partial R}{\partial x}\right)\vec{j}+\left(\frac{\partial Q}{\partial x}-\frac{\partial R}{\partial y}\right)\vec{k}.$$

Bu tushunchalar, irlandiyalik olim U.Gamilton tomonidan taklif qilingan simvolik vektor ∇ (nabla) operatori yordamida, soddagina qilib yoziladi:

$$\nabla u=\left(\frac{\partial}{\partial x}\vec{i}+\frac{\partial}{\partial y}\vec{j}+\frac{\partial}{\partial z}\vec{k}\right)u=\frac{\partial u}{\partial x}\vec{i}+\frac{\partial u}{\partial y}\vec{j}+\frac{\partial u}{\partial z}\vec{k}=grad\,u.$$

Demak, $\nabla u=grad u$. Xuddi shunga o‘xshash, $\nabla\cdot\vec{a}=div\,\vec{a}$. ∇(nabla) vektorning $\vec{a}(M)=P(x,y,z)\vec{i}+Q(x,y,z)\vec{j}+P(x,y,z)\vec{k}$ ector – funksiyaga vektorli ko‘paytmasi shu vektorning rotori (uyurmasi)ni beradi:

$$[\nabla,\vec{a}]=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ P & Q & R\end{vmatrix}=rot\,\vec{a}.$$

Gradiyent, divergensiya, rotor (uyurma)ni olish amallari, *birinchi tartibli differensial* amallardir.

Ma'lumki, $\overrightarrow{grad\,u}, rot\,\vec{a}$ amallar vektorli maydonlarni, $div\,\vec{a}$ amali esa, skalyar maydonni vujudga keltiradi. Bu amallarning quyidagi: $div\,grad\,u$, $grad\,div\,\vec{a}$, $rot\,grad\,u$, $rot\,rot\,\vec{a}$, $div\,rot\,\vec{a}$ kombinasiyalari, ikkinchi tartibli *differensial amallar* deyiladi. Bu differensial amallarni quyidagi jadval ko'rinishida tasvirlash mumkin:

↑	u skalyar maydon	$\vec{a}$ vektorli maydon	
	grad	*div*	*rot*
grad	—	$grad\,div\,\vec{a}$	—
div	$div\,grad\,u = \Delta u$	—	$div\,rot\,\vec{a} = 0$
rot	$rot\,grad\,u = 0$	—	$rot\cot\vec{a} =$ $= grad\,div\,\vec{a} - \Delta\vec{a}$

1-misol. $a)\, div(u\,grad\,v) = grad\,u \cdot grad\,v + u\Delta v$,

$b)$ $div(u\,grad u) = |grad u|^2 + u\Delta u$ ni isbotlang.

Yechilishi: $a)$ Ma'lumki, $\vec{a}(M)$ vektorning divergensiyasi ushbu

$$div\,\vec{a} = \frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z}$$

formula orqali, $v(x,y,z)$–skalyar maydonning gradiyenti esa,

$$grad\,v = \frac{\partial v}{\partial x}\vec{i} + \frac{\partial v}{\partial y}\vec{j} + \frac{\partial v}{\partial z}\vec{k}$$

formula orqali topiladi. U holda,

$$div(u\,grad\,v) = \frac{\partial}{\partial x}\left(u\frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial y}\left(u\frac{\partial v}{\partial y}\right) + \frac{\partial}{\partial z}\left(u\frac{\partial v}{\partial z}\right) =$$

$$= \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + u\frac{\partial^2 v}{\partial x^2} + \frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial y} + u\frac{\partial^2 v}{\partial y^2} + \frac{\partial u}{\partial z}\cdot\frac{\partial v}{\partial z} + u\frac{\partial^2 v}{\partial z^2} =$$

$$= u\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right)+\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial y}+\frac{\partial u}{\partial z}\frac{\partial v}{\partial z}=$$

$$= gradu\cdot gradv + u\Delta v,\ \text{бунда}\,\Delta=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{dy^2}+\frac{\partial^2}{\partial z^2}.$$

b) ∇ – operatorning xossalaridan foydalanib, quyidagicha isbotlash mumkin:

$$div(u\,gradu)=(\nabla, ugradu)=(gradu, \nabla u)+$$
$$+u(\nabla, gradu)=(gradu, gradu)+u\,div(gradu)=$$
$$=|gradu|^2+u\Delta u.$$

16-bob bo'yicha nazariy materiallarni mustahkamlash uchun topshiriqlar

16.1. Skalyar va vektorli maydonlarning ta'riflari va ularga misollar ([10], 2-q., 198-199 betlar; [17], 219-221 betlar; [30], 11- bo'lim).

16.2. Sirt sathi tushunchasi ([17], 214-215 betlar; [30], 14- bo'lim).

16.3. Vektor chizig'i tushunchasi va uning har xil ko'rinishdagi tenglamalari([17], 220-221 betlar; [30], 14- bo'lim).

16.4. Yo'nalish bo'yicha hosila, skalyar maydon ta'riflari ([12], 2-q., 71-75 betlar, [17], 216-219 betlar, [28], 577-580 betlar).

16.5. Yo'nalish bo'yicha hosilani topish formulasi([12], 2-q.,73-74 betlar; [17], 217 bet; [30], 16- bo'lim).

16.6. Skalyar maydonning gradiyenti va uning sodda xossalari([17], 217-219 betlar; [30], 16- bo'lim).

16.7. Potensial maydon ta'rifi va unga doir misollar ([17], 221-223 betlar; [28], 591-592 betlar; [9], 2-t., 9- bo'lim; [30], 16- bo'lim).

16.8. Vektorli maydonning divergensiyasi va uning fizik ma'nosi ([10], 2-q., 203 bet, [17], 221-223 betlar, [28], 580 bet; [9], 2-t., 9- bo'lim; [30], 16- bo'lim).

16.9. Vektorli maydonning rotori va uning fizik ma'nosi. Vektorli maydon potensiali ([10], 2-q., 203 bet, [17], 233-237 betlar, [28], 580-581betlar; [30], 16- bo'lim).

16.10. Solenoidal maydon va uning fizik ma'nosi. Misollar ([17], 229-230 betlar; [30], 16- bo'lim).

16.11. Gamilton operatori va uning sodda xossalari ([10], 2-q., 204-206 betlar, [17], 239-240 betlar, [28], 581-582 betlar; [30], 16- bo'lim).

16.12. Gradiyent, divergensiya va rotorni Gamilton operatori orqali yozish ([17], 224-243 betlar; [30], 16- bo'lim).

16.13. Vektorli maydonning sirkulyasiyasi va oqimi([17], 233-234 betlar; [30], 16- bo'lim).

16.1-amaliy mashg'ulot.
Maydonlar nazariyasi

1-misol. 1) $u=\sqrt{x^2+y^2+z^2}$ skalyar maydonning $P_1(-2;3;6)$ nuqtadagi $\vec{l}=\overrightarrow{P_1P_2}$, $\vec{l}$ yo'nalish bo'yicha hosilasini, toping, $P_2(-1;1;4)$,

Yechilishi ([9], 2-t., 7.1-bo'lim, [30], 14.5-bo'lim). Berilgan u – skalyar maydonning P_1 nuqtadagi $\vec{e}=\overrightarrow{P_1P_2}$ yo'nalish bo'yicha hosilasini, ushbu

$$\frac{\partial u(P_1)}{\partial \vec{e}}=\frac{\partial u(P_1)}{\partial x}\cos\alpha+\frac{\partial u(P_1)}{\partial y}\cos\beta+\frac{\partial u(P_1)}{\partial z}\cos\gamma \qquad (1)$$

formula orqali hisoblaymiz:

$$\frac{\partial u(P_1)}{\partial x}=\left.\frac{x}{\sqrt{x^2+y^2+z^2}}\right|_{P_1}=-\frac{2}{7},\quad \frac{\partial u(P_1)}{\partial y}=\left.\frac{y}{\sqrt{x^2+y^2+z^2}}\right|_{P_1}=\frac{3}{7},$$

$$\frac{\partial u(P_1)}{\partial z}=\left.\frac{z}{\sqrt{x^2+y^2+z^2}}\right|_{P_1}=\frac{6}{7}.$$

$\overrightarrow{P_1P_2}$ – vektor yo'nalishi bo'yicha yo'nalgan $\vec{l}^0$ birlik vektor

$$\vec{l}^0=\frac{\overrightarrow{P_1P_2}}{\left|\overrightarrow{P_1P_2}\right|}=\left(\frac{1}{3};\frac{-2}{3};-\frac{2}{3}\right),\ \text{bunda } \cos\alpha=\frac{1}{3},\ \cos\beta=-\frac{2}{3},\ \cos\gamma=-\frac{2}{3}.$$

1) formulaga asosan, $\dfrac{\partial u(P_1)}{\partial \vec{l}}=\left(-\dfrac{2}{7}\right)\cdot\dfrac{1}{3}+\dfrac{3}{7}\left(-\dfrac{2}{3}\right)+\dfrac{6}{7}\cdot\left(-\dfrac{2}{3}\right)=-\dfrac{20}{21}.$

2) $u=\sqrt{x^2+y^2+z^2}$ skalyar maydonning $P_1(-2;3;6)$ nuqtadagi gradiyentini toping.

Yechilishi. Berilgan u skalyar maydonning $P_1(-2;3;6)$ nuqtadagi gradiyenti ushbu

$$gradu(P_1)=\frac{\partial u(P_1)}{\partial x}\vec{i}+\frac{\partial u(P_1)}{\partial y}\vec{j}+\frac{\partial u(P_1)}{dz}\vec{k}$$

formula bo'yicha hisoblanadi. Ravshanki,

$$\frac{\partial u(P_1)}{\partial x}=-\frac{2}{7},\quad \frac{\partial u(P_1)}{\partial y}=\frac{3}{7},\quad \frac{\partial u(P_1)}{\partial z}=\frac{6}{7}.$$

Demak, $gradu(P_1)=-\frac{2}{7}\vec{i}+\frac{3}{7}\vec{j}+\frac{6}{7}\vec{k}$.

3) $u=\sqrt{x^2+y^2+z^2}$ skalyar maydonning $P_1(-2;3;6)$ va $P_2(-1;1;4)$ nuqtalardagi gradiyentlari orasidagi burchakni toping.

Yechilishi. $gradu(P_1)$ va $gradu(P_2)$ lar orasidagi burchak ushbu

$$\cos\varphi=\frac{(gradu(P_1), gradu(P_2))}{|gradu(P_1)|\cdot|gradu(P_2)|} \qquad (2)$$

formula bo'yicha hisoblanadi:

Skalyar maydonning P_2 nuqtadagi gradiyentini topamiz:

$$grad(P_2)=\left.\frac{x}{\sqrt{x^2+y^2+z^2}}\right|_{P_2}\vec{i}+\left.\frac{y}{\sqrt{x^2+y^2+z^2}}\right|_{P_2}\vec{j}+$$

$$+\left.\frac{z}{\sqrt{x^2+y^2+z^2}}\right|_{P_2}\vec{k}=\frac{-1}{3\sqrt{2}}\vec{i}+\frac{1}{3\sqrt{2}}\vec{j}+\frac{4}{3\sqrt{2}}\vec{k}.$$

Endi (2) formulaga asosan,

$$\cos\varphi=\frac{\frac{2}{21\sqrt{2}}+\frac{3}{21\cdot\sqrt{2}}+\frac{24}{21\cdot\sqrt{2}}}{1\cdot 1}=\frac{29}{21\sqrt{2}}$$

yoki

$$\varphi=arc\cos\frac{29}{21\sqrt{2}}\approx arc\cos\ 0{,}9764.$$

2-misol. Berilgan $\vec{a}=(2xy+z)\vec{i}+(x^2-2y)\vec{j}+x\vec{k}$ vektorli maydonning solenoidal maydon emasligini ko‘rsating.

Yechilishi. Ta’rifga ko‘ra, vektorli maydon solenoidal maydon bo‘lishi uchun uning divergensiyasi nulga teng, ya’ni $div\,\vec{a}=0$ bo‘lishi kerak:

$$div\,\vec{a}=\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}=2y-2+0\neq 0.$$

Demak, berilgan vektorli maydon solenoidal maydon emas ekan.

3-misol. $\vec{a}=x\vec{i}+z\vec{k}$ vektorli maydonning $(S):z=x^2+y^2,\ z=4$ sirt bo‘yicha vektor oqimini toping.

Yechilishi. $(S_1):z=x^2+y^2$ va $(S_2):z=4$ sirtlar bo‘yicha vektor oqimlarini alohida hisoblaymiz:

(S_1)–sirtga o‘tkazilgan normal vektorlarni topamiz:

$$\vec{n^0}=\pm\frac{grad(z-x^2-y^2)}{|grad(z-x^2-y^2)|}=\pm\frac{-2x\vec{i}-2y\vec{j}+\vec{k}}{\sqrt{1+4x^2+4y^2}}$$

$\left(\vec{n}^0,z\right)>90^0$ bo‘lgani uchun ildiz ishorasini « - » qilib olamiz, bundan

$$\left(\vec{a},\vec{n^0}\right)=-\frac{z-2x^2}{\sqrt{1+4x^2+4y^2}}=\frac{2x^2-z}{\sqrt{1+4x^2+4y^2}},\ dS_1=\frac{dxdy}{|\cos\gamma|}=\frac{dxdy}{\sqrt{1+4x^2+4y^2}}$$

Demak,

$$O_1=\iint_{(S_1)}\left(\vec{a},\vec{n^0}\right)dS_1=\iint_{(D_{xy})}(2x^2-z)\Big|_{z=x^2+y^2}dxdy=\iint_{x^2+y^2\le 4}(x^2-y^2)dxdy,$$

keyingi integralni hisoblash uchun $x=\rho\cos\varphi, y=\rho\sin\varphi$ almashtirishni bajaramiz. Unda

$$\iint_{x^2+y^2\le 4}(x^2-y^2)dxdy=\int_0^{2\pi}d\varphi\int_0^2\rho^3\cdot\cos 2\varphi\, d\rho=4\cdot\int_0^{2\pi}\cos 2\varphi\, d\varphi=0.$$

Xuddi shunday (S_2) sirt bo‘yicha vektor oqimining hisoblaymiz:

$$\vec{n}^0=\pm\frac{grad(z-4)}{|grad(z-4)|}=\pm\vec{k},\quad \left(\widehat{\vec{n}^0,z}\right)<90^0\Rightarrow \vec{n}^0=\vec{k},$$

bo'lgani uchun

$$\left(\vec{a},\vec{n}^0\right)=z\vec{k},\ dS_2=\frac{dxdy}{|\cos\gamma|}=dxdy,$$

bunda $|\cos\gamma|=1$.

Shunday qilib,

$$O_2=\iint\limits_{(S_2)}\left(\vec{a},\vec{n}^0\right)dS_2=\iint\limits_{(D_{xy})} z\Big|_{z=4}dxdy=4\int_0^{2\pi}d\varphi\int_0^2\rho d\rho=16\pi.$$

Demak, $O=O_1+O_2=16\pi$.

4–masala. $\vec{a}(M)=xz^2\vec{i}+yx^2\vec{j}+zy^2\vec{k}$ vektorli maydonning $x^2+y^2\le z^2$, $0\le z\le 1$ konus yon sirtining tashqi tomoni bo'yicha o'tayotgan O vektor oqimini toping.

Yechilishi. Konusning butun sirti bo'yicha olingan integralni J_1 orqali, uning asosining yuqori tomoni bo'yicha olingan integralni J_2 bilan belgilaymiz. U holda konusning yon sirti bo'yicha olingan integral $J=J_1-J_2$. bo'ladi. J_1 integralga Gauss – Ostrogradskiy formulasini qo'laymiz:

$$J_1=\oiint\limits_{(S_1)}xz^2dydz+yx^2dzdx+zy^2dxdy=\iiint\limits_{(v)}\left(z^2+x^2+y^2\right)dxdydz,$$

bunda (S_1)–konusning tashqi to'la sirti.

Bu integralni hisoblash uchun silindrik koordinatlar sistemasiga o'tamiz:

$$x=\rho\cos\varphi,\ y=\rho\sin\varphi,\ z=1,\ 0\le\varphi\le 2\pi,\ 0\le\rho\le z,\ 0\le z\le 1,$$

bunda $I=\rho$,

$$J_1=\int_0^1dz\int_0^{2\pi}d\varphi\int_0^z\left(z^2+\rho^2\right)\rho d\rho=\frac{3}{10}\pi.$$

Endi konusning (S_2) asosi bo'yicha olingan J_2 sirt integrali ni hisoblaymiz:

$$J_2=\iint\limits_{(S_2)} xz^2dydz+yx^2dxdz+zy^2dxdy=\iint\limits_{(D_{xy})} y^2dxdy=$$

$$=\int_0^{2\pi}d\varphi\int_0^1\rho^3\sin^2\varphi d\rho=\frac{1}{4}\int_0^{2\pi}\sin^2\varphi d\varphi=\frac{\pi}{4}.$$

Shunday qilib, talab qilingan oqim

$$O=J=J_1-J_2=\frac{3}{10}\pi-\frac{\pi}{4}=\frac{\pi}{20}.$$

5-masala. $\vec{a}(M)=(x^3+y^2)\vec{i}+(y^2+z^3)\vec{j}+(z^2+x^3)\vec{k}$ vektorli maydonning $M_0(0;1;2)$ nuqtadagi divergensiyasini toping.

Yechilishi. Vektorli maydonning divergensiyasi quyidagi

$$div\,\vec{a}(M)=\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}$$

formula orqali topiladi. Bu holda

$$P=x^3+y^2, Q=y^2+z^3, R=z^2+x^3, div\left(\vec{a}(M_0)\right)=\left(3x^2+2y+2z\right)\Big|_{M_0}=6.$$

berilgan maydon potensial maydon bo‘lar ekan.

Mustaqil yechish uchun misollar

1. Quyidagi berilgan skalar maydonlarning sath chiziqlarini aniqlang.

1) $U=xy$. 2) $U=e^{x^2-y^2}$. 3) $U=\frac{y}{x}$.

4) $U=\vec{r}^{\,2}$, bunda $\vec{r}=x\vec{i}+y\vec{j}$. 5) $U=\arcsin(x+y)$.

2. Quyidagi berilgan skalar maydonlarning sath sirtlarini toping.

1) $U=x^2+y^2+z^2$. 2) $U=x^2+y^2$. 3) $U=arctg\frac{x^2+y^2}{z^2}$

4) $U=\frac{x^2}{4}+\frac{y^2}{9}+\frac{z^2}{16}$. 5) $U=x+y+z$.

3. Quyidagi funksiyaning nuqtada $\overrightarrow{M_0M}$ yo‘nalish bo‘yicha hosilasini toping.

1) $f(x,y)=3x^2+5y^2$, $M_0(1;-1)$, $M(2;1)$.

2) $f(x,y)=xy^2z^3$, $M_0(3,2,1)$, $M(7,5,1)$.

4. Ushbu $f(x,y)=3x^2+5y^2$ funksiyaning $M_0(1;1)$ nuqtada, $\vec{l}=\{-1/\sqrt{2},1/\sqrt{2}\}$ vektorning yo'nalishi bo'yicha hosilasini toping.

5. Ushbu $f(x,y)=x^3+2xy^2+3yz^2$ funksiyaning $M_0(3;3;1)$ nuqtada, $\vec{l}=\{2/3;2/3;2/3\}$ vektorning yo'nalishi bo'yicha hosilasini toping.

6. Ushbu $f(x,y)=3x^4+y^3+xy$ funksiyaning $M_0(1,2)$ nuqtada, Ox o'q bilan 135° burchak tashkil qilgan numing yo'nalishi bo'yicha hosilasini toping.

7. Ushbu $f(x,y)=arctg\frac{y}{x}$ funksiyaning $x^2+y^2=2x$ aylananing $M_0\left(\frac{1}{2},\frac{\sqrt{3}}{2}\right)$ nuqtasiga o'tkazilgan tashqi normalning yo'nalishi bo'yicha hosilasini toping.

8. A(1;0;-1) nuqtada $U=x^2-y^2+yz-x$ skalar maydonning gradiyent yo'nalishi va uzunligini aniqlang.

9. $U(x,y,z)$ skalar maydonning gradiyentini toping.

1) $U(x,y,z)=x\cdot y\cdot z.$ 2) $U=e^{xy}-yz^2.$

3) $U=x^2-y^2-z^2.$ 4) $U=\ln(x^2-2y^2+z^2).$

10. $A(x_0,y_0,z_0)$ nuqtada $V(x,y,z)$ skalar maydon gradiyentini toping.

1) $V=2x^2-3xy-xz-z^2$, $A(1,1,1)$. 2) $V=z\sin(x-y)$, $A\left(\frac{\pi}{2};\frac{\pi}{6};1\right)$.

3) $V=arctg(x+2y+z^2)$, $A(1;1;0)$. 4) $V=z\cos(x-y)$, $A\left(\frac{\pi}{2};\frac{\pi}{3};1\right)$.

11. Qaysi nuqtalarda $U=\vec{r}^{\,2}\left(\vec{r}=x\vec{i}+y\vec{j}+z\vec{k},\ r=\left|\vec{r}\right|\right)$ skalar maydon gradiyenti

1) Ox o'qqa parallel; 2) Ox o'qqa perpendikular;

3) x-$y+z=z$ tekislikka perallel:

4) $x-3y+2z-1=0$ tekislikka perpendikular;

5) raven 0; 6) uzunligi 4 ga teng bo'ladi?

12. 1) $U=arctg\frac{x}{y}$ skalar maydonning $P_1(1;1)$ va $P_2(-1;-1)$ nuqtalaridagi gradiyentlari orasidagi burchakni toping.

2) $A(3;0;1)$ va $B(1;-1;0)$ nuqtalarda $U=\dfrac{x}{x^2+y^2}$ skalar maydon gradiyentlari orasidagi burchakni toping.

3) $U=\sqrt{x^2+y^2+z^2}$ va $V=\ln\left(x^2+y^2+z^2\right)$ skalar maydonlar gradiyentlarining $P_0(0;0;1)$ nuqtadagi qiymatini hisoblab ular orasidagi burchakni toping.

4) $U=x^2+2y^2-z^2$ skalar funksiyaning $P_1(2;3;-1)$ va $P_2(1;-1;2)$ nuqtalardagi gradiyentlari orasidagi burchagini toping.

13. Quyidagi tengliklarni isbotlang, bunda U, V, f differensiallanuvchi, C- o‘zgarmas, $\left|\vec{r}\right|=r=\sqrt{x^2+y^2+z^2}$:

1) $grad\,(U+C)=gradU$;

2) $grad(U\cdot C)=C\cdot gradU$;

3) $grad\left(\dfrac{U}{V}\right)=\dfrac{1}{V^2}(U\,gradU-U\cdot gradV)\ (V\neq 0)$;

4) $grad\left(U^n\right)=n\,U^{n-1}gradU\ (n\in N)$;

5) $grad\left(\varphi'(U)\right)=\varphi'(U)\cdot gradU$;

6) $gradf(U,V)=\dfrac{\partial f}{\partial U}\cdot gradU+\dfrac{\partial f}{\partial V}\cdot gradV$;

7) $grad r^2=2\vec{r}$; 8) $grad r=\dfrac{\vec{r}}{r}$.

14. $\vec{r}=x\vec{i}+y\vec{j}+z\vec{k}$, $r=\left|\vec{r}\right|$, $\vec{a}$ va $\vec{b}$– o‘zgarmas vektorlar. $V=V(r)$ skalar funksiyaning $gradV$ ni toping.

1) $V=\left(\vec{a},\vec{r}\right)$. 2) $V=\left(\vec{a},\vec{r}\right)\cdot\left(\vec{b},\vec{r}\right)$.

3) $V=\vec{r}^{\,2}$. 4) $V=\left[\vec{a},\vec{r}\right]^2$. 5) $V=\ln r$.

15. Quyidagi $\vec{a}$ vektor maydonlarining vektor chiziqlarini toping.

1) $\vec{a}=x\vec{i}+2y\vec{j}$. 2) $\vec{a}=x\vec{i}-y\vec{j}$.

3) $\vec{a}=2z\vec{i}+4y\vec{k}$. 4) $\vec{a}=x^2\vec{i}+y^2\vec{j}$.

5) $\vec{a}=z\vec{j}-y\vec{k}$. 6) $\vec{a}=x\vec{i}+y\vec{j}+z\vec{k}$.

16. M nuqta orqali o‘tuvchi $\vec{a}$ vektor maydonning vektor chizig‘ini toping.

1) $\vec{a} = x\vec{i} + y\vec{j} + z\vec{k}$, $M(3,1,2)$.

2) $\vec{a} = -y\vec{i} + x\vec{j} + 4\vec{k}$, $M(1;0,0)$.   3) $\vec{a} = x^2\vec{i} - y^3\vec{j} + z^2\vec{k}$, $M\left(\frac{1}{2};-\frac{1}{2};1\right)$.

4) $\vec{a} = xz\vec{i} + yz\vec{j} + (x^2+y^2)\vec{k}$, $M(1;1;0)$.

16. $\vec{a} = x\vec{i} + z\vec{k}$ vektor maydonning (S): $z = x^2 + y^2$, $z = 4$ sirt bo‘yicha vektor oqimini toping.

18. Markazi koordinatalar boshida, radiusi R ga teng bo‘lgan $x^2 + y^2 + z^2 = R^2$ sferaning tashqi normal yo‘nalishi orqali o‘tuvchi $\vec{a} = \frac{\vec{r}}{|\vec{r}|}$ $\left(\vec{r} = \{x,y,z\}\right)$ vektor maydonning oqimini toping.

19. $x + y = 4$ va $y = 1$ tekisliklar bilan chegaralangan $x^2 + y^2 = R^2$ silindrning tashqi sirtlari orqali o‘tuvchi $\vec{a} = \{x;-xy,z\}$ vektor maydonning oqimini toping.

20. $0 \le x \le b$, $0 \le y \le b$, $0 \le z \le b$ kubning tashqi normal yo‘nalishi orqali o‘tuvchi $\vec{a} = \{x^3, y^3, z^3\}$ vektorning oqimini toping.

21. $z = 0$ va $z = 1$ tekisliklar bilan chegaralangan $x^2 + y^2 = 1$ silindrikning tashqi sirtlari orqali o‘tuvchi $\vec{a} = \{-y, x, z\}$ vektor maydoning oqimini toping.

22. $\vec{n}$ vektor Oy o‘q bilan o‘tkir burchak tashkil qilib, $x = 0$ va $x = 1$ tekisliklar bilan chegaralangan birinchi oktantada joylashgan $y + z = 1$ tekislikning sirti orqali o‘tuvchi $\vec{a} = \{x, y, 0\}$ vektor maydonning oqimini toping.

23. $x^2 + y^2 + z^2 = R^2$sferaning birinchi oktantada joylashgan, tashqi normal yo‘nalishi orqali o‘tuvchi $\vec{a} = \{xy, yz, xz\}$ vektor maydon oqimini toping.

24. $\vec{n}$ vektor Oz o‘q bilan o‘tmas burchak tashkil qilib, $z = 0$ va $z = 2$ tekisliklar bilan chegaralangan $S: x^2 + y^2 = z^2$ konus sirt orqali o‘tuvchi $\vec{a} = \{y - z,\ z - x,\ x - y\}$ vektor maydon oqimini toping.

25. $\vec{n}$ tashqi normal, $(x-1)^2+(y-2)^2+(z-3)^2=9$ sfera orqali o'tuvchi $\vec{a}=\{x^2,-2xy,z\}$ vektor maydon oqimini toping.

26. $\vec{n}$ tashqi normal, $z=0$ va $z=2$ tekisliklar orasida joylashgan $x^2+y^2=R^2$ silindrning yon sirti orqali o'tuvchi $\vec{a}=\{x^2,y^2,z^2\}$ vektor maydon oqimini toping.

27. Ostrogradskiy teoremasidan foydalanib, ko'rsatilgan yopiq (S) sirt orqali o'tuvchi vektor maydonning oqimini toping.

1) $\vec{a}=(1+2x)\vec{i}+y\vec{j}+z\vec{k}$, $(S): x^2+y^2=z^2$, $z=4$.

2) $\vec{a}=x\vec{i}+xz\vec{j}+y\vec{k}$, $(S): x^2+y^2=4-z, z\geq 0$.

3) $\vec{a}=x^3\vec{i}+y^3\vec{j}+z^3\vec{k}$, $(S): x^2+y^2+z^2=x$.

4) $\vec{a}=x^3\vec{i}+y^3\vec{j}+z\vec{k}$, $(S): x^2+y^2\leq z\leq 1$

5) $\vec{a}=x^2\vec{i}+y^2\vec{j}+z^2\vec{k}$, $(S): (x-1)^2+(y-3)^2+(z-4)^2=1$.

28. Tashqi normal yo'nalishiga ega bo'lgan, yopiq (S) sirt orqali o'tuvchi $\vec{a}=\{P,Q,R\}$ vektor maydon oqimini toping.

1) $\vec{a}=\{x,z,y\}$, $(S): x^2+y^2=R^2$, $z=0$, $z=4$.

2) $\vec{a}=\{x^3;y^3;z^3\}$, $(S): |x|\leq a$, $|y|\leq a$, $|z|\leq a$.

3) $\vec{a}=\{xz;y^2;x\}$, $(S): x+y=1$, $x=0, y=0, z=0, z=1$.

4) $\vec{a}=\{y^2-z;xy;-(y+x)\}$, $(S): x+y+z=1$, $x=0, y=0$, $z=0$.

5) $\vec{a}=\{2x;2y;-z\}$, $(S): \sqrt{x^2+y^2}\leq z\leq H$.

29. $\vec{a}$ vektor maydon divergensiyasining M nuqtadagi qiymatini toping.

1) $\vec{a}=x^2\vec{i}+y^2\vec{j}+z^2\vec{k}$, $M(1;1,1)$.

2) $\vec{a}=xy^2\vec{i}-yz\vec{j}+z^2\vec{k}$, $M(2,1,1)$.

3) $\vec{a}=xy\vec{i}+yz\vec{j}+xz\vec{k}$, $M(2,1,3)$.

30. Agar $\vec{r}=x\vec{i}+y\vec{j}+z\vec{k}$, $|\vec{r}|=\sqrt{x^2+y^2+z^2}$ bo'lsa, $div\,\vec{a}$ ni toping:

1) $\vec{a}=\vec{r}$. 2) $\vec{a}=\left|\vec{r}\right|\cdot\vec{r}$. 3) $\vec{a}=\dfrac{\vec{r}}{\left|\vec{r}\right|}$.

4) $\vec{a}=\dfrac{-x\,\vec{i}+y\,\vec{j}+z\,\vec{k}}{\sqrt{x^2+y^2}}$. 5) $\vec{a}=\frac{1}{2}\left(y^2+z^2\right)\vec{i}+\frac{1}{2}\left(z^2+x^2\right)\vec{j}+\frac{1}{2}\left(x^2+y^2\right)\vec{k}$.

6) $\operatorname{div}[\vec{a},\vec{r}]$, $\vec{a}-$ o‘zgarmas vektor. 7) $div\left(f\left(\vec{r}\right)\cdot\vec{r}\right)$.

31. Quyidagi ifodalarni koordinatalar shaklida yozing:

1) $div(gradU)$. 2) $div(U\cdot gradU)$. 3) $div(grad(UV))$.

32. $\vec{r}=x\,\vec{i}+y\,\vec{j}+z\,\vec{k}$, $\left|\vec{r}\right|=\sqrt{x^2+y^2+z^2}$ bo‘lsa, quyidagi ifodalarni toping.

1) $div\left(grad r^2\right)$. 2) $div\,(\vec{c}\cdot\vec{r})\left(\vec{c}=const\right)$. 3) $div\,grad\,f(r)$.

4) $div\left(f(r)\cdot\vec{r}\right)$. 5) $div[\vec{c},\vec{r}]$, $\vec{c}=const$.

33. Quyidagi tengliklarni isbotlang.

1) $div\left(\vec{a}+\vec{b}\right)=div\,\vec{a}+div\,\vec{b}$, 2) $div\left(U\cdot\vec{a}\right)=U\,div\,\vec{a}+\vec{a}\,gradU$.

34. Quyidagi berilgan vektorli maydonlarning qaysilari solenoidal maydon bo‘lishini aniqlang.

1) $\vec{a}=x\left(z^2-y^2\right)+y\left(x^2-y^2\right)\vec{j}+z\left(y^2-x^2\right)\vec{k}$.

2) $\vec{a}=\{xy,-y-x,z-xy\}$.

3) $\vec{a}=\left\{y^2,-\left(x^2+y^2\right),z\left(3y^2+1\right)\right\}$.

4) $\vec{a}=\left(x^2yz-x^3\right)\vec{i}+yx^3\,\vec{j}+\left(x^2z-y\right)\vec{k}$.

5) $\vec{a}=x^2yz\,\vec{i}+2xyz\,\vec{j}-z^2(xy+x)\,\vec{k}$.

6) $\vec{a}=\dfrac{x}{\sqrt{x^2+y^2}}\vec{i}-\dfrac{y}{\sqrt{x^2+y^2}}\vec{j}+\dfrac{\left(x^2-y^2\right)z}{\sqrt{\left(x^2+y^2\right)^3}}\vec{k}$.

7) $\vec{a}=\left(x^2-yz+2\right)\vec{i}-2xy\,\vec{j}+\left(yx^3-1\right)\vec{k}$.

35. Qanday shartda $\vec{a}=\varphi(r)\vec{r}$ $\left(r=\sqrt{x^2+y^2+z^2}\right)$ maydon solenoidal maydon bo‘ladi?

36. $\vec{a} = \{P,Q\} = P\vec{i} + Q\vec{j}$ vektor maydonning sirkulatsiyasini berilgan (L) chiziq bo'yicha hisoblang.

1) $\vec{a} = \{y;-x\}$, $(L): x^2 + y^2 = R^2$ – aylana.

2) $\vec{a} = \{y;-x\}$, $(L): (x-1)^2 + (y+1)^2 = R^2$ – aylana.

3) $\vec{a} = \{xy;1\}$, $(L): 0 \le x \le 1,\ 0 \le y \le 1$ – kvadrat chegarasi.

4) $\vec{a} = \{y;2x\}$, (L): uchlari $A(1;1)$, $B(4;2)$, $C(0,4)$ nuqtalarda bo'lgan ABC uchburchak chegarasi bo'yicha.

5) $\vec{a} = \{-y,x\}$, (L): $x = 2\cos t - \cos 2t$, $y = 2\sin t - \sin 2t$ kardioida (parametrning o'sish tartibida).

37. $\vec{a} = \{P,Q,R\} = P\vec{i} + Q\vec{j} + R\vec{k}$ vektor maydonning sirkulatsiyasini berilgan (L) chiziq bo'yicha hisoblang:

1) $\vec{a} = \{xz + y,\ yz - x,\ -x^2 - y^2\}$, $(L): \begin{cases} x^2 + y^2 = 1, \\ z = 3. \end{cases}$

2) $\vec{a} = \{-y;x;c\}$, (bunda c - o'zgarmas son), $(L): \begin{cases} x^2 + y^2 = 1, \\ z = 0. \end{cases}$

3) $\vec{a} = \{y^2;z^2;x^2\}$, $(L): \begin{cases} x^2 + y^2 + z^2 = R^2, \\ x^2 + y^2 = Rx, \end{cases} \quad z \ge 0.$

4) $\vec{a} = \{x - y;y - z;z - x\}$, (L): uchlari $A(1,0,0)$, $B(1,1,0)$, $C(1,0,1)$ nuqtalarda bo'lgan ABC uchburchak konturi.

5) $\vec{a} = \{x^2,y^2,z^2\}$, bunda $(L): \begin{cases} x^2 + y^2 = 1, \\ z = 1. \end{cases}$

6) $\vec{a} = \{y + z;z + x;x + y\}$, bunda $(L): \begin{cases} x^2 + y^2 + z^2 = R^2, \\ x + y + z = 0. \end{cases}$

7) $\vec{a} = \left\{-\dfrac{y}{x^2 + y^2};\ \dfrac{x}{x^2 + y^2};\ 0\right\}$, bunda $(L): \begin{cases} x^2 + y^2 = 1, \\ z = H. \end{cases}$

38. Quyidagi berilgan vektorlarning sirkulatsiyasini mos konturlar yo'nalishida avval to'g'ridan - to'g'ri, keyin esa, Stoks formulasidan foydalanib hisoblang.

1) $\vec{a} = \{z;x;y\}$, bunda $(L): \begin{cases} x^2 + y^2 = 4, \\ z = 0. \end{cases}$

2) $\vec{a}=\{y;-x;z\}$, bunda $(L):\begin{cases} x^2+y^2+z^2=4, \\ x^2+y^2=z^2, z\geq 0. \end{cases}$

3) $\vec{a}=\{y;-x;-(x+y)\}$, bunda $(L):\begin{cases} z=x^2+y^2, \\ z=1. \end{cases}$

4) $\vec{a}=\{z^2;0;0\}$ bunda $(L):\begin{cases} x^2+y^2+z^2=16, \\ x=0,\ y=0,\ z=0. \end{cases}$

39. Quyidagi $\vec{a}$ vektor maydoning rotorini toping:

1) $\vec{a}=x\vec{i}+y\vec{j}+z\vec{k}$.

2) $\vec{a}=(x+z)\vec{i}+(y+z)\vec{j}+(x^2+z)\vec{k}$.

3) $\vec{a}=(x^2+y^2)\vec{i}+(y^2+z^2)\vec{j}+(z^2+x^2)\vec{k}$.

4) $\vec{a}=z^3\vec{i}+y^3\vec{j}+x^3\vec{k}$.

5) $\vec{a}=\frac{1}{2}(-y^2\vec{i}+x^2\vec{j})$

40. Quyidagi $\vec{a}$ vektor maydonning M_0 nuqtadagi $rot\,\vec{a}$ hisoblang:

1) $\vec{a}=xyz\,\vec{i}+(2x+3y-z)\vec{j}+(x^2+z^2)\vec{k}$, $M_0(1;3;2)$.

2) $\vec{a}=\frac{y}{z}\vec{i}+\frac{z}{x}\vec{j}+\frac{x}{y}\vec{k}$, $M_0(1;2;-2)$.

41. Quyidagi tengliklarni isbotlang.

1) $rot\left(\vec{a_1}\pm\vec{a_2}\right)=rot\,\vec{a_1}\pm rot\,\vec{a_2}$.

2) $rot\left(C\vec{a}\right)=C\,rot\,\vec{a}$ $(C=const)$.

3) $rot\left(f\cdot\vec{a}\right)=f\cdot rot\,\vec{a}+[grad\,f,\vec{a}]$.

4) $div[\vec{a},\vec{b}]=\vec{b}\,rot\,\vec{a}-\vec{a}\cdot rot\,\vec{b}$.

5) $rot(grad\,u)=0$.

6) $div\left(rot\,\vec{a}\right)=0$.

42. Quyidagi berilgan vektor maydonlarning potensial maydon bo'lish - bo'lmasligini tekshiring:

1) $\vec{a}=x\vec{i}+y\vec{j}+z\vec{k}$.

2) $\vec{a}=x\vec{i}+yx\vec{j}+zy\vec{k}$.

3) $\vec{a}=x^2\vec{i}+y^2\vec{j}+z^2\vec{k}$.

4) $\vec{a}=xz\vec{i}+2y\vec{j}+xy\vec{k}$.

5) $\vec{a}=xy\vec{i}-z\vec{j}+x\vec{k}$.

6) $\vec{a} = y^2(1-z)\vec{i} + 2xy(1-z)\vec{j} - (xy^2 - 3z^2)\vec{k}$.

43. $\vec{a} = f(r)\vec{r}$ (bu yerda $\vec{r} = \{x, y, z\}$, $f(r) = f\left(\sqrt{x^2 + y^2 + z^2}\right)$ differensiallanuvchi funksiya) vektor maydonning potensial maydon ekanligini isbotlang.

44. $\vec{a}(M)$ vektor maydonning potensial maydon ekanligini isbotlang va uning potensialini toping.

1) $\vec{a} = (3x^2y - y^3)\vec{i} + (x^3 - 3xy^2)\vec{j}$. 2) $\vec{a} = \frac{\vec{r}}{r^2}$.

3) $\vec{a} = x^2\,\vec{i} + y^2\,\vec{j} + z^2\,\vec{k}$. 4) $\vec{a} = yz\,\vec{i} + xz\,\vec{j} + yx\,\vec{k}$.

5) $\vec{a} = (z - 2x)\,\vec{i} + (z - 2y)\,\vec{j} + (x + y)\vec{k}$.

6) $\vec{a} = 2xyz\,\vec{i} + x^2z\,\vec{j} + x^2y\,\vec{k}$.

7) $\vec{a} = (yz - xy)\,\vec{i} + \left(xz - \frac{x^2}{2} + yz^2\right)\vec{j} + (xy + y^2z)\vec{k}$.

8) $\vec{a} = \{2xy^3 + 2xy\sin(x^2y);\ 3x^2y^2 + x^2\sin(x^2y)\}$.

45. Ushbu $\left[\vec{r}, \vec{c}\right]$ ($\vec{c}$ – nolmas o'zgarmas vektor, $\vec{r} = \{x, y, z\}$,) vektor maydon: 1) potensialmi? 2) solenoidalmi? .

46. Agar $\vec{r} = x\,\vec{i} + y\,\vec{j} + z\,\vec{k}$, $\left|\vec{r}\right| = \sqrt{x^2 + y^2 + z^2}$ bo'lsa, quyidagi $\vec{a}$ vektor maydonlarning qaysi biri potensial, solenoidal?

1) $\vec{a} = \frac{\vec{r}}{r^3}$; 2) $\vec{a} = \frac{1}{r}\vec{r}$.

47. A nuqtadan B nuqtagacha $\vec{F}$ vektor maydonning ishini hisoblang.

1) $\vec{F} = 3x^2\,\vec{i} + 2y\,\vec{j} + \vec{k}$, $A(1; 2; -1)$, $B(0; 1; 1)$.

2) $\vec{F}(y^2 + 2xz)\vec{i} + (z^2 + 2xy)\vec{j} + (x^2 + 2yz)\vec{k}$, $A(0,0,0)$, $B(1;-1;1)$.

3) $x^2 + y^2 = 1$ aylananing $A(1;0)$ nuqtasidan $B(0;1)$ nuqtasigacha bo'lgan yoyi bo'yicha olingan $\vec{F} = 2xy\,\vec{i} + x\,\vec{j}$ kuch maydonining bajargan ishini hisoblang.

4) $y = x^2$ parabolaning $O(0,0)$ nuqtadan $A(1,1)$ nuqtasigacha bo'lgan yoyi bo'yicha olingan $\vec{F} = \{x^2 + 2xy;\ x^2 + y^2\}$ kuch maydoning ishini hisoblang.

5) $x = a\cos t,\ y = a\sin t,\ z = bt$ $(0 \le t \le 2\pi)$ vint chizig'ining t parametrning o'sish yo'nalishida $\vec{F} = \vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ kuch maydonining ishini hisoblang.

6) $\vec{F} = \left\{\frac{1}{y}; \frac{1}{z}; \frac{1}{x}\right\}$, $A(1;1;1)$ $B(2;4;8)$.

7) $\vec{F} = \left\{e^{y-z}; e^{z-x}; e^{x-y}\right\}$, $A(0;0;0)$, $B(1;3;5)$.

8) $\vec{F}\left\{x^3 - y^3, -3xy^2; 3xz^2\right\}$, $A(1;1;1)$, $B(2;0;1)$.

9) $(x+1)^2 + y^2 + z^2 = 1$ sferaning $A(-1,1;0)$ dan $B(-1;-1;0)$ nuqtasigacha bo'lgan yarim aylanasining katta radius bo'yicha $\vec{F}\left\{y + 2xz^2; x - 2y; 2x^2z\right\}$ kuch maydonining ishini hisoblang.

10) $\vec{F} = \left\{\frac{1}{y+z}; \frac{1}{z+x}; \frac{1}{x+y}\right\}$, $A(-1;0;3)$, $B(0;-1;2)$,

11) $\vec{F} = \dfrac{y\vec{i} + z\vec{j} + x\vec{k}}{\sqrt{x^2 - y^2 + z^2 - x + z}}$ $A(1;1;1)$, $B(6,6,6)$.

48. Gamilton operatori yordamida quyidagi tengliklarni isbotlang.

1) $div(rot(\vec{a})) = 0$.　　2) $div(grad(U)) = \nabla^2 U = \Delta U$.

3) $rot(gradU)) = 0$.　　4) $grad(div(\vec{a})) = \nabla(\nabla, \vec{a})$.

Mustaqil yechish uchun misollarning javoblari

1. 1) $y = \frac{C}{x}$ - teng tomonli giperbola. 2) $x^2 - y^2 = C$ - giperbolalar oilasi. 3) $y = Cx$ koordinatalar boshidan o'tgan to'g'ri chiziqlar dastasi. 4) $x^2 + y^2 = C^2$ aylanalar oilasi. 5) $x + y = C$ $(|C| < 1)$to'g'ri chiziqlar oilasi. **2.** 1) $x^2 + y^2 + z^2 = C^2$ markazi koordinatalar boshida bo'lgan konsentrik sferalar oilasi. 2) $x^2 + y^2 = C, Oz$ simmetriya o'qi bo'lib, doiraviy silindrlar oilasi. 3) Oz o'qi $O(0,0,0)$ nuqtasiz $\frac{x^2}{c^2} + \frac{y^2}{c^2} = z^2$, doiraviy konus. 4) $\frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{16} = C$ ellipsoidalar oilasi. 5)$x + y + z = C$ parallel tekisliklar oilasi. **3.** 1) $-\frac{14}{\sqrt{5}}$.2) $\frac{52}{5}$. **4.** $2\sqrt{2}$.5. 62. **6.** $-\frac{1}{\sqrt{2}}$. 7. $\frac{\sqrt{3}}{2}$.

8. $gdradU(A)=\{1;-1;0\}$, $|gradU(A)|=\sqrt{2}$. **9.** 1) $gradU = yz\vec{i}+xz\vec{j}+xy\vec{k}$. 2) $gradU = ye^{xy}\vec{i}+\left(xe^{xy}-z^2\right)\vec{j}-2yz\vec{k}$. 3) $gradU = 2x\vec{i}-2y\vec{j}-2z\vec{k}$ .4) $gradU = \frac{2x}{x^2-2y^2+z^2}\vec{i}-\frac{4y}{x^2-2y^2+z^2}\vec{j}+\frac{2z}{x^2-2y^2+z^2}\vec{k}$.

10. **1)** $gradV(A)=\{0;-3;-3\}$. **2)** $gradV(A)=\left\{\frac{1}{2};-\frac{1}{2};\frac{\sqrt{3}}{2}\right\}$. **3)** $gradV(A)=\left\{\frac{1}{10};\frac{1}{5};0\right\}$.

4) $gradV(A)=\left\{-\frac{1}{2};\frac{\sqrt{3}}{2};\frac{\sqrt{2}}{2}\right\}$.**11.** 1) Ox o'qqa; 2) yOz tekislikka; 3) $x-y+z=0$ tekislikka; 4) $\vec{S}=\{1;-3;-2\}$ yo'naltiruvchi vektorga ega bo'lib, $O(0,0,0)$ koordinatalar boshi orqali o'tgan to'g'ri chiziq nuqtalari; 5) $O(0,0,0)$ nuqtada; 6) $x^2+y^2+z^2=4$ sfera nuqtalarida. **12.** 1) $\varphi=\pi$. 2) $\varphi=\arccos\frac{1}{\sqrt{5}}$.

3) $\varphi=0$. 4) $\varphi=\pi-\arccos\frac{4}{\sqrt{41}}$. **14. 1)** $\vec{a}$. **2)** $\vec{a}\left(\vec{b},\vec{r}\right)+\vec{b}\left(\vec{a},\vec{r}\right)$.

3) $2\vec{r}$. **4)** $2\left|\vec{a}\right|^2\vec{r}-2\left(\vec{a},\vec{r}\right)\vec{a}=\left[\vec{a},\left[\vec{r},\vec{a}\right]\right]$. **5)** $gradV=\frac{1}{\left|\vec{r}\right|}\vec{r}$.

15. 1) $x^2=c_1y,\ z=c_2$.**2)** $xy=c_1,\quad z=c_2$. **3)** $x=c_1, xy^2-z^2=c_2$.

4) $\frac{1}{x}-\frac{1}{y}=c_1,\ z=c_2$. **5)** $y^2+z^2=c_1,\ x=c_2$. **6)** $x=\frac{y}{c_1}=\frac{z}{c_2}$.

16. 1) $x=3t+C_1, y=t+C_2, z=2t+C_3$. 2) $x=\cos t, y=\sin t,\ z=4t$.

3) $\frac{1}{x}-\frac{1}{z}=1,\ \frac{1}{x}-\frac{1}{2y^2}=4$. 4) $y=x, z^2=2\left(x^2-1\right)$. 16. 6π.

18. $4\pi R^2$. **19.** $6\pi R$. **20.** b^5. **21.** 0. **22.** $\frac{1}{2}$. **23.** $\frac{3}{16}\pi R^4$.

24. 0.**25.** 36π. **26.** 0 **27. 1)** $\frac{-256}{3}\pi$. **2)**8π. **3)** $\frac{\pi}{5}$. **4)**π. **5)**$\frac{32}{3}\pi$.

28. 1) $4\pi R^2$. **2)** $24a^5$. **3)** $\frac{1}{12}$. **4)**$\frac{1}{24}$. **5)** πH^3. **29.** 1) $div\,\vec{a}(M)=3$. 2) $div\,\vec{a}(M)=2$.

3)$div\,\vec{a}(M)=6$.1.30. 1) $div\,\vec{a}=3$. 2) $div\,\vec{a}=4\left|\vec{r}\right|$. 3) $div\,\vec{a}=\frac{2}{\left|\vec{r}\right|}$. 4) $div\,\vec{a}=\frac{2x^2}{\left(x^2+y^2\right)^{3/2}}$.

5) $div\vec{a}=0$. 6) $div[\vec{a},\vec{r}]=0$. 7) $div\left(f\left(\vec{r}\right)\cdot\vec{r}\right)=3f\left(\vec{r}\right)+\vec{r}\,f'(\vec{r})$.

31. 1) $\frac{\partial^2U}{\partial y^2}+\frac{\partial^2U}{\partial y^2}+\frac{\partial^2U}{\partial z^2}$. 2) $\left(\frac{\partial U}{\partial x}\right)^2+\left(\frac{\partial U}{\partial y}\right)^2+\left(\frac{\partial U}{\partial z}\right)^2+U\left(\frac{\partial^2U}{\partial x^2}+\frac{\partial^2U}{\partial y^2}+\frac{\partial^2U}{\partial z^2}\right)$.

3) $U\,div\,gradV+2gradU\,gradV+2VgradU$. **32.** 1) 6. 2) $\left(\vec{c}\cdot\vec{r}\right)/\dot{r}$.

3) $f'(r)+2\frac{f(r)}{r}$. 4) $2f(r)$. 5) 0.

34. 1) ha. 2) ha. 3) yo'q.

4) yo'q. 5) ha. 6) ha. 7)ha. **35.** $\varphi(r)=\frac{C}{r^3}$, $r\neq 0, C=const$.

36. 1)$-2\pi R^2$. 2)$-2\pi R^2$. 3)$-\frac{1}{2}$. 4) 5. 5)12π. **37.1)** -2π .2) 2π. 3) $-\frac{\pi R^3}{4}$. 4)$\frac{3}{2}$.

5) 0. 6) 0. 7)2π. **38.** 1) 4π. 2) -4π. 3) -2π. 4) $\frac{128}{3}$. **39.** 1) $rot\,\vec{a}=\{0;0;0\}$.

2)$rot\,\vec{a}=\{-1;1-2x;0\}$. 3) $rot\,\vec{a}=-2(z\,\vec{i}+x\,\vec{j}+y\,\vec{k})$. 4) $rot\,\vec{a}=\{0;3z^2-3x^2,0\}$. 5)

$rot\,\vec{a}=(x+y)\vec{k}$.**40.** 1) $rot\,\vec{a}(M_0)=\vec{i}+\vec{j}$. 2)$rot\,\vec{a}(M_0)=\frac{-5}{4}\vec{i}-\vec{j}+\frac{5}{2}\vec{k}$. **42.** 1) ha.

2) yo'q. 3) ha. 4) yo'q.5) yo'q. 6) ha. **44.** 1) $U(x,y)=x^3+C$. 2) $U=\ln r+C$.

3)$U=\frac{1}{3}\left(x^3+y^3+z^3\right)+C$. 4) $U=xyz+C$. 5) $U=xz+yz-x^2+C$.

6) $U(x,y,z)=x^2yz+C$. 7) $U=xyz-\frac{x^2y}{2}+\frac{y^2z^2}{2}+C$.8) $U=x^2y^3-\cos\left(x^2y\right)+C$.

45. 1) ha. 2) yo'q. **46.** 1) ham potensial, ham solenoidal maydon. 2) potensial. solenoidal maydon emas. **47.** 1) $A=-2$. 2) $A=1$.3) $A=0$.4) $A=\frac{5}{3}$.5) $A=2\pi^2b^2$6) $A=8\frac{20}{21}\cdot\ln z$.7) $A=\frac{3}{4}\left(3+e^4-12e^{-2}\right)$.8) $A=8$ .9) $A=2$.10) $A=\frac{1}{2}(1+\ln 3)$.11) $A=15$.

16-bob bo‘yicha amaliy mashg‘ulotlarni mustahkamlash uchun nazorat topshiriqlari

16.1-masala. Skalyar maydonning sath chiziqlari va sath sirtlarini topish. Vektorli maydonning vektor chiziqlarini topish.

Quyida berilgan skalyar maydonning sath chiziqlarini toping.

16.1.1. $u = x + y^2$. **16.1.2.** $u = \frac{x}{y}$.

16.1.3. $u = e^{x^2 - y^2}$. **16.1.4.** $u = \ln\sqrt{\frac{y}{2x}}$.

16.1.5. $u = x^2 + y^2$. **16.1.6.** $u = y^2 + x$.

16.1.7. $u = \frac{y}{x}$.

Quyida berilgan maydonning vektor chiziqlarini toping.

16.1.8. $\vec{A}(P) = y\vec{i} - x\vec{j}$.

16.1.9. $\vec{A}(P) = x\vec{i} - y\vec{j}$.

16.1.10. $\vec{A}(P) = (y+z)\vec{i} + x\vec{j} - x\vec{k}$.

16.1.11. $\vec{A}(P) = (z-y)\vec{i} + (x-z)\vec{j} + (y-x)\vec{k}$.

16.1.12. $\vec{A}(P) = x(y^2 - z^2)\vec{i} + y(z^2 + x^2)\vec{j} + z(x^2 + y^2)\vec{k}$.

Quyida berilgan skalyar maydonning sath sirtlari topilsin:

16.1.16. $u = x^2 + y^2 - z^2$.

16.1.14. $u = x + y + z$.

16.1.15. $u = \frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{16}$.

16.1.16. $u = x^2 - y^2 - z^2$.

16.1.16. $u = 2x + 3y + 4z$.

16.1.18. $u = \frac{x^2}{25} + \frac{y^2}{16} + \frac{z^2}{9}$.

Berilgan nuqtalarni o‘zida saqlovchi quyidagi maydonning sath sirtlarini toping:

16.1.19. $u = x^2 - y^2 + z^2$, $(1;1;1)$.

16.1.20. $u = x^2 - y^2 + z^2$, $(1;2;1)$.

16.1.21. $u=\frac{x^2}{2}+\frac{y^2}{4}+\frac{z^2}{9}$, $(0;1;1)$.

16.1.22. $u=\frac{x^2}{4}-\frac{y^2}{16}-\frac{z^2}{9}$, $(2;2;2)$.

16.1.23. $\vec{A}(P)=-y\vec{i}+x\vec{j}+z\vec{k}-$ maydonning $P(1;0;0)$ nuqtadan o'tuvchi vektor chizig'ini toping.

16.1.24. $\vec{A}(P)=x^2\vec{i}-y^3\vec{j}+z^2\vec{k}-$ maydonning $P(1/2;-1/2;1)$ nuqtadan o'tuvchi vektor chizig'ini toping.

16.1.25. $\vec{A}(P)=x^3\vec{i}-y^3\vec{j}+z^3\vec{k}-$ maydonning $P(1;-1;1)$ nuqtadan o'tuvchi vektor chizig'ini toping.

16.1.26. *a*) $u(x,y)=xy$ skalyar maydonning sath chizig'ini aniqlang;

b) $u(x,y,z)=\sqrt{R\ \ -x^2-y^2-z^2}$ skalyar maydonning sath sirtini aniqlang;

c) $\vec{a}=[\vec{r},\vec{c}]$. $\left(\vec{r}=x\vec{i}+y\vec{j}+z\vec{k},\ \vec{c}=c_1\vec{i}+c_2\vec{j}+c_3\vec{k}\right)$ vektorli maydonning vektor chizig'ini aniqlang.

Yechilishi ([9], 2-t., 7.1-bo'lim, [30], 14.1-bo'lim). *a*) Sath chizig'i quyidagi tenglama orqali aniqlanadi: $xy=c$, bunda $c=const$.

Demak, berilgan skalyar maydonning sath chiziqlari giperboloidlar oilasidan iborat (16.-chizma).

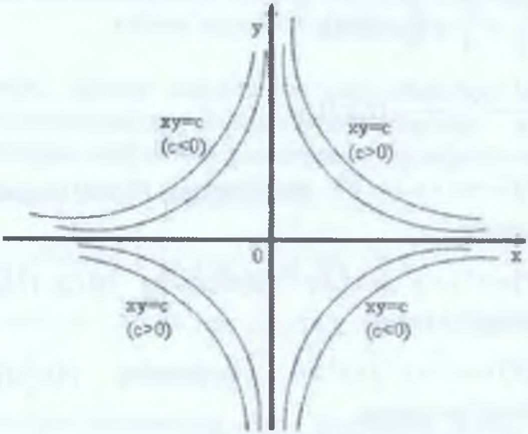
Agar $c=0$ bo'lsa,

$$xy=0 \text{ yoki } x=0,\ y=0$$

b) Berilgan skalyar maydonning sath sirti quyidagi tenglamalardan aniqlanadi:

$$\sqrt{R^2-x^2-y^2-z^2}=c \text{ yoki } x^2+y^2+z^2=R^2-c^2.$$

Xususiy holda, $c=0$ da $x^2+y^2+z^2=R^2$ sfera, $c=R$ bo'lganda esa $x^2+y^2+z^2=0$. Bu holda sfera nuqtaga aylanadi, ya'ni koordinata boshini ifodalaydi.



16.1-chizma.

c) Berilgan $\vec{a}-$ vektorli maydonning vektor chizig'i ushbu

$$\frac{dx}{a_x(x,y,z)}=\frac{dy}{a_y(x,y,z)}=\frac{dz}{a_z(x,y,z)}$$

formula orqali aniqlanadi. Ravshanki,

$$\vec{a}=[\vec{r},\vec{c}]=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ x & y & z\\ a & b & c\end{vmatrix}=(cy-bz)\vec{i}+(az-cx)\vec{j}+(bx-ay)\vec{k},\text{ bunda}$$

$$a_x(x,y,z)=(cy-bz),\quad a_y(x,y,z)=(az-cx),\quad a_z(x,y,z)=(bx-ay).$$

Bu holda $\vec{a}$-vektorli maydonning differensial tenglamasi, ushbu

$$\frac{dx}{cy-bz}=\frac{dy}{az-cx}=\frac{dz}{bx-cy}$$

ko'rinishga ega bo'ladi. Birinchi kasrning surat va maxrajini x ga, ikkinchi kasrni y ga, uchinchi kasrni esa z ga ko'paytirib, hadma–had qo'shamiz, hamda proporsiyalar xossasiga asosan, quyidagiga ega bo'lamiz:

$$\frac{dx}{cy-bz}=\frac{dy}{az-cx}=\frac{dz}{bx-ay}=\frac{xdx+ydy+zdz}{0}$$

Shunday qilib,

$$xdx+ydy+zdz=0 \text{ yoki } d\left(x^2+y^2+z^2\right)=0, \quad x^2+y^2+z^2=C_1^2$$

Xuddi shunday, birinchi kasrning surat va maxrajini a ga, ikkinchini b ga, uchinchini c ga ko'paytirib, so'ngra hadma – had qo'shish natijasida,

$$\frac{dx}{cy-bz}=\frac{dy}{az-cx}=\frac{dz}{bx-ay}=\frac{adx+bdy+cdz}{0}$$

ega bo'lamiz. Bundan

$$adx+bdy+cdz=0 \text{ yoki } ax+by+cz=C_2$$

Shunday qilib, vektorli maydon vektor chizig'i tenglamasi quyidagi ko'rinishga keladi:

$$\begin{cases} x^2+y^2+z^2=C_1^2 \quad (C_1\ge 0), \\ ax+by+cz=C_2. \end{cases}$$

Demak, $\vec{a}$ - vektorli maydonning vektor chizig'i $x^2+y^2+z^2=C_1^2$ sfera bilan $\vec{c}$ vektorga perpendikulya bo'lgan $ax+by+cz=C_2$ tekislikning kesishishidan hosil bo'lgan aylanani ifodalar ekan.

16.2-masala. $\vec{a}(M)$ vektorli maydonning (S) sirt orqali o'tuvchi O-vektor oqimi va divergeinsiyasini toping.

16.2.1. $\vec{a}=(1+2x)\vec{i}+y\,\vec{j}+z\,\vec{k}$, $(S): x^2+y^2=z^2,\ z=4.$

16.2.2. $\vec{a}=x\,\vec{i}+xz\,\vec{j}+y\,\vec{k}$, $(S): x^2+y^2=4-z,\ z=0,\ \ z\ge 0.$

16.2.3. $\vec{a}=x^3\,\vec{i}+y^3\,\vec{j}+z^3\,\vec{k}$, $(S): x^2+y^2+z^2=x.$

16.2.4. $\vec{a}=y\,\vec{j}+z\,\vec{k}$, $(S): z=x^2+y^2,\ z=4.$

16.2.5. $\vec{a}=2x\,\vec{i}+3y\,\vec{j}+2z\,\vec{k}$, $(S): x^2+y^2=z^2,\ z=9.$

16.2.6. $\vec{a}=x\,\vec{i}+z\,\vec{j}+y\,\vec{k}$, $(S): x^2+y^2=z,\ 0\le z\le 4.$

16.2.7. $\vec{a}=x^3\,\vec{i}+y^3\,\vec{j}+z^3\,\vec{k}$, $(S): x^2+y^2+z^2=x.$

16.2.8. $\vec{a} = y\vec{j} + z\vec{k}$, $(S): z = x^2 + y^2$, $z = 4$.

Quyidagi berilgan $\vec{a}(M)$ vektorli maydonning $\vec{n}$ normal bo'yicha oriyentrlangan (S) sirtdan o'tuvchi vektor oqimi O topilsin.

16.2.9. $\vec{a} = \vec{r}$ $\left(\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}\right)$, $(S): \sqrt{x^2 + y^2} \le z \le h$ konusning tashqi tomoni.

16.2.10. $\vec{a} = \vec{r}\left(\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}\right)$, $(S): x^2 + y^2 \le R^2$, $0 \le z \le h$

silindrning tashqi tomoni

16.2.11. $\vec{a} = f(r)\vec{r}$, $(S): x^2 + y^2 + z^2 = R^2$ sferaning tashqi tomoni.

16.2.12. $\vec{a} = \frac{\vec{r}}{r^3}$ $\left(\vec{r} = x\vec{i} + y\vec{j} + z\vec{k},\ r = \sqrt{x^2 + y^2 + z^2}\right)$, $(S): x^2 + y^2 + z^2 = R^2$

sferaning tashqi tomoni.

16.2.13. $\vec{a} = \{x^2; y^2; z^2\}$, $(S): x + y + z = 1$, $x = 0$, $y = 0$, $z = 0$ tekisliklar bilan chegaralangan to'liq sirtning tashqi tomoni.

16.2.14. $\vec{a} = \{0; y^2; z\}$, $(S): z = x^2 + y^2$ paraboloidning $z = 2$ tekislik bilan kesilgan qismining tashqi tomoni.

16.2.15. $\vec{a} = \left\{x; y; \sqrt{x^2 + y^2 - 1}\right\}$, $(S): x^2 + y^2 - z^2 = 1$ giperboloidning $z = 0$ va $z = \sqrt{3}$ tekisliklar bilan chegaralangan qismining tashqi tomoni.

Quyidagi $\vec{a}(M)$ vektorli maydonning (S) sirtdan o'tuvchi O vektor oqimini Gauss – Ostrogradskiy formulasi yordamida toping.

16.2.16. $\vec{a} = (x - y)\vec{i} + (x - z)\vec{j} + (y - x)\vec{k}$, $(S): x + y + z = 1$, $x + y - z = 1$, $y = 0$, $x = 0$ tekisliklar bilan chegaralangan tetraedr to'liq sirtining tashqi tomoni.

16.2.17. $\vec{a} = y^2 z\vec{i} - yz^2\vec{j} + x\left(y^2 + z^2\right)\vec{k}$, $(S): y^2 + z^2 \le a^2$, $0 \le x \le a$ silindr to'liq sirtining tashqi tomoni.

16.2.18. $\vec{a} = 2x\vec{i} + 2y\vec{j} - z\vec{k}$, $(S): \sqrt{x^2 + y^2} \le z \le H$ konus to'liq sirtining tashqi tomoni.

16.2.19. $\vec{a} = (x + z)\vec{i} + (y + x)\vec{j} + (z + y)\vec{k}$, $(S): x^2 + y^2 \le R^2$, $0 \le z \le y$ jism sirtining tashqi tomoni.

16.2.20. $\vec{a} = x^2 y\vec{i} + xy^2\vec{j} + xyz\vec{k}$, $(S): x^2 + y^2 + z^2 \le R^2$,

$x\ge 0,\ y\ge 0, z\ge 0$ jism sirtining tashqi tomoni.

16.2.21. $div\left(a\vec{r}\right)$ ni hisoblang, bunda $a-$o‘zgarmas miqdor, $\vec{r}=\{x,y,z\}$.

16.2.22. $div\left(\varphi\vec{A}\right)=\varphi\, div\,\vec{A}+\left(\vec{A}, grad\,\varphi\right)$ $(\varphi=\varphi(x,y,z)$ скаляр функция$)$ tenglikni isbotlang.

16.2.23. $div[\vec{r}\, f(r)]$ $\left(r=\sqrt{x^2+y^2+z^2}\right)$ ni hisoblang.

16.2.24. $\vec{a}=\dfrac{-x\,\vec{i}+y\,\vec{j}+z\,\vec{k}}{\sqrt{x^2+y^2}}$ vektorli maydonning $M(3;4;5)$ nuqtadagi divergensiyasini toping.

16.2.25. $\vec{a}=\dfrac{-x\,\vec{i}+y\,\vec{j}+z\,\vec{k}}{\sqrt{x^2+y^2+z^2}}$ vektorli maydonning $M(1;1;1)$ nuqtadagi divergensiyasini toping.

16.2.26. $\vec{a}=y^2\,\vec{j}+z\,\vec{k}$ vektorli maydonning $(S):z=x^2+y^2, z=2$ sirt bo‘yicha vektor oqimini toping.

Yechilishi ([30], 16.8-bo‘lim). $(S_1):z=x^2+y^2$ va $(S_2):z=2$ sirtlar bo‘yicha vektor oqimlarini alohida hisoblaymiz:

$(S_1)-$sirtga o‘tkazilgan normal vektorlarni topamiz:

$$\vec{n}^0=\pm\frac{grad\left(z-x^2-y^2\right)}{\left|grad\left(z-x^2-y^2\right)\right|}=\pm\frac{-2x\,\vec{i}-2y\,\vec{j}+\vec{k}}{\sqrt{1+4x^2+4y^2}}$$

$\left(\vec{n}^0,z\right)>90^0$ bo‘lgani uchun ildiz ishorasini « - » qilib olamiz, bundan

$$\left(\vec{a},\vec{n}^0\right)=-\frac{z-2y^3}{\sqrt{1+4x^2+4y^2}}=\frac{2y^3-z}{\sqrt{1+4x^2+4y^2}},$$

$$dS_1=\frac{dxdy}{|\cos\gamma|}=\sqrt{1+4x^2+4y^2}\,dxdy$$

Demak,

$$Q_1=\iint\limits_{(S_1)}\left(\vec{a},\vec{n}^0\right)dS_1=\iint\limits_{(D_{xy})}\left(2y^3-z\right)\Big|_{z=x^2+y^2}dxdy=\iint\limits_{x^2+y^2\le 2}\left(2y^3-x^2-y^2\right)dxdy,$$

keyingi integralni hisoblash uchun $x=\rho\cos\varphi, y=\rho\sin\varphi$ almashtirishni bajaramiz. Unda

$$\iint\limits_{x^2+y^2\le 2}\left(2y^3-x^2-y^2\right)dxdy=\int\limits_0^{2\pi}d\varphi\int\limits_0^{\sqrt{2}}\left(2\rho^4\cdot\sin^3\varphi-\rho^3\right)d\rho=-2\pi.$$

Xuddi shunday, (S_2) sirt bo‘yicha vektor oqimini hisoblaymiz:

$$\vec{n}^0=\pm\frac{grad(z-2)}{|grad(z-2)|}=\pm\vec{k},\quad \left(\widehat{\vec{n}^0,z}\right)<90^0\Rightarrow\vec{n}^0=\vec{k},$$

bo‘lgani uchun

$$\left(\vec{a},\vec{n}^0\right)=z,\quad dS_2=\frac{dxdy}{|\cos\gamma|}=dxdy,$$

bunda $|\cos\gamma|=1$.

Shunday qilib,

$$O_2=\iint\limits_{(S_2)}\left(\vec{a}.\vec{n}^0\right)dS_2=\iint\limits_{(D_{xy})}z\Big|_{z=2}dxdy=2\int\limits_0^{2\pi}d\varphi\int\limits_0^{\sqrt{2}}\rho d\rho=4\pi.$$

Demak, $O=O_1+O_2=2\pi$.

16.3-masala. Vektorli maydon sirkulyasiyasi. Stoks formulasi. Vektorli maydon uyurmasi (rotori) .

Quyidagi $\vec{a}(M)$ vektorli maydonning (K) yopiq chiziq bo‘yicha hisoblang:

16.3.1. $\vec{a}=(xz+y)\vec{i}+(yz-x)\vec{j}-\left(x^2+y^2\right)\vec{k};\quad (K): x^2+y^2=1, z=3.$

16.3.2. $\vec{a}=-y\,\vec{i}+x\,\vec{j}+c\,\vec{k}$ (c – o‘zgarmas son), $(K): x^2+y^2=1, z=0.$

16.3.3. $\vec{a}=-y\,\vec{i}+x\,\vec{j}+c\,\vec{k}$ (c-o‘zgarmas son), $(K): (x-2)^2+y^2=1, z=0.$

16.3.4. $\vec{a}=y^2\,\vec{i}+z^2\,\vec{j}+x^2\,\vec{k}, (K): x^2+y^2+z^2=R^2,\ x^2+y^2=Rx\,(z\ge 0).$

16.4. $\vec{a}=(xz+y)\vec{i}+(yz-x)\vec{j}-\left(x^2+y^2\right)\vec{k};\quad (K): x^2+y^2=4, z=9.$

16.5. $\vec{a} = -y^2\vec{i} + x^2\vec{j} + c\vec{k}$ (c – o'zgarmas son), $(K): x^2 + y^2 = 4, z = 0$.

16.3.7. $\vec{a} = -y\vec{i} + x\vec{j} + c\vec{k}$ (c – o'zgarmas son), $(K): (x-2)^2 + y^2 = 4, z = 0$.

16.3.8. $\vec{a} = y^2\vec{i} + z^2\vec{j} + x^2\vec{k}$, $(K): \frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{25} = 1$, $x^2 + y^2 = 4x\,(z \geq 0)$.

16.3.9. $\vec{a} = (2x+z)\vec{i} + (2y-z)\vec{j} + xyz\vec{k}$, $(K): x^2 + y^2 = 1 - z$ aylanma paraboloidning koordinatalar tekisliklari bilan kesishishidan hosil bo'lgan chiziq.

Quyidagi berilgan vektorli maydonning mos konturlar yo'nalishida avvalo to'g'ridan-to'g'ri, so'ngra esa, Stoks formulasidan foydalanib hisoblang:

16.3.10. $\vec{a} = z\vec{i} + x\vec{j} + y\vec{k}$, $(K): x^2 + y^2 = 4,\ z = 0$.

16.3.11. $\vec{a} = y\vec{i} - x\vec{j} + z\vec{k}$, $(K): x^2 + y^2 + z^2 = 4,\ x^2 + y^2 = z^2,\ z \geq 0$.

16.3.12. $\vec{a} = y\vec{i} - x\vec{j} + (x+y)\vec{k}$, $(K): z = x^2 + y^2, Bz = 1$.

16.3.16. $\vec{a} = z^2\vec{i}$, $(K): x^2 + y^2 + z^2 = 16,\ x = 0,\ y = 0,\ z = 0$.

16.3.14. $\vec{a} = z^2\vec{i} + x^2\vec{j} + y^2\vec{k}$, $(K): x^2 + y^2 + z^2 = R^2$ sferaning $x + y + z = R$ tekislik bilan kesishish natijasida hosil bo'lgan, $\vec{k}$ ga nisbatan musbat yo'nalishida olingan chiziq.

16.3.15. $\vec{a} = z^3\vec{i} + x^3\vec{j} + y^3\vec{k}$, $(K):\ 2x^2 - y^2 + z^2 = R^2$ giperboloidning $x + y = 0$ tekislik bilan kesishishi natijasida hosil bo'lgan, $\vec{i}$ ga nisbatan musbat yo'nalishida olingan chiziq.

16.3.16. $\vec{a} = y^2\vec{i} + xy\vec{j} + (x^2 + y^2)\vec{k}$, $(K): x^2 + y^2 = Rz$ paraboloidning birinchi oktantdagi qismining $x = 0, y = 0, z = R$ tekisliklar bilan kesilgan qismidan hosil bo'lgan, paraboloidning tashqi normaliga nisbatan musbat yo'nalishidan olingan chiziq.

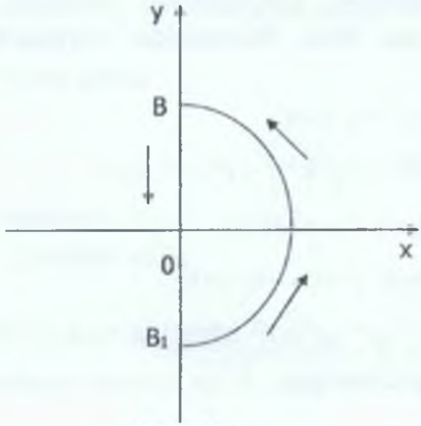
16.3.17. $\vec{a} = y^2\vec{i}$; $(K): \vec{r} = a_1\cos t\,\vec{i} + b_1\sin t\,\vec{j}$ ellipsning o'ng yarim tomoni va Oy o'qning kesmasi (16.2-chizma)

16.3.18. $\vec{a} = y\vec{i} - x\vec{j}$; $(K): x = R\cos^3 t,\ \ y = R\sin^3 t$ astroidaning birinchi chorakdagi qismi, hamda koordinatalar o'qlari kesmalari yordamida hosil qilingan chiziq (16.3-chizma).

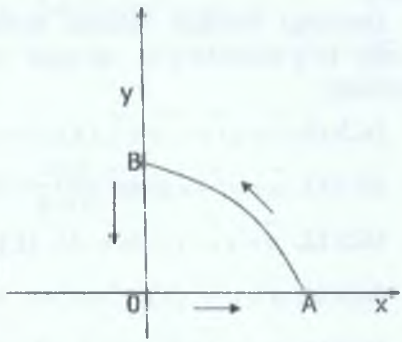
16.3.19. $\vec{a} = z^2\vec{i} + x^2\vec{j} + y^2\vec{k}$; $(K): x^2 + y^2 + z^2 = 1,\ x + y + z = 1$ yo'nalish, koordinatalar boshidan qaraganda, soat mili yo'nalishi bo'yicha olingan.

16.3.20. $\vec{a} = x^3\vec{i} + y^3\vec{j} + z^3\vec{k}$; $(K): z = x^2 + y^2,\ z + y = 2$ yo'nalish, koordinatalar boshidan qaralganda soat mili yo'nalishibo'yicha olingan

16.3.21. $rot\left(\vec{r}\cdot\vec{a}\right)\cdot\vec{b}$ hisoblang, bunda $\vec{a}$ va $\vec{b}$ o'zgarmas vektorlar, $\vec{r} = \{x, y, z\}$- nuqtaning radius–vektori.



16.2-chizma



16.3-chizma

16.3.22. $\vec{a} = \dfrac{x\vec{i} + y\vec{j} + z\vec{k}}{\sqrt{x^2 + y^2 + z^2}}$ vektorning rotorini hisoblang.

16.3.23. $rot(grad u)$ ni toping.

16.3.24. $rot(xyz(x\vec{i} + y\vec{j} + z\vec{k}))$ ni toping.

16.3.25. $rot(x^2\vec{i} + y^2\vec{j} + z^2\vec{k})$ ni toping.

16.3.26. $a)\ \vec{a} = ye^{xy}\vec{i} + xe^{xy}\vec{j} + xyz\vec{k}$, vektorli maydonning $(K): x^2 + y^2 = (z-1)^2$ konus bilan $Oxyz$ koordinatalar tekisligining kesishish chizig'i bo'yicha olingan sirkulyasiyasini ta'rif bo'yicha toping.

Yechilishi ([30], 16.7-bo'lim). (K): Oyz va Oxz tekisliklarda joylashgan BC va CA kesmalar va AB aylana yoyidan, ya'ni $x^2 + y^2 = 1,\ z = 0$ iborat bo‘lgan yopiq chiziq. Yo‘nalish 16.4-chizmada ko‘rsatilgan. Berilgan $\vec{a}$ vektorli maydon sirkulyasiyasi ushbu $S = \oint\limits_{(K)} (\vec{a}, d\vec{r})$ formula bilan topiladi:

Demak, bu holda sirkulyasiya

$$S = \int\limits_{(K)} (\vec{a}, d\vec{r}) = \int\limits_{(BC)} (\vec{a}, d\vec{r}) + \int\limits_{(CA)} (\vec{a}, d\vec{r}) + \int\limits_{(\breve{AB})} (\vec{a}, d\vec{r}).$$

1) BC kesma bo‘yicha olingan egri chiziqli integralni hisoblaymiz: $x = 0,\ dx = 0, z = 1 - y, dz = -dy,\ 0 \le y \le 1$.

Demak, $\int\limits_{(BC)} (\vec{a}, d\vec{r}) = \int\limits_{(BC)} ydx = 0$.

2) CA kesma bo‘yicha olingan egri chiziqli integralni hisoblaymiz:

$$y = 0, dy = 0, z = 1 - x,\ dz = -dx, 0 \le x \le 1,$$

Demak, $\int\limits_{(CA)} (\vec{a}, d\vec{r}) = \int\limits_{(CA)} xdy = 0.$

3) $\begin{cases} x^2 + y^2 = 1, \\ z = 0 \end{cases}$ aylananing $(\breve{AB})$ yoyi bo‘yicha olingan egri chiziqli integralni hisoblaymiz: $z = 0, dz = 0$.

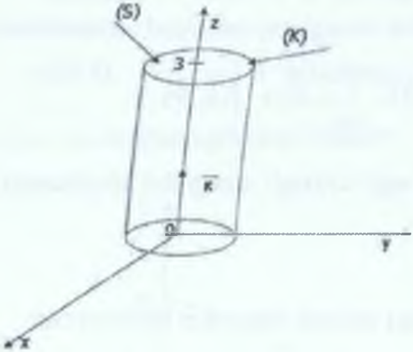
$$\int\limits_{(\breve{AB})} (\vec{a}, d\vec{r}) = \int\limits_{(\breve{AB})} e^{xy}(ydx + xdy) = \int\limits_{(\breve{AB})} e^{xy} d(xy) = e^{xy}\Big|_{A(1;0)}^{B(0,1)} = 1 - 1 = 0.$$

Shunday qilib, berilgan $\vec{a} = ye^{xy}\,\vec{i} + xe^{xy}\,\vec{j} + zxy\,\vec{k}$ vektorli maydonning sirkulyasiyasi $S = 0$ ekan.

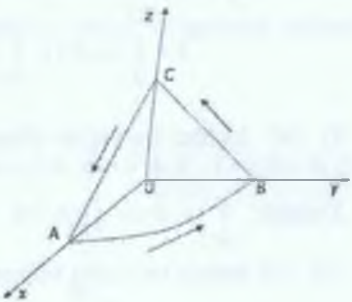
b) $\vec{a} = y\,\vec{i} + x^2\,\vec{j} - z\,\vec{k}$ vektorli maydonning $(K): x^2 + y^2 = 4, z = 3$ chiziq bo‘ylab $\vec{k}$ birlik vektorga nisbat musbat yo‘nalishida olingan sirkulyasiyasini. Ctoks formulasidan foydalanib, toping.

Yechilishi. $(S) = \{x^2 + y^2 \le 4,\ z = 3\}$, $\vec{n}^0 = \vec{k}$ bo‘lgani uchun Stoks formulasiga ko‘ra,

$$S=\iint\limits_{(S)}(rot\,\vec{a},\vec{n}^{\circ})ds=\iint\limits_{(D)}(2x-1)dxdy.$$



16.4-chizma.



16.5-chizma.

Oxirgi integralni hisoblash uchun qutb koordinatlar sistemasiga o'tamiz (16.5-chizma):

$$\iint\limits_{(D)}(2x-1)dxdy=\iint\limits_{(\Delta)}(2\rho\cos\varphi-1)\rho d\rho d\varphi=\int\limits_0^{2\pi}d\varphi\int\limits_0^2(2\rho\cos\varphi-1)\rho d\rho=-4\pi\,.$$

16.4-masala. Quyida berilgan vektorli maydonlarning solenoidal maydon bo'lishi yoki bo'lmasligini aniqlang:

16.4.1. $\vec{a}=x\left(z^2-x^2\right)\vec{i}+y\left(x^2-y^2\right)\vec{j}+z\left(y^2-x^2\right)\vec{k}\,.$

16.4.2. $\vec{a}=y^2\,\vec{i}-\left(x^2-y^2\right)\vec{j}+z\left(3y^2+1\right)\vec{k}.$

16.4.3. $\vec{a}=(1+2xy)\,\vec{i}-y^2z\,\vec{j}+\left(z^2y-2zy+1\right)\vec{k}\,.$

16.4.4. $\vec{a}=\dfrac{x}{\sqrt{x^2+y^2}}\vec{i}+\dfrac{y}{\sqrt{x^2+y^2}}\vec{j}+\dfrac{\left(x^2-y^2\right)z}{\left(x^2+y^2\right)^{3/2}}\vec{k}.$

16.4.5. $\vec{a}=xy^2\,\vec{i}+x^2y\,\vec{j}-\left(x^2+y^2\right)z\,\vec{k}\,.$

16.4.6. Qanday shartda $\vec{a}=\varphi(r)\vec{r}$, bunda $\vec{r}=x\,\vec{i}+y\,\vec{j}+z\,\vec{k}$, $r=\left|\vec{r}\right|=\sqrt{x^2+y^2}$ vektorli maydon solenoidal maydon bo'ladi?

Quyida berilgan vektorli maydonlarning potensial maydon bo'lish yoki bo'lmasligini aniqlang:

16.4.7. $\vec{a} = x^2\vec{i} + y^2\vec{j} + z^2\vec{k}$.

16.4.8. $\vec{a} = xz\vec{i} + zy\vec{j} + yx\vec{k}$.

16.4.9. $\vec{a} = (ax + y + bz)\vec{i} + (2x + cy + dz)\vec{j} + (bx + dy + cz)\vec{k}$.

16.4.10. $\vec{a} = yz\cos xy\,\vec{i} + xz\cos xy\,\vec{j} + \sin xy\,\vec{k}$.

16.4.11. $\vec{a} = (y + z)\vec{i} + (z + x)\vec{j} + (x + y)\vec{k}$.

16.4.12. $\vec{a} = \dfrac{yz\vec{i} + zx\vec{j} + y\vec{k}}{1 + x^2y^2z^2}$.

16.4.16. $\vec{a} = y\vec{i} + x\vec{j} + e^z\vec{k}$.

16.4.14. $\vec{a} = \dfrac{\vec{r}}{r}$, bunda $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$, $r = |\vec{r}| = \sqrt{x^2 + y^2}$.

16.4.15. $\vec{a} = r \cdot \vec{r}$, bunda $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$, $r = |\vec{r}| = \sqrt{x^2 + y^2}$.

16.4.16. $\vec{a} = x^4\vec{i} + y^4\vec{j} + z^4\vec{k}$.

16.4.16. $\vec{a} = xz^2\vec{i} + zy^2\vec{j} + yx^2\vec{k}$.

16.4.18. $\vec{a} = (ax^2 + y + bz)\vec{i} + (2x + cy^2 + dz)\vec{j} + (bx + dy + cz^2)\vec{k}$.

16.4.19. $\vec{a} = z\cos xy\,\vec{i} + x\cos xy\,\vec{j} + y\sin xy\,\vec{k}$.

16.4.20. $\vec{a} = (y + z)\vec{i} + (z + x)\vec{j} + (x + y)\vec{k}$.

Quyida berilgan $\vec{a}$ vektorli maydon solenoidal maydon bo'ladimi?

16.4.21. $\vec{a} = x(z^2 - y^2)\vec{i} + y(x^2 - z^2)\vec{j} + z(y^2 + x^2)\vec{k}$.

16.4.22. $\vec{a} = (1 + 2xy)\vec{i} - y^2z\vec{j} + (z^2y - 2yz + 1)\vec{k}$.

16.4.23. $\vec{a} = x^2yz\vec{i} + xy^2z\vec{j} - xyz^2\vec{k}$.

16.4.24. $\vec{a} = \dfrac{-y\vec{i} + x\vec{j}}{x^2 + y^2} + xy\vec{k}$.

16.4.25. Berilgan $\vec{a} = \dfrac{\vec{r}}{r^3}$, bunda $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$,

$r=\left|\vec{r}\right|=\sqrt{x^2+y^2+z^2}$ maydon potensial maydon, solenoidal maydon bo'ladimi?

16.4.26. Berilgan $\vec{a}=(2xy+z)\vec{i}+\left(x^2-2y\right)\vec{j}+x\vec{k}$ vektorli maydonning potensial maydon ekanligini, lekin solenoidal maydon emasligini ko'rsating.

Yechilishi. 1) Ta'rifga ko'ra, vektorli maydon potensial maydon bo'lishi uchun maydonning har bir nuqtasida $rot\,\vec{a}=0$ bo'lishi kerak. Shuning uchun, bu shartni tekshirib ko'ramiz:

$$P(x,y,z)=2xy+z,\quad Q(x,y,z)=x^2-2y,\quad R(x,y,z)=x,$$

$$rot\,\vec{a}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ 2xy+z & x^2-2y & x\end{vmatrix}=(0-0)\vec{i}+(1-1)\vec{j}+(2x-2x)\vec{k}=0.$$

Demak, berilgan maydon potensial maydon bo'lar ekan.

2) Ta'rifga ko'ra, vektorli maydon solenoidal maydon bo'lishi uchun uning divergensiyasi nolga teng, ya'ni $div\,\vec{a}=0$ bo'lishi kerak:

$$div\,\vec{a}=\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}=2y-2+0\neq 0.$$

Demak, berilgan vektorli maydon solenoidal maydon emas ekan.

16.5-masala. Vektrli yoki skalyar maydonda ikkinchi tartibli differensial amallar yordamida quyidagi berilgan tengliklarni isbotlang.

16.5.1. $div(grad u)=\nabla^2 u$.

16.5.2. $grad\left(div\,\vec{a}\right)=\nabla\left(\nabla,\vec{a}\right)$.

16.5.3. $\nabla^2\vec{a}=\nabla^2 a_x\vec{i}+\nabla^2 a_y\vec{j}+\nabla^2 a_z\vec{k}$.

16.5.4. Agar $\vec{a}=x^3\vec{i}+y^3\vec{j}+z^3\vec{k}$ bo'lsa $grad\left(div\,\vec{a}\right)$ ni toping.

16.5.5. Agar $\vec{a}=xy^2\vec{i}+xz^2\vec{j}+xz^2\vec{k}$ bo'lsa $rot\left(rot\,\vec{a}\right)$ni toping.

16.5.6. $rot\left(rot\,\vec{a}\right)=0$.

16.5.7. Agar $\vec{a}=\left(y^2+z^2\right)x\,\vec{i}+\left(x^2+z^2\right)y\,\vec{j}+\left(x^2+y^2\right)z\,\vec{k}$ bo'lsa $\nabla^2\vec{a}$ ni toping.

16.5.8. $rot\left(u\,\vec{a}\right)=u\,rot\,\vec{a}+grad\left(u\,\vec{a}\right)$.

16.5.9. $rot\left(\vec{a}+\vec{a}\right)=rot\,\vec{a}+rot\,\vec{b}$.

16.5.10. $\nabla^2(uv)=u\nabla^2v+v\nabla^2u+2\nabla u\nabla v$, bunda $\nabla=\vec{i}\,\dfrac{\partial}{\partial x}+\vec{j}\,\dfrac{\partial}{\partial y}+\vec{k}\,\dfrac{\partial}{\partial z}$,

$\nabla^2=\nabla\nabla=\Delta=\dfrac{\partial^2}{\partial x^2}+\dfrac{\partial^2}{\partial y^2}+\dfrac{\partial^2}{\partial z^2}$.

16.5.11. $div[grad\,f(r)]$ ni toping, bunda $r=\sqrt{x^2+y^2+z^2}$.

16.5.12. $div\left([\vec{a},\vec{b}]\right)=\vec{b}\,rot\,\vec{a}-\vec{a}\,rot\,\vec{b}$.

16.5.15. $rot(grad u)=0$.

16.5.14. $div\left(rot\,\vec{a}\right)=0$.

16.5.15. $grad\,f(u,v)=\dfrac{\partial f}{\partial u}grad u+\dfrac{\partial f}{\partial v}grad v$.

16.5.16. $div\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ \omega_x & \omega_y & \omega_z\end{vmatrix}=0$.

16.5.16. $div[f(r)\vec{r}\,]=\dfrac{f'(r)}{r}\left(\vec{c},\vec{r}\right)$.

16.5.18. $rot\left(\vec{r}\,\vec{a}\right)=\dfrac{1}{r}[\vec{r},\vec{a}]$, bunda $\vec{a}$ - o'zgarmas vektor.

16.5.19. $div(u\nabla u)=u\Delta u+(\nabla u)^2$.

16.5.20. $div(u\nabla v)=u\Delta v+\nabla u\cdot\nabla v$.

16.5.21. Agar $\vec{a}=x^4\,\vec{i}+y^4\,\vec{j}+z^4\,\vec{k}$ bo'lsa $grad\left(div\,\vec{a}\right)$ ni toping.

16.5.22. Agar $\vec{a}=xy\,\vec{i}+yz\,\vec{j}+xz\,\vec{k}$ bo'lsa $rot\left(rot\,\vec{a}\right)$ni toping.

16.5.23. Agar $\vec{a}=2yx^4\vec{i}+xy^4\vec{j}+yz^4\vec{k}$ bo'lsa $grad\left(div\,\vec{a}\right)$ ni toping.

16.5.24. Agar $\vec{a}=x^5\vec{i}+y^5\vec{j}+z^5\vec{k}$ bo'lsa $rot\left(rot\,\vec{a}\right)$ ni toping

16.5.25. Agar $\vec{a}=x^4\vec{i}+y^4\vec{j}+z^4\vec{k}$ bo'lsa $div\left(div\,\vec{a}\right)$ ni toping

16.5.26. $a)\, div(u\, grad\, v)=grad\, u\cdot grad\, v+u\Delta v$,

$b)$ $div(u\, grad u)=|grad u|^2+u\Delta u$ ni isbotlang.

Yechilishi ([30], 16.4-bo'lim). $a)$ Ma'lumki, $\vec{a}(M)$ vektorning divergensiyasi ushbu

$$div\,\vec{a}=\frac{\partial a_x}{\partial x}+\frac{\partial a_y}{\partial y}+\frac{\partial a_z}{\partial z}$$

formula orqali, $v(x,y,z)$ – skalyar maydonning gradiyenti esa,

$$grad\, v=\frac{\partial v}{\partial x}\vec{i}+\frac{\partial v}{\partial y}\vec{j}+\frac{\partial v}{\partial z}\vec{k}$$

formula orqali topiladi. U holda,

$$div(u\, grad\, v)=\frac{\partial}{\partial x}\left(u\frac{\partial v}{\partial x}\right)+\frac{\partial}{\partial y}\left(u\frac{\partial v}{\partial y}\right)+\frac{\partial}{\partial z}\left(u\frac{\partial v}{\partial z}\right)=$$

$$=\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}+u\frac{\partial^2 v}{\partial x^2}+\frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial y}+u\frac{\partial^2 v}{\partial y^2}+\frac{\partial u}{\partial z}\cdot\frac{\partial v}{\partial z}+u\frac{\partial^2 v}{\partial z^2}=$$

$$=u\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right)+\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial y}+\frac{\partial u}{\partial z}\frac{\partial v}{\partial z}=$$

$$=grad u\cdot grad v+u\Delta v,\ \text{бунда}\,\Delta=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{dy^2}+\frac{\partial^2}{\partial z^2}.$$

$b)$ ∇–operatorning xossalaridan foydalanib, quyidagicha isbotlash mumkin:

$div(u\, grad u)=(\nabla,u grad u)=(grad u,\nabla u)+$

$+u(\nabla,grad u)=(grad u,grad u)+u\, div(grad u)=$

$=|grad u|^2+u\Delta u.$

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