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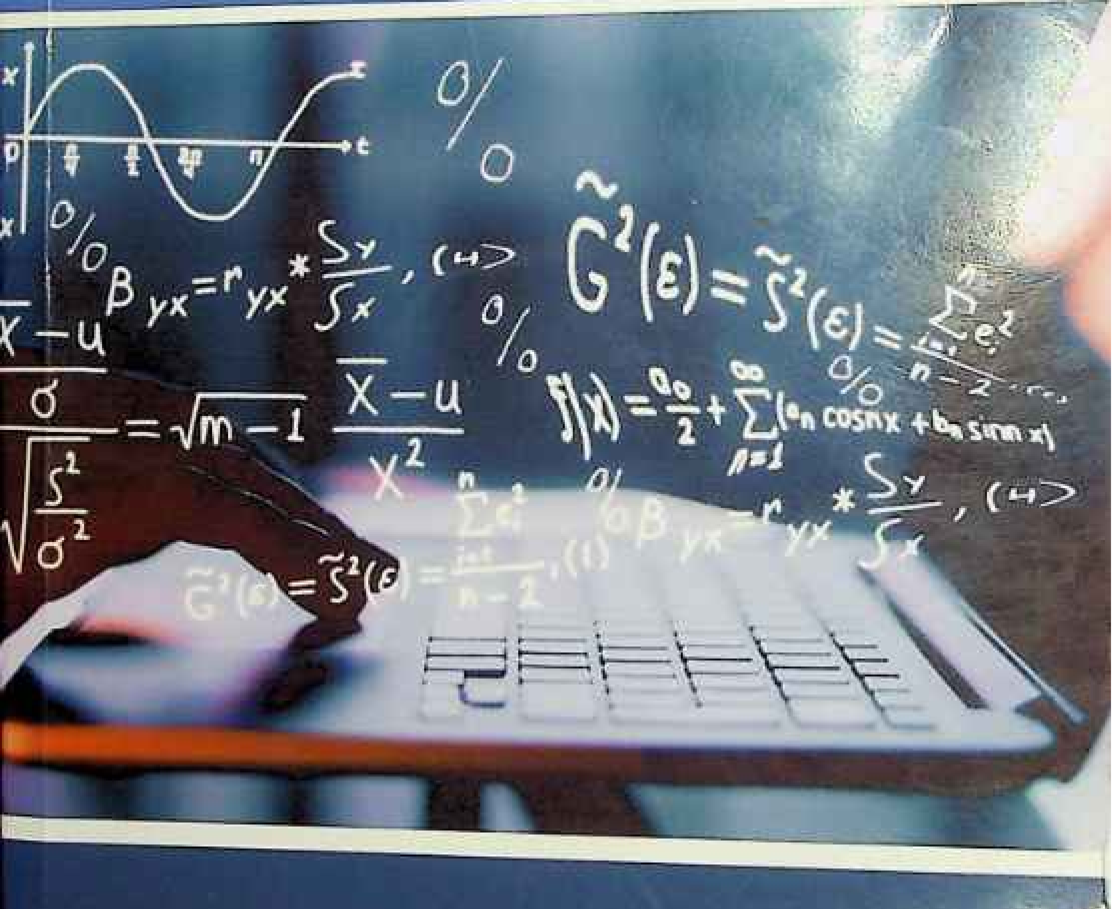

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NAGOYA UNIVERSITETI

**Ishida Yukio, Sh.T. Pirmatov,
D.N. Shamsiyev, O‘.S. Raxmonov, R.N. Shamsiyev**

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**Toshkent
“Shafoat Nur Fayz”
2022**

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Ushbu o'quv qo'llanma Nagoya Universiteti tomonidan tayyorlangan Pre-kollej matematikasi videodarslarini tarjimasi asosida yozilgan. O'quv qo'llanma matematikaning o'rta va o'rta maxsus ta'lim (akademik litsey, kollej) dasturlariga asoslangan bo'lib, funksiyalar, integrallar va ularning tadbiqlari, hamda chiziqli algebra bo'limlarini qamrab olgan. Uning ba'zi bo'limlaridan oliy ta'limning texnika yo'nalishlari bo'yicha birinchi bosqich talabalari foydalanishi ham mumkin.

O'quv qo'llamadan litsey talabalari va o'rta maktabning yuqori sinf o'quvchilari ham foydalanishi mumkin.

Taqrizchilar:
A.Abdukarimov-TDTU, "Oliy matematika" kafedrasi dotsenti, f.-m.f.n.
U.Safarov-TTPU, "Tabiy-matematika fanlari" kafedrasi dotsenti, f.-m.f.n.

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KIRISH

Nagoya Universiteti Pre-kollej matematikasi

Sizga havola etilayotgan o'quv qo'llanma uchta qismdan iborat bo'lib, ular quyidagicha:
1-kurs: Funksiyalar va tenglamalar;
2-kurs: Hisoblashlar;
3-kurs: Chiziqli algebra.

1 kurs. Har bir mavzu ikki qismdan iborat bo'lib, ma'ruza A va ma'ruza B ga bo'linadi. Ularning har birida avval mavzu tushuntiriladi, so'ngra misollar yechib ko'rsatiladi. Barcha ma'ruzalardagi misol va mashqlar imkon darajasida amaliyotga yaqin qilib tanlangan.

1-kurs quyidagi mavzulardan tashkil topgan:

1. Ko'phadlar va birhadlar
2. Algebraik tenglamalar va funksiyalar
3. Chiziqli va kvadratik tengsizliklar
4. Trigonometrik funksiya (1-qism)
5. Trigonometrik funksiya (2-qism)
6. Trigonometrik funksiya (3-qism)
7. Kompleks sonlar
8. Eksponensial funksiyalar
9. Kasr va irratsional funksiyalar
10. Teskari funksiyalar
11. Logarifm va uning xossalari
12. Sinus va kosinus teoremalari
13. Trigonometrik funksiyalarning grafiklari
14. Funksiyalar va tenglamalar (1-qism)
15. Funksiyalar va tenglamalar (2-qism)

2-kurs. 1-kursda biz trigonometrik funksiyalar, ko'rsatkichli va logarifmik funksiyalar kabi turli xil funksiyalarni o'rgangan edik. Ushbu funksiyalarni bilish 2-kursni tushunishda juda muhim hisoblanadi. 2-kurs hisob-kitob (calculus) larga bag'ishlangan.

Birinchidan, biz hisob-kitob (calculus) so'zi haqida tushuntirish beramiz. Hisob-kitob (calculus) matematikaning funksiyalar, limitlar, hosilalar, integrallar, sonli qator va boshqalarga yo'naltirilgan bo'limi hisoblanadi. Masalan, harakatlanayotgan to'pning holatini bilsangiz, hisobni aniqlash orqali uning qanchalik tez harakatlanishini va to'p qayergacha borishini topishingiz mumkin. Qisqa qilib aytganda, hisoblash o'zgarishlarni o'rganadi. Xuddi shu tarzda, "geometriya" - shaklni o'rganish, "algebra" - bu ishlashni o'rganishdir.

Hisob-kitob (calculus) fan, texnika, iqtisodiyot, biologiya va boshqa sohalarda keng qo'llaniladi. Hisob-kitob (calculus) ikki tarmoqqa ega. Ular differensial va integral hisob. Oddiy qilib aytganda, differensial hisoblashda "o'zgarish" va integral hisoblashda "yig'ish" muhokama qilinadi.

2-kurs quyidagi mavzulardan tashkil topgan:

1. Funksiyalar limiti va hosilalari.
2. Hosilalar va grafiklar.
3. Differensiallash formulalari.
4. Trigonometrik funksiyalarning hosilalari.
5. Logarifmik funksiyalar va eksponentalar hosilalari.
6. Hosilaning tenglama va tengsizliklarga tadbiqi.
7. Fizikaga tadbiq etish.
8. Funksiyaning yaqinlashishi.
9. Boshlang'ich funksiya.
10. Aniq integrallar.
11. To'rtburchaklar bo'yicha yuzani taqribiy hisoblash.
12. Integrallarni tadbiqlari (1-qism).
13. Integrallarni tadbiqlari (2-qism).
14. Differensial tenglamalar (1-qism).
15. Differensial tenglamalar (2-qism).

3-kurs quyidagi mavzulardan tashkil topgan:

1. Vektorlarning asosiy qoidalari.
2. Skalyar ko'paytma.
3. Vektor tenglamalar.
4. Matrisalarning asosiy qoidalari.
5. Matrisalarni ko'paytirish.
6. Teskari matrisa va birgalikdagi tenglamalar sistemasi.
7. Chiziqli almashtirish.

1-KURS. FUNKSIYALAR VA TENGLAMALAR

1.1. Ko‘phadlar va ularni ko‘paytuvchilarga ajratish

1A. Ko‘phadlarni qo‘shish, ayirish va ko‘paytirish

Chekli sondagi birhadlar yig‘indisi, ayirmasi va ko‘paytmasidan tashkil topgan ifodaga ko‘phad deyiladi. O‘zgaruvchilarning ko‘paytmasidan iborat ifodaga esa birhad deyiladi.

Kophadlar ustida qo‘shish, ayirish va ko‘paytirish amallarni bajarish mumkin.

Dastlab quyidagi savollarga javob topamiz:

Savol: Radiusi r sm bo‘lgan doiraning yuzi nimaga teng?

J: $\pi r^2\ (sm^2)$ ---------bu birhadga misol.

Savol: x yenalik tort bo‘lagidan 5 tasi va y yenalik batonlar ikkalasining birgalikdagi narxi qancha?

J: $5x + 2y$ ----------bu esa ko‘phad.

Mana endi ba’zi matematik terminlar bilan tanishamiz:

$-5x^3y^2$- bu birhad, bu yerda -5 soni birhadning koeffitsiyenti, darajadagi 3 soni x ning darajasi, 2 esa y ning darajasi deyiladi.

Endi ba’zi qo‘shimcha ma’lumotlarni keltiramiz.

1-eslatma: Ko‘phadning darajasi – darajasi eng yuqori bo‘lgan hadning darajasi. M: $2x^3+2x+1$ darajasi- 3.

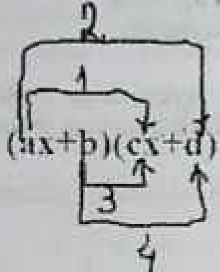
2-eslatma: Quyidagi ifodalar ko‘phad emas $3x^{\frac{1}{2}}+1(=3\sqrt{x}+1)$, $2x^{-2}=\frac{2}{x^2}$.

Biz yuqorida ifoda ko‘phad bo‘lishi uchun unda yig‘indi, ayirma va ko‘paytma bo‘lishi kerakligini ko‘rgan edik. Keltirilgan ifodalarning ko‘phad bo‘lmasligiga sabab darajada manfiy yoki kasr son turganligi, bu esa bo‘lish amali degani.

3-eslatma: Ko‘phad ifodalarning kombinatsiyasidan tuzilgan bo‘lishi ham mumkin: $5x^3+2x+1-2x^2+x \to 5x^3-2x^2+3x+1$

4-eslatma: Odatda, ko‘phadlarning darajalarini kamayish tartibda yoziladi va unga standart shakl deyiladi: $2x^3+2x+1-x^2 \to 2x^3-x^2+2x+1$

Endi ko'phadni ko'phadga ko'paytirishni ko'rib chiqamiz: (1-ifoda)×(2-ifoda).



$$=acx^2+adx+bcx+bd=acx^2+(ad+bc)x+bd$$

Uni quyidagicha talqin qilishimiz mumkin:
1-bosqich: 1-hadlar
2-bosqich: tashqi hadlar
3-bosqich: ichki hadlar
4-bosqich: oxirgi hadlar

Ko'paytirishda odatda distributivlik xossasidan va guruhlashdan foydalaniladi.

1-misol. Ushbu $(2x^2+x-3)(x+2)$ ko'paytmani toping.

Yechimi: 1-usul 2-usul:

$$(2x^2+x-3)(x+2)$$
$$=(2x^2+x-3)x+(2x^2+x-3)\cdot 2$$
$$=2x^3+x^2-3x+4x^2+2x-6$$
$$=2x^3+5x^2-x-6$$

$$\begin{array}{r} 2x^2+x-3 \\ \times\ x+2 \\ \hline 2x^3+x^2-3x \\ 4x^2+2x-6 \\ \hline 2x^3+5x^2-x-6 \end{array}$$

Qisqa hisoblash uchun quyidagi **qisqa ko'paytirish formulalarini** eslab qoling.

1. $(a+b)^2 = a^2 + 2ab + b^2$
$(a-b)^2 = a^2 - 2ab + b^2$ $\quad b \to (-b)$

2. $(a+b)(a-b) = a^2 - b^2$

3. $(x+a)(x+b) = x^2 + (a+b)x + ab$

4. $(ax+b)(cx+d) = acx^2 + (ad+bc)x + bd$

5. $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$
$(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$ $\quad b \to (-b)$

1-mashq. Berilgan ko'phadlar uchun quyidagi amallarni bajaring: $A = 2x^3 - x^2 + 1$, $B = -x^2 + 2x + 1$.

1) A va B ko'phadlarni qo'shing hamda ayiring;
2) A ko'phadni B ko'phadga ko'paytiring.

1B. Ko'phadlarni ko'paytuvchilarga ajratish usullari

Ko'paytuvchiga ajratish qoidalari

Ko'phadni ko'paytuvchiga ajratish, bu ko'paytmani yoyishga teskari amal hisoblanadi.

Ko'phadlarga yoyish

$$(x+2)(x+3) = x^2 + 5x + 6$$

Ko'paytuvchilarga ajratish

Ko'phadlarga yoyishni yuqorida qaragan edik. Endi teskari amalni qarab chiqamiz. Buning uchun eng sodda holdan boshlaymiz.

Sonni tub ko'paytuvchilarga ajratish.

tub sonlar

$$12 = 2 \times 6 \qquad 12 = \frac{1}{2} \times 24 \qquad 12 = \underbrace{2 \times 2 \times 3}_{\text{tub ko'paytuvchilari}}$$

Ko'phadni ko'paytuvchilarga ajratish:

$$x^4 - 16 = (x^2 + 4)(x^2 - 4) = \underbrace{(x^2 + 4)(x + 2)(x - 2)}_{\text{ko'paytuvchilarga ajratildi}}$$

Ko'paytuvchilarga ajratish uslubi mukammal emas, ya'ni ko'paytuvchilarga ajratishning aniq uslubi yo'q. Ba'zan o'ylab topish yordam berishi mumkin. Buning uchun bir nechta foydali qoidalar bor. Ular quyidagicha:

1) Ko‘paytmadan umumiy ko‘paytuvchini qavsdan tashqariga olib chiqish:

$$2x^3+10x^2+12x=2x(x^2+5x+6)$$

2) Quyidagi formuladan foydalanish mumkin:

$$\boxed{(a+b)}\ \text{1- qoida}$$

$$\boxed{ab}$$

$$2x^3+10x^2+12x=2x(x^2+5x+6) \quad \text{2-qoida}$$

$$=2x(x+2)(x+3)$$

$$\boxed{a} \qquad \boxed{b}$$

3) Eng past darajali birhadni qavsdan chiqarib qayta guruhlash:

$$x^3-3x^2-2xy+6y=-(2x-6)y+(x^3-3x^2) \quad \text{1-qoida}$$

$$=-2y(x-3)+x^2(x-3) \quad \text{2-qoida}$$

$$=(x^2-2y)(x-3) \quad \text{3-qoida}$$

Ko‘paytma qoidasidan foydalanib yechishni keyinroq ko‘rib chiqamiz.

2-misol. Quyidagi ko‘phadlarni ko‘paytuvchilarga ajrating

1) $2x^2+5xy+3y^2-3x-5y-2$

2) x^4-10x^2+9

Yechimi: 1) bir xil darajali o‘zgaruvchilarni yonma-yon yozamiz quyidagicha shakl almashtiramiz:

$$2x^2+5xy+3y^2-3x-5y-2$$
$$=2x^2+(5y-3)x+(3y^2-5y-2)$$
$$=2x^2+(5y-3)x+(y-2)(3y+1)$$
$$=\{x+(y-2)\}\{2x+(3y+1)\}$$
$$=(x+y-2)(2x+3y+1)$$

$$\begin{array}{ccccc} 1 & \times & y-2 & \to & 2y-4 \\ 2 & & 3y+1 & \to & 3y+1 \\ \hline & & & & 5y-3 \end{array}$$

2) bu ko‘phad 4-darajali bo‘lgani sababli, unda darajani pasaytirish qoidasini qo‘llaymiz. Bu yerda quyidagi belgilashni kiritib hisoblaymiz:

$$X=x^2$$
$$x^4-10x^2+9=X^2-10X+9=(X-1)(X-9)$$
$$=(x^2-1)(x^2-9)=(x+1)(x-1)(x+3)(x-3)$$

Ko‘paytuvchilarga ajratish teoremasi:

$f(x)$ ko‘phad uchun $f(k)=0$ tenglik o‘rinli bo‘lsa, uning $(x-k)$ ko‘paytuvchisi ham bo‘ladi.

Bu teoremaning isbotini ko'phadli tenglamalar mavzusida keltiramiz.

Bu teoremani tadbiq qilish quyidagicha:

1) $f(k)=0$ bo'ladigan k soni topiladi;

2) $f(x)$ funksiya $(x-k)$ga bo'linadi va $g(x)=\dfrac{f(x)}{x-k}$ topiladi;

3) Keyin $f(x)$ funksiya $f(x)=(x-k)g(x)$ kabi yoziladi.

4) Endi $f(x)$ funksiyaga qaraganda $g(x)$ funkssiyai ko'paytuvchilarga ajratish osonlashadi.

3-misol. Ko'phadni ko'paytuvchilarga ajrating:

$$f(x) = x^3 + 4x^2 + x - 6.$$

Yechimi: $f(1) = 0$ tenglikdan foydalanamiz:

$x^3 + 4x^2 + x - 6 =$
$= (x-1)(x^2 + 5x + 6) =$
$= (x-1)(x-2)(x-3).$

$$\begin{array}{r|l} -\,x^3 + 4x^2 + x - 6 & x - 1 \\ \cline{2-2} \underline{x^3 - x^2 \qquad\qquad} & x^2 + 5x + 6 \\ -\,5x^2 + x - 6 & \\ \underline{5x^2 - 5x \quad} & \\ -\,6x - 6 & \\ \underline{6x - 6} & \\ 0 & \end{array}$$

4-misol. Quyidagi ko'phadlarni ko'paytuvchilarga ajrating:

1) $4x^2 + 8x - 21$; 2) $x^3 + 7x^2 + 8x + 2$.

Yechimi:

1) $(ax + b)(cx + d) = acx^2 + (ad + bc)x + bd$ formuladan foydalanamiz. Bu yerda koeffisiyentlar $a = 2, b = -3, c = 2, d = 7$ bo'ladi. O'ng tomondagi hisoblashlar orqali quyidagi ko'paytmani hosil qilamiz:

$$4x^2 + 8x - 21 = (2x - 3)(2x + 7). \qquad \begin{array}{ccc} 2 & -3 & \rightarrow & -6 \\ 2 & 7 & \rightarrow & 14 \end{array}$$

2) Ko'paytma teoremasidan foydalanamiz.

$x = -1$ nuqtada funksiya nolga aylanadi: $f(-1) = 0$. Shuning uchun berilgan $x^3 + 7x^2 + 8x + 2$ ko'phadni $x + 1$ ikki hadga bo'lamiz:

$$\frac{x^3 + 7x^2 + 8x + 2}{x + 1} = x^2 + 6x + 2.$$

Demak,

$$x^3 + 7x^2 + 8x + 2 = (x^2 + 6x + 2)(x + 1).$$

1.2. Algebraik tenglamalar va funksiyalar
2A. Algebraik tenglamalar

Chiziqli tenglamalar. Kvadrat tenglamalar.

Dastlab ushbu masalani qaraylik: *qutida 1500 gr lik 8 ta olma bor. Qutining bo'sh holatidagi og'irligi 300 gr bo'lsa, har bir olmaning og'irligini toping.*

Javob: 1ta olmaning og'irligini x deb belgilaylik va shu tariqa $8x + 300 = 1500$ tenglama hosil bo'ladi. Bu tenglamada noma'lum $x = 150$ gr ga teng bo'ladi.

Tenglama- deb ikkita mulohazaning harfli yoki sonli tengligiga aytiladi: $f(x) = g(x)$.

Tenglamaning turlari ko'p bo'lib, agar $f(x)$ va $g(x)$ lar ko'phadlar bo'lsa, bu holda tenglama algebraik yoki ko'phad ko'rinishidagi tenglama deyiladi.

$$a_0\ x^n + a_1\ x^{n-1} + a_2\ x^{n-2} + \cdots \ldots . + a_{n-1}\ x + a_n\ = 0$$

Eng sodda tenglama bu birinchi tartibli ko'phadli tenglama bo'lib, u chiziqli tenglama deyiladi.

$$ax + b = 0 \quad (a \neq 0)$$

Bu tenglamani yechish uchun, dastlab o'zgarmasni tenglamaning o'ng tomoniga o'tkazib keyin a koeffisiyentga bo'lamiz.

$$ax = -b \qquad x = -\frac{b}{a}$$

$x = -\frac{b}{a}$ qiymat tenglamaning ildizi deyiladi.

Endi birgalikda bo'lgan chiziqli tenglamalar sistemasini qaraymiz.

Uni o'rganishda quyidagi misolni qaraymiz:

1-misol. Quyidagi tenglamalr sitemasini yeching. $\begin{cases} 2x + y = 8 \\ x + y = 6 \end{cases}$

Bu sistemani yechishda bir necha usullar mavjud.

Yechimi: 1-usul. O'rniga qo'yish usuli: bunda 2-tenglamadan $y = -x + 6$ ni topamiz, va 1-tenglamadagi y noma'lum o'rniga qo'yamiz. Natijada $2x + (-x + 6) = 8$ ni hosil qilamiz. Bu bir noma'lumli tenglamadan $x = 2$ qiymat hosil bo'ladi, bu qiymat yordamida $y = -2 + 6 \ \rightarrow y = 4$ kelib chiqadi.

2-usul. Qo'shish usuli: bunda tenglamalrni qo'shish orqali bir noma'lum yo'qotiladi, ya'ni tenglamalardagi noma'lumlar sonini kamaytiriladi. Bu misolda biz 1-tenglamani 2-tenglamadan ayiramiz. Natijada $x = 2$ tenglik kelib chiqadi.

Ikkinchi tartibli ko'phad ko'rinishidagi tenglamaga **kvadrat tenglama** deyiladi.

$$ax^2 + bx + c = 0, \quad (a \neq 0)$$

bu yerda x noma'lum, a, b, c koeffisiyentlar esa o'zgarmas sonlar va $a \neq 0$ sonlar.

Kvadrat tenglamani yechishning 3xil usuli bor.

Birinchisi ko'paytuvchilarga ajratish usuli.

Agar biz tenglamadagi kvadrat uchhadni ko'paytuvchiga ajratsak, tenglama $a(x-p)(x-q)=0$ ko'rinishga keladi. Demak, bu kvadrat tenglmaning ildizlari $x=p$, $x=q$ sonlari bo'ladi.
Ushbu misolni qaraylik.

2-misol. Quyidagi tenglamani yeching:
1) $x^2+8x+12=0$ 2) $6x^2-x-15=0$

Yechimi: 1-tenglamada $p+q=8$, $pq=12$. Biz ular orqali 1-tenlamani $(x+2)(x+6)=0$ kabi ko'paytuvchiga ajrata olamiz. Bundan tenglamaning ildizlari:
$x+2=0$, $x+6=0$ $x_1=-2$, $x_2=-6$ kelib chiqadi.

Ikkinchi tenglamani biz $(2x+3)(3x-5)=0$ kabi ko'paytuvchiga ajrata olamiz. Bundan ildizlar

$$2x+3=0, \quad 3x-5=0 \quad x_1=-\frac{3}{2}, \quad x_2=\frac{5}{6}$$

qiymatlarga teng bo'ladi.

Endi ko'paytuvchiga ajratmasdan kvadrat tenglamani yechamiz.

Mana bu tenglamani qaraylik,

$$x^2+5x+3=0$$

Uni yechish uchun quyidagi formuladan foydalanamiz:

$$ax^2+bx+c=0 \ (a \neq 0)$$

Agar $b^2-4ac>0$ bo'lsa, ikkita ildizi bor:

$$x=\frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

Agar $b^2-4ac=0$ bo'lsa, bitta ildizi bor(ikkita ildiz bir xil)

$$x=-\frac{b}{2a}$$

Agar $b^2-4ac<0$ bo'lsa haqiqiy ildiz yo'q (buni keying darslarimizda isbotlaymiz).

Endi kvadratik formulani keltirib chiqaramiz.

$$ax^2+bx+c=0$$

Bu tenglamani barcha koeffisiyentlarini a ga bo'lsak, va ozod hadni tenglamaning o'ng tomoniga o'tkazsak

$$x^2+\frac{b}{a}x=-\frac{c}{a}$$

tenglik hosil bo'ladi.
Endi bu tenglamani to'la kvadrat tenglamaga keltiramiz.

$$(x+\frac{b}{2a})^2=-\frac{c}{a}+\frac{b^2}{4a^2}=\frac{b^2-4ac}{4a^2}$$

Ikkala tomonini ildiz ostiga olsak,

$$x+\frac{b}{2a}=\pm\frac{\sqrt{b^2-4ac}}{2a} \qquad x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

kvadrat formula kelib chiqadi.

3-misol. Uzunligi 100sm bo'lgan ipni yuzasi 400sm^2 bo'lgan to'g'ri to'rtburchak shaklga keltirildi. Hosil bo'lgan to'g'ri to'rtburchakning eni va bo'yini toping.

Yechimi: To'g'ri to'rtburchakning bir tomonini x deb belgilaylik, ikkinchi tomonini esa perimeter formulasidan foydalanib aniqlasak $50-x$ ga teng bo'ladi. U holda yuzani topish formulasidan

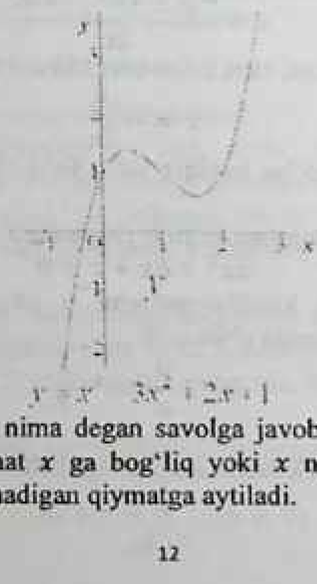
$$x(50-x)=400$$

kvadrat tenglama kelib chiqadi. Bu kvadrat tenglamani kvadrat formuladan foydalanib yechsak, $x=10,40$ qiymatlari kelib chiqadi. Bu ukkala qiymatlar ham kvadrat tenglamani ildizi yoki to'g'ri to'rtbuchak tomonlarining uzunligi bo'ladi.

Eslatma: *Agar shunday kvadrat tenglamaga keladigan uzunlikni aniqlovchi noma'lumlarda ikkita qiymat har qanday chiqqanda ham uning qiymatlari 0, 50 o'rtasida bo'lishi kerak. Chunki uzunlik nol bo'la olmaydi, agar 0 bo'lsa u kesma emas nuqta bo'lib qoladi.*

2B. Funksiya tushunchasi

Chiziqli funksiyalar. Kvadratik funksiyalar.



$y = x^3 - 3x^2 + 2x + 1$

Dastlab, funksiya nima degan savolga javob topamiz. Matematikada, funksiya –deb y qiymat x ga bog'liq yoki x ning x_1, x_2, x_3, ……..x_n qiymatlari orqali aniqlanadigan qiymatga aytiladi.

y qiymatni x yoki x_1, x_2, x_3, ……..x_n qiymatlarning funksiyasi deb aytamiz. $y = f(x)$ tenglikda y o'zgaruvchini x o'zgaruvchining funksiyasi deyiladi yoki $y = f(x_1, x_2, x_3, \ldots\ldots.x_n)$ tenglikda esa y o'zgaruvchini x_1, x_2, x_3, ……..x_n o'zgaruvchilarning funksiyasi deyiladi.

Bu yerda y –erksiz o'zgaruvchi x- esa erkli o'zgaruvchi deyiladi. Endi mana bu misolni qaraylik.

4-misol. Uzunligi 100sm bo'lgan ipdan to'g'ri to'rtburchak shakli yasaldi. Bu to'g'ri to'rtburchakning y yuzini to'g'ri to'rtburchak tomonini x uzunligining funksiyasi sifatida aniqlang.

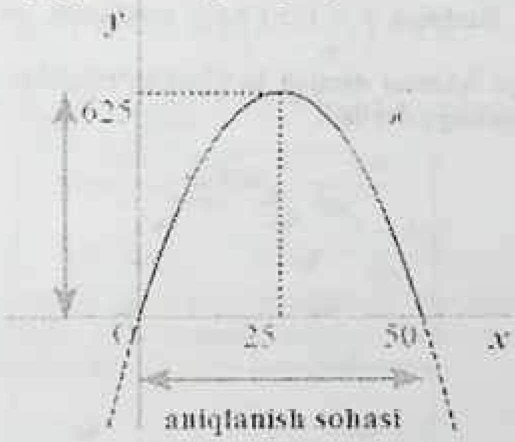
Yechimi: To'g'ri to'rtburchakning bir tomoni x bo'lsa ikkinchi tomoni $50 - x$ ga teng bo'ladi. Endi yuzani topish formuladidan

$y = x(50 - x) = -x^2 + 50x \qquad (50 > x > 0)$

tenglikni hosil qilamiz. Bu yerda y to'g'ri to'rtburchakning yuzasi x esa tomon uzunligini aniqlaydi.

Aniqlanish sohasi deb, erkli o'zgaruvchilar qiymatlari to'plamiga aytiladi.

Qiymatlar sohasi deb esa, barcha erkli o'zgaruvchilari orqali aniqlanadigan y ning qiymatlar to'plamiga aytiladi.



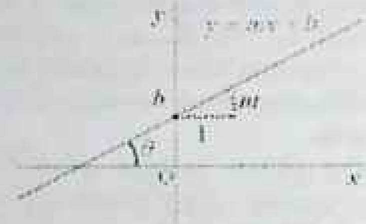
Agar funksiya birinchi darajali ko'phaddan tashkil topgan bo'lsa, bunday funksiya chiziqli funksiya deyiladi.

Chiziqli funksiyaning bir necha shakllari bor.

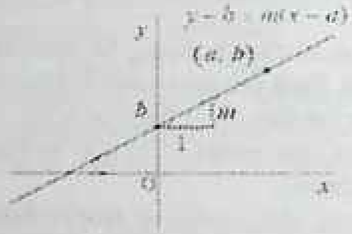
1) Burchak koeffisiyentli ko'rinishidagi funksiya

$$y = mx + b \ (m \neq 0)$$

$$bu\ yerda\ m - qiyaligi, b - esa\ y\ bo'yicha\ \ siljishi$$



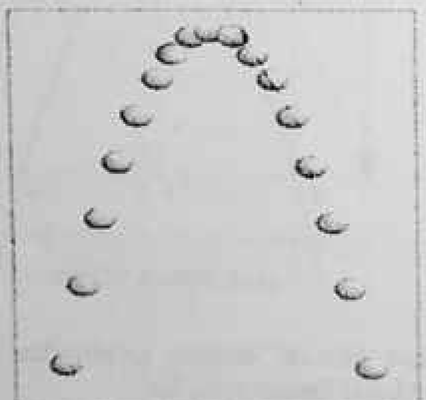
2) Koordinatada ajratgan nuqtalari bo‘yicha $y - b = m(x - a), m \neq 0$. Bu to‘g‘ri chiziq o‘tadigan nuqta $(a; b)$.



3) Umumiy tenglama ko‘rinishi $ax + by + c = 0$.

Qisqa qilib aytganda, uchinchi ko‘rinishi to‘g‘ri chiziq tenglamasi, lekin funksiya emas. Funksiya $y = f(x)$ kabi aniqlanadi, ya’ni x dan y kelib chiqishi kerak.

Agar funksiya ikkinchi darajali ko‘phad ko‘rinishida aniqlangan bo‘lsa, unga **kvadrat funksiya** deyiladi.



Bu to‘lqin biz uchun juda yaxshi tanish, chunki uning trayektoriyasi otilgan topning harakatini aniqlaydi.

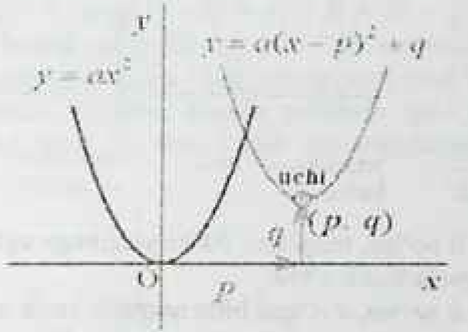
Kvadrat funksiyaning ham bir necha xil ko‘rinishlari bor.

1) Umumiy holatda kvadrat funksiyaning ko‘rinishi quyidagicha

$$y = ax^2 + bx + c \quad (a \neq 0)$$

Shuningdek, umumiy ko'rinishni biz tenglamaning xarakteristik shakli, ya'ni standart shakliga aylantirib o'rganamiz.

2) Standart shakli $y = a(x-p)^2 + q \quad (a \neq 0)$, bu yerda p va q lar berilgan parobalaning uchini aniqlaydi.



•p = gorizantal uchi
•q = vertikal uchi

Grafikdan ko'rinib turibdiki, p- gorizontal o'qdagi nuqtasi q –esa vertikal o'qdagi nuqtasi.

Kvadrat funksiyaning umumiy ko'rinishidan standart shakldagi formulani keltirib chiqaramiz va parabola uchining koordinatalarini aniqlaymiz:

$$ax^2 + bx + c = a\left(x^2 + \frac{b}{a}x\right) + c =$$

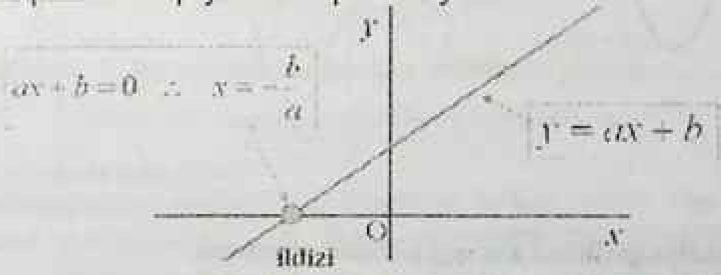
$$= a\left\{x^2 + 2\frac{b}{2a}x + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2\right\} + c = a\left\{\left(x + \frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2\right\} + c =$$

$$= a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a^2}.$$

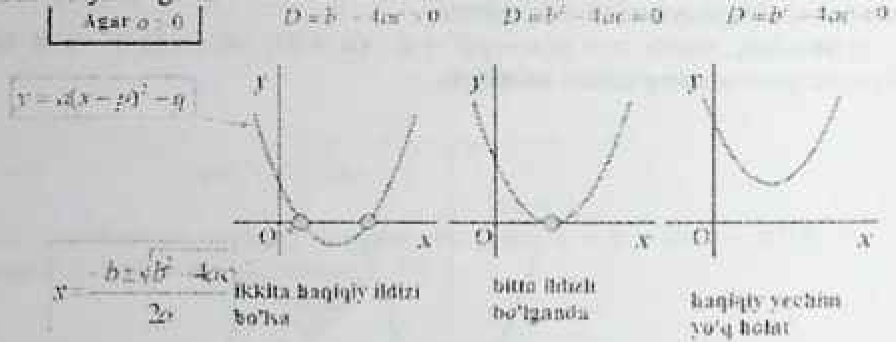
$$\therefore \quad p = -\frac{b}{2a}, \qquad q = -\frac{b^2 - 4ac}{4a^2}.$$

Endi funksiyaning grafigi va tenglamasi orasidagi bog'lanishni ko'ramiz.

Aytishimiz mumkinki, tenglamaning ildizlari funksiyaning x o'qdagi kesishish nuqtalarini aniqlaydi. Chiziqli funksiya uchun



Endi kvadrat funksiyani qaraymiz. Lekin bizda 3xil shartlar bor edi. Shu bo'yicha grafiklar chizamiz.

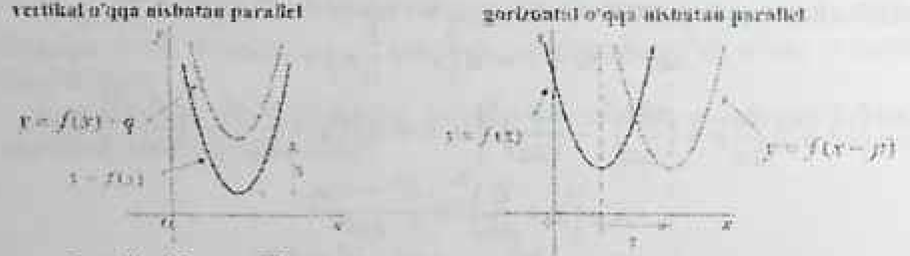


1) Agar $D > 0$ bo'lsa, tenglama ikkita yechimga ega bo'ladi va parabola x o'qini ikkita nuqtada kesib o'tadi.

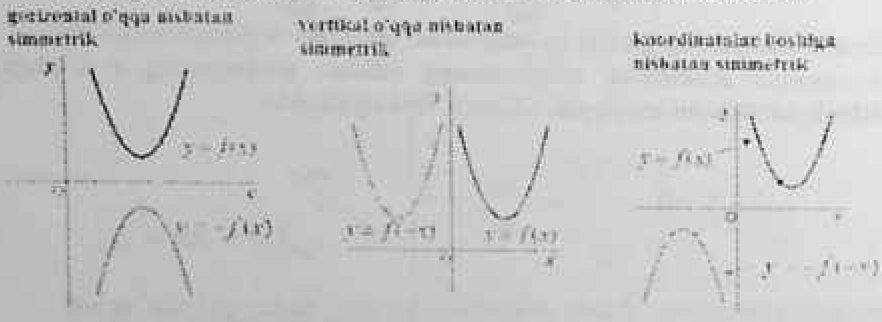
2) Agar $D = 0$ bo'lsa, x o'qini bitta nuqtada kesib o'tadi(ya'ni urinadi).

3) $D < 0$ bo'lganda, x o'qda haqiqiy yechim yo'qligi uchun kesib o'tmaydi.

Endi funksiya grafigini koordinata o'qlari bo'ylab parallel ko'chirish formulalari va chizmalarini keltiramiz.



Quyidagi 3ta grafik esa o'qqa nisbatan simmetriklikni ko'rsatadi.



1) Funksiya grafiklari x o'qqa nisbatan simmetrik

2) Funksiya grafiklari y o'qqa nisbatan simmetrik

3) Funksiya grafiklari (0;0) nuqtaga nisbatan simmetrik bo'ladi.
Yuqorida qaralganlarni misolda ko'ramiz.

5-misol. Quyidagi funksiyaning eng katta va eng kichik qiymatlarini toping.

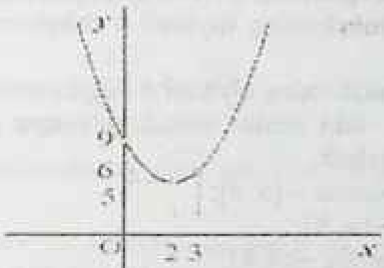
$$y = x^2 - 4x + 9 \quad (0 \le x \le 3)$$

Yechimi: Bu funksiyani standart shaklga keltiramiz:

$$y = x^2 - 4x + 9 \ = (x^2 - 4x + 4) + 5 = (x-2)^2 + 5$$

Bu shakldan bizga ma'lumki, parobalaning uchi (2;5) nuqtada bo'ladi.

Funksiya aniqlanish sohasining chap chegarasiga mos keluvchi va o'ng chegarasiga mos keluvchi nuqtalar, hamda parabola uchi koordinatalariga ko'ra funksiyaning eng katta va eng kichik qiymatlarini aniq topishimiz mumkin.

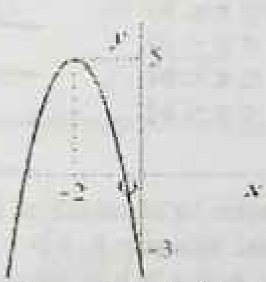


Demak, eng katta qiymati 9 ga teng, eng kichik qiymati 5 ga teng.

6-misol.

1) $y = 2x^2 - 4x + 5$ parabola uchining koordinatalarini toping.

2) $y = ax^2 + bx + c$ kvadrat funksiyaning grafigidan foydalanib, koeffisiyentlarini toping.



Yechimi: 1) Bu misolda biz standart shaklga keltiramiz.

$$2x^2 - 12x + 17 = 2(x^2 - 6x) + 17 = 2[(x-3)^2 - 3^2] + 17 = 2(x-3)^2 - 1$$

Demak, parobala uchi (3;-1)

2) Koordinata uchini grafik orqali topadigan bo'lsak, (-2;5). Shu orqali biz tenglamani $y = a(x+2)^2 + 5$ kabi yozishimiz mumkin. Endi bizda a ni

topish qoldi. Uni topish uchun vertikal o'qni (0;-3) nuqtani kesib o'tganidan foydalansak, $-3 = a(0+2)^2 + 5$ $a = -2$ ekani kelib chiqadi.
Demak, $y = -2(x+2)^2 + 5$ $y = -2x^2 - 8x - 3$

1.3. Chiziqli va kvadrat tengsizliklar
3A. Sonli tengsizliklar. Chiziqi tengsizliklar

Matematikada ikki sonni taqqoslaganda biz tengsizliklardan foydalanamiz. Bunda 4 xil ko'rinishdagi tengsizliklardan foydalaniladi:

$a < b$ bu tengsizlik a miqdorning qiymati b miqdorning qiymatiga qaraganda kichik;

$a \le b$ esa a miqdorning qiymati b miqdorning qiymatidan katta emas;

$a > b$ esa a miqdorning qiymati b miqdorning qiymatiga qaraganda katta;

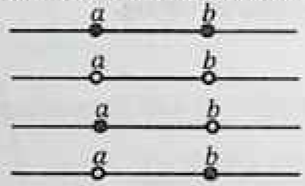
$a \ge b$ esa a miqdorning qiymati b miqdorning qiymatidan kichik emas.

Sonli oraliq - ikki sonlar orasidagi yotgan haqiqiy sonlar. Oraliqlar quyidagi turlarga ajraladi:

Yopiq oraliq-kesma $-[a,b]$;
Ochiq oraliq $-(a,b)$;
Yarim ochiq oraliq $-[a,b]$;
Yarim ochiq oraliq $-(a,b]$.

[ ] qavs—oxirgi nuqtalari kirishini ham bildiradi.
( ) qavs esa - oxirgi nuqtalardan tashqari barcha nuqtalar kirishini bildiradi.
Bu intervallar sonlar o'qida quyidagi tengsizliklar kabi belgilanadi.

$\{x \in R:\ a \le x \le b\}$
$\{x \in R:\ a < x < b\}$
$\{x \in R:\ a \le x < b\}$
$\{x \in R:\ a < x \le b\}$



Endi tengsizliklar haqida ba'zi xossalarni ko'rib chiqamiz.

1) Tranzitivlik xossasi: agar $a > b$, $b > c$ bo'lsa, u holda $a > c$ bo'ladi.
2) Qo'shish xossasi: agar $a > b$ bo'lsa, u holda $a + c > b + c$ bo'ladi.
3) Ayirish xossasi: agar $a > b$ bo'lsa, u holda $a - c > b - c$ bo'ladi.
4) Ko'paytirish va bo'lish xossasi: agar $a > b$, $c > 0$ bo'lsa, u holda

$$ac > bc,\ \frac{a}{c} > \frac{b}{c}$$

Bu xossalar juda muhim, shuning uchun ularni yaxshi eslab qolishingiz kerak.

Uchinchi xossada, $b = c$ deb olsak, $a > c$ ekanligidan $a - c > 0$ o'rinli bo'ladi. Boshqacha qilib aytganda, sonni bir tomondan ikkinchi tomonga o'tkazganda o'zining ishorasini o'zgartiradi.

1-misol. Quyidagi tengsizlikni isbotlang:

$$\frac{a+b}{2} \geq \sqrt{ab} \quad (a \geq 0,\ b \geq 0)$$

Bu yerda a, b sonlar musbat sonlar yoki nolga teng bo'lishi mumkin.

Yechimi: Bu masalani yechish uchun quyidagilarni bajaramiz:

$$\frac{a+b}{2} - \sqrt{ab} = \frac{a+b-2\sqrt{ab}}{2} = \frac{\sqrt{a}^2+\sqrt{b}^2-2\sqrt{ab}}{2} = \frac{(\sqrt{a}-\sqrt{b})^2}{2} \geq 0.$$

Ildiz belgisi ostidagi son ham, ularning ayirmasining kvadrati ham musbat, shuning uchun oxirgi ifoda ham musbat.

Demak,

$$\frac{a+b}{2} \geq \sqrt{ab}$$

Agar $a = b$ bo'lsa, tenglik o'rinli bo'ladi.

Matematikada, $\frac{a+b}{2}$ o'rta arifmetik, va $\sqrt{ab}$ esa o'rta geometrik qiymat deyiladi.

$a \neq b$ bo'lganda, har doim o'rta arifmetik qiymat o'rta geometrik qiymatga qaraganda katta bo'ladi va $a = b$ bo'lgandagina teng bo'ladi.

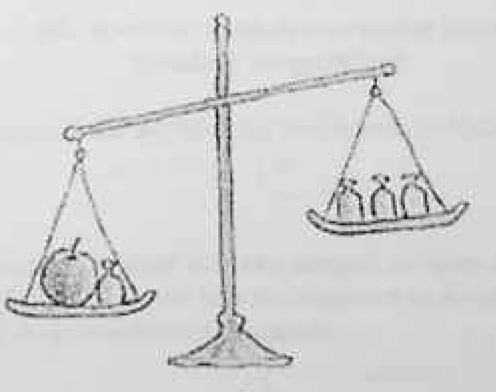
Sǿngi tasdiqni quyidagi misollarda siz yaxshi tushunib olishingiz mumkin:

$$\frac{12+3}{2} = 7{,}5 \qquad \sqrt{12 \times 3} = 6$$

$$\frac{8+7}{2} = 7{,}5 \qquad \sqrt{8 \times 7} = 7{,}48$$

$$\frac{7{,}5+7{,}5}{2} = 7{,}5 \qquad \sqrt{7{,}5 \times 7{,}5} = 7{,}5$$

Tengsizlikka doir bitta misol keltiramiz. Mana bu rasmga qarang.



Bir toshning og'irligi 100gr ga teng. Biz bunda ko'rishimiz mumkinki, bitta olma va bitta tosh uchta toshga qaraganda og'riroq. Agar bitta olmaning og'riligini x deb belgilasak, ushbu tengsizlikka ega bo'lamiz.

$$x + 100 > 3 \times 100$$

100 ni o'ng tomonga o'tkazib ayiradigan bo'lsak, $x > 200$ kelib chiqadi. Bu tenglikka chiziqli tengsizlik deyiladi.

Umumiy holda $ax + b > cx + d$ chiziqli tengsizlikni yechihda, noma'lumlarni bir tomonga sonlarni bir tomonga o'tkazamiz. $ax - cx > d - b$

$$(a - c)x > d - b$$

Keyin, tengsizlikning ikkala tomonini $(a - c)$ ko'paytuvchiga bo'lamiz; agar $(a - c)$ koeffitsiyent manfiy bo'lsa, tengsizlik qarama-qarshi tomonga o'zgaradi.

Mana bu misolni qaraylik.

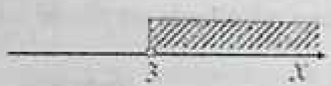
2-misol. $4x - 2 > 10$ tengsizlikni yeching.

Yechimi: Tengsizlikning ikkala tomoniga ham 2 ni qo'shsak,

$$4x > 12$$

Keyin ikkala tomonini ham 4ga bo'lamiz

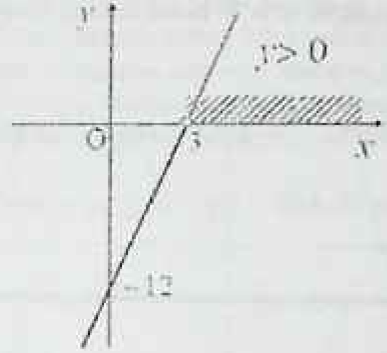
$$x > 3$$



Bu oraliq sonlar o'qidagi x ning qabul qilishi mumkin bo'lgan qiymatlarni ko'rsatadi.

Endi tengsizlik va grafiklarlarni orasidagi bog'lanishni qaraymiz.

2-Misoldagi $4x - 2 > 10$ tengsizlikga mos ravishda $y = 4x - 12$ funksiyani yozsak, va $y > 0$ sohani chizadigan bo'lsak, Ox o'qda $x = 3$ bo'lsa, $y = 0$ bo'ladi va $y > 0$ tengsizlikga mos soha $x > 3$ bo'ladi.



Shunіng uchun, yechim $x > 3$ bo'ladi.

3-misol. Ikkita tengsizliklarni (tengsizliklar sistemasini) yeching:

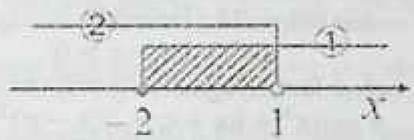
$$\begin{cases} 7x - 1 \geq 4x - 7 \\ x + 5 > 3(1 + x) \end{cases}$$

Yechimi: Bu sistemani yechish uchun har bir tengsizlikni o'zini alohida alohida yechib olamiz.

Dastlab, biz har bir tengsizlikni yechib olamiz:

1) $7x - 1 \geq 4x - 7 \Rightarrow 7x - 4x \geq -7 + 1 \Rightarrow 3x \geq -6 \Rightarrow x \geq -2$

2) $x + 5 > 3(1 + x) \Rightarrow -2x > -2 \Rightarrow x < 1$



Demak, bu ikki tengsizliklarning umumiy sohasi $[-2,1)$ yarim ochiq oraliqdan iborat bo'ladi.

3B. Kvadrat funksiya va uning ildizi
Kvadrat tengsizliklar

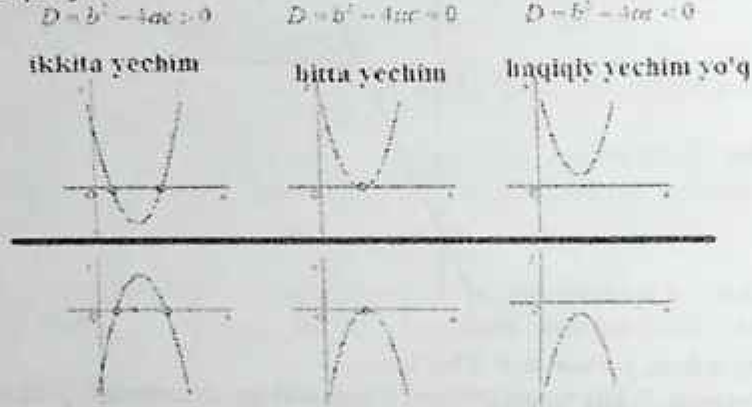
Kvadrat tengsizliklar quyidagi ko'rinishlarda bo'ladi.

$$ax^2 + bx + c > 0$$

$$\geq, \; <, \; \leq$$

Chap tomondagi ko'phad ikkinchi darajali bo'lgani uchun unga kvadrat tengsizlik deyiladi. Biz yuqorida kvadrat funksiya va kvadrat tenglamaning ildizlari orasidagi bog'lanishni o'rgangandik.

Birinchi qatordagi grafiklar $a > 0$ holat uchun, bunda parobalaning shoxlari yuqoriga qaraydi.



Ikkinchi qatordagi grafikda esa $a < 0$ holat uchun, bunda parobalaning shoxlari pastga qaragan.

x - o'qdagi kesishish nuqtalari kvadrat funksiyaning ildizlari nechta ekanligini ko'rsatadi.

1) $D > 0$ holatda ikkita ildizga ega.
2) $D = 0$ holatda bitta yechimga ega.
3) $D < 0$ bunda esa haqiqiy yechim ega emas.

Endi kvadrat tengsizlikni yechish bosqichlarini ko'rib chiqamiz.

1-bosqich. Kvadrat tengsizlikni standart shaklga keltiramiz.

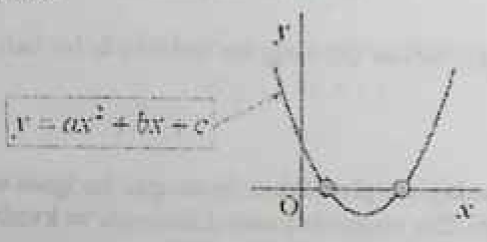
$$ax^2 + bx + c > 0$$

2-bosqich. Kvadrat funksiyaga mos grafik chizamiz.

$$y = ax^2 + bx + c = a(x - p)^2 + q$$

3-bosqich. $ax^2 + bx + c = 0$ kvadrat tenglamani yechib va x_1 va x_2 yechimlarini topamiz.

4-bosqich. Son o'qini kvadrat tenglamaning x_1 va x_2 ildizlari yordamida oraliqlarga ajratamiz va har bir oraliqda y o'zgaruvchining ishorasini aniqlaymiz va $ax^2 + bx + c > 0$ tengsizlikni qanoatlantiradigan oraliqlarni tanlaymiz.



Endi har bir bosqichni quyidagi misol orqali tushunib olamiz.

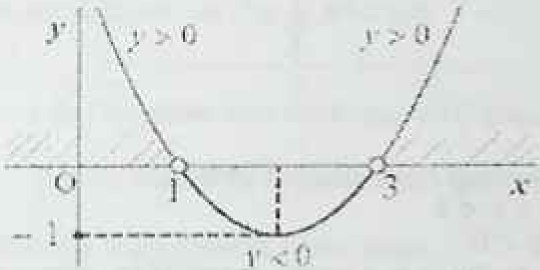
$a > 0$ va $D > 0$ holatdagi misolni qaraylik,

4-misol. $x^2 - 4x + 3 > 0$ tengsizlikni yeching.

Yechimi: Tengsizlikni yechish uchun biz kvadrat tenglamani mos standart shaklga keltiramiz. $y = (x-2)^2 - 1$ standart shakli.

Funksiya grafigi Ox o'qni abssissasi 1 va 2 bo'lgan nuqtalarda kesib o'tadi. Bundan, kesmadagi bo'yalgan soha yechim ekanini ko'rishimiz mumkin.

Biz kvadrat funksiyani $y = (x-1)(x-3)$ kabi yozib olishimiz mumkin.



Grafikdan berilgan tengsizlikning yechimi $x < 1, x > 3$ ekanligi ko'rinib turibdiki.

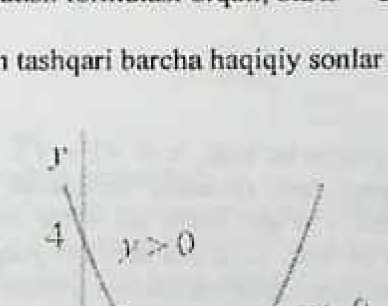
Endi $a > 0$ va $D = 0$ bo'lgandagi misolni ko'raylik.

5-misol. $x^2 - 4x + 4 > 0$ tengsizlikni yeching.

Yechimi: Kvadrat funksiyani mos standart shaklga keltiramiz: $y = (x-2)^2$. Demak, funksiya grafigi Ox o'qqa abssissasi $x = 2$ bo'lgan nuqtada urinadi.

Ko'paytmaga ajratish formulasi orqali, biz $x = 2$ ekanligini hosil qilamiz.

Shuningdek, $x = 2$ dan tashqari barcha haqiqiy sonlar tengsizlik yechimi bo'ladi.

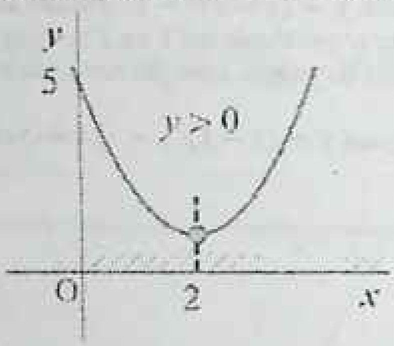


Endi $a > 0$ va $D < 0$ holatni qaraylik.

6-misol. $x^2 - 4x + 5 > 0$ tengsizlikni yeching.

Yechimi: Uni ham standart shaklga keltiramiz. $y = (x-2)^2 + 1$

Bundan ko'rinadiki, funksiya grafigi Ox o'qini biror nuqtada ham kesib o'tmaydi. Ya'ni tengsizlikning yechimi barcha haqiqiy sonlar.



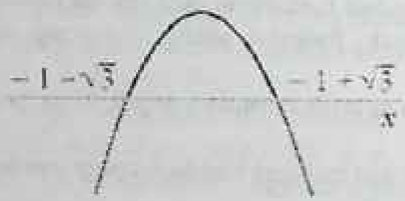
7-misol. Quyidagi tengsizliklarni yeching:

1) $-x^2 - 2x + 2 < 0$

2) $x^2 + x + 2 < 0$

Yechimi: 1) Dastlab kvadrat tengsizlikga mos $-x^2 - 2x + 2 = 0$ kvadrat tenglamaning yechimlarini aniqlaymiz:

$$x = \frac{2 \pm \sqrt{(-2)^2 - 4(-1)(2)}}{2(-1)} = -1 \pm \sqrt{3}$$



Grafik asosida qaraydigan bo'lsak, $x < -1 - \sqrt{3}$ $x > -1 + \sqrt{3}$

2) Kvadrat funksiyaga mos ravishda hisoblasak $D = -7 < 0$ bo'ladi. Grafikni chizsak, u Ox - o'qni kesib hech bir nuqtada kesib o'tmaydi. Qo'shimcha qiladigan bo'lsak, parabola shoxlari yuqoriga qaraydi. Umumiy qilib aytganda, berilgan tengsizlikning haqiqiy yechimi yo'q.

8-misol. Quyidagi birgalikdagi tengsizliklarni yeching:

$$\begin{cases} x^2 + 4x + 3 > 0 \\ 2x^2 + x - 6 \le x^2 + 2x \end{cases}$$

Yechimi: Birinchi tengsizlikni ko'paytuvchilarga ajratsak, $x^2 + 4x + 3 = (x+3)(x+1) > 0$ bo'ladi. Uning yechimi esa $x < -3$, $x > -1$ bo'ladi. Uni grafikda qora chiziq bilan chizamiz.

Ikkinchi tengsizlikni ham ko'paytuvchilarga ajratsak,

$$2x^2 + x - 6 - x^2 - 2x = x^2 - x - 6 = (x+2)(x-3) \le 0$$

Uning yechimlari esa $-2 \leq x \leq 3$ intervalda yotadi. Uni esa chizmada ko‘k bilan chizamiz.

Bu ikkala tengsizlikning yechimlarini kesishish nuqtalarni birga yozsak, sistemaning yechimi $-1 < x \leq 3$ bo‘ladi.

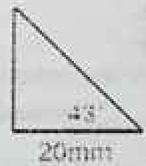
1.4. Trigonometrik funksiyalar (1-qism)

4A Burchakning trigonometrik funksiyalari

- O‘tkir burchaklarning trigonometrik funksiyalari
- Trigonometrik funksiyalar o‘rasidagi ba’zi munosabatlar
- $sin\theta$, $cos\theta$, $tg\theta$ larning o‘zaro bog‘liqligi.

Balandlikni o‘lchash masalasi

Qiyalik burchagi ma’lum bo‘lsa, katetni o‘lchash orqali balandlikni topishimiz mumkin.



18.7mm

Masala:

Daraxtning balandligi H qancha?

(L masofani va balandlik burchagini θ ni o‘lchab topishimiz mumkin.)

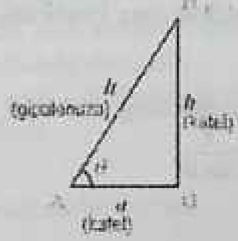
Bu masalani hal qilish uchun burachaklar uchun maxsus tushunchlarga ega bo‘lishimiz zarur bo‘ladi. Biz hozir ular bilan tanishamiz.

O‘tkir burchakning trigonometrik funksiyasi

Sinus funksiyasi $\sin\theta = \frac{b}{h}$

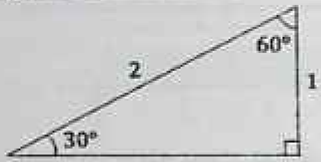
Kosinus funksiyasi $\cos\theta = \frac{a}{h}$

Tangens funksiyasi $tg\theta = \frac{b}{a}$



Maxsus trigonometrik qiymatlar

Yodda saqlash uchun maxsus holatlarni qarab chiqamiz.



$$\sin 30^\circ = \frac{1}{2};\quad \sin 45^\circ = \frac{1}{\sqrt{2}};\quad \sin 60^\circ = \frac{\sqrt{3}}{2};$$

$$\cos 30^\circ = \frac{\sqrt{3}}{2};\quad \cos 45^\circ = \frac{1}{\sqrt{2}};\quad \cos 60^\circ = \frac{1}{2};$$

$$tg 30^\circ = \frac{1}{\sqrt{3}};\quad tg 45^\circ = 1;\quad tg 60^\circ = \sqrt{3};$$

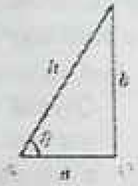
Trigonometrik funksiyalar orasidagi ba'zi bog'lanishlar.

Trigonometrik funksiyalar ta'rifiga ko'ra

$\sin\theta = \frac{b}{h}$

$\cos\theta = \frac{a}{h} \quad \Longrightarrow \quad \boxed{tg\theta = \frac{\sin\theta}{\cos\theta}} \quad (1)$

$tg\theta = \frac{b}{a}$



Pifagor teoremasidan foydalanamiz

$a^2 + b^2 = h^2 \quad \Longrightarrow \quad \boxed{\sin^2\theta + \cos^2\theta = 1} \quad (2)$

Yuqoridagi ikki formuladan foydalanib quyidagi formulani hosil qilamiz:

$(1), (2) \quad \Longrightarrow \quad \boxed{1 + tg^2\theta = \frac{1}{\cos^2\theta}}$

Izoh. $(\sin\theta)^2$ *ni odatda qulaylik uchun* $\sin^2\theta$ *kabi yoziladi.*

$\sin\theta, \cos\theta, tg\theta$ larning teskari shakllari.

Har bir θ - burchakda $\sin\theta, \cos\theta, tg\theta$ larga teskari qiymat qabul qiladigan quyidagi trigonometrik funksiyalarni qarab chiqamiz:

θ - burchakning kosekansi deb, birning $\sin\theta$ ga nisbatiga aytiladi, ya'ni

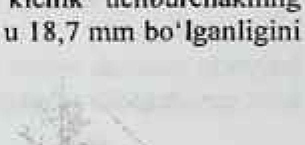
$$\csc\theta = \frac{1}{\sin\theta};$$

θ - burchakning sekansi deb, birning $\cos\theta$ ga nisbatiga aytiladi, ya'ni $\sec\theta = \frac{1}{\cos\theta};$

θ - burchakning kotangensi deb, birning $tg\theta$ ga nisbatiga aytiladi, ya'ni $\operatorname{ctg}\theta = \dfrac{1}{tg\theta}$.

1-misol. Yuqorida berilgan masala uchun quyidagicha masalalar tuzish mumkin.

(1) Biz balandlik burchagini 20 m uzoqlikdan o'lchab, 43^0 ekanligini topdik. Keyin biz shunga o'xshash burchagi $\theta = 43^0$ va mos kateti 20 mm bo'lgan kichik uchburchakni tasvirladik. Ushbu kichik uchburchakning berilgan burchagiga qarama-qarshi tomonni o'lchab, u 18,7 mm bo'lganligini aniqladik. Daraxtning balandligi qancha?



(2) Balandlik burchagi 60^0 bo'lgan joyni qidirdik. Keyin, biz ushbu joy bilan daraxt orasidagi masofani o'lchadik va uning 10,8 m ekanligini bildik. Bu daraxtning balandligi qancha?

Yechimi:

(1) $20\times\dfrac{18.7}{20} = 18.7m$;

(2) $tg\theta = \dfrac{H}{L}$;

$H = L\cdot tg\theta = 10.8\times tg60^\circ = 10.8\times\sqrt{3} = 10.8\times 1.732 = 18.7\,m.$

2-misol. Biz balandlik burchaklari 30^0 va 60^0 bo'lgan ikkita A va B pozitsiyalarini qidirdik. A va B orasidagi masofa 12,6 m edi. Ushbu daraxtning balandligi qancha?

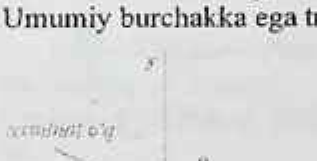
Yechimi: BQ va PQ mos ravishda a va h bo'lsin.

$tg30^0 = \dfrac{h}{a+12.6}$ => $a+12.6 = \dfrac{h}{tg30^0} = \sqrt{3}h$ va $tg60^0 = \dfrac{h}{a}$ => $a = \dfrac{h}{tg60^0} = \dfrac{h}{\sqrt{3}}$

$\dfrac{h}{\sqrt{3}} + 12.6 = \sqrt{3}h$ => $h = 12.6\times\dfrac{\sqrt{3}}{2} \approx 12.6\times\dfrac{1.73}{2} \approx 11.0\,m.$

4B. Trigonometrik funksiyalar. Umumiy burchak

Umumiy burchakka ega trigonometrik funksiyalar



Umumiy burchak

Bosh o'q- bu burchak boshlangan o'q.

Terminal o'q- bu aylanadigan nur.

Agar $\theta > 0$ bo'lsa, burchak soat strelkasi harakatiga qarama-qarshi yo'nalishda ochiladi,

$\theta < 0$ bo'lsa, burchak soat strelkasi harakati yo'nalishi bo'yicha ochiladi.

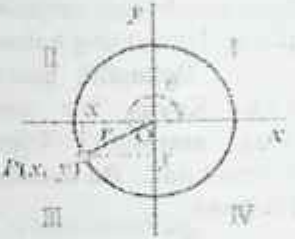
Misol:

α va θ lar koterminal burchaklar bo'lsa,

$$\boxed{\begin{array}{l}\theta=\alpha+360^0\times n\\ (n=0,\pm 1,\pm 2,...).\end{array}}$$

Umumiy burchakka ega bo'lgan trigonometrik funksiyalar

P(x, y) nuqta θ - burchakli terminal o'qdagi nuqta bo'lsin. *OP* kesmaning uzunligi esa: $r=\sqrt{x^2+y^2}$.



Ixtiyoriy burchak uchun trigonometrik funksiyalarni quyidagicha aniqlaymiz:

$$\sin\theta=\frac{y}{r},$$

$$\cos\theta=\frac{x}{r},$$

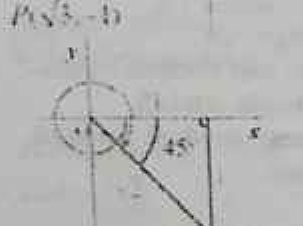
$$tg\theta=\frac{y}{x}.$$

Aniqlanishiga ko'ra trigonometrik funksiyalarning ishoralari:

Chorak	*I*	*II*	*III*	*IV*
	$x>0,\ y>0$	$x<0,\ y>0$	$x<0,\ y<0$	$x>0,\ y<0$
$\sin\theta$	+	+	-	-
$\cos\theta$	+	-	-	+
$tg\theta$	+	-	+	-

3-misol. (1) 570 ° va (2) -405 ° burchaklarning trigonometrik funksiyalarining qiymatlarini toping.

Yechimi:

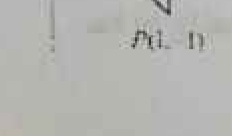


$$570^0=360^0+210^0$$

$$\sin 570^0=\sin(180^0+30^0)=-\sin 30^0=-\frac{1}{2};$$

$$\cos 570^0=\cos(180^0+30^0)=-\cos 30^0=-\frac{\sqrt{3}}{2};$$

$$tg 570^0=tg(180^0+30^0)=tg 30^0=\frac{1}{\sqrt{3}};$$



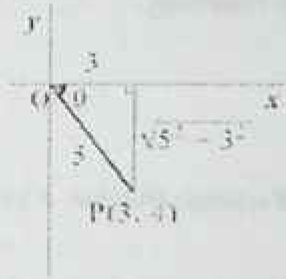
$$-405^0=-360^0-45^0;$$

$$\sin\left(-405^0\right)=\sin(-45^0)=-\frac{1}{\sqrt{2}};$$

$$\cos\left(-405^0\right)=\cos(-45^0)=\frac{1}{\sqrt{2}};$$

$$tg\left(-405^{\circ}\right)=tg(-45^{\circ})=-1.$$

4-misol. Faraz qilaylik, θ – burchak IV-chorakda bo'lib, $\cos\theta=\frac{3}{5}$ $\sin\theta$ va $tg\theta$ qiymatlarini toping.



Yechimi:

$$\sin^2\theta=1-\cos^2\theta=1-\left(\frac{3}{5}\right)^2=\frac{16}{25}.$$

θ – burchak IV-chorakda joylashganligi sababli,

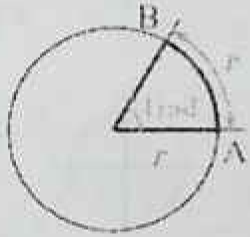
$$\sin\theta<0 \Rightarrow \quad \sin\theta=-\sqrt{\frac{16}{25}}=-\frac{4}{5}, \quad tg\theta=\frac{\sin\theta}{\cos\theta}=-\frac{4}{3}.$$

1.5. Trigonometrik funksiyalar (2-qism)

5A Burchakning radian o'lchov birligi. Trigonometrik funksiyalar grafiklari

Gradus (°) - aylananing to'liq aylangan markaziy burchagi 1/360 qismidir.

Radian (o'lchovsiz)- burchak yoyining radiusga *nisbati* bilan tavsiflanadi.



$$360^{0} \Leftrightarrow 2\pi=\frac{2\pi r}{r}$$

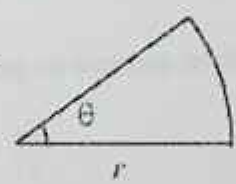
(Aylana uzunligining radiusga nisbati).

Radianda birlik yo'q, lekin "*rad*" belgisi ishlatiladi.

1rad = 57.29 •••° (360 ° = 2π rad yodlang).

Radianning qo'llanilishi

1-misol. Radianning oddiy ifodalarda qo'llanilishi.



Burchagi θ va radiusi r bo'lgan sektorning yuzi:

θ [gradus]: $\pi r^2\frac{\theta}{360}$;

θ [radian]: $r^2\frac{\theta}{2}\left(=\pi r^2\frac{\theta}{2\pi}\right)$.

2-misol. Kichik burchakli trigonometrik funksiyalarning qiymatlarini taqribiy hisoblang.

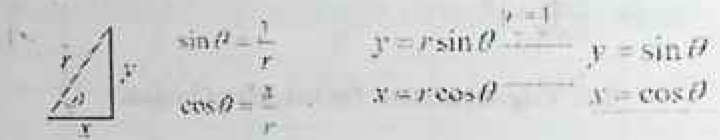
Yechimi: Burchak θ kichik bo'lganda

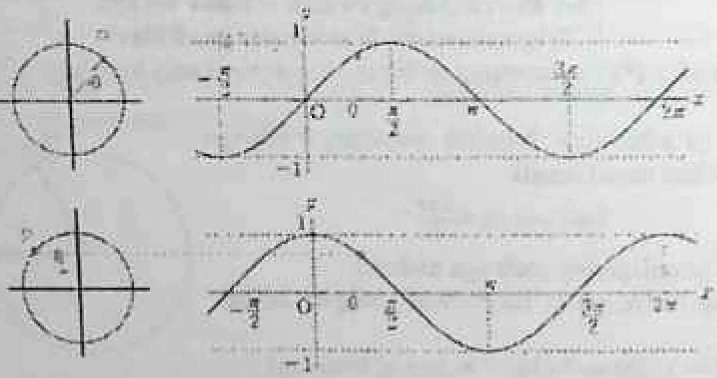
$$\sin\theta = \frac{BH}{OB} \approx \frac{arcAB}{OB} = \theta,$$

Masalan $\sin\frac{\pi}{180} \approx \frac{\pi}{180} \approx \frac{3.1416}{180} = 0.01745.$ (Jadvaldan, $1^\circ \approx 0.0175$)

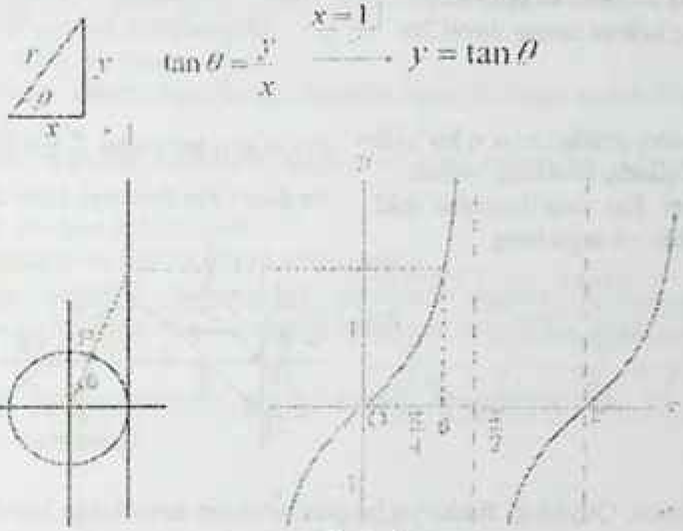
Sinus va kosinus funksiyalarning grafiklari

Bu funksiyalarning aniqlanishidan foydalanib ularning grafiklarini quyidagicha quramiz:





Tangens funksiyasining grafigi



Davriy funksiya

Agar

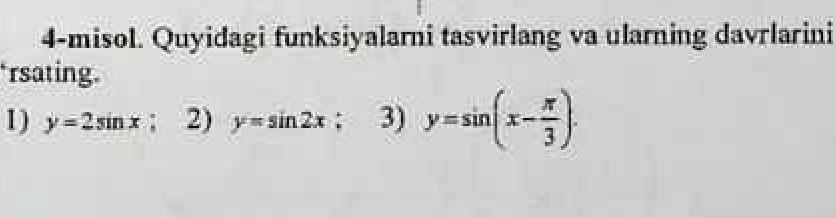
$$f(x+p)=f(x)$$

tenglik o'rinli bo'lsa, $f(x)$ funksiyaga *davriy* funksiya deyiladi. Bu yerda p-davr.

Davriy funksiya qiymatlari muntazam ravishda takrorlanib turadi.

3-misol. $y = \sin x$ funksiyaning davrini toping.

Yechimi: $\sin(x+2\pi)=\sin x$, $\quad Davr = 2\pi$

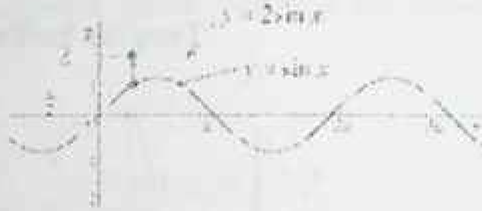


4-misol. Quyidagi funksiyalarni tasvirlang va ularning davrlarini ko'rsating.

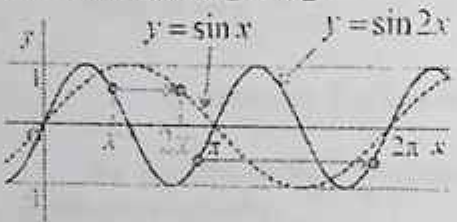
1) $y=2\sin x$; 2) $y=\sin 2x$; 3) $y=\sin\left(x-\frac{\pi}{3}\right)$.

Yechimi:

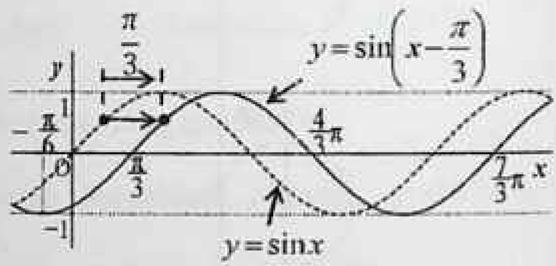
(1) Bu funksiyaning grafigi y- o'q bo'yicha 2 marta kengayadi, shunung uchun uning davri 2π



(2) Uning grafigi x- o'q bo'yicha 2 marta siqiladi, shunung uchun uning davri Sin ning davridan ikki marta kichik va π ga teng.



(3) x- o'q bo'yicha $\frac{\pi}{3}$ ga o'nga siljiydi va davri Sin funlsiya kabi 2π ga teng.



5-misol. Quyidagi funksiya haqida berilgan savollarga javob bering:

$$y=2\sin\left(2x-\frac{\pi}{3}\right),\ (0\le x\le 2\pi)$$

(1) Ushbu funksiya qachon nolga teng bo'ladi?
(2) Ushbu funksiyaning $x=0, 2\pi$ nuqtalardagi qiymatlari qanday?
(3) Ushbu funksiyani tasvirlab bering.

Yechimi:

(1) Berilgan funksiya argumenti o'zgarish oralig'ini topamiz

$$0\le x\le 2\pi,\ \therefore\ 0\le 2x\le 4\pi,\ \therefore\ -\frac{\pi}{3}\le 2x-\frac{\pi}{3}\le 4\pi-\frac{\pi}{3}$$

demak, uning nollari quyidagicha bo'ladi

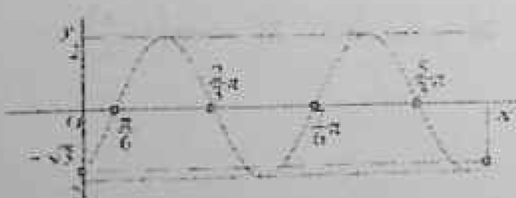
$$2x-\frac{\pi}{3}=0,\ \pi,\ 2\pi,\ 3\pi\quad \therefore\ x=\frac{\pi}{6},\ \frac{2\pi}{3},\ \frac{7\pi}{6},\ \frac{5\pi}{3}.$$

(2) Bu nuqtalardagi qiymatlarni bevosita hisoblab topamiz

$$x=0:\ \ y=2\sin\left(-\frac{\pi}{3}\right)=-\sqrt{3};$$

$$x=2\pi:\ \ y=2\sin\left(4\pi-\frac{\pi}{3}\right)=2\sin\left(-\frac{\pi}{3}\right)=-\sqrt{3}.$$

(3)



5B. Trigonometrik tenglama. Trigonometrik tengsizlik

Trigonometrik tenglama- bu noma'lum trigonometrik funksiya qatnashgan har qanday tenglamadir.

Masalan: $2\sin^2 x + 3\cos x - 3 = 0$.

Bunday tenglama, albatta barcha tenglamalar kabi, ma'lum burchaklar uchun o'rinli bo'ladi.

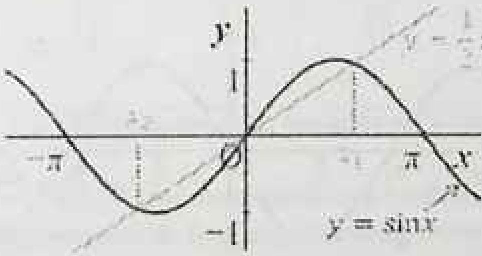
Izoh. *Har qanday burchak uchun to'g'ri keladigan trigonometrik tenglikka*
trigonometrik ayniyat deb ataladi.

Keyingi darslarda biz bu ularni o'rganamiz.

Ba'zi trigonometrik tenglamani algebraik usullar yordamida *oson* yechish mumkin, ba'zilarini esa *taqribiy* hisoblash orqali hal qilish mumkin.

6-misol. $2\sin x - x = 0$.

Yechimi: Bu tegnlamani grafik usulda yechib x_1 va x_2 taqribiy ildizlarini topish mumkin

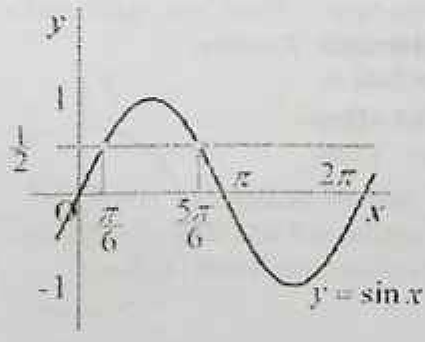


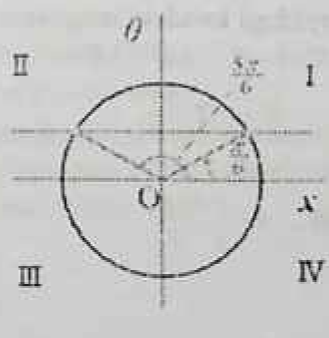
7-misol. $2\sin x - 1 = 0$.

Yechimi: 1-qadam. Biz avval $\sin x$ ni tenglamaning o'zgaruvchisi sifatida olamiz va unga nisbatan yechamiz $\sin x = \frac{1}{2}$.

2-qadam. $y = \sin x$ grafigidan yoki birlik aylanasidan 0 dan 2π gacha bo'lgan oraliqda yuqoridagi tenglikni qanoatlantiradigan x ning qiymatlarini topamiz.

$\therefore x = \frac{\pi}{6}, \frac{5\pi}{6}$





3-qadam.
Davriylikni hisobga olgan holda, $2\pi n$ ni qo'shamiz, u holda

$$\therefore\ x=\frac{\pi}{6}+2\pi n,\ \frac{5\pi}{6}+2\pi n$$

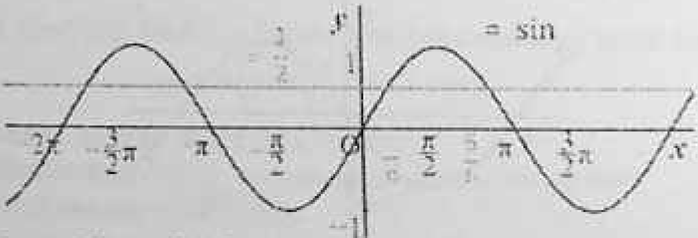
Trigonometrik tengsizlik

Trigonometrik tengsizlik deganda, trigonometrik funksiya noma'lum bo'lgan har qanday tengsizlik tushuniladi. Uni trigonometrik tenglama asosida yechish mumkin.

8-misol.. Quyidagi trigonometrik tengsizlikni yeching: $2\sin x-1>0$.

Yechimi: 1-qadam. Berilgan tengsizlikni, tengsizlik belgisini tenglik belgisiga almashtirish orqali trigonometrik tenglamaga aylantiramiz $2\sin x-1=0$.

2-qadam. Hosil bo'lgan tenglamani $[0,2\pi]$ oralig'ida yechamiz $x=\frac{\pi}{6},\ \frac{5\pi}{6}$.



3-qadam. Olingan ildizlar orqali ajralgan intervallar orasidan trigonometrik tengsizlikni qanoatlantiradigan intervallarni topamiz: $\frac{\pi}{6}<x<\frac{5\pi}{6}$.

4-qadam. Yechimni butun o'q bo'yicha yoyib chiqamiz

$$\frac{\pi}{6}+2\pi n<x<\frac{5\pi}{6}+2\pi n.$$

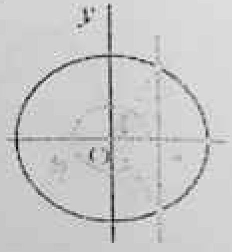
9-misol. Quyidagi trigonometrik tenglamani yeching:

$$2\sin^2 x+3\cos x-3=0.$$

Yechimi: Quyidagi almashtirishni bajaramiz: $X=\cos x$.
Natijada quyidagi kvadrat tenglama hosil bo'ladi

$$2(1-X^2)+3X-3=0\ \therefore\ 2X^2-3X+1=0\ \therefore\ (X-1)(2X-1)=0.$$

$$X=1,\ X=\frac{1}{2}.$$



$\cos x=1$ dan $\therefore\ x=0,\ 2\pi$

$\cos x=\frac{1}{2}$ dan $\therefore\ x=\frac{\pi}{3},\ \frac{5\pi}{3}$.

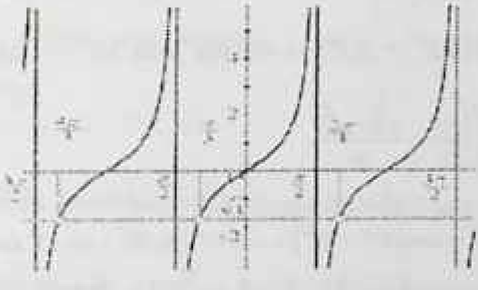
10-misol. Quyidagi trigonometrik tengsizlikni yeching: $tgx \geq -\sqrt{3}$.

Yechimi: Tengsizlik belgisining o'rniga tenglik qo'yib, tenglama tuzamiz:

$$tgx = -\sqrt{3}.$$

Tangens rasmda ko'rsatilgandek π davrga ega. U holda $x = -\frac{\pi}{3}$ ildiz $\left[-\frac{\pi}{2}; \frac{\pi}{2}\right]$ oralig'idagi ildiz bo'ladi.

Grafikdan va davriylikni hisobga olgan holda, yechimni quyidagicha topamiz: $-\frac{\pi}{3} + \pi n \leq x < \frac{\pi}{2} + \pi n$.



1.6. Trigonometrik funksiya (3-qism)

6A. Trigonometrik ayniyatlar. Ba'zi trigonometrik funksiyalarning aniq qiymatlari

Biz bu mavzuda trigonometrik funksiyalar uchun muhim tengliklarni keltirib chiqramiz. Dastlab burchak yig'indi va ayirmasining sinus va kosinuslarini qaraymiz.

Ayniyat - bu o'zgaruvchining *har qanday* qiymati uchun o'rinli bo'ladigan tenglik.

Hisoblashlarni bajarishda biz tenglamaning har qanday qismini boshqa o'zgaruvchiga almashtirishimiz mumkin. Ifodani qulayroq shaklga keltirish uchun bizga ma'lum bo'lgan ayniyatlardan foydalanamiz.

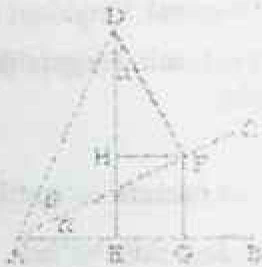
$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$
$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$
$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$
$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

Ushbu formulalar muhim hisoblanadi, chunki trigonometriyagi qolgan barcha ayniyatlar shu formulalardan foydalangan holda keltirib chiqariladi.

1-misol. Isbotlang: $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$.

Yechimi:

Chizmadan foydalanib DE ni topamiz:



$$DE = AD\sin(\alpha+\beta) = FG + DH =$$

$$= FA\sin\alpha + DF\cos\alpha = AD\cos\beta\sin\alpha + AD\sin\beta\cos\alpha\ .$$

Demak,

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta\,.$$

Biz ma'lum burchaklar uchun ba'zi trigonometrik funksiyalarning qiymatlarini topishimiz mumkin.

2-misol. (a) $\sin 15^\circ$ va (b) $\cos 15^\circ$ qiymatlarini toping.

Yechimi:

$$\sin 15^\circ = \sin(45^\circ - 30^\circ) = \sin 45^\circ \cos 30^\circ - \cos 45^\circ \sin 30^\circ =$$

$$= \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \times \frac{1}{2} = \frac{\sqrt{3}-1}{2\sqrt{2}} = \frac{\sqrt{6}-\sqrt{2}}{4}.$$

$$\cos 15^\circ = \cos(45^\circ - 30^\circ) = \cos 45^\circ \cos 30^\circ + \sin 45^\circ \sin 30^\circ =$$

$$= \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \times \frac{1}{2} = \frac{\sqrt{3}+1}{2\sqrt{2}} = \frac{\sqrt{6}+\sqrt{2}}{4}.$$

3-misol. Burchak yig'indisi va ayirmasining tangenslarini toping.

Yechimi: Sinus va kosinus formulalaridan foydalanamiz.

$$tg(\alpha+\beta) = \frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)} = \frac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\cos\alpha\cos\beta - \sin\alpha\sin\beta} = \frac{\dfrac{\sin\alpha}{\cos\alpha} + \dfrac{\sin\beta}{\cos\beta}}{1 - \dfrac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}} = \frac{tg\alpha + tg\beta}{1 - tg\alpha tg\beta},$$

$$tg(\alpha-\beta) = \frac{\sin(\alpha-\beta)}{\cos(\alpha-\beta)} = \frac{\sin\alpha\cos\beta - \cos\alpha\sin\beta}{\cos\alpha\cos\beta + \sin\alpha\sin\beta} = \frac{\dfrac{\sin\alpha}{\cos\alpha} - \dfrac{\sin\beta}{\cos\beta}}{1 + \dfrac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}} = \frac{tg\alpha - tg\beta}{1 + tg\alpha tg\beta}.$$

Demak,

$$tg(\alpha+\beta) = \frac{tg\alpha + tg\beta}{1 - tg\alpha tg\beta}, \quad tg(\alpha-\beta) = \frac{tg\alpha - tg\beta}{1 + tg\alpha tg\beta}$$

4-misol. (a) $\sin 75^\circ$, (b) $\cos 75^\circ$ va (c) $tg 75^\circ$ qiymatlarini toping.

Yechimi:

(a) $\sin 75^\circ = \sin(45^\circ + 30^\circ) = \sin 45^\circ \cos 30^\circ + \cos 45^\circ \sin 30^\circ =$

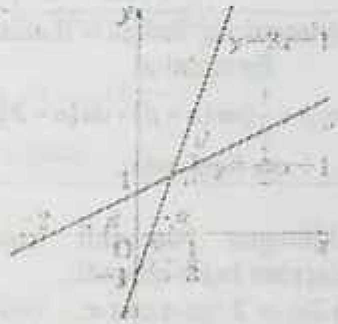
$$= \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \times \frac{1}{2} = \frac{\sqrt{3}+1}{2\sqrt{2}} = \frac{\sqrt{6}+\sqrt{2}}{4};$$

(b) $\cos 75^\circ = \cos(45^\circ + 30^\circ) = \cos 45^\circ \cos 30^\circ - \sin 45^\circ \sin 30^\circ =$

$$= \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \times \frac{1}{2} = \frac{\sqrt{3}-1}{2\sqrt{2}} = \frac{\sqrt{6}-\sqrt{2}}{4};$$

$$(c)\ \operatorname{tg} 75° = \operatorname{tg}(45° + 30°) = \frac{\operatorname{tg} 45° + \operatorname{tg} 30°}{1 - \operatorname{tg} 45° \operatorname{tg} 30°} = \frac{1 + \frac{1}{\sqrt{3}}}{1 - 1 \cdot \frac{1}{\sqrt{3}}} = \frac{\sqrt{3} + 1}{\sqrt{3} - 1}.$$

5-misol. $y = 3x - 1$ va $y = \frac{1}{2}x + 1$ ikkita to'g'ri chiziqlar berilgan.



Quyidagi savollarga javob bering.

(1) Ushbu chiziqlar va x o'qi orasidagi α va β burchaklarni aniqlang;

(2) Ushbu ikki to'g'ri chiziq orasidagi $\theta \left(0 < \theta < \frac{\pi}{2}\right)$ burchakni toping.

Yechimi:

(1) To'g'ri chiziq burchak koeffisiyentli tenglamasidan $tg\alpha = 3$, $tg\beta = \frac{1}{2}$ larga ega bo'lamiz;

(2) Chizmaga ko'ra uchburchak ichki va tashqi burchaklarini bog'lasak

$$\theta = \alpha - \beta$$

U holda

$$tg\theta = tg(\alpha - \beta) = \frac{tg\alpha - tg\beta}{1 + tg\alpha tg\beta} = \frac{3 - \frac{1}{2}}{1 + 3 \times \frac{1}{2}} = 1, \quad \therefore \quad \theta = \frac{\pi}{4}.$$

6B. Trigonometrik ayniyatlarni keltirib chiqarish.

Biz yuqorida isbotlangan ayniyatlardan foydalangan holda quyidagi ayniyatlarni keltirib chiqaramiz:

- Ikkilangan burchak;
- Yarim burchak;
- Yig'indini ko'paytmaga o'tkazish;
- Ko'paytmani yig'indiga o'tkazish;

Trigonometrik ayniyatlarni keltirib chiqarish

Uchlangan burchak formulalari:	Ikkilangan burchak formulalari:	Yarim burchak formulalari:
$\sin 3\alpha = -4\sin^3\alpha + 3\sin\alpha$ va boshqalar	$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$ va boshqalar.	$\sin^2\frac{\alpha}{2} = \frac{1-\cos\alpha}{2}$ va boshqalar.

$$\beta = 2\alpha \ , \qquad \alpha = \beta$$
$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$
$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$

Teskari hisoblash $\alpha + \beta = A, \ \alpha - \beta = B$

Ko'paytmani yig'indiga o'tkazish formulalari:	Yig'indini ko'paytmaga o'tkazish formulalari:
$\sin\alpha\cos\beta = \frac{1}{2}\{\sin(\alpha+\beta)+\sin(\alpha-\beta)\}$	$\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$
va boshqalar.	va boshqalar.

Ikkilangan burchakli trigonometrik ayniyatlar bevosita yig'indi formulalaridan kelib chiqadi:

$$\sin 2\alpha = 2\sin\alpha\cos\alpha, \quad \cos 2\alpha = \cos^2\alpha - \sin^2\alpha = 1 - 2\sin^2\alpha = 2\cos^2\alpha - 1$$

$$\operatorname{tg} 2\alpha = \frac{2\operatorname{tg}\alpha}{1-\operatorname{tg}^2\alpha}$$

6-misol. $tg2\alpha = \frac{2tg\alpha}{1-tg^2\alpha}$ ni isbotlang.

Yechimi:

$$tg2\alpha = \frac{\sin 2\alpha}{\cos 2\alpha} = \frac{\sin(\alpha+\alpha)}{\cos(\alpha+\alpha)} = \frac{2\sin\alpha\cos\alpha}{\cos^2\alpha - \sin^2\alpha} = \frac{2tg\alpha}{1-tg^2\alpha}.$$

Yarim burchakli trigonometrik ayniyatlar

$$\sin^2\frac{\alpha}{2} = \frac{1-\cos\alpha}{2} \qquad \cos^2\frac{\alpha}{2} = \frac{1+\cos\alpha}{2} \qquad tg^2\frac{\alpha}{2} = \frac{1-\cos\alpha}{1+\cos\alpha}$$

7-misol. $\sin 22.5^\circ$, $\cos 22.5^\circ$, $tg 22.5^\circ$ larning qiymatlarini toping.

Yechimi: 22.5° burchak I-chorakda bo'lgani uchun, bu qiymatlarning barchasi musbatdir.

$$\sin^2 22.5^\circ = \frac{1-\cos 45^\circ}{2} = \left(1 - \frac{1}{\sqrt{2}}\right)\Big/2 = \frac{\sqrt{2}-1}{2\sqrt{2}} = \frac{2-\sqrt{2}}{4} \quad \therefore \quad \sin 22.5^\circ = \frac{\sqrt{2-\sqrt{2}}}{2}$$

$$\cos^2 22.5^\circ = \frac{1+\cos 45^\circ}{2} = \frac{2+\sqrt{2}}{4} \qquad \therefore \quad \cos 22.5^\circ = \frac{\sqrt{2+\sqrt{2}}}{2}$$

Shuning uchun

$$tg 22.5^\circ = \frac{\sin 22.5^\circ}{\cos 22.5^\circ} = \frac{\sqrt{2-\sqrt{2}}}{\sqrt{2+\sqrt{2}}} = \sqrt{\frac{\left(2-\sqrt{2}\right)^2}{4-2}} = \frac{2-\sqrt{2}}{\sqrt{2}} = \sqrt{2}-1.$$

Yig'indini ko'paytmaga o'tkazish bo'yicha ayniyatlarni hosil qilish uchun mos funksiyalar yig'indisi va ayirmasini qo'shish yoki ayirish kerak

$$\sin\alpha\cos\beta=\frac{1}{2}\{\sin(\alpha+\beta)+\sin(\alpha-\beta)\}$$

$$\cos\alpha\sin\beta=\frac{1}{2}\{\sin(\alpha+\beta)-\sin(\alpha-\beta)\}$$

$$\cos\alpha\cos\beta=\frac{1}{2}\{\cos(\alpha+\beta)+\cos(\alpha-\beta)\}$$

$$\sin\alpha\sin\beta=-\frac{1}{2}\{\cos(\alpha+\beta)-\cos(\alpha-\beta)\}$$

8-misol. $\sin 15^\circ \sin 75^\circ$ ning qiymatini toping.

Yechimi:

$$\sin 15^\circ \sin 75^\circ=-\frac{1}{2}\{\cos(15^\circ+75^\circ)-\cos(15^\circ-75^\circ)\}=$$

$$=-\frac{1}{2}\{\cos 90^\circ-\cos(-60^\circ)\}=\frac{1}{2}\cos(60^\circ)=\frac{1}{4}.$$

Ko'paytmani yig'indiga o'tkazish bo'yicha ayniyatlarni hosil qilish uchun $\alpha+\beta=A$, $\alpha-\beta=B$ kabi almashtirishlarni bajarish zarur bo'ladi.

$$\sin A+\sin B=2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$$

$$\sin A-\sin B=2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$$

$$\cos A+\cos B=2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$$

$$\cos A-\cos B=-2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$$

9-misol. $\cos 10^\circ+\cos 110^\circ+\cos 230^\circ$ qiymatini toping.

Yechimi:

$$\cos 10^\circ+\cos 110^\circ+\cos 230^\circ=(\cos 10^\circ+\cos 230^\circ)+\cos 110^\circ=$$

$$=2\cos 120^\circ\cos(-110^\circ)+\cos 110^\circ=$$

$$=-\cos 110^\circ+\cos 110^\circ=0$$

10-misol. $s=\sin\theta$ va $c=\cos\theta$ kabi belgilash berilgan. Quyidagi funksiyalarni s va c orqali ifodalang.

(1) $\sin 4\theta$; (2) $\cos 4\theta$.

Yechimi: Ikkala hol uchun ham ikkilangan burchak formulalaridan foydalanamiz.

$$(1)\ \sin 4\theta=\sin(2\times 2\theta)=2\sin 2\theta\cos 2\theta=4sc(1-2s^2);$$

$$(2)\ \cos 4\theta=\cos(2\times 2\theta)=2\cos^2 2\theta-1=2(2\cos^2\theta-1)^2-1=$$

$$=2(2c^2-1)^2-1==2(4c^4-4c^2+1)^2-1=8c^4-8c^2+1.$$

11-misol. $tgx=-2$ bo'lsa, $\sin 2x+\cos 2x$ning qiymatini toping.

Yechimi: $sin^2x+cos^2x=1$ ayniyatdan foydalanamiz.

$$\sin 2x+\cos 2x=\frac{\sin 2x+\cos 2x}{1}=\frac{2\sin x\cos x+\cos^2 x-\sin^2 x}{\sin^2 x+\cos^2 x}=$$

$$=\frac{2tgx+1-tg^2x}{1+tg^2x}=\frac{2(-2)+1-(-2)^2}{(-2)^2+1}=-\frac{7}{5}.$$

1.7. Kompleks sonlar

7A. Kompleks sonlar. Kvadrat tenglamalar

Bizga ma'lumki kvadrat tenglamaning umumiy ko'rinishi quyidagicha bo'ladi:

$$ax^2+bx+c=0.$$

Bu tenglamaning ildizlari uchun quyidagilar o'rinli edi:

Agar $D=b^2-4ac>0$ bo'lsa, kvadrat tenglama ikkita har xil haqiqiy ildizlarga ega bo'ladi: $x_{1,2}=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$.

Agar $D=b^2-4ac=0$ bo'lsa, kvadrat tenglama bitta haqiqiy ildizga ega bo'ladi: $x=-\dfrac{b}{2a}$.

Agar $D=b^2-4ac<0$ bo'lsa, kvadrat tenglama haqiqiy ildizga ega emas.

Oxirgi holda ham tenglamaning ildizlari mavjud bo'lsin degan shartni qo'ysak, quyidagi yangi kompleks son tushunchga kelamiz.

Biz shunday i sonni kiritamizki, uning kvadrati -1ga ya'ni

$$i=\sqrt{-1}$$

bo'lsin. Bu i sonini mavhum birlik, deb ataymiz.

Bu tushunchadan foydalangan holda

$$ax^2+bx+c=0$$

kvadrat tenglama uchun quyidagi xulosaga kelamiz:

Kvadrat tenglamaning ildizlari

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

ko'rinishda topilar edi.

Agar $D=b^2-4ac>0$ bo'lsa, kvadrat tenglama ikkita har xil haqiqiy ildizlarga ega bo'ladi.

Agar $D=b^2-4ac=0$ bo'lsa, kvadrat tenglama bitta haqiqiy ildizga ega bo'ladi.

Agar $D=b^2-4ac<0$ bo'lsa, kvadrat tenglama ikkita har xil kompleks ildizlarga ega bo'ladi.

Kompleks son

$a+bi$ ko'rinishdagi ifodaga kompleks son deyiladi, bu yerda a va b haqiqiy sonlar bo'lib, a –kompleks sonning haqiqiy qismi, b – kompleks sonning mavhum qismi, i –esa mavhum birlikdir.

Maxsus holatlar:

a: haqiqiy son ($a\neq 0$, $b=0$ bo'lsa);

$a+bi$: mavhum son ($b\neq 0$ bo'lsa);

bi: sof mavhum son ($a=0$, $b\neq 0$ bo'lsa).

Demak: Kompleks son=haqiqiy son+mavhum son

Kompleks sonlar ustida amallar

Tengliklar

$a+bi=c+di$ bo'lsa, u holda $a=c$ va $b=d$ bo'ladi.

$a+bi=0$ bo'lsa, u holda $a=0$ va $b=0$ bo'ladi.

Qo'shish

$$(a+bi)+(c+di)=(a+c)+(b+d)i$$

Ayirish

$$(a+bi)-(c+di)=(a-c)+(b-d)i$$

Ko'paytirish

$$(a+bi)(c+di)=(ac-bd)+(ad+bc)i$$

Bo'lish

$$\frac{(c+di)}{(a+bi)}=\frac{(ac+bd)}{a^2+b^2}+\frac{(ad-bc)}{a^2+b^2}i$$

Qo'shma kompleks son

$a-bi$ kompleks son $a+bi$ kompleks sonning qo'shmasi deyiladi.

Bu yerda a kompleks sonning haqiqiy qismi, $-bi$ esa mavhum qismining qarama-qarshisidir.

Odatda, $z=a+bi$ kompleks songa qo'shma kompleks son $\overline{z}=a-bi$ ko'rinishida belgilanadi.

Kompleks sonni kompleks songa bo'lganda maxrajdagi kompleks sonning qo'shmasini surat va maxrajiga ko'paytiramiz (qo'shma kompleks son haqida quyida ma'lumot beramiz).

Qo'shma kompleks sonning afzalligi

Kompleks sonni o'zining qo'shmasiga ko'paytirsak, haqiqiy son hosil bo'ladi:

$$z\bar{z}=(a+bi)(a-bi)=a^2-(bi)^2=a^2+b^2$$

Shuning uchun, kompleks sonlarni bir-biriga bo'lishda yuqoridagi formuladan foydalanish qulay.

Masalan,

$$\frac{(c+di)}{(a+bi)}=\frac{(c+di)(a-bi)}{(a+bi)(a-bi)}=\frac{(ac+bd)}{a^2+b^2}+\frac{(ad-bc)}{a^2+b^2}i$$

1-misol. $x^2-2x+4=0$ ko'rinishdagi kvadrat tenglamani kompleks sonlarda yeching.

Yechimi: Kvadrat tenglamani yechish formulasidan foydalanamiz:

$$x_{1,2}=\frac{+2\pm\sqrt{2^2-4\times1\times4}}{2}=1\pm3i.$$

2-misol. $ax^2+bx+c=0$ ko'rinishdagi kvadrat tenglama va uning ikkita α va β ildizlari berilgan bo'lsin. Quyidagi tengliklarni isbotlang:

$$\alpha+\beta=-\frac{b}{a},\ \alpha\beta=\frac{c}{a}.$$

Yechimi: Kvadrat tenglamaning ildizlari formulasidan foydalanamiz:

$$\alpha+\beta=\frac{-b+\sqrt{b^2-4ac}}{2a}+\frac{-b-\sqrt{b^2-4ac}}{2a}=-\frac{b}{a},$$

$$\alpha\beta=\left(\frac{-b+\sqrt{b^2-4ac}}{2a}\right)\left(\frac{-b-\sqrt{b^2-4ac}}{2a}\right)=\frac{(-b)^2-(b^2-4ac)}{4a^2}=\frac{c}{a}.$$

Misol

3-misol. Quyidagi ifodalarni soddalashtiring:

(1) $\left(\sqrt{3}+\sqrt{-2}\right)\left(\sqrt{2}-\sqrt{-3}\right)$ (2) $1+i+i^2+\cdots+i^{10}$ (3) $\frac{2+i}{2-i}+\frac{3-i}{3+i}$

Yechimi:

(1) $\left(\sqrt{3}+\sqrt{-2}\right)\left(\sqrt{2}+\sqrt{-3}\right)=\left(\sqrt{3}+\sqrt{2}i\right)\left(\sqrt{2}-\sqrt{3}i\right)=$

$=\left(\sqrt{3}\sqrt{2}+\sqrt{2}\sqrt{3}\right)+\left(\sqrt{2}\sqrt{2}-\sqrt{3}\sqrt{3}\right)i=2\sqrt{6}-i$

(2) $1+i+i^2+\cdots+i^{10}=(1+i-1-i)+(1+i-1-i)+1+i-1=i$

(3) $\frac{2+i}{2-i}+\frac{3-i}{3+i}=\frac{(2+i)^2}{(2-i)(2+i)}+\frac{(3-i)^2}{(3+i)(3-i)}=\frac{3+4i}{5}+\frac{8-6i}{10}=\frac{7+i}{5}$

7B Kompleks sonlar tekisligi

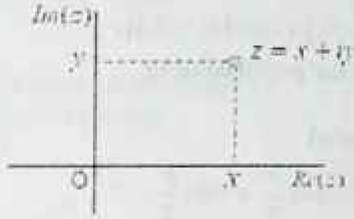
Biz bu mavzuda kompleks sonning geometric tasviri, kompleks sonlarning afzallilar haqida to'xtalamiz.

Kompleks sonlar tekisligi (Argand diagrammasi)

Kompleks sonlar tekisligi kompleks sonlar orqali tasvirlanadi. Kompleks sonlar tekisligida haqiqiy va mavhum o'qlari asos qilib olinadi.

Kompleks son — Kompleks sonlar tekisligi

$\boxed{z = x + iy} \Rightarrow$ *x o'qida ko'chish*, *y o'qida ko'chish*



Kompleks sonlar ustida amallar bajarishning eng qulay shakli uning trigonometrik ko'rinishidir. Buninh uchun sonni qutb koordinatalar sistemasida ifodalaymiz.

Qutb koordinata sistemasida ifodalanishi

$(x, y) \quad z = x + iy$ — Algebdaik ifodalanishi

$\Downarrow - \boxed{x = r\cos\theta, \quad y = r\sin\theta}$

$(r, \theta) \quad z = r(\cos\theta + \sin\theta)$ — Qutb koordinata sistemasida ifodalanishi

Kompleks sonning moduli deb

$$r = |z| = \sqrt{z\bar{z}} = \sqrt{x^2 + y^2}$$

ga aytiladi.

θ ga kompleks sonning argumenti deyiladi va

$$\cos\theta = \frac{x}{r} \qquad \text{va} \qquad \sin\theta = \frac{y}{r}$$

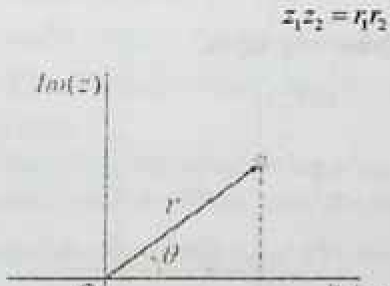
tengliklardan aniqlanadi.

Agar ikkita kompleks sonlar

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1), \qquad z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$$

trigonometrik shaklda berilgan bo'lsa, ularning ko'paytmasi quyidagicha bo'ladi:

$$z_1 z_2 = r_1 r_2 \{\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\}.$$



Kompleks sonlar tekisligida

ikkita kompleks son ko'paytmasining moduli shu kompleks sonlar modullarining ko'paytmasiga teng;

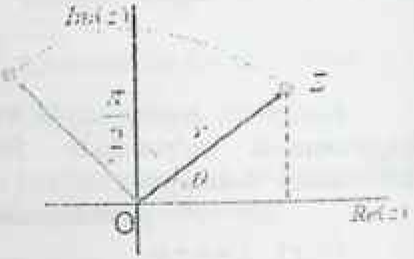
ikkita kompleks son ko'paytmasining argumenti shu kompleks sonlar argumentlarining yig'indisiga teng.

4-misol. Ko'paytma formulasini isbotlang.

Isbot:

$$z_1z_2 = r_1r_2(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) =$$
$$= r_1r_2(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2) =$$
$$= r_1r_2\{\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\}.$$

"i" ga ko'paytirishning geometrik ma'nosi:



$z_1 = z, \quad z_2 = i$ ya'ni

$z = r(\cos\theta + \sin\theta), \quad i = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$

bo'lsin.

Ularning ko'paytmasi quyidagicha bo'ladi:

$$zi = r\left\{\cos\left(\theta + \frac{\pi}{2}\right) + i\sin\left(\theta + \frac{\pi}{2}\right)\right\}$$

Demak kompleks sonni, "i" *ga ko'paytirish degani modulini o'zgartirmasdan 90° ga burish tushuniladi*

Muavr formulasi

Ko'paytirish qoidasidan, quyidagiga ega bo'lamiz:

$z^2 = r(\cos\theta + i\sin\theta)r(\cos\theta + i\sin\theta) = r^2(\cos 2\theta + i\sin 2\theta)$.

Xuddi shunday,

$z^3 = r^2(\cos 2\theta + i\sin 2\theta)r(\cos\theta + i\sin\theta) = r^3(\cos 3\theta + i\sin 3\theta)$.

....

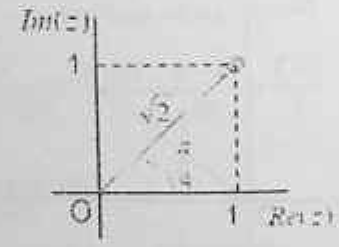
Natijada, quyidagi formula hosil bo'ladi:

$$\boxed{(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta}$$

Muavr formulasi

Bu tenglik kompleks sonlarni va trigonometriyani bog'laydi.

5-misol. Quyidagi qiymatni toping: $(1+i)^8$.



Yechimi: Kompleks sonlar tekisligidagi joylashgan o'rnidan quyidagini topamiz:

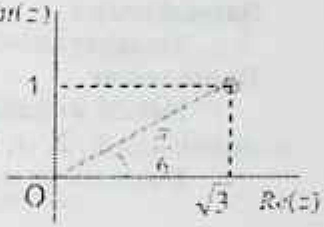
$$1 + i = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right).$$

Muavr formulasiga ko'ra quyidagiga ega bo'lamiz:

$$(1+i)^8=\sqrt{2}^8\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)^8=16\left(\cos\frac{8\pi}{4}+i\sin\frac{8\pi}{4}\right)=$$
$$=16(\cos 2\pi+i\sin 2\pi)=16.$$

6-misol. Quyidagi qiymatni toping:

$$\left(\sqrt{3}+i\right)^8.$$



Yechimi: Kompleks sonlar tekisligidagi joylashgan o'rnidan quyidagini topamiz:

$$\sqrt{3}+i=2\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right).$$

Muavr formulasiga ko'ra quyidagiga ega bo'lamiz:

$$\left(\sqrt{3}+i\right)^8=2^8\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)^8=256\left(\cos\frac{4\pi}{3}+i\sin\frac{4\pi}{3}\right)=$$
$$=256\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)=-128-128\sqrt{3}i.$$

7-misol. Quyidagi tenglikni qanoatlantiruvchi A, B, C koeffisientlarni qiymatlarini toping:

$$\cos 5\theta=A\cos^5\theta+B\cos^3\theta+C\cos\theta.$$

Yechimi: Muavr formulasiga ko'ra quyidagiga ega bo'lamiz:

$$(\cos\theta+i\sin\theta)^5=\cos 5\theta+i\sin 5\theta.$$

Ikkinchi tomondan

$$(x+y)^5=x^5+5x^4y+10x^3y^2+10x^2y^3+5xy^4+y^5.$$

formulaga ko'ra,

$$(\cos\theta+i\sin\theta)^5=(\cos\theta)^5+5(\cos\theta)^4(i\sin\theta)+10(\cos\theta)^3(i\sin\theta)^2+$$
$$+10(\cos\theta)^2(i\sin\theta)^3+5(\cos\theta)(i\sin\theta)^4+(i\sin\theta)^5=$$
$$=\cos^5\theta-10\cos^3\theta\sin^2\theta+5\cos\theta\sin^4\theta+$$
$$+i\left(5\cos^4\theta\sin\theta-10\cos^2\theta\sin^3\theta+\sin^5\theta\right).$$

bo'ladi.

Yuqoridagi ikki tenglikning o'ng tomonlarini tenglashtirsak,

$$\cos 5\theta=\cos^5\theta-10\cos^3\theta\sin^2\theta+5\cos\theta\sin^4\theta=$$
$$=\cos^5\theta-10\cos^3\theta\left(1-\cos^2\theta\right)+5\cos\theta\left(1-\cos^2\theta\right)^2=$$
$$=16\cos^5\theta-20\cos^3\theta+5\cos\theta\Rightarrow A=16,\ B=-20,\ C=5.$$

1.8. Ko‘rsatkichli funksiya
8A Ratsional va irratsional sonlar

Biz dastlab haqiqiy sonlar haqida to‘xtalamiz.

Natural sonlar

Sanashda ishlatiladigan sonlar: 1, 2, 3, 4, 5, ...

Butun sonlar

Natural sonlardan tashkil topgan sonlar, ularning manfiy sonlari va nol: ..., -3, -2, -1, 0, 1, 2, 3, ...

Butun sonlar sonlar o‘qida quyidagi nuqtalar bilan ifodalanadi.

Ratsional sonlar

Ratsional son bo‘lishi mumkin bo‘lgan har qanday son ikkita butun sonning bo‘linmasi sifatida ifodalanadi: $\frac{m}{n}$.

Maxraj n nolga teng bo‘lmasligi kerak:

[Masalan]: $\frac{1}{2}=\left(\frac{2}{4}\right), \frac{3}{7}.$

Har qanday butun son maxraji 1 ga teng bo‘lgan ratsional sondir.

Har qanday ratsional son o‘nli kasrda ikki xil ko‘rinishda ifodalanadi:

Cheklli o‘nli kasr [Masalan]: $\frac{1}{25}=0,04.$

Davriy o‘nli kasr [Masalan]: $\frac{1}{27}=0,037|037|037|037....$

Irratsional sonlar

* Har qanday irratsional son ratsional son emas, ya’ni irratsional sonni $\frac{m}{n}$. kasr ko‘rinishda yozib bo‘lmaydi.

* Irratsional sonlarning o‘nli kasr ko‘rinishi cheksiz bo‘lib, hech qachon davriy kasr ko‘rinishda yozib bo‘lmaydi.

[Masalan]: $\sqrt{2}=1,41421356..., \pi=3,14159265...$

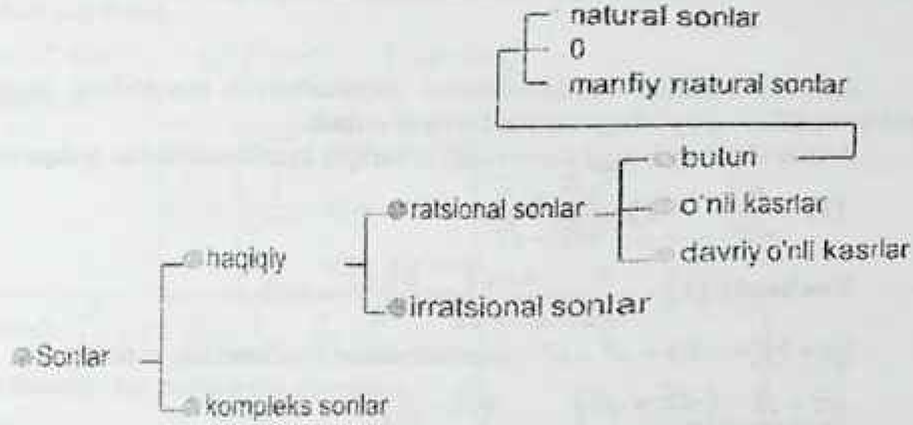
Haqiqiy sonlar

* Haqiqiy sonlarga barcha ratsional sonlar va barcha irratsional sonlar kiradi.

* Haqiqiy sonlar son o‘qi bilan ifodalanadi.

Matematikada bizga ma’lum sonlarni quyidagicha tizimlashtirish mumkin.

Sonlar sistemasi



1-misol. $\sqrt{2}$ ni irratsional son ekanligini isbotlang.

Yechimi: Agar $\sqrt{2}$ ratsional son bo'lsa, uni $\frac{m}{n}$ ko'rinishdagi kasr orqali ifodalashimiz mumkin.

U holda

$$\frac{m}{n}=\sqrt{2}\Rightarrow\frac{m}{n}\cdot\frac{m}{n}=\frac{mm}{nn}=2$$

Lekin, m va n 1 dan tashqari umumiy bo'luvchiga ega emas, bundan tashqari, mm va nn hech qanday umumiy bo'luvchisi yo'q. Shuning uchun,

$$\frac{mm}{nn}=2$$

bo'lishi mumkin emas. Shunday ekan, $\sqrt{2}$ ratsional son emas, ya'ni irratsional son.

2-misol. Quyidagi davriy o'nli kasrni oddiy kasr sifatida ifodalang:

0.037037037037……

Yechimi: Raqam har uchta raqamda takrorlangani uchun, biz x va $1000x$ o'rtasidagi farqni ko'rib chiqamiz.

1000x=37.037037037037…

\- x=0.037037037037…

999x=37

Shuning uchun, $x=\frac{37}{999}=\frac{1}{27}$.

Maxrajni irratsionallikdan qutqarish

Ifodani sodda va hisoblashni osonlashtirish maqsadida, maxrajdagi irratsionallikni quyidagi tarzda bartaraf etiladi.

3-misol. Quyidagi sonlarning maxrajni irratsionallikdan qutqaring:

(1) $\frac{6}{\sqrt{2}}$, (2) $\frac{\sqrt{2}+\sqrt{3}}{\sqrt{2}-\sqrt{3}}$.

Yechimi: (1) $\frac{6}{\sqrt{2}}=\frac{6\times\sqrt{2}}{\sqrt{2}\times\sqrt{2}}=\frac{6\times\sqrt{2}}{2}=3\sqrt{2}$.

$(a+b)(a-b)=a^2-b^2$ munosabatdan foydalansak, u holda

$$\frac{\sqrt{2}+\sqrt{3}}{\sqrt{2}-\sqrt{3}}=\frac{\left(\sqrt{2}+\sqrt{3}\right)}{\left(\sqrt{2}-\sqrt{3}\right)}=\frac{\left(\sqrt{2}+\sqrt{3}\right)^2}{\left(\sqrt{2}-\sqrt{3}\right)\left(\sqrt{2}+\sqrt{3}\right)}=$$

$$=\frac{2+2\sqrt{2}\sqrt{3}+3}{2-3}=-5-2\sqrt{6}.$$

4-misol. Quyidagi ifodani hisoblang:

$$\frac{\sqrt{5}}{\sqrt{3}-1}-\frac{\sqrt{3}}{\sqrt{5}+\sqrt{3}}.$$

Yechimi:

$$\frac{\sqrt{5}}{\sqrt{3}-1}-\frac{\sqrt{3}}{\sqrt{5}+\sqrt{3}}=\frac{\sqrt{5}\left(\sqrt{3}+1\right)}{\left(\sqrt{3}-1\right)\left(\sqrt{3}+1\right)}-\frac{\sqrt{3}\left(\sqrt{5}-\sqrt{3}\right)}{\left(\sqrt{5}+\sqrt{3}\right)\left(\sqrt{5}-\sqrt{3}\right)}=$$

$$=\frac{\sqrt{5}\sqrt{3}+\sqrt{5}}{2}-\frac{\sqrt{3}\sqrt{5}-3}{2}=\frac{3+\sqrt{5}}{2}.$$

8B. Ko'rsatkichli funksiyalar

Butun musbat ko'rsatkichli daraja

Asoslari teng bo'lgan ko'paytma.

[Masalan]:

$2^1=2,$

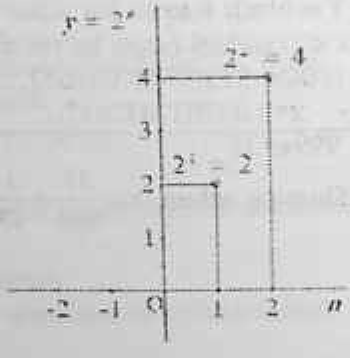
$2^2=2\times2=4,$

$2^3=2\times2\times2=8,...$

daraja

$$a^n=\underbrace{a\times a\times...\times a}_{n}$$

asos butun son



Daraja haqidagi teoremalar

Ko‘paytirish

$a^n a^m = a^{n+m}, \quad (a^n)^m = a^{nm}, \quad (ab)^n = a^n b^n.$

Bo‘lish

$$\boxed{\frac{a^n}{a^m} = \begin{cases} a^{n-m}, & agar\ \ n > m; \\ \dfrac{1}{a^{m-n}}, & agar\ \ n < m; \\ 1, & agar\ \ n = m; \end{cases}} \quad \Rightarrow \quad \boxed{\frac{a^n}{a^m} = a^{n-m}}.$$

[Masalan]: $\dfrac{a^5}{a^3} = a^2, \ \dfrac{a^3}{a^8} = \dfrac{1}{a^5}, \ \dfrac{a^6}{a^6} = 1.$

Butun manfiy ko‘rsatkichli daraja

$$\frac{a^n}{a^m} = a^{n-m}$$

Agar $m = n$ bo‘lsa, $\dfrac{a^n}{a^n} = a^0$ ga ega bo‘lamiz.

Shuning uchun, $a^0 = 1$ deb aniqlaymiz.

Agar $n = 0$ bo‘lsa, $\dfrac{a^0}{a^m} = a^{-m}$ ga ega bo‘lamiz.

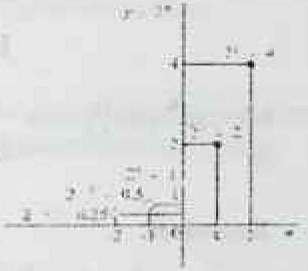
Shuning uchun, $a^{-n} = \dfrac{1}{a^n}$ deb aniqlaymiz.

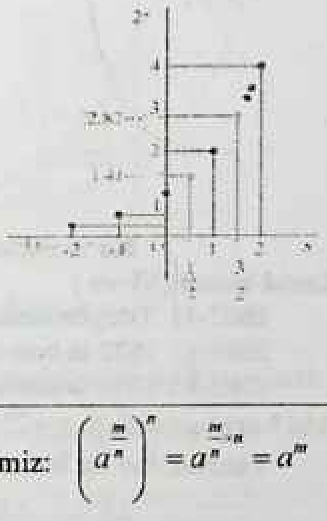
[Masalan]: $2^0 = 1, \ \ 2^{-1} = \dfrac{1}{2}, \ \ 2^{-2} = \dfrac{1}{2^2} = \dfrac{1}{4}.$

Ratsional ko‘rsatkichli daraja

ratsional son

$a^r \longleftarrow a^{\frac{m}{n}}$





$(a^n)^m = a^{nm}$ qoidadan, quyidagiga ega bo‘lamiz: $\left(a^{\frac{m}{n}}\right)^n = a^{\frac{m}{n}\cdot n} = a^m$

Shuning uchun, $\quad a^{\frac{m}{n}} = (a^m)^{\frac{1}{n}} = \sqrt[n]{a^m}.$

[Masalan]:

$2^{\frac{1}{2}} = \sqrt{2} = 1{,}414...,$

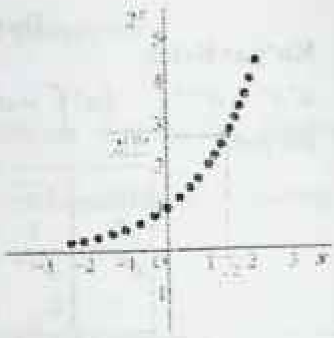
$2^{\frac{3}{2}} = \sqrt{2^3} = 2\sqrt{2} = 2{,}828...$

Irratsional ko'rsatkichli daraja

irratsional son

a^x

Irratsional ko'rsatgichli darajaning qiymati ratsional ko'rsatgichli darajaning limiti sifatida aniqlanadi [Masalan]:



$2^1 = 2.00000$

$2^{1.4} = 2.63902$

$2^{1.41} = 2.65737$

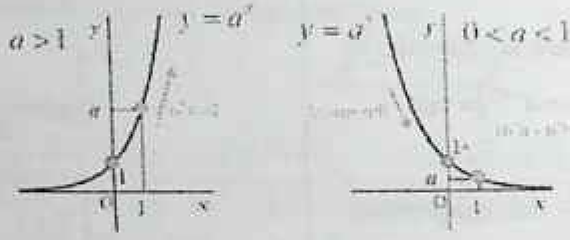
$\dots$

Ko'rsatkichli funksiya

Musbat haqiqiy son — Barcha haqiqiy son

$$y = a^x \quad (a > 0,\ a \neq 1)$$

Xossalari



Ko'rsatkichli o'sishga ega bo'lgan hodisalar (1)

Kovid holati (Virus)

2002-11 Xitoyda birinchi bemor topildi

2003-03 1622 ta bemorlar (58 ta vafot etgan)

2003-05 8860 ta bemorlar (792 ta vafot etgan)

Ikki barobar o'sish $y = 2^x$

Taxmin: har bir bakteriyalar har bir soatda ikki hujayraga bo'linadi.

[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]

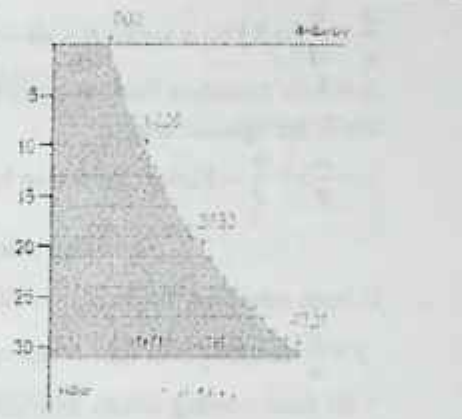
Murakkab foizlarni hisoblash

$$y = y_0(1+r)^x$$ Yil

Asosiy Foiz

[Masalan]

Asosiy	**$1000**
Yillik aralashma foiz stavkasi 5 %	
1 yil o'tib	**$1050.00**
10 yil o'tib	**$1628.89**
20 yil o'tib	**$2653.30**
30 yil o'tib	**$4321.94**

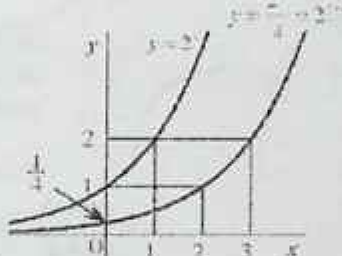


5-misol. Berilgan koordinatalar bo'yicha quyidagi funksiyalarning grafiklarini chizing va ular orasidagi munosabatlarni muhokama qiling:

(1) $y=2^x$ (2) $y=\dfrac{2^x}{4}$

Yechimi:

$y=\dfrac{2^x}{4}=2^{x-2}$ bo'lgani uchun $y=2^x$ grafigidan 2 birlik o'ng tomonidan joylashgan bo'ladi, ya'ni $y=2^x$ funksiya grafigini 2 birlik o'ngga surishdan hosil bo'ladi.



1.9. Ratsional va irratsional funksiya

9A. Chiziqli ratsional funksiya. Oddiy ratsional funksiya. Tengsizlik

Ratsional funksiya quyidagi ko'rinishdagi funksiyaga aytiladii:

$$y=\frac{g(x)}{f(x)},$$

bu yerda, $g(x)$ va $f(x)$ ko'phadlar.

Chiziqli ratsional funksiya quyidagi ko'rinishda bo'ladi:

$$y=\frac{ax+b}{cx+d}, \ (ad-bc\neq 0,\ c\neq 0).$$

$ad-bc=0$ bo'lganda,

$$\frac{a}{c}=\frac{b}{d}(=k) \Rightarrow a=ck,\ b=dk \Rightarrow y=\frac{ckx+dk}{cx+d}=k$$

$y=k$ -o'zgarmas funksiyaga aylanib qoladi.

$c=0$ bo'lganda esa,

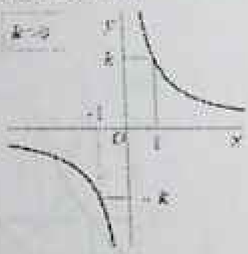
$y=\frac{a}{d}x+\frac{b}{d}$ -chiziqli funksiya hosil bo'ladi.

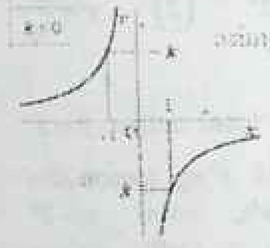
$y=\frac{k}{x}$ -ko'rinishdagi funksiya

U ham ratsional funksiya bo'lib, bu holda $a=0, d=0$

$y=\frac{k}{x}$ **funksiyada** ikki o'zgaruvchi o'zaro teskari proporsionaldir.

* Bi funksianing ikkita asimptotasi bo'lib, $x=0$ gorizontal asimptota, $y=0$ vertikal asimptota hisoblanadi.
* Grafigi giperboladan iborat, k ning qiymatiga qarab I va III yoki II va IV choraklarda joylashadi.





$y=\frac{ax+b}{cx+d}$ - ko'rinishdagi funksiya

Funksiyani quyidagicha o'zgartiramiz:

$$y=\frac{ax+b}{cx+d} \Rightarrow \boxed{y=\frac{k}{x-p}+q}$$

Unga chiziqli ratsional funksiyaning standart shakl deb ataladi. Standart shakldan quyidagilarni xulosa qilishimiz mumkin:

* $x=p$ vertikal asimptota $y=q$ gorizontal asimptota hisoblanadi.
* Aniqlanish sohasida $x \neq p$, qiymatlar sohasida $y \neq q$ bo'ladi.

1-misol. $y=\frac{3x-6}{x-1}$ ratsional funksiyada quyidagilarni bosqichma-bosqich bajaring:

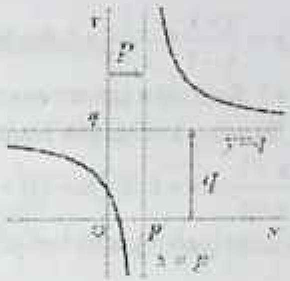
[1-bosqich] Funksiyani standart shaklga aylantiring;

[2-bosqich] Gorizontal va vertikal asimptotalarni toping;

[Qadam 3] Ox o'qi va Oy o'qi bilan kesishish nuqtalarini toping;

[4-bosqich] Grafigini chizing.

Yechimi:



(1) Funksiyani standart shaklga keltiramiz:

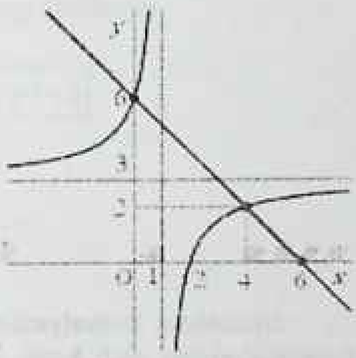
$$y=\frac{3x-6}{x-1}=\frac{3x-3-3}{x-1}=3-\frac{3}{x-1}.$$

(2) Demak standart shaklga ko'ra
$y=3$ gorizontal asimptotasi,
$x=1$ esa vertikal asimptotasi bo'ladi.

(3) $y=0$ ni qo'yib, $x=2$ ni va
$x=0$ ni qo'yib, $y=6$ ni topamiz.

(4) Yuqoridagilarga ko'ra grafikni chizamiz.

Tengsizlik



2-misol. Quyidagi tengsizlikni yeching:

$$\frac{3x-6}{x-1}\geq -x+6.$$

Yechimi: 1-misolning natijasiga ko'ra bizda tengsizlikning chap tomonida turgan funksiyaning grafigi bor. Shuning uchun $y=\frac{3x-6}{x-1}$ va $y=-x+6$ funksiya grafiklarining keshishgan nuqtasini topamiz:

$$\frac{3x-6}{x-1}=-x+6\Rightarrow 3x-6=(x-1)(-x+6)\Rightarrow$$
$$\Rightarrow x(x-4)=0\Rightarrow x=0,\ x=4.$$

Funksiyalar grafigi joylashuvi va kesishish nuqtalaridan tengsizlik yechimi haqida quyidagiga ega bo'lamiz:

$$0\leq x\leq 1,\ 4\leq x.$$

3-misol. Tengsizlikni yeching:

$$\frac{x+1}{x-1}>x+1.$$

Yechimi: Ikkita funksiyani ko'rib chiqamiz: $y=\frac{x+1}{x-1}=\frac{2}{x-1}+1$ va $y=x+1$.

Birinchi funksiya uchun gorizontal va vertikal asimptotalar mos ravishda y=1 va x=1 bo'ladi.

$y=\frac{x+1}{x-1}=\frac{2}{x-1}+1$ funksiya uchun x=-1 bo'lganda y=0,

$y=x+1$ funksiya uchun esa x=-1 da y=0 bo'ladi.

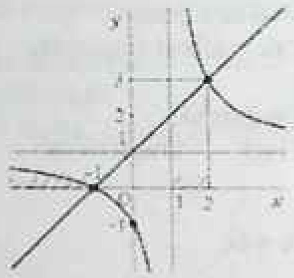
Ularning kesishgan nuqtalarini topamiz:

$$\frac{x+1}{x-1}=x+1 \Rightarrow (x+1)(x-2)=0 \Rightarrow x=-1,2.$$

Funksiyalar grafiklari joylashuvidan berilgan tengsizlik yechimi

$$x<-1,\ 1<x<2,$$

ekanligini hosil qilamiz.

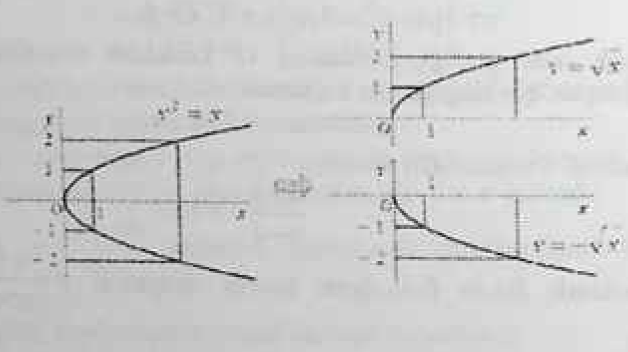


9B. Irratsional funksiyalar

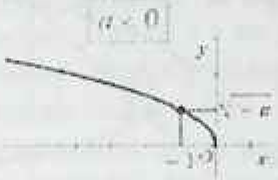
Irratsional funksiyaning qat'iy ta'rifi yo'q, chunki bu ko'rinishdagi funksiyalarning sinfi keng. Irratsional funksiya deb, ikki ko'phadning ko'-paytmasi shaklida yozib bo'lmaydigan funksiyalarni tushunushimiz mumkin.

Ildizda o'zgaruvchilarni o'z ichiga olgan funksiya irratsional funksiyaga misol bo'la oladi.

Misol.



$y=\sqrt{ax}$ ko‘rinishdagi funksiya



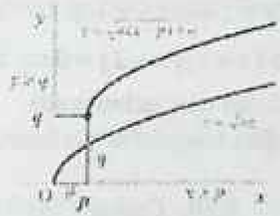
Aniqlanish sohasi $x \le 0$ Aniqlanish sohasi $x \ge 0$

Qiymatlar sohasi $y \ge 0$ Qiymatlar sohasi $y \ge 0$

$y=\sqrt{ax+b}+q$ ko‘rinishdagi funksiya

Standart ko‘rinishi quyidagicha bo‘ladi:

$$y=\sqrt{ax+b}+q \Rightarrow \boxed{y=\sqrt{a(x-p)}+q}$$



Tenglama va tengsizlik

4-misol. Quyidagi savollarga javob bering:

(1) $y=\sqrt{2-x}$ va $y=-x+\sqrt{2}$ funksiyalarning grafiklarini chizig;

(2) Tenglamani yeching: $\sqrt{2-x}=-x+\sqrt{2}$;

(3) Tengsizlikni yeching: $\sqrt{2-x} \ge -x+\sqrt{2}$;

Yechimi: (1) Ularning grafiklari parabolaning qismi va to‘g‘ri chiziq bo‘ladi.

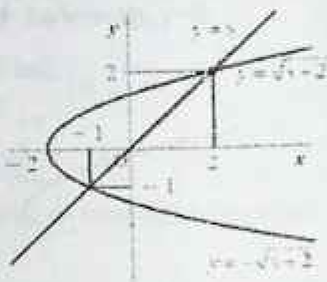
$$(2)\ \sqrt{2-x}=-x+\sqrt{2} \Rightarrow 2-x=\left(-x+\sqrt{2}\right)^2 \Rightarrow$$

$$\Rightarrow x\left(x-2\sqrt{2}+1\right)=0 \Rightarrow x_1=0,\ x_2=2\sqrt{2}-1.$$

Grafikdan ko'rinadiki $y(\geq 0$), ildiz faqat $x=0$ bo'ladi.

(3) (1) va (2) dan quyidagiga ega bo'lamiz: $0 \leq x \leq 2$.

5-misol. Tengsizlikni yeching: $\sqrt{x+2} > x$.



Yechimi: Ikkita funksiyani qaraymiz: $y=\sqrt{x+2}$ *va* $y=x$.

$y=\sqrt{x+2}$ funksiya grafigi parabolaning qismi bo'lib

$x=0 \Rightarrow y=\sqrt{2}$ *va* $y=0 \Rightarrow x=-2$ bo'ladi.

$y=\sqrt{x+2}$ *va* $y=x$ funksiyalarning o'zaro kesishish nuqtalarini topamiz:

$$\sqrt{x+2}=x \Rightarrow x+2=x^2 \Rightarrow$$
$$\Rightarrow (x+1)(x-2)=0 \Rightarrow x_1=-1,\ x_2=2.$$

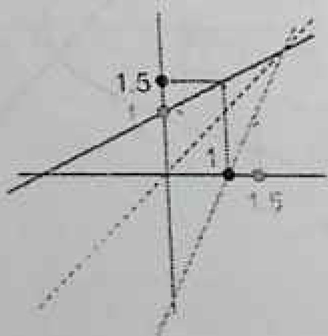
Faqat $x = 2$ bu shartni qanoatlantiradi.

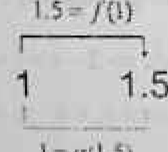
Grafikdan ko'rinadiki tengsizlik yechimi, $-2 \leq x \leq 2$ bo'ladi.

1.10. Teskari funksiya

10A Teskari funksiya. Ko'rsatkichli funksiya va logaritmik funksiya

Misol. Ushbu $y=f(x)=\frac{1}{2}x+1$ funksiyani qaraylik. Teskari funksiya x va y ni almashtirish orqali topiladi. $x=\frac{1}{2}y+1$ bundan esa $y=g(y)=2x-2$





Teskari $g(x)$funksiya $f(1)$ni 1 ga qaytaradi.

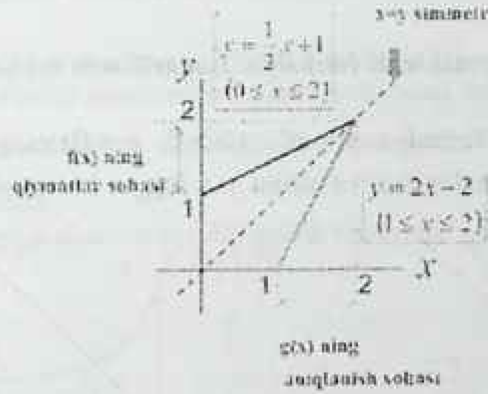
Teskari funksiya

Teskari funksiyaning ta'rifi. Agar aniqlanish sohasiga kiruvchi barcha x lar uchun $f(g(x))=x$ va $g(f(x))=x$ tengliklar bajarilsa, $f(x)$ va $g(x)$ funksiyalar bir-biriga o'zaro teskari funksiyalar deyiladi.

O'zaro teskari bo'lgan $f(x)$ va $g(x)$ funksiyalar shunday xususiyatga egaki, birining aniqlanish sohasi ikkinchisining qiymatlar sohasi bilan ustma ust tushadi va aksincha. Bundan tashqari o'zaro teskari funksiyalarning grafiklari $y=x$ to'g'ri chiziqqa nisbatan simmetrik.

Teskari funksiya qanday topiladi?

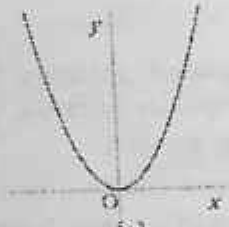
1. O'zgaruvchilar almashtiriladi: $y=f(x) \rightarrow x=f(y)$.
2. O'zgaruvchilar qayta almashtiriladi (ya'ni tenglamadan y topiladi): $x=f(y) \rightarrow y=g(x)$
3. $g(x)$ ning aniqlanish sohasi $f(x)$ning qiymatlar sohasi bilan bir xilligini tekshirish kerak.

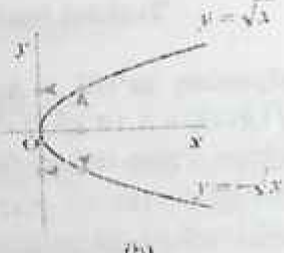


Teskari funksiya

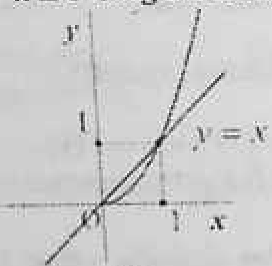
Parabola

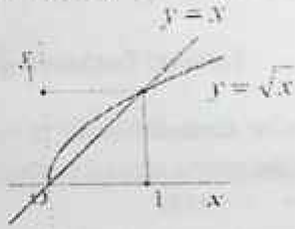
1. Quyidagi $y=x^2$ funksiyaga teskari funksiya mavjud emas chunki, (b) funksiya emas.





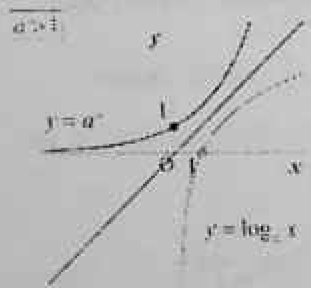
2. $y=x^2,\ x\geq 0$ ning esa teskari funksiyasi mavjud.

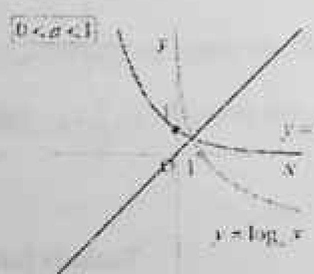




Ko'rsatkichli funksiya. Logarifimik funksiya

Ko'rsatkichli funksiya $y=a^x$ $(a>0,\ a\neq 0)$ ning teskari fuksiyasi logarifimik funksiya deyiladi va ushbu $y=\log_a x$ ko'rinishda yoziladi:





Logarifimik funksiya xossalari:

1. Aniqlanish sohasi barcha musbat haqiqiy sonlar.
2. Qiymatlar sohasi barcha haqiqiy sonlar.
3. Grafigi doim (1, 0) nuqtadan o'tadi.

1-misol. $y=\sqrt{1-x}$ funksiyani va unga teskari funksiyani kordinata tekkisligida tasvirlang.

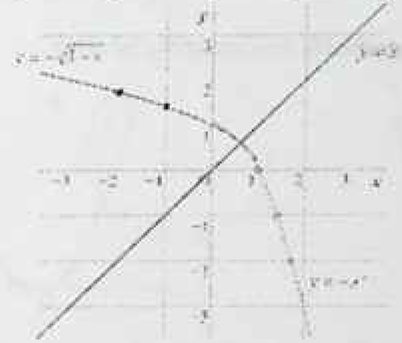
Yechimi: a) Funksiya grafigidan nuqtalarni topamiz

x	1	0	−1	−2	−3

y	0	1	$\sqrt{2}$	$\sqrt{3}$	2

b) Ularga $x = y$ ga nisbatan simmetrik nuqtalarni olamiz, ya'ni ularni qiymatlarini almashtiramiz

x	0	1	$\sqrt{2}$	$\sqrt{3}$	2
y	1	0	-1	-2	-3



2-misol. 1) Ushbu $y = \dfrac{x}{x+2}$ $(x \geq 0)$ funksiyani tasvirlang.

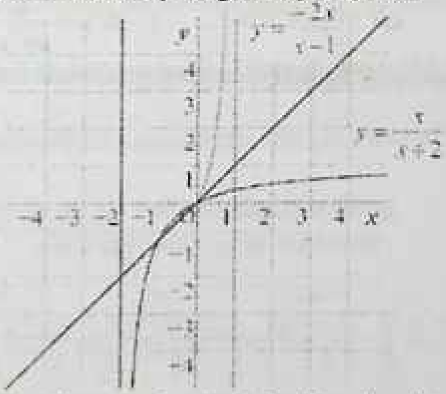
2) $y = x$ simmetrik akslantirish yordamida teskari funksiyani tasvirlang.

3) Teskari funksiya aniqlanish sohasini toping.

4) x va y ni o'rnini almashtirib teskari funksiyani toping.

Yechimi: 1) Funksiyani quyidagicha almashtiramiz $y = 1 - \dfrac{2}{x+2}$ va uning grafigini chizamiz

2) Uning grafigiga simmetrik qilib grafik yasaymiz



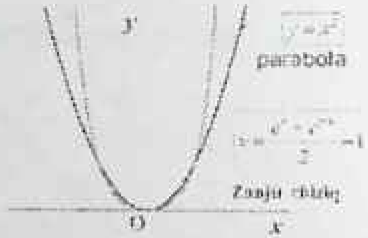
3) Grafikdan foydalanib topamiz: Aniqlanish sohasi $0 \leq x < 1$, qiymatlar sohasi $y \geq 0$.

4) Berilgan funksiyadan x va y ni o'rnini almashtiriramiz $x=\dfrac{y}{y+2}$

bundan esa $y=\dfrac{-2x}{x-1}$.

10B. Zanjir chiziq. Giperbolik funksiya

Zanjir chiziq- bu giperbolik kosinusning grafigi



Giperbolik funksiyalar

$$shx=\frac{e^x-e^{-x}}{2},\quad chx=\frac{e^x+e^{-x}}{2},\quad thx=\frac{e^x-e^{-x}}{e^x+e^{-x}}$$

Giperbolik funksiyalarni trigonometrik nomlar bilan atalishining sababi, ularning ko'pchilik xossalari o'xshsashdir. Masalan:

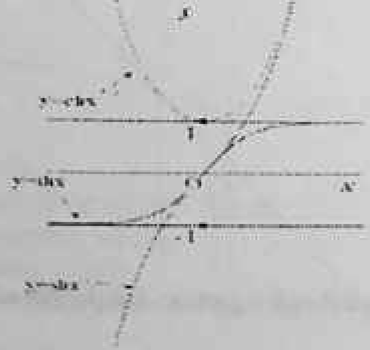
Trigonometrik funksiya

$$tgx=\frac{\sin x}{\cos x},\quad \cos^2 x+\sin^2 x=1,\quad 1+tg^2x=\frac{1}{\cos^2 x}$$

Giperbolik funksiya

$$thx=\frac{shx}{chx},\quad ch^2x-sh^2x=1,\quad 1-th^2x=\frac{1}{ch^2x}$$

Quyida giperbolik funksiyalarning grafiklari keltirilgan.



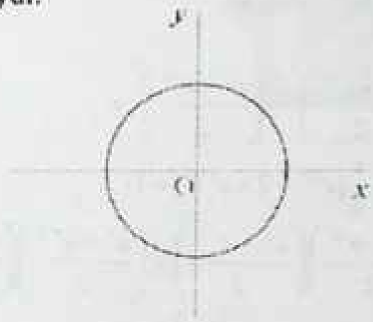
Ikki parametrli ifodalarni taqqoslash

1. $(x, y) = (\cos t, \sin t)$

Parametrni yo'qotib, biz quyidaginga ega bo'lamiz

$$x^2 + y^2 = 1$$

Bu esa aylanani ifodalaydi:



2. $(x, y) = (cht, sht)$

Oldingi ayniyatdan foydalanib biz quyidaginga ega bo'lamiz

$$x^2 - y^2 = 1$$

Bu esa giperbolani ifodalaydi:

Yuqorida o'rganilganlarga asoslanib quyidagi xulosalarga kelishimiz mumkin:

- ➢ Shaklidagi o'xshashlikka asoslanib, ularni giperbolik funktsiyalar deymiz.
- ➢ Xarakterdagi o'xshashlikka asoslanib, biz shx va boshqa belgilaridan foydalanamiz.

3-misol. Quyidagi qo'shish formulasini isbotlang

$$ch(x + y) = chxchy + shxshy.$$

Yechimi:

$$ch(x+y) = \left(\frac{e^x + e^{-x}}{2}\right)\left(\frac{e^y + e^{-y}}{2}\right) + \left(\frac{e^x - e^{-x}}{2}\right)\left(\frac{e^y - e^{-y}}{2}\right) =$$

$$= \frac{1}{4}\left(e^x e^y + e^x e^{-y} + e^{-x} e^y + e^{-x} e^{-y}\right) + \frac{1}{4}\left(e^x e^y - e^x e^{-y} - e^{-x} e^y + e^{-x} e^{-y}\right) =$$

$$=\frac{1}{2}\left(e^{x+y}+e^{-x-y}\right)=ch(x+y).$$

4-misol. Quyidagi formulalarni isbotlang

1) $ch^2x-sh^2x=1$

2) $sh(x+y)=shxchy+chxshy$.

Yechimi:

1) $$ch^2x-sh^2x=\left(\frac{e^x+e^{-x}}{2}\right)^2-\left(\frac{e^x-e^{-x}}{2}\right)^2=$$

$$=\frac{1}{4}\left(e^{2x}+2+e^{-2x}\right)-\frac{1}{4}\left(e^{2x}-2+e^{-2x}\right)=1.$$

2) $$sh(x+y)=\left(\frac{e^x-e^{-x}}{2}\right)\left(\frac{e^y+e^{-y}}{2}\right)+\left(\frac{e^x+e^{-x}}{2}\right)\left(\frac{e^y-e^{-y}}{2}\right)=$$

$$=\frac{1}{4}\left(e^xe^y+e^xe^{-y}-e^{-x}e^y-e^{-x}e^{-y}\right)+\frac{1}{4}\left(e^xe^y-e^xe^{-y}+e^{-x}e^y-e^{-x}e^{-y}\right)=$$

$$=\frac{1}{2}\left(e^{x+y}-e^{-x-y}\right)=sh(x+y).$$

1.11. Logarifm va o‘nli logarifm

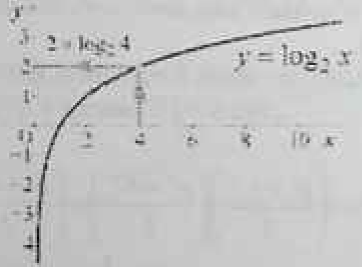
11A. Sonning logarifmi. Logarifmni afzalliklari. Logarifm xossalari

$M=a^p$ berilgan bo‘lsin.

Ta’rif. *M sonining a asosli logarifmi deb, bu sonni hosil qilish uchun a asosni ko‘tarish kerak bo‘lgan p darajaga aytiladi va u quyidagicha yoziladi*

$$p=\log_a M.$$

Boshqacha qilib aytganda, p ning qiymati “M ni hosil qilish uchun a ni qanday darajaga ko‘tarish kerak?” degan savolga javobdir.

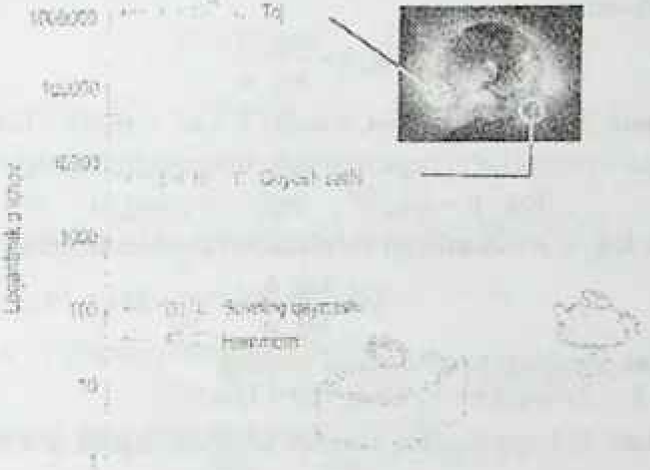


Logarifmlar juda ko‘p masalalarni yechishni yengillashtiradi, xususan ularning ba’zi afzalliklarini keltiramiz:

- Logarifm katta sonni kichik son bilan ifodalashi mumkin.

Misol uchun, $x = 100000$ $\log_{10} x = 5$

- Logarifmik o'lchov turli xil tafsilotlar bilan katta o'lchamdagi qiymatlarni ifodalashda yordam beradi.



Logarifm xossalari

1) $\log_a MN = \log_a M + \log_a N$

2) $\log_a \frac{M}{N} = \log_a M - \log_a N$

3) $\log_a M^k = k \log_a M$

1-misol. Yuqoridagi 1) Ko'paytirish xossasini isbotlang.

Yechimi: Aytaylik $M = a^p$ va $N = a^q$ bo'lsin. U holda $p = \log_a M$, $q = \log_a N$.

Ularni ko'paytmasini hisoblaymiz $M \cdot N = a^p \cdot a^q = a^{p+q}$.

Shuning uchun,

$$\log_a MN = \log_a M + \log_a N.$$

Maxsus qiymatlar

$$a^1 = a \quad \rightarrow \quad \log_a a = 1$$

$$a^0 = 1 \quad \rightarrow \quad \log_a 1 = 0$$

$$a^{-1} = \frac{1}{a} \quad \rightarrow \quad \log_a \frac{1}{a} = -1.$$

Maxsus xossalar

$$\log_a \frac{M}{N} = \log_a M - \log_a N, \quad M = 1, \quad \log_a \frac{1}{N} = -\log_a N$$

$$\log_a M^k = k\log_a M, \quad k=\frac{1}{n}, \quad \log_a \sqrt[n]{M} = \frac{1}{n}\log_a M.$$

2-misol. $a(\neq 1)$, $b(\neq 1)$, $c(\neq 1)$ musbat haqiqiy sonlar uchun quyidagi formulani isbotlang

$$\log_a b = \frac{\log_c b}{\log_c a}.$$

Yechimi: $p=\log_a b$ bo'lsin, u holda $b=a^p$ o'rinli bo'ladi. Tenglikni ikki tomonini c asosga ko'ra logarifmlasak, biz quyidaginga ega bo'lamiz

$$\log_c p = \log_c a^p, \quad \log_c p = p\log_c a$$

$a\neq 1$ da biz $\log_c a\neq 0$ ekanligini ko'rishimiz mumkin. Shuning uchun

$$p=\frac{\log_c b}{\log_c a}.$$

3-misol. Quyidagi tenglamalarni yeching

1) $\log_2 x = 3$ 2) $\log_2(x+3)+\log_2(2x-1)=2$

Yechimi: 1) Logarifmning bevosita ta'rifidan topsak $x=2^3=8$

2) Aniqlanish sohasiga ko'ra $x+3>0$ va $2x-1>0$. Shuning uchun $x>\frac{1}{2}$.

So'ngra logarifmni xossalaridan foydalanib quyidagilarni bajaramiz

$$\log_2(x+3)(2x-1)=\log_2 4$$

$$(x+3)(2x-1)=4, \quad 2x^2+5x-7=0$$

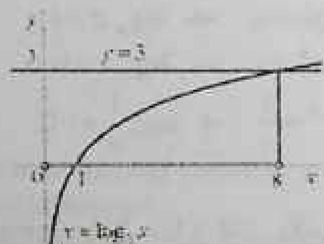
$$(x-1)(2x+7)=0, \quad x=1, \ x=-\frac{7}{2}$$

$x>\frac{1}{2}$ ekanligidan biz berilgan tenglamani yechimi $x=1$ ekanligini topamiz.

4-misol. Quyidagi tengsizlikni yeching

$$\log_2 x<3$$

Yechimi: Ushbu $\log_2 x<3=\log_2 2^3=\log_2 8$ tengsizlikka ega bo'lamiz. $y=\log_2 x$ funksiyaning grafigidan biz $0<x<8$ ekanligini hosil qilamiz.



5-misol. Quyidagilarni qiymatini toping

1) $\log_5 125$ 2) $\log_{10} 10\sqrt{10}$

Yechimi: 1) Biz $\log_5 125 = p$ deb belgilaylik. U holda biz quyidagiga ega bo'lamiz

$$5^p = 125 = 5^3, \quad p = 3.$$

2) $\log_{10} 10\sqrt{10} = \log_{10} 10^{\frac{3}{2}} = \frac{3}{2}\log_{10} 10 = \frac{3}{2}$

6-misol. Quyidagi amallarni bajaring

1) $\log_{10} 2 + \log_{10} 5$ 2) $\log_3 7 - \log_3 63$

Yechimi: 1) $\log_{10} 2 + \log_{10} 5 = \log_{10}(2\cdot 5) = \log_{10} 10 = 1$

2) $\log_3 7 - \log_3 63 = \log_3 \frac{7}{63} = \log_3 \frac{1}{9} = \log_3 3^{-2} = -2.$

7-misol. 1) Quyidagi amallni bajaring

$$(\log_2 9 + \log_4 3)(\log_3 2 + \log_9 4)$$

2) Quyidagi logarifmik tenglamani yeching

$$\log_2 x + \log_2(x-7) = 3$$

Yechimi:

1) $(\log_2 9 + \log_4 3)(\log_3 2 + \log_9 4) = \left(\log_2 9 + \frac{\log_2 3}{\log_2 4}\right)\left(\frac{\log_2 2}{\log_2 3} + \frac{\log_2 4}{\log_2 9}\right) =$

$$= \left(2\log_2 3 + \frac{\log_2 3}{2}\right)\left(\frac{1}{\log_2 3} + \frac{2}{2\log_2 3}\right) = \frac{5}{2}\log_2 3 \cdot \frac{2}{\log_2 3} = 5$$

2) $\log_2 x + \log_2(x-7) = 3$ tenglamaning aniqlanish sohasi $x > 0$ va $x - 7 > 0$ dan $x > 7$ ekanligini topamiz.

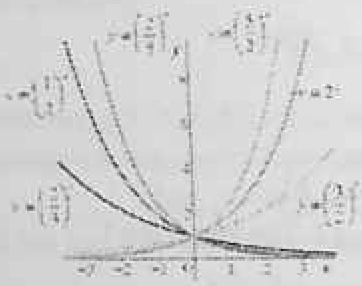
So'ngra

$$\log_2 x(x-7) = \log_2 2^3, \quad x(x-7) = 8, \quad (x+1)(x-8) = 0$$

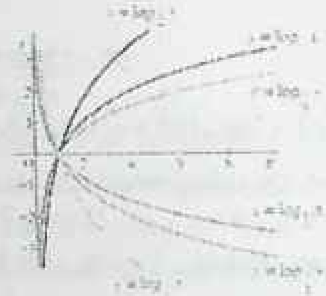
bundan, $x = -1$ va $x = 8$. Aniqlanish sohasidan foydalanib $x = 8$ ekanligini topamiz.

11B. Maxsus logarifm. O'nli logarifm

Turli ko'rsatkichli funksiyalar



Turli logarifmik funksiyalar



Yuqorida bir necha logarifmik funksiyalar grafiklari keltirilgan. Ulardan qaysi birini ishlatamiz?

➢ Biz o'nli kasrni yaxshi bilamiz, ya'ni ixtiyoriy sonni yoyishimiz mumkin, masalan.

$$2356 = 2\cdot10^3 + 3\cdot10^2 + 5\cdot10 + 6.$$

Suning uchun logarifm xossalarini e'tiborga olib maxsus logarifmni kiritamiz.

O'nli logarifm

$a=10,\qquad y=\log_{10}x\qquad$ yoki $\qquad y=\lg x$

➢ Buning ilm-fan va muhandislik sohalarida qo'llanilishi mavjud.

Natural logarifm

$a=e=2.718...,\qquad y=\log_e x\qquad$ yoki $\qquad y=\ln x$

➢ Bu $y=\ln x$ ko'rinishda yoziladi

➢ Bu sof matematikada qo'llaniladi.

➢ Bu keyinchalik differensiyallash darsida tushuntiriladi.

Ikkilik logarifm

$$a=2,\qquad y=\log_2 x$$

➢ Bu kompyuter fanida qo'llaniladi.

O'nli logarifm

➢ O'nli logarifm qiymatlari logarifmik jadvalda keltirilgan.

➢ Bu jadval $1<M<9.99$ uchun berilgan.

➢ Ixtiyoriy $M=a\cdot10^n$ $(1\le n<10)$ uchun, quyidagi formula bo'yicha topiladi

$$\log_{10}M=\log_{10}\left(a\cdot10^n\right)=n+\log_{10}a.$$

O'nli logarifm jadvali

	0	1	2	3	4	5	6	7	8	9
1.0	.0000	.0043	.0086	.0128	.0170	.0212	.0253	.0294	.0334	.0374
1.1	.0414	.0453	.0492	.0531	.0569	.0607	.0645	.0682	.0719	.0755
1.2	.0792	.0828	.0864	.0899	.0934	.0969	.1004	.1038	.1072	.1106
1.3	.1139	.1173	.1206	.1239	.1271	.1303	.1335	.1367	.1399	.1430
1.4	.1461	.1492	.1523	.1553	.1584	.1614	.1644	.1673	.1703	.1732
1.5	.1761	.1790	.1818	.1847	.1875	.1903	.1931	.1959	.1987	.2014
⋮										⋮
9.5	.9777	.9782	.9786	.9791	.9795	.9800	.9805	.9809	.9814	.9818
9.6	.9823	.9827	.9832	.9836	.9841	.9845	.9850	.9854	.9859	.9863
9.7	.9868	.9872	.9877	.9881	.9886	.9890	.9894	.9899	.9903	.9908
9.8	.9912	.9917	.9921	.9926	.9930	.9934	.9939	.9943	.9948	.9952
9.9	.9956	.9961	.9965	.9969	.9974	.9978	.9983	.9987	.9991	.9996

8-misol. Quyidagi sonlarning o'nli logarifmini toping

$$0.01,\quad 0.1,\ 1,\ 10,\ 100.$$

Yechimi: $\log_{10} 0.01 = \log_{10} 10^{-2} = -2$, $\log_{10} 0.1 = -1$

$\log_{10} 1 = 0, \quad \log_{10} 10 = 1, \quad \log_{10} 100 = 2.$

9-misol. Logarifmik jadvaldan 11910 ning o'nli logarifmni toping.

Yechimi:

$$\log_{10} 11910 = \log_{10}\left(1.191 \cdot 10^4\right) \approx 4 + \log_{10} 1.19 = 4 + 0.0755 = 4.0755.$$

10-misol. 6^{25} ning raqamlari sonini toping. Bunda quyidagi tenglikdan foydalaning

$$\log_{10} 2 = 0.3010, \quad \log_{10} 3 = 0.4771$$

Yechimi: $N = 6^{25}$ deb belgilaymiz va ikkala tomondan o'nli logarifmini topamiz.

$$\log_{10} N = \log_{10} 6^{25} = 25(\log_{10} 2 + \log_{10} 3) =$$

$$= 25(0.3010 + 0.4771) = 19.4525.$$

U holda $N = 10^{19.4525}$. Shuning uchun $10^{19} < N < 10^{20}$. Bu esa 6^{25}ni raqamlari soni 20 ekanligini bildiradi.

Logarifmning afzalliklari

Biz ko'paytirish qoidasiga ko'ra, $\log_a MN = \log_a M + \log_a N$ ni ko'paytirishni oson qo'shish bilan almashtirishimiz mumkin.

11-misol. $x = 123111$, $y = 9663189$ lar berilgan. Logaritmik jadval yordamida xy ning taqribiy qiymatini toping.

Yechimi:

$x = 1.23111 \cdot 10^5$, $\log_{10} x = \log_{10} 1.23111 + \log_{10} 10^5 \approx 5 + \log_{10} 1.23$

$y = 9.663189 \cdot 10^6$, $\log_{10} y = \log_{10} 9.663189 + \log_{10} 10^6 \approx 6 + \log_{10} 9.66$

Jadvaldan $\log_{10} 1.23 = 0.0899$, $\log_{10} 9.66 = 0.9850$

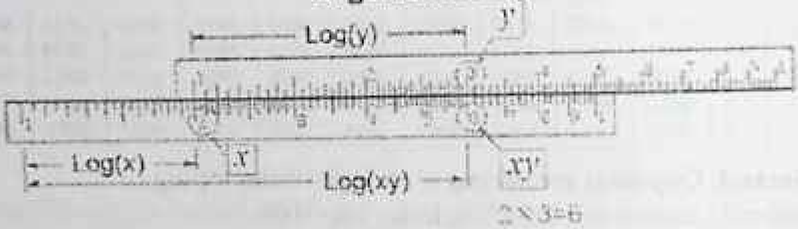
Shuning uchun

$$\log_{10} xy = \log_{10} x + \log_{10} y \approx (\log_{10} 1.23 + 5) + (\log_{10} 9.66 + 6) \approx$$

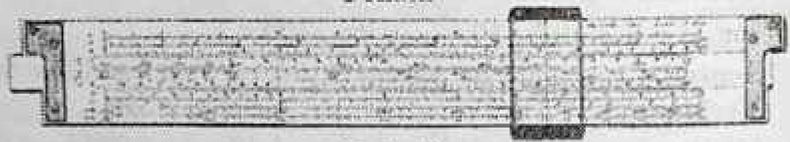
$$\approx 5.0899 + 6.9850 \approx 12 + 0.749 \approx 12 + \log_{10} 1.19 = \log_{10}(1.19 \cdot 10^{12})$$

Demak, $xy \approx 1.19 \cdot 10^{12}$

Logarifmik lineyka



1 shakl



2 shakl

12-misol. Biz 5 % stavka bilan 1 million yenni bankda saqlaganimizda, 5 yildan keyin qancha pul olishimiz mumkin? Jadvaldan foydalaning.

Yechimi: $y = 1000000 \cdot 1.05^5$

$$\log_{10} y = 6 + \log_{10} 1.05^5 = 6 + 5\log_{10} 1.05 = 6 + 5 \cdot 0.0212 = 6 + 0.106$$

Jadvaldan $0.106 \approx \log_{10} 1.28$. U holda

$$\log_{10} y \approx 6 + \log_{10} 1.28 = \log_{10}(1.28 \cdot 10^6) = \log_{10}(128 \cdot 10^4)$$

Taxminan 1.28 million yenani olishimiz mumkin.

Aniq qiymat $1000000 \cdot 1.05^5 = 1276281.5$

13-misol. $\log_{10} 2 = 0.3010$ va $\log_{10} 3 = 0.4771$ berilgan.

Quyidagilarni qiymatini toping

1) $\log_{10} 125$ 2) $\log_2 5$

Yechimi:

1) $\log_{10} 125 = \log_{10} 5^3 = 3\log_{10} 5 = 3\log_{10}(10/2) =$

$= 3(1 - \log_{10} 2) = 3 \cdot (1 - 0.3010) = 2.0970$

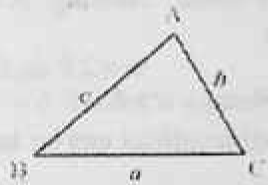
$$2)\ \log_2 5 = \frac{\log_{10} 5}{\log_{10} 2} = \frac{1-\log_{10} 2}{\log_{10} 2} = \frac{1-0.3010}{0.3010} \approx 2.322$$

1.12. Sinuslar va kosinuslar teoremasi

12A Uchburchaklarning asosiy xossalari. Sinuslar qonuni

Ixtiyoriy uchburchak uchun o'rinli bo'lgan xossalar

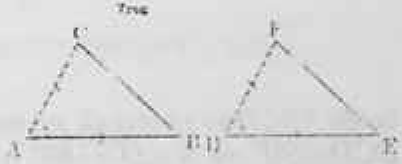
1) $\angle A + \angle B + \angle C = 180°$.
2) $a < b + c,\ \ b < a + c,\ \ c < a + b$

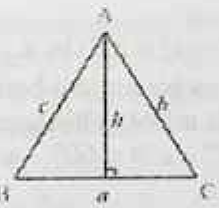


Ixtiyoriy uchburchakni quyidagilar bo'yicha aniqlash mumkin:

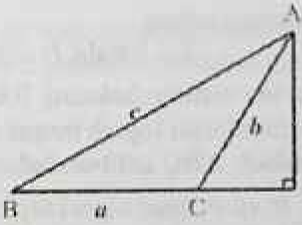
1) Uch tomon bo'yicha;
2) Ikki tomon va ularga yopishgan burchak bo'yicha;
3) Ikki ichki burchak va yopishgan tomon bo'yicha;

O'xshash va teng uchburchaklar





C o'tkir



C o'tmas

Sinuslar teoremasi

Uchburchak tomoni uzunligining u tomonga qarama-qarshi burchakning sinusiga nisbati barcha tomonlar va burchaklar uchun bir xil

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Sinuslar teoremasi isboti

1. Uchburchakning burchaklari o'tkir bo'lgan hol. Uchburchakka tashqi aylana chizamiz. Ushbu aylananing diametri BD ga teng bo'lsin. Bitta vatarga tiralgan burchaklar tengligidan, biz $D = A$, $< BCD = 90°$ ega bo'lamiz. Shuning uchun biz quyidagi tenglikni olamiz

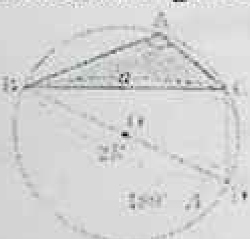
$$a = 2R\sin D = 2R\sin A$$

Shunga o'xshash $b = 2R\sin B$, $c = 2R\sin C$. Hosil qilinganlardan biz quyidagi tenglikka ega bo'lamiz

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R.$$

1-misol. $A > 90°$ uchun $a = 2R\sin A$ tenglikni isbotlang.

Yechimi: Biz O markazidan o'tgan BD chiziqni chizamiz:



U holda ABCD to'rtburchak aylanaga ichki chizilganligi uchun $< BCD = 90°$ va $A + D = 180°$ ga teng.

Shuning uchun

$$a = 2R\sin D = 2R\sin(180° - A) = 2R\sin A.$$

Quyida uchburchakning ikki burchagi va bir tomoni berilganida, boshqa tomonlarini topish mumkinligi haqida misol keltirilgan.

2-misol. ABC uchburchakda $\angle A = 45°$, $\angle B = 60°$ va $a = 10$ berilgan. b va c tomonlarni toping.

Yechimi: $\angle C = 180° - 60° - 45° = 75°$. Sinuslar teoremasidan

$$\frac{10}{\sin 45°} = \frac{b}{\sin 60°} = \frac{c}{\sin 75°}$$

Har bir sinusning qiymatini hisoblaymiz

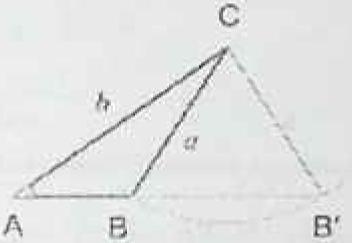
$$\sin 45° = \frac{\sqrt{2}}{2},\ \sin 60° = \frac{\sqrt{3}}{2},\ \sin 75° = \sin(30° + 45°) = \frac{\sqrt{6} + \sqrt{2}}{4}$$

Shuning uchun

$$b = 5\sqrt{6},\quad c = 5(\sqrt{3} + 1).$$

Agar "ikki tomon va ular orasidagidan farqli burchakni" bilsak, oxirgi tomonni aniqlay olmaymiz.

Burchak $\angle A$ va a, b tomonlar berilgan bo'lsin. Lekin bu ucburchak ABC yoki $AB'C$ shaklida bo'lishi mumkin:



Ya'ni uchinchi tomon bir qiymatli aniqlanmaydi.

3-misol. ABC uchburchakda $b=12$, $\angle C=135°$ va tashqi chizilgan aylana radiusi $R=12$ ga teng. Burchak $\angle B$ va c tomonlarni toping.

Yechimi: Sinuslar teoremasi va tashqi chizilgan aylanani radiusiga bog'langan formuladan

$$\frac{a}{\sin A}=\frac{12}{\sin B}=\frac{c}{\sin 135°}=24$$

Shuning uchun

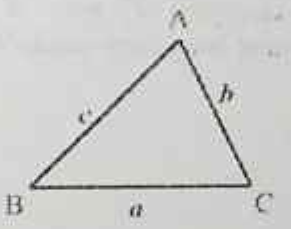
$$\sin B=\frac{1}{2} \Rightarrow \angle B=30°, \ c=24\sin 135°=24\sin 45°=12\sqrt{2}.$$

12B Kosinuslar teoremasi

Agar biz uchburchakning ikkita tomoni va ular orasidagi burchakni bilsak, bu burchakka qarama-qarshi tomonni topishimiz mumkin ekan. U quyidagi tengliklar bilan aniqlanadi.

Kosinuslar teoremasi

$$a^2=b^2+c^2-2bc\cos A,$$
$$b^2=a^2+c^2-2ac\cos B$$
$$c^2=a^2+b^2-2ab\cos C$$

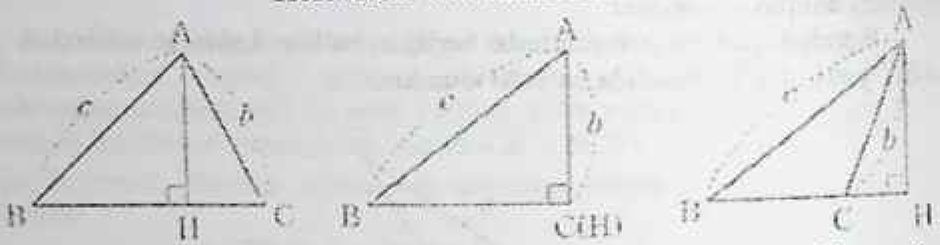


Boshqa tomondan esa biz uch tomonga ko'ra burchaklarni topishimiz mumkin:

$$\cos A=\frac{b^2+c^2-a^2}{2bc}, \ \cos B=\frac{a^2+c^2-b^2}{2ac},$$

$$\cos C=\frac{a^2+b^2-c^2}{2ab}$$

Kosinuslar teoremasi isboti



A nuqtadan BC tomonga balandlik tushuramiz va bu nuqtani H orqali belgilaymiz.

$$a = CH + BH = b\cos C + c\cos B \tag{1}$$

Shunga o'xshash

$$b = c\cos A + a\cos C \tag{2}$$
$$c = a\cos B + b\cos A \tag{3}$$

Agar $(2)\times b + (3)\times c - (1)\times a$ kabi amalni bajarsak biz quyidagiga ega bo'lamiz

$$b^2 + c^2 - a = 2bc\cos A \Rightarrow \quad a^2 = b^2 + c^2 - 2bc\cos A.$$

4-misol. ABC uchburchakda $b = 2,\ c = 1+\sqrt{3}$ va $\angle A = 60^\circ$ lar berilgan. a tomon va burchak $\angle B, \angle C$ larni toping.

Yechimi: Kosinuslar teoremasidan

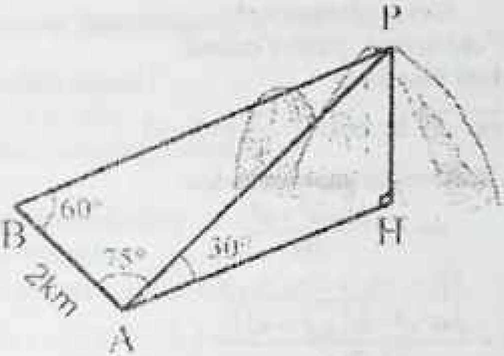
$$a^2 = 2^2 + \left(1+\sqrt{3}\right)^2 - 2\cdot 2\left(1+\sqrt{3}\right)\cos 60^\circ = 6 \Rightarrow a = \sqrt{6}$$

Sinuslar teoremasidan

$$\frac{\sqrt{6}}{\sin 60^\circ} = \frac{2}{\sin B},\ \sin B = \frac{1}{\sqrt{2}} \Rightarrow \angle B = 45^\circ \text{ yoki } \angle B = 135^\circ$$

Biz $a = \sqrt{6} > b = 2$ dan $\angle B < \angle A = 60^\circ$ olamiz. Shuning uchun, $\angle B = 45^\circ$ va $\angle C = 180^\circ - \angle A - \angle B = 75^\circ$.

5-misol. A va B nuqtalar orasidagi masofa 2 km. Tog'ning P tepasidan qaraganimizda $\angle PAB = 75^\circ$, $\angle PBA = 60^\circ$. Bundan tashqari, tog' cho'qqisi P ning A nuqtadan ko'tarilish burchagi 30 ° ga teng. Tog'ning balandligi qancha?



Yechimi: $\angle APB = 180° - 75° - 60° = 45°$. Sinuslar teoremasidan

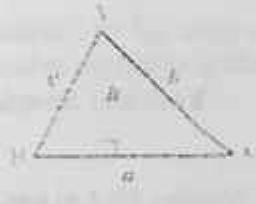
$$\frac{AP}{\sin 60°} = \frac{2}{\sin 45°} \Rightarrow AP = \sqrt{6}.$$

Uchburchak APH to'g'ri uchburchakli bo'lgani uchun,

$$PH = AP\sin 30° = \frac{\sqrt{6}}{2} \text{ km}$$

1.13. Trigonometrik funksiyalarning tadbiqi
13A Uchburchaklarning yuzalari. Geron formulasi. Tashqi chizilgan aylana. Ichki chizilgan aylana

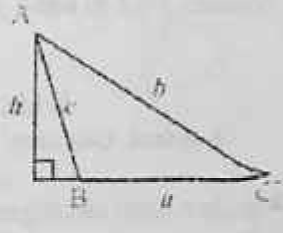
Uchburchakning yuzasi



Berilgan ikki tomon va ular orasidagi burchakka ko'ra

$$S = \frac{1}{2}bc\sin A = \frac{1}{2}ca\sin B = \frac{1}{2}ab\sin C$$

Isbot.



Ong tomonda ko'rsatilgan belgilashlardan foydalangan holda uchburchak balandligini topamiz.

$h = c\sin B \quad (0° \le B \le 90°)$ yoki

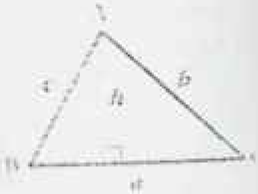
$h = c\sin(180° - B) = c\sin B \quad (90° \le B \le 180°)$

U holda

$$S = \frac{1}{2}ah = \frac{1}{2}ac\sin B$$

Geron formulasi

1-misol. Uchburchak yuzasi uchun quyidagi formulani isbotlang



$S=\sqrt{s(s-a)(s-b)(s-c)}$ bu erda $s=\frac{1}{2}(a+b+c)$

Yechimi: Kosinuslar teoremasidan

$$\cos A=\frac{b^2+c^2-a^2}{2bc}$$

Unga asosan

$$\sin A=\sqrt{1-\cos^2 A}=\frac{\sqrt{4b^2c^2-\left(b^2+c^2-a^2\right)^2}}{2bc}$$

Bu qiymatni yuzani hisoblash formulasiga qo'ysak

$$S=\frac{1}{2}bc\sin A=\frac{1}{4}\sqrt{4b^2c^2-(b^2+c^2-a^2)^2}=$$

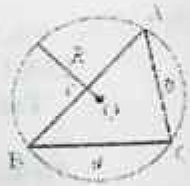
$$=\frac{1}{4}\sqrt{\{2bc-(b^2+c^2-a^2)\}\{2bc+(b^2+c^2-a^2)\}}=$$

$$=...=\sqrt{s(s-a)(s-b)(s-c)}$$

Ushbu formula – **Geron formulasi** deb nomlanadi.

Tashqi chizilgan aylana

2-misol. Quyidagi formulani isbotlang



$$S=\frac{abc}{4R}$$

bu erda a,b va c uchburchakning tomonlari, R esa tashqi chizilgan aylana radiusi.

Yechimi: Sinuslar teoremasidan

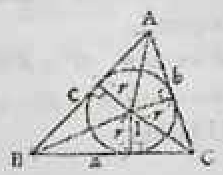
$$\frac{a}{\sin A}=2R$$

Bu yerdan $SinA$ ni topib yuza formulasiga qo'yib topamiz

$$S=\frac{1}{2}bc\sin A=\frac{1}{2}bc\frac{a}{2R}=\frac{abc}{4R}$$

Ichki chizilgan aylana

3-misol. Quyidagi formulani isbotlang



$$S=rs$$

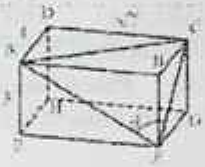
Bu erda r ichki chizilgan aylana radiusi va $s=\frac{1}{2}(a+b+c)$, a,b va c uch tomonning uzunliklari.

Yechimi: Aylana markazini I nuqta deb belgilasak, uchburchak yuzasi quyidagi yig'indi ko'rinishiga keladi.

$$S_{\Delta ABC}=S_{\Delta IBC}+S_{\Delta ICA}+S_{\Delta IAB}=\frac{1}{2}ra+\frac{1}{2}rb+\frac{1}{2}rc=rs$$

13B Fazoviy jismlarga doir masalalar

To'g'ri parallelepiped



4-misol. ABCD-EFGH to'g'ri parallelepipedda ACF uchburchakning yuzasini toping.

(1) Kosinuslar teoremasidan foydalaning

(2) Geron formulasidan foydalaning

Yechimi: (1) Pifagor teoremasidan $CF=5, AF=4, AC=\sqrt{23}$

Kosinuslar teoremasidan $23=4^2+5^2-2\cdot4\cdot5\cos\theta \quad \therefore \cos\theta=\frac{9}{20}$

Shunda $\sin\theta=\sqrt{1-\cos^2\theta}=\frac{\sqrt{319}}{20}$

Yuza esa $S=\frac{1}{2}5\cdot4\sin\theta=\frac{\sqrt{319}}{2}$ ga teng bo'ladi.

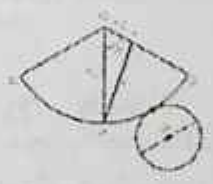
(2) $s=\frac{1}{2}\left(5+4+\sqrt{23}\right)=\frac{1}{2}\left(9+\sqrt{23}\right) \quad \therefore s-CF=\frac{1}{2}\left(\sqrt{23}-1\right)$

$s-AF=\frac{1}{2}\left(1+\sqrt{23}\right) \qquad s-AC=\frac{1}{2}\left(9-\sqrt{23}\right)$

$S=\sqrt{\frac{1}{2}\left(9+\sqrt{23}\right)\frac{1}{2}\left(\sqrt{23}-1\right)\frac{1}{2}\left(1+\sqrt{23}\right)\frac{1}{2}\left(9-\sqrt{23}\right)}=\frac{\sqrt{319}}{2}$

Konus sirtidagi eng qisqa egri chiziq

5-misol. Konus quyidagi o'lchamlar bilan berilgan: Asosnig diametri $AB=20$см, yasovchi $OB=30$см, $OC=10$см Konus sirtidagi AC ikki nuqtani bog'lovchi eng qisqa egri chiziq uzunligini toping.



Yechimi: Konusning yoyilmasini qaraymiz Sektor burchagini quyidagi nisbatlardan topamiz

$$\frac{\theta}{360^0}=\frac{20\pi/2}{2\pi\cdot30}, \quad \therefore \theta=60^0$$

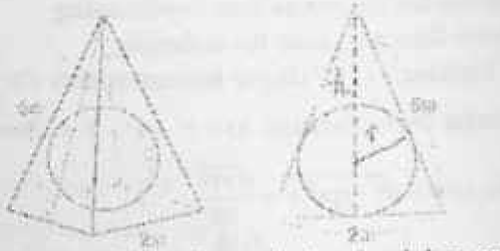
Kosinuslar teoremasidan $x^2=10^2+30^2-2\cdot10\cdot30\cos60^0=700$

Demak, $x=10\sqrt{7}\approx26{,}5$см

Sfera va muntazam to'rtburchakli piramida

6-misol. Sfera muntazam to'rtburchakli piramidaga ichki chizilgan. Piramida asosinig yuzasi 4 ga, yon yog'inig yuzasi 5 ga tengligi ma'lum. Sfera radiusining uzunliga nimaga teng?

Yechimi: Asosdagi kvadrat tomoni 2 ga va piramidaning yon yog'ini hosil qiluvchi uchburchakning balandligi (piramidaning apofemasi) 5 ga teng. Piramidaning qizil bilan belgilangan o'q kesimining balandligi

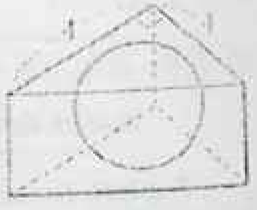


$$h = \sqrt{5^2 - 1^2} = 2\sqrt{6}$$

ga teng. Uchburchak yuzasi uchun 1-misoldagi formuladan foydalanamiz

$$S = rs = r\left(\frac{a+b+c}{2}\right),\ \frac{1}{2}\cdot 2\cdot 2\sqrt{6} = r\left(\frac{2+5+5}{2}\right),\ \therefore r = \frac{\sqrt{6}}{3}$$

Sfera va uchburchakli to'g'ri prizma



7-misol. Sfera uchburchakli to'g'ri prizmaga ichki chizilgan. Prizmaning asosi to'g'ri burchakli teng yonli uchburchak.

(1) sfera radiusining uzunligi qanday?

(2) sfera sirti yuzasining qiymati nimaga teng?

(3) sharning hajmi nimaga teng?

Yechimi:

(1) 6-misoldagi kabi sfera radiusini topamiz

$$\frac{1}{2}\cdot 4\cdot 4 = r\left(\frac{4+4+4\sqrt{2}}{2}\right),\ \therefore r = \frac{4}{2+\sqrt{2}} = 4-2\sqrt{2}$$

U holda (2)

(2) $S = 4\pi r^2 = 4\pi(4-2\sqrt{2})^2 = 32\pi(3-2\sqrt{2})$

(3) $V = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi(4-2\sqrt{2})^3 = \frac{64}{3}\pi(10-7\sqrt{2})$

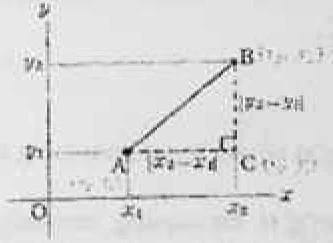
1.14. Grafik va tenglamalar

14A Ikki nuqta orasidagi masofa. Kesmani berilgan nisbatda bo'lish.To'g'ri chiziqlar

Masofa formulasi

Agar ikkita $A(x_1,y_1)$ va $B(x_2,y_2)$ nuqtalar berilgan bo'lsa, ular orasidagi masofa quyidagi formula bilan topiladi

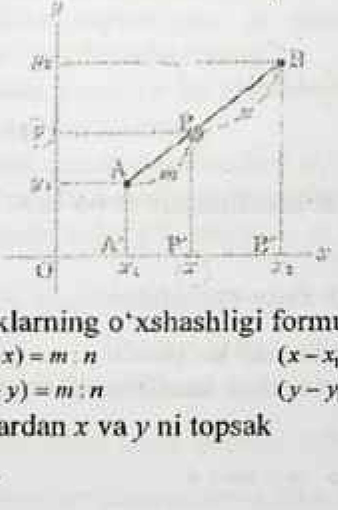
$$AB=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

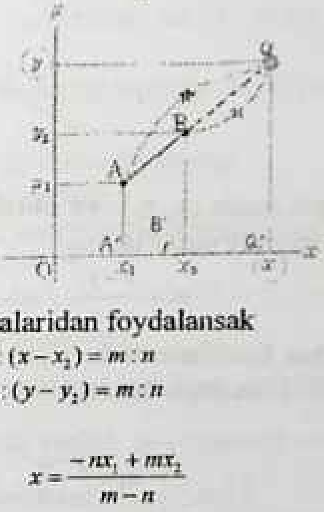


Kesmani berilgan nisbatda bo'lish

Ko'p geometrik masalalarda kesmalarni biror nisbatda bo'lish holatlari uchraydi. Bizga ikkita $A(x_1,y_1)$ va $B(x_2,y_2)$ nuqtalar berilgan bo'lsin.

Ichki bo'luvchi nuqta. **Tashqi bo'luvchi nuqta.**





Uchburchaklarning o'xshashligi formulalaridan foydalansak

$(x-x_1):(x_2-x)=m:n$ $\qquad$ $(x-x_1):(x-x_2)=m:n$

$(y-y_1):(y_2-y)=m:n$ $\qquad$ $(y-y_1):(y-y_2)=m:n$

Bu tengliklardan x va y ni topsak

$$x=\frac{nx_1+mx_2}{m+n} \qquad x=\frac{-nx_1+mx_2}{m-n}$$

$$y=\frac{ny_1+my_2}{m+n} \qquad y=\frac{-ny_1+my_2}{m-n}$$

1-misol. X o'qida $A(-1,3)$ va $B(2,4)$ nuqtalardan bir xil uzoqlikda joylashgan C nuqtani toping.

Yechimi: C nuqtaning koordinatalari $(x,0)$ bo'lsin. U holda masofalar tengligiga asosan

$$\sqrt{(x+1)^2+(0-3)^2}=\sqrt{(x-2)^2+(0-4)^2}$$

Ikkala tomonni kvadratga oshirsak $(x+1)^2+9=(x-2)^2+16$ va $x=\frac{5}{3}$.

To'g'ri chiziq

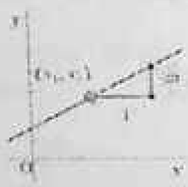
To'g'ri chiziq tenglamasining turli ko'rinishlari bor bo'lib,

$$ax+by+c=0$$

ga to'g'ri chiziqning umumiy tenglamasi deyiladi. Birinchi darajali har qanday ax+by+c=0 tenglama Dekart koordinatalar sistemasida to'gri chiziqning tenglamasidir.

Boshqa ko'rinishlari

(1)Berilgan nuqtadan o'tuvchi va berilgan burchak koeffisientli ko'rinishi



Berilgan nuqta (x_1,y_1) va burchak koeffitsienti m bo'lsin. To'g'ri chiziqdagi ixtiyoriy nuqta uchun

$$m=\frac{y-y_1}{x-x_1} \quad \Rightarrow \quad y-y_1=m(x-x_1)$$

(2) Burchak koeffitsientli berilgan ordinatali ko'rinishi

y o'qini $(0,n)$ nuqtada kesuvchi va m burchak koeffisientli to'g'ri chiziq uchun

$$m=\frac{y-n}{x-0} \quad \Rightarrow \quad y=mx+n$$

(3) Ikkita (x_1,y_1) va (x_2,y_2) nuqtadan o'tuvchi to'g'ri chiziq tenglamasini burchak koeffisiyentli tenglamadan topamiz

$$y-y_1=\frac{y_2-y_1}{x_2-x_1}(x-x_1) \quad \text{(bu yerda } x_2\neq x_1)$$

chunki burchak koeffitsienti $m=\frac{y_2-y_1}{x_2-x_1}$

Agar oxirgi tenglamada xususan

$x_2=x_1$ bo'lsa, $x=x_1$ (vertikal to'g'ri chiziq)

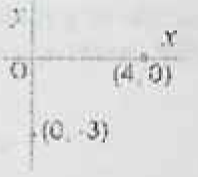
$y_2 = y_1$ bo'lsa $y = y_1$ (gorizontal to'g'ri chiziq)
kabi tenglamalar hosil bo'ladi.

2-misol. Ikkita $(4,0)$ va $(0,-3)$ nuqtadan o'tuvchi to'g'ri chiziq tenglamasini tuzing.

Yechimi: Yuqorida keltirilgan tenglamadan

foydalanamiz $y-0=\dfrac{-3-0}{0-4}(x-4)$

Shunda $3x-4y=12$ yoki $\dfrac{x}{4}+\dfrac{y}{(-3)}=1$.



Endi to'gri chiziqning kesmalardagi tenglamasini keltiramiz.

Agar to'gri chiziq X o'qidan a-kesma, Y o'qidan b-kesma ajratsa uning

tenglamasi $\dfrac{x}{a}+\dfrac{y}{b}=1$

bo'ladi.

3-misol. Uchta $A(-1,1), B(3,a), C(a+3,7)$ nuqtalar l to'gri chiziqda shu ketma-ketlikda joylashgan. a ning qiymatini va l to'gri chiziqning tenglamasini quydagicha toping:

1) Tenglamani $y=mx+n$ ko'rinishda qidiring va nuqtalar bir to'gri chiziqda joylashish shartini toping;
2) Shu shartni qanoatlantiruvchi a,m,n qiymatlarini toping;
3) Misolga to'g'ri keladigan qiymatlarni tanlang.

Yechimi: 1)To'g'ri chiziq A, B, C nuqtalardan o'tganligi uchun, ularni to'g'ri chiziq tenglamasiga qo'ymiz

$$1=-m+n$$
$$a=3m+n$$
$$7=m(a+3)+n$$

2) Ikkinchi va uchinchi tenglamalardan birinchini ayirib $\begin{matrix} a=4m+1 \\ 6=m(a+4) \end{matrix}$ tenglamalarni hosil hosil qilamiz. Ularni yechib $(m+2)(4m-3)=0$ ga kelamiz va uchala parametrlarni topamiz $(m,a,n)=(-2,-7,-1)$, $\left(\dfrac{3}{4},4,\dfrac{7}{4}\right)$

3) Nuqtalar ketma-ketligi sababli $a+3>3$ bo'ladi. Faqat ikkinchi javob to'g'ri keladi ,va $a=4$. $y=\dfrac{3}{4}x+\dfrac{7}{4}$ to'gri chiziqning kabi tenglamasiga ega bo'lamiz.

14B Parallel to‘g‘ri chiziqlar. Perpendikulyar to‘g‘ri chiziqlar. Simmetriya To‘g‘ri chiziqgacha bo‘lgan masofa

$y = m_1x + n_1$

$y = m_2x + n_2$

Berilgan ikki to‘g‘ri chiziqlar uchun **paralellik sharti** $m_1 = m_2$

Berilgan ikki to‘g‘ri chiziqlar uchun **perpendikulyarlik sharti** $m_1 \cdot m_2 = -1$

Isbot:

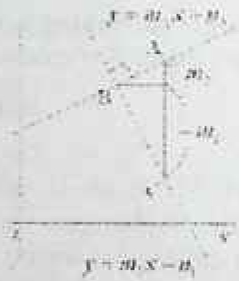
$\Delta ABC: \; AB^2 + BC^2 = AC^2 \qquad (i)$

$\Delta ABD: \; AB^2 = BD^2 + AD^2 = 1 + m_2^2 \qquad (ii)$

$\Delta BCD: \; BC^2 = 1 + (-m_1)^2 \qquad (iii)$

Bu uchta tenglikdan

$1 + m_2^2 + 1 + (-m_1)^2 = (m_2 - m_1)^2 \quad \therefore m_1 \cdot m_2 = -1$



Simmetriya

4-misol. $P(2,3)$ nuqtaga $y = 2x + 5$ to‘g‘ri chiziqqa nisbatan simmetrik nuqtaning koordinatalarini toping.

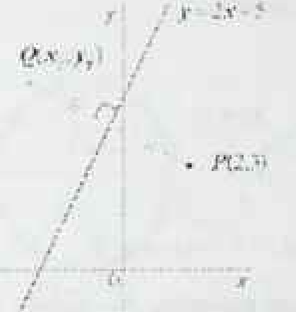
Yechimi: $Q(x_q, y_q)$bo‘lsin. PQ t'o‘g‘ri chiziq berilgan to‘g‘ri chiziqqa perpendikulyar, shuning uchun

$$2 \cdot \left(\frac{y_q - 3}{x_q - 2}\right) = -1 \quad (i).$$

PQ kesmaning o‘rtasi berilgan to‘g‘ri chiziqda yotaganligi uchun

$$\frac{3 + y_q}{2} = 2 \cdot \left(\frac{2 + x_q}{2}\right) + 5 \quad (ii)$$

(i) va (ii) dan $x_q = -\frac{14}{5}, y_q = \frac{27}{5}$

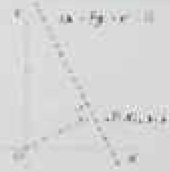


To'g'ri chiziqgacha bo'lgan masofa

5-misol. Koordinatalar boshidan $ax+by+c=0$ to'g'ri chiziqqacha bo'lgan masofani toping.

Yechimi: To'g'ri chiziq tenlamasini $y=-\frac{a}{b}x-\frac{c}{b}$ ko'rinishida yozib olamiz.

Shu to'g'ri chiziqdagi koordinatalar boshiga eng yaqin $A(x_1, y_1)$ nuqtani tanlaymiz.



Shunda $ax_1+by_1+c=0 \quad (i)$

OA kesma to'g'ri chiziqqa perpendikulyar bo'lgani uchun

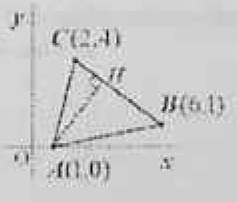
$$\left(-\frac{a}{b}\right)\cdot\left(\frac{y_1}{x_1}\right)=-1 \quad (ii)$$

(i) va (ii) tenglamalardan $x_1=\frac{-ac}{a^2+b^2}$, $y_1=\frac{-bc}{a^2+b^2}$ kelib chiqadi.

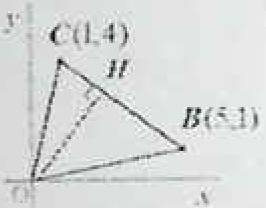
Masofa esa $d=\sqrt{x_1^2+y_1^2}=\frac{|c|}{\sqrt{a^2+b^2}}$ ga teng.

6-misol. Uchburchakning yuzasini quyidagi qadamlar bilan toping.

1) Uchburchakni bir birlik chapga siljiting va yangi koordinatalarni aniqlang;
2) 5-misolni qo'llab AH kesmaning uzunligini toping;
3) BC tomonning uzunligini toping;
4) Uchburchak yuzasini toping.



Yechimi: Nuqtalarning yangi koordinatalari $A(0,0), B(5,1), C(1,4)$.



1) BC ning tenglamasi $y-4=\frac{1-4}{5-1}(x-1)$ $\therefore 3x+4y-19=0$

2) 5-misolni qo'llab AH ning uzunligini topamiz, $d=\frac{|-19|}{\sqrt{3^2+4^2}}=\frac{19}{5}$.

3) Ikki nuqta orasidagi masofani topsak $BC=\sqrt{(5-1)^2+(1-4)^2}=5$.

4) Uchburchak yuzasi esa $S=\frac{1}{2}\cdot\frac{19}{5}\cdot 5=\frac{19}{2}$ bo'ladi.

7-misol. (4,0) va (0,-3) nuqtalardan o'tuvchi to'g'ri chiziq tenglamasini tuzing.

Yechimi: Yuqorida keltirilgan tenglamadan $y-0=\frac{-3-0}{0-4}(x-4)$

Shunda $3x-4y=12$ yoki

$$\frac{x}{4}+\frac{y}{-3}=1.$$

1.15. Grafik va tenglamalar
15A Aylana. Aylana va to'g'ri chiziq

Aylana ta'rifidan $\sqrt{(x-a)^2+(y-b)^2}=r$ $(r>0)$ yoki

$$(x-a)^2+(y-b)^2=r^2$$

hosil bo'ladi, unga aylananing kanonik tenglamasi deb ataladi.

Soddalashtirishlardan so'ng

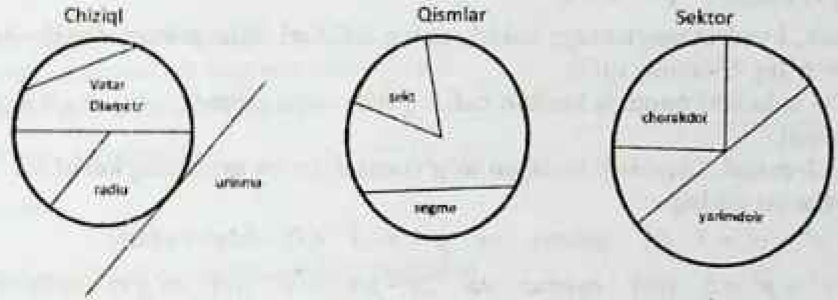
$$x^2-2ax+y^2-2by+a^2+b^2=r^2$$

$$x^2+y^2+lx+my+n=0 \quad \left(l^2+m^2>4n\right)$$

$$\left(x+\frac{l}{2}\right)^2+\left(y+\frac{m}{2}\right)^2=\frac{l^2+m^2-4n}{2}$$

ga umumiy tenglama deyiladi.

Aylanaga doir tushunchalar



1-misol. Quydagi misollarni yeching.

(1) $A(-1,7), B(2,-2), C(6,0)$ nuqtalardan o'tuvchi aylana tenglamasini tuzing.

(2) Aylana markazini toping.

Yechimi: $x^2+y^2+lx+my+n=0$

(1) Aylana A, B, C nuqtalaridan o'tgani uchun ularning koordinatalari aylana tenglamasini qanoatlantiradi.

$$-l+7m+n=-50$$
$$2l-2m+n=-8$$
$$6l+n=-36$$

U holda $l=-4, m=-6, n=-12$ va $x^2+y^2-4x-6y-12=0$

(2) Bu tenglamani kanonik ko'rinishga keltirsak

$$(x^2-4x+4)+(y^2-6y+9)-4-9-12=0$$
$$\therefore (x-2)^2+(y-3)^2=5^2$$

Demak aylana markazi $(2,3)$ nuqtada bo'ladi.

Aylana va to'g'ri chiziq joylashuvi

Ularning o'zaro joylashuvi uchun uchta holat mavjud:

(a) kesib o'tadi (b) urinadi (c) kesib o'tmaydi

Joylashuv masalasini aniqlash uchun ularning tenglamalarini birgalikda yechamiz:

$$\left.\begin{array}{l} Aylana: \quad x^2+y^2+lx+my+n=0 \\ To'g'ri\ chiziq: \quad y=px+q \end{array}\right\} \Rightarrow ax^2+bx+c=0$$

Demak, kvadrat tenglamaga keldik, uning ildizlari diskriminant $D=b^2-4ac$ ga bo'g'liq. Shunday qilib:

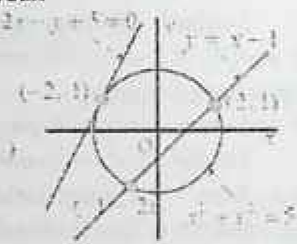
(a) $D>0$ da ikki nuqtada kesib o'tadi, (b) $D=0$ da urinadi, (c) $D<0$ kesib o'tmaydi

2-misol. Quyidagi berilgan to'g'ri chiziqlar va aylananig kesishish nuqtalarini toping

(1) $x^2+y^2=5$ (i) *aylana va* $y=x-1$ (ii) *to'g'ri chiziq*

(2) $x^2+y^2=5$ (iii) *aylana va* $2x-y+5=0$ (iv) *to'g'ri chiziq*

Yechimi: (1) (ii) tenlamani (i) tenglamaga qo'ysak



$x^2-x-2=(x-2)(x+1)=0$

$\therefore x=-1,2 \quad \therefore y=-2,1$

Aylana va to'g'ri chiziqning umumiy nuqtalari $(-1,-2)$ $(1,2)$.

(2) (iv) tenlamani (iii) tenglamaga qo'ysak

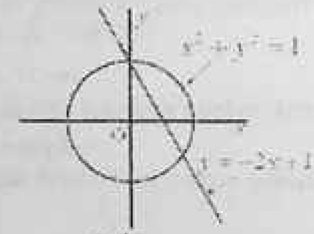
$x^2+4x+4=(x+2)^2=0$

$\therefore x=-2 \quad \therefore y=1$

Demak, to'g'ri chiziq aylanaga urinadi va urinish nuqtasi $(-2,1)$.

3-misol. $y=-2x+1$ to'g'ri chiziq $x^2+y^2=1$ aylanani kesib o'tadi. Hosil bo'lgan vatar uzunligini toping.

Yechimi: Ikki tenglamadan y o'zgaruvchini yo'qotsak

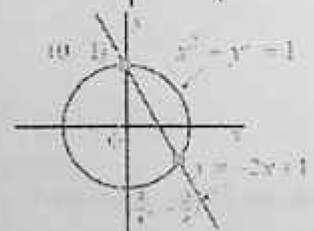


$$x^2+(-2x+1)^2=1$$

$$\therefore 5x^2-4x=5x\left(x-\frac{4}{5}\right)=0$$

$$\therefore x=0,\frac{4}{5}$$

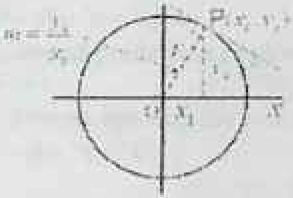
Shunda kesishish nuqtalari $(0,1)$ $\left(\frac{4}{5},-\frac{3}{5}\right)$ bo'ladi.



Vatar uzunligi esa $\sqrt{\left(0-\frac{4}{5}\right)^2+\left(1+\frac{3}{5}\right)^2}=\frac{4\sqrt{5}}{5}$ ga teng.

15B Aylanaga urinma.Berilgan nuqtadan o‘tuvchi to‘g‘ri chiziq. Berilgan nuqtadan o‘tuvchi aylana. Tengsizlik bilan berilgan soha

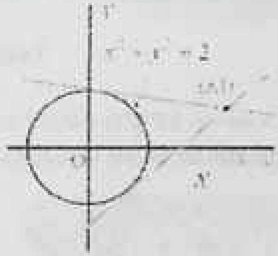
Urinma quyidagi xossalarga ega



- Urinma aylanani bir nuqtada kesib o‘tadi
- Urinma aylana radiusiga perpendikulyar
- Uinma tenglamasi $y - y_1 = -\frac{x_1}{y_1}(x - x_1)$,

  shunda

$$yy_1 + xx_1 = r^2$$



4- misol. $x^2 + y^2 = 2$ aylanaga $(3,1)$ nuqtadan o‘tuvchi urinma tenglamasini tuzing.

Yechimi: $yy_1 + xx_1 = 2$ urinma $(3,1)$ nuqtadan o‘tgani uchun

$$y_1 + 3x_1 = 2 \quad (i).$$

Aylana ham shu nuqtadan o‘tgani uchun

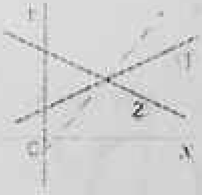
$$x_1^2 + y_1^2 = 2 \quad (ii).$$

(i) va (ii) dan $5x_1^2 - 6x_1 + 1 = (5x_1 - 1)(x_1 - 1) = 0$

$x_1 = \frac{1}{5}, 1$ shunda urinish nuqtalari $(x_1, y_1) = \left(\frac{1}{5}, \frac{7}{5}\right), (1, -1)$.

Urinmalar tenglamalari $\frac{1}{5}x + \frac{7}{5}y = 2$, $x - y = 2$ kabi bo‘ladi.

Berilgan nuqtadan o‘tuvchi aylana va to‘g‘ri chiziq



Ikki

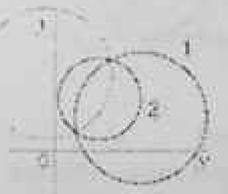
$$a_1x + b_1y + c_1 = 0 \quad (1)$$

$$a_2x + b_2y + c_2 = 0 \quad (2)$$

to‘g‘ri chiziq berilgan.

Bir nuqtadan o‘tuvchi to‘g‘ri chiziqlar dastasi tenglamasi

$$k(a_1x + b_1y + c_1) + a_2x + b_2y + c_2 = 0$$



Ikki

$$f(x, y) = x^2 + y^2 + l_1x + m_1y + n_1 = 0 \quad (1)$$

$$g(x, y) = x^2 + y^2 + l_2x + m_2y + n_2 = 0 \quad (2)$$

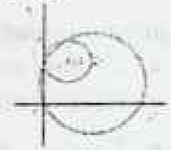
aylanalar berilgan.

Ikki nuqtadan o‘tuvchi aylanalar tenglamasi

$$k \cdot f(x, y) + g(x, y) = 0$$

5-misol. $(1,2)$ nuqtadan va $x^2+y^2-2x-4y+4=0$, $x^2+y^2-4x-2y=0$ aylanalar kesishgan nuqtalardan o'tuvchi aylana tenglamasini tuzing.

Yechimi: Ikki nuqtadan o'tuvchi aylana tenglamasi



$$k\cdot\left(x^2+y^2-2x-4y+4\right)+x^2+y^2-4x-2y=0$$

Berilgan nuqtaning koordinatalarini tenglamaga qo'ysak

$$-k-3=0 \quad \therefore k=-3$$

va k ni tenglamaga qoysak

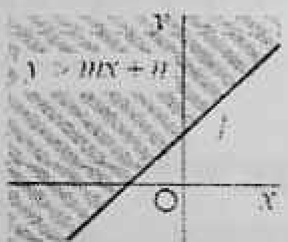
$$-3\cdot\left(x^2+y^2-2x-4y+4\right)+x^2+y^2-4x-2y=0$$

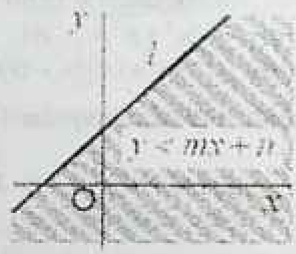
$$x^2+y^2-x-5y+6=0$$

kabi aylana tenglamasi bhosil bo'ladi.

Tengsizlik bilan berilgan soha

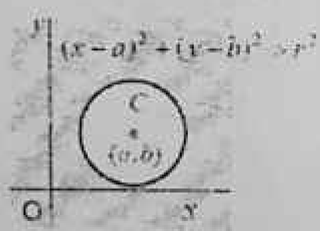
l: $y=mx+n$ to'g'ri chiziq tekislikni ikkita yarim tekislikka bo'ladi. Ular tengsizliklar bilan ifodalanadi.

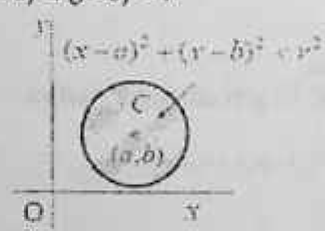




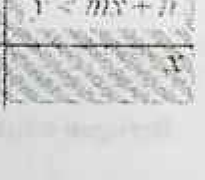
Aylana

$$(x-a)^2+(y-b)^2=r^2$$





6-misol. $x^2+y^2-4x-6y+9<0$ tengsizlik bilan berilgan sohani chizing.



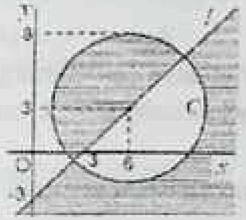
Yechimi: Ifodani kanonik ko'rinishga keltirib olamiz

$$(x-2)^2+(y-3)^2=2^2$$

Soha radiusi 2 ga teng bo'lgan, markazi $(2,3)$ nuqtada yotuvchi aylana bilan chegaralangan ekan. Bu bo‘yalgan soha chizmada ko‘rsatilgan doiradan iborat.

7- misol. $(y-x+3)(x^2+y^2-12x-6y+20)<0$ tengsizlik bilan berilgan sohani chizing.

Yechimi:



$AB<0 \iff A>0, B<0$ yoki $A<0, B>0$ Demak,

(1) $y>x-3, \quad (x-6)^2+(y-3)^2<5^2$

(2) $y<x-3, \quad (x-6)^2+(y-3)^2>5^2$

$y=x-3$ chegarani l deb belgilaymiz

$(x-6)^2+(y-3)^2=5^2$ chegarani C deb belgilaymiz

Shunda soha

(1) l dan yuqori va C ning ichi,

(2) l dan past va C ning tashqarisidan iborat.

2-KURS. HISOBLASHLAR

2.1. Funksiya limiti va hosilasi

1A. Limit ta'rifi

Agar $x \to a$ (x o'zgaruvchi a songa yaqinlashganda) bo'lganda, $f(x)$ funksiya L qiymatga yaqinlashsa, u holda biz " $f(x)$ funksiya $x \to a$ yaqinlashganda L limitga ega " deymiz va

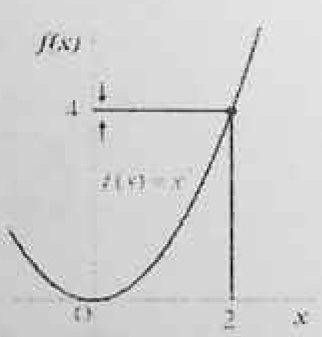
$$\lim_{x \to a} f(x) = L$$

ko'rinishda yozamiz.

Biz yuqoridagi limitni, shunday o'qishimiz mumkin: x ozgaruvchi a songa yaqinlashganda $f(x)$ funksiya L qiymatga yaqinlashadi.

Misol. $f(x) = x^2$ funksiyaning $x = 2$ nuqtadaga yaqin qiymatlarini hisoblashlarni tekshirib ko'ramiz. x ning $1 < x < 2$ va $2 < x < 3$ oraliqdagi qiymatlarida funksiyaning qiymatlari quyidagi jadvalda keltirilgan.

x	$f(x)$	x	$f(x)$
1.0	1.000000	3.0	9.000000
1.5	2.250000	2.5	6.250000
1.8	3.240000	2.2	4.840000
1.9	3.610000	2.1	4.410000
1.99	3.960000	2.01	4.040100
1.999	3.996001	2.001	4.004001



Shunday qilib, biz ayta olamizki, $f(x) = x^2$ funksiya x argument 2 ga yaqinlashganda funksiya qiymati 4 ga teng bo'ladi.

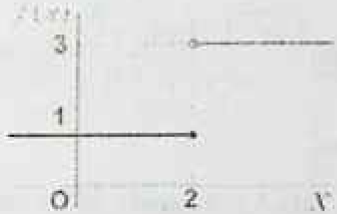
Limitlar haqida ba'zi tushunchalar (izohlar)

Birinchidan, a nuqtaga yaqinlashishning ikkita yo'nalishi mavjud. "$a-$" belgi, x chap tomondan a nuqtaga yaqinlashishini va "$a+$" belgi x o'ng tomondan a nuqtaga yaqinlashishini anglatadi.

1-izoh. *Funksiya limiti mavjud bo'lishi uchun funksiyanin chap va o'ng tomonlama limitlari teng bo'lishrai zarur, ya'ni*

$$\lim_{x\to a-} f(x) = L \text{ va } \lim_{x\to a+} f(x) = L.$$

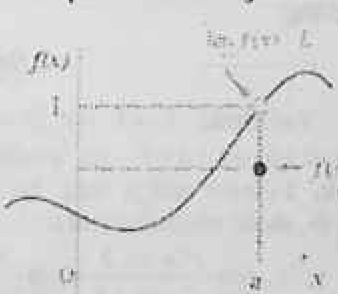
Misol. Grafikda ko'rsatilgan funksiyaning $x=2$ nuqtada limiti mavjud emas.



Ushbu misolda funksiyaning chap limiti 1 ga va o'ng limiti 3 ga teng. Chap va o'ng limitlar teng bo'lmaganligi uchun $x=2$nuqtada limit mavjud emas.

1-izoh. *Limitning qiymati har doim ham funksiyaning shu nuqtadagi qiymati $f(a)$ ga teng bo'lmaydi.*

Quyidagi rasmda keltirilgan-misolda $f(x)$ funksiya limiti mavjud va u L ga teng. Biroq, bu qiymat $f(a)$dan farq qiladi.

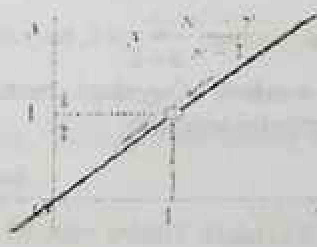


1-misol. Quyidagi funksiyaning limitini hisoblang.

$$\lim_{x\to 1}\frac{x^2-x}{x-1}$$

Yechimi:

$$\lim_{x\to 1}\frac{x^2-x}{x-1} = \lim_{x\to 1}\frac{x(x-1)}{x-1} = \lim_{x\to 1} x = 1.$$



Ushbu misolda keltirilganidek, funksiya $x=a$ nuqtada qiymatiga ega bo'lmasa ham, uning limiti mavjud bo'lishi mumkinligi ko'rsatilgan.

Aniqmasliklar. $0^0, \frac{0}{0}, 1^\infty, \infty-\infty, \frac{\infty}{\infty}, 0\times\infty, \infty^0$ va boshqa shakllarda limitning qiymatini aniqlab bo'lmaganligi uchun, ularga **aniqmasliklar** deyiladi. Bu ko'rinishdagi limitlarni hisoblashga esa **aniqmasliklarni ochish** deyiladi.

2-misol. Quyidagi funksiyaning limitini hisoblang:

$$\lim_{x\to 0}\frac{\sqrt{x+1}-1}{x}.$$

Yechimi: Irratsionallik qatnashgan bu aniqmasliklarni ochish uchun, suratning qo'shmasiga quyidagi tarzda ko'paytirib, limitni topishimiz mumkin.

$$\lim_{x\to 0}\frac{\sqrt{x+1}-1}{x}=\lim_{x\to 0}\frac{\left(\sqrt{x+1}-1\right)\left(\sqrt{x+1}+1\right)}{x\left(\sqrt{x+1}+1\right)}=\lim_{x\to 0}\frac{x+1-1}{x\left(\sqrt{x+1}+1\right)}=$$

$$=\lim_{x\to 0}\frac{1}{\sqrt{x+1}+1}=\frac{1}{2}$$

Yuqoridagi ikkita misol yordamida biz noaniq shakl 0/0 ning limit qiymatini topishning ikkita usulini o'rganib chiqdik.

3-misol. Quyidagi ifodani qanoatlantiruvchi a va b ning qiymatlarini aniqlang:

$$\lim_{x\to 1}\frac{x^2+x-2}{x^2+ax+b}=1.$$

Yechimi: $x\to 1$ da ifodaning surati $x^2+x-2\to 0$ va maxraji esa $x^2+ax+b\to 1+a+b$ ga yaqinlashadi. Shu sababli, limit mavjud bo'lishi uchun $1+a+b$ nolga teng bo'lishi kerak. Ya'ni, $b=-a-1$. Buni ifodaga qo'yib, shakl almashtiramiz:

$$\lim_{x\to 1}\frac{x^2+x-2}{x^2+ax+b}=\lim_{x\to 1}\frac{(x-1)(x+2)}{(x-1)(x+a+1)}=\lim_{x\to 1}\frac{x+2}{x+a+1}=\frac{3}{a+2}.$$

Shuning uchun, $\frac{3}{a+2}=1$, bu yerdan $a=1$ va $b=-2$.

4-misol. Quyidagi munosabatni qanoatlantiruvchi a va b ning qiymatlarini aniqlang:

$$\lim_{x\to 1}\frac{ax^2+bx+1}{x-1}=3.$$

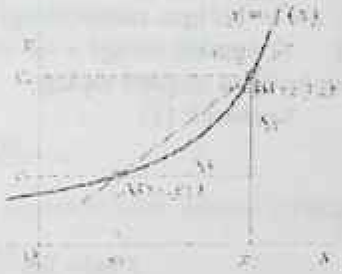
Yechimi: Ushbu misolni quyidagicha yechishimiz mumkin. $x\to 1$ da $x-1$ maxraj 0 ga yaqinlashadi va ax^2+bx+1 surat $1+a+b$ ga yaqinlashadi. Limit mavjud bo'lishi uchun, a + b + 1 nolga teng bo'lishi kerak, ya'ni, $b=-a-1$. Buni ifodaga qo'yib, shakl almashtirishdan quyidagiga ega bo'lamiz:

$$\lim_{x\to1}\frac{ax^2+bx+1}{x-1}=\lim_{x\to1}\frac{(x-1)(ax-1)}{x-1}=\lim_{x\to1}(ax-1)=a-1.$$

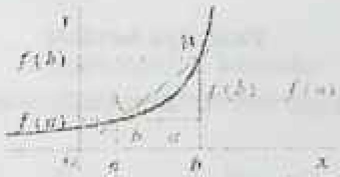
$a-1=3$, bu yerdan $a=4$ va $b=-5$.

1B. Funksiyaning hosilasi

Ushbu grafikda, x erkli o'zgaruvchi, x_1 dan x_2 gacha o'zgarganda, y o'zgaruvchi, y_1 dan y_2 ga o'zgaradi. Bu yerda x_2-x_1 ayirmani Δx bilan, y_2-y_1 o'zgarishni esa Δy bilan belgilaymiz.

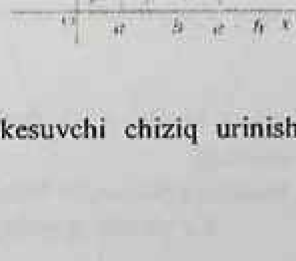


Δy ning Δxga nisbati o'rtacha o'zgarish tezligidan iborat. Ushbu grafikda dy/dx qiyalik o'rtacha o'zgarish tezligini aks ettiradi.



U $\dfrac{f(b)-f(a)}{b-a}$ ifoda bilan ham ifodalanishi mumkin.

Bu yerda, biz A va B nuqtada egri chiziqni kesib o'tuvchi to'g'ri chiziqni ko'rib chiqamiz. Bu chiziqning qiyaligi yani o'zgarishi quyidagi $\dfrac{f(b)-f(a)}{b-a}$ nisbat bilan aniqlanadi.



Hosila

Agar B nuqta A nuqtaga yaqinlashsa, bu kesuvchi chiziq urinish chizig'iga yaqinlashadi va uning qiyaligi

$$\lim_{b\to a}\frac{f(b)-f(a)}{b-a}=f'(a)$$

deb yoziladi.

Ushbu limit qiymati $f(x)$ funksiyaning a nuqtada **hosilasi** deb nomlanadi va $f'(a)$ bilan belgilanadi.

Agar $b = a + h$ deb olsak, hosila

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = f'(a)$$

kabi yoziladi.

5-misol. $f(x) = x^2$ funksiya haqida quyidagi savollarga javob bering.
(1) $x = 1$ va $x = 2$ orasidagi o'rtacha o'zgarish tezligini toping.
(2) $x = a$ bo'lgan momentdagi o'zgarish tezligini toping.
(3) O'zgarish tezligi $x = 1$ va $x = 2$ o'rtasidagi o'rtacha o'zgarish tezligiga teng bo'lgan nuqtani toping.

Yechimi: (1)

$$\frac{f(2) - f(1)}{2 - 1} = 4 - 1 = 3.$$

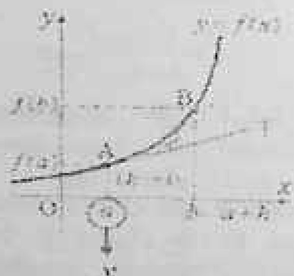
(2)

$$f'(a) = \lim_{h \to 0} \frac{(a+h)^2 - a^2}{h} = \lim_{h \to 0} (2a + h) = 2a.$$

(3) (1) va (2) natijalaridan foydalanib, biz $2a = 3$ ega bo'lamiz, bundan $a = 3/2$ bo'ladi.

Funksiya hosilasi

Chiziqning qiyaligi, agar "a" ning vaziyati o'zgarsa o'zgaradi.



Agar a qiymatni o'zgaruvchan deb uni x bilan almashtirsak, u holda

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

$f'(x)$ hosila $f(x)$ funksiyaning hosilasi yoki $f(x)$ning hosila funksiyasi deyiladi.
Funksiya hosilasini belgilanishing turli ko'rinishlari mavjud.

Ko'pincha quyidagi belgilashlar ishlatiladi:

$$\boxed{f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx} f(x) = Df(x) = D_x f(x)}$$

Ta'rif bo'yicha dy/dx, "dy ning dx ga nisbati", "dy dan dx" yoki "dy dx" sifatida o'qiladi.

6-misol. Ushbu funksiyalar hosilasini toping:

(1) $f(x)=x$ (2) $f(x)=x^2$ (3) $f(x)=x^3$.

Yechimi: Bu yerda biz ta'rifdan foydalanamiz.

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}.$$

(1) $f'(x)=\lim\limits_{h\to 0}\dfrac{x+h-x}{h}=\lim\limits_{h\to 0}(1)=1$.

(2) $f'(x)=\lim\limits_{h\to 0}\dfrac{(x+h)^2-x^2}{h}=\lim\limits_{h\to 0}(2x+h)=2x$.

(3) $f'(x)=\lim\limits_{h\to 0}\dfrac{(x+h)^3-x^3}{h}=\lim\limits_{h\to 0}(3x^2+3xh+h^2)=3x^2$.

Ushbu turdagi funksiyalarni hosilasini topish tez-tez uchrab turganligi sababli, ushbu x^n funksiyaning hosilasini

$$\frac{d}{dx}(x^n)=nx^{n-1}$$

formulasi sifatida qabul qilamiz.

Yuqori tartibli hosilalar

$f'(x)$ ham x ning funksiyasi bo'lgani uchun, u ham o'z hosilasiga ega bo'lib,

$$\frac{d^2y}{dx^2}=f''(x)=f^{(2)}(x),$$

bilan belgilanadi. Bu hosila funksiyaning ikkinchi tartibli hosilasi deyiladi. Biz ushbu operatsiyani davom ettirishimiz mumkin va

$$\frac{d^3y}{dx^3}=f'''(x)=f^{(3)}(x)$$

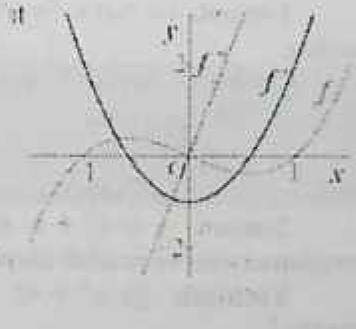
ega bo'lamiz. Umuman olganda, funksiya hosilasini topish jarayoni **differensiallash** deb ataladi.

7-misol. Agar $f(x)=x^3-x$ bo'lsa, $f''(x)$ toping va izohlang.

Yechimi:

$$\frac{d^2y}{dx^2}=f''(x)=f^{(2)}(x)$$

formulasidan foydalanib, $f'(x)=3x^2-1$ va $f''(x)=6x$ ni topamiz.



Ushbu funksiya hosilalari grafigi quyidagi rasmda tasvirlangan.

Ushbu grafiklarni ko'rib turganingizdek,

$f''(x)$ - $y=f'(x)$ egri chiziqning qiyaligi,

$f''(x)$ - $y=f'(x)$ ning o'zgarish tezligi.

8-misol. $f(x)=x^3+ax^2+bx+c$ funksiya $f(1)=3$, $f(0)=1$ va $f'(-1)=16$ shartlarini qanoatlantirsa a,b,c – o'zgarmaslarni toping?

Yechimi: Funksiyaning hosilasi $f'(x)=3x^2+2ax+b$ teng. Berilgan shartlardan

$$f(1)=1+a+b+c=3,$$
$$f(0)=c=1,$$
$$f'(-1)=3-2a+b=16$$

aniqlaymiz. Ushbu tenglamalardan $a=-4$, $b=5$, $c=1$ ekanligini topamiz.

Ushbu funksiyaning grafigi qizil egri chiziq bilan rasmda ko'rsatilgan.

2.2. Hosila va Grafiklar
2A Funksiya grafigiga urinma chiziqlar

Urinma tenglamasi

To'g'ri chiziq m qiyalik bilan $P(a,b)$ nuqtadan o'tsa,

$$m=\frac{y-b}{x-a}.$$

Shuning uchun $y-b=m(x-a)$.

$y=f(x)$ funksiyaning $(a,f(a))$ nuqtadan o'tuvchi urinma tenglamasi $y-f(a)=f'(a)(x-a)$ kabi bo'ladi.

1-misol. $y=f(x)=x^2+1$ funksiyaning $x=1$ nuqtada urinma tenglamasini toping.

Yechimi: $x=1$ da y ning qiymati $f(1)=2$ ga teng. Shuningdek $f'(x)=2x$ hosilasi $f'(1)=2$ ga teng.

Shuning uchun, urinma tenglamasi $y-2=2(x-1)$ yoki $y=2x$ bo'ladi.

2-misol. $y=x^2+4$ funksiyaning $(1,1)$ nuqtadan o'tuvchi urinma tenglamalarini va urinish nuqtalarini toping.

Yechimi: (a,a^2+4) urinish nuqtasi bo'lsin. Berilgan funksiya hosilasi

$$f'(x)=2x.$$

Shuning uchun, urinma tenglamasi

$$y - (a^2 + 4) = 2a(x - a)$$

kabi bo'lib, urinma to'g'ri chizig'i (1,1) nuqtadan o'tgani uchun

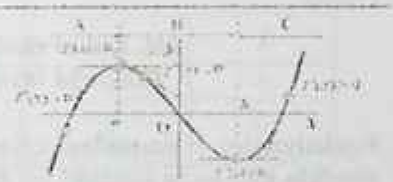
$$1 - (a^2 + 4) = 2a(1 - a),$$

$$a^2 - 2a - 3 = 0,$$

$$a = -1, 3.$$

$a = -1$ da urinish nuqta (-1,5) bo'ladi, urinma tenglamasi esa $y = -2x + 3$.
$a = 3$ da urinish nuqtasi (3,13), urinma tenglamasi $y = 6x - 5$ bo'ladi.

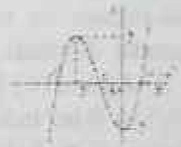
Funksiya grafigining uning hosilalariga bog'liq o'zgarishi



A va C ochiq kesmada $f'(x) > 0$ va $f(x)$ o'suvchi.
B ochiq kesmada $f'(x) < 0$ va $f(x)$ kamayuvchi.

3-misol. $y = x^3 + 3x^2 - 2$ funksiyaning o'zgarishini o'rganing va grafigini tasvirlang.

Yechimi: Funksiya hosilasi $y' = 3x^2 + 6x$.



Funksiya hosilasining ishorani saqlash oraliqlarini topamiz:

$y' = 3x(x + 2) = 0 \;\; x = -2{,}0,$

$y' > 0$ agar $x < -2, x > 0,$

$y' < 0$ agar $-2 < x < 0.$

Grafikning y o'qini kesish nuqtalari: (0,2) va (0,-2).
x o'qini kesish nuqtalari:

$y = x^3 + 3x^2 - 2 = (x + 1)(x^2 + 2x - 2) = 0$

Demak $(-1{,}0)$, $(-1 + \sqrt{3}, 0)$, $(-1 - \sqrt{3}, 0)$. Shunday qilib quyidagi jadvalni hosil qilamiz.

x	......	-2	......	0	......
y'	+	0	−	0	+
y	↗	2	↘	−2	↗

3-misol. $y = x^3 - 2x^2 - 3x$ funksiya haqida quyidagi savollarga javob bering.

1) Funksiyaning (-1,0) nuqtada urinma tenglamasini tuzing.
2) Funksiya va urinmaning kesishish nuqtasini toping.

Yechimi: 1) Funksiya hosilasi $y' = 3x^2 - 4x - 3$.
Berilgan nuqtadagi qiymati $f'(-1) = 4$.

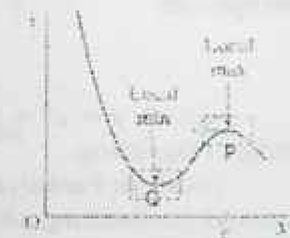
Formulaga ko'ra urinma tenglamasi $y - 0 = 4(x + 1)$
yoki $y = 4x + 4$.

2) Urinma va egri chiziqning kesishgan nuqtasi $x^3 - 2x^2 - 3x = 4x + 4$

Bu yerdan, $(x + 1)^2(x - 4) = 0$. Shuning uchun, $x = 4$ va kesishish nuqtasi (4;20).

2B. Lokal ekstremum. Global ekstremum. Ikkinchi tartibli hosilaning qo'llanilishi

Funksiyaning c nuqtadagi qiymati $x = c$ nuqta atrofida eng katta (kichik) bo'lsa, ya'ni bu nuqtada maksimal (minimal) qiymatga ega bo'lsa, funksiya $f(c)$ lokal maksimumga (lokal minimumga) ga ega deyiladi.



Maksimum va minimum nuqtalarga ekstrmum nuqtalar deyiladi.

Agar $f(c)$ lokal ekstremum bo'lsa u holda $f'(c) = 0$ bo'ladi.

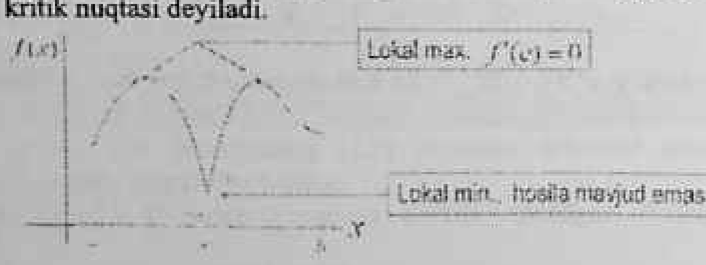
Absolyut maksimum (yoki global maksimum)- bu butun $[a, b]$ sohadagi maksimal qiymatdir.

Absolyut minimum (yoki global minimum) - bu butun $[a, b]$ sohadagi minimal qiymatdir.

Izohlar:

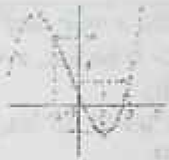
(1) Hosila nolga teng $f'(c) = 0$ bo'lishi, funksiya albatta, nuqtada lokal maximum yoki minimumga ega ekanligini anglatmaydi;

(2) Agar $f'(c) = 0$ yoki $f'(c)$ mavjud bo'lmasa, c son $f(x)$ ning kritik nuqtasi deyiladi.



4-misol. $-1 \leq x \leq 2$ sohada $f(x) = 2x^3 + 3x^2 - 12x + 1$ ning maximum va minimum qiymatini toping.

Yechimi: (1) Extremumlarini topamiz:



$6x^2 + 6x - 12 = 6(x+2)(x-1)$, $x = -2$ va 1.

Ulardan faqat $x = 1$ berilgan oraliqda yotadi, $f(1) = -6$.

(2) Oraliq chetidagi qiymatlarini topamiz:

$$f(-1) = 14, f(2) = 5.$$

(3) y -to'g'ri chiziqni kesish nuqtasini topamiz:

$y = f(0) = 1$

(4)Jadval tuzamiz:

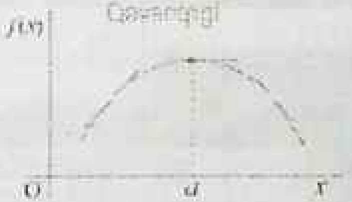
x	-1	....	1	....	2
$f'(x)$	$-$	$-$	0	$+$	$+$
$f(x)$	14	↘	Local Min.-6	↗	5

(5) Grafikni chizamiz.

(6) Maksimum va minimum qiymatlarni topamiz:

Maksimum qiymati 14 (x=-1 da erishadi). Minimum qiymati -1 (x=1 da erishadi).

Grafikning qavariq va botiqligi



$f'(x) > 0 \to f'(a) = 0 \to f'(x) < 0$

$f''(a) < 0$

$x = a$ da qavariq

$f'(x) < 0 \to f'(b) = 0 \to f'(x) > 0$

$f''(b) > 0$

$x = b$ da botiq

Aytaylik, $x = c$ da $f'(c) = 0$ bo'lsin. U holda: Agar $f''(c) > 0$ bo'lsa, $x = c$ da $f(x)$ local minimumga erishadi;
Agar $f''(c)<0$ bo'lsa, $x = c$ da $f(x)$ local maksimumga erishadi;
Agar $f''(x) = 0$ bo'lsa, u holda ekstremum mavjudligi aniqmas. Masalan, $x=0$ nuqtada

$y = x^2$	L.Min.
$y = -x^2$	L.Maks
$y = x^3$	Noaniq

Egilish nuqtasi

Yuqoridagilardan xulosa qilishimiz mumkinki, shunday nuqta mavjudki, bu nuqtada funksiya o'zining qavariq yoki botiqligini o'zgartiradi. Agar funksiya qavariqlakdan botiqlikka o'tsa yoki aksincha bo'lsa, unga **egilish nuqtasi** deb ataladi.

5-misol. Shaklda ko'rsatilgandek qalin kvadrat qog'ozning to'rtta burchagini kesib kvadrat chashka tayyorlandi. Ushbu idishning hajmi qachon maksimal bo'ladi?

Yechimi: Kvadratlarning yon uzunligi burchakdan x uzunlikda kesilgan.

U holda,

- Qiymat $V=(12-2x)^2x=4(x^3-12x^2+36x)$
- Shartga ko'ra $0<x<6$
- Hosilasi $V'=12(x-2)(x-6)$

x	0	...	2	...	6
V'		+	0	−	
V		↗	128	↘	

$$x = 2, x = 6$$

Bu funksiyaning maksimumini topsak

$V(2) = 128, V(6) = 0$. Demak maksimum qiymat 128.

2.3. Differensiallash formulalari

3A. Daraja pasaytirish va chiziqlilik xossasi. Ko'paytma va bo'linmadan hosila olish qoidasi

Daraja pasaytirish xossasi. Darajali funksiya uchun hosila olishning quyidagi xossasi o'rinli:

$$\frac{d}{dx}x^n = nx^{n-1}.$$

Ushbu qoida ixtiyoriy haqiqiy sonlar uchun o'rinli bo'ladi.

1-misol. Quyidagi funksiyalar hosilasini toping.

1. $f(x)=\frac{1}{x^2}$ 2. $f(x)=\sqrt{x^3}$

Yechimi:

1) $f'(x)=\frac{d}{dt}\left(x^{-2}\right)=-2x^{-2-1}=-2x^{-3}=-\frac{2}{x^3}$,

2) $f'(x)=\frac{d}{dt}\left(\sqrt{x^3}\right)=\frac{d}{dt}\left(x^{\frac{3}{2}}\right)=\frac{3}{2}x^{\frac{1}{2}}=\frac{3}{2}\sqrt{x}$.

Chiziqlilik xossasi

$f(x)$ va $g(x)$ differensiallanuvchi funksiyalar bo'lsin. U holda hosila olish uchun quyidagi formulalar o'rinli:

1. Konstantani hosila belgisidan chiqarish mumkin $(cf)'=cf'$
2. Yig'indining hosilasi: $(f+g)'=f'+g'$
3. Ayirmaning hosilasi: $(f-g)'=f'-g'$

Yig'indi hosilasining isboti. Ta'rifga ko'ra

$$\{f(x)+g(x)\}'=\lim_{h\to 0}\frac{(f(x+h)+g(x+h))-(f(x)+g(x))}{h}$$

Limit ostidagi ifodani shakl almashtirsak,

$$\{f(x)+g(x)\}'=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}+\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}=f'(x)+g'(x)$$

ga ega bo'lamiz.

Ko'paymadan hosila qoidasi

$f(x)$ va $g(x)$ funksiyalar differensiallanuvchi funksiyalar bo'lsin, u holda

$$(fg)'=f'g+fg'$$

Isbot.

$$\{f(x)g(x)\}'=\lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x)}{h}=$$

$$=\lim_{h\to 0}\left\{\frac{f(x+h)-f(x)}{h}\cdot g(x+h)+f(x)\cdot\frac{g(x+h)-g(x)}{h}\right\}=$$

$$=f'(x)g(x)+f(x)g'(x)$$

2-misol. Berilgan funksiyaning hosilasini toping $h(x)=3x^2(5x+1)$.

Yechimi: Ko'paytmadan hosila olish qoidasiga ko'ra,

$h'(x)=6x(5x+1)+3x^2(5)=45x^2+6x$

Bo'linmadan hosila olish qoidasi

$$\left(\frac{f}{g}\right)'=\frac{f'g-fg'}{g^2} \quad \text{xususiy holda} \quad \left(\frac{1}{g}\right)'=-\frac{g'}{g^2}$$

Ikkinchi formulani quyidagi tartibda isbot qilamiz.

$$\left\{\frac{1}{g(x)}\right\}'=\lim_{h\to 0}\frac{1}{h}\cdot\left\{\frac{1}{g(x+h)}-\frac{1}{g(x)}\right\}=$$
$$=\lim_{h\to 0}\left\{-\frac{g(x+h)-g(x)}{h}\cdot\frac{1}{g(x+h)g(x)}\right\}=-\frac{g'(x)}{g^2(x)}$$

Yuqoridagilardan foydalanib,

$$\left\{\frac{f(x)}{g(x)}\right\}'=\left\{f(x)\cdot\frac{1}{g(x)}\right\}'=\frac{f'(x)}{g(x)}+f(x)\cdot\frac{-g'(x)}{\{g(x)\}^2}=$$
$$=\frac{f'(x)g(x)-f(x)g'(x)}{\{g(x)\}^2}$$

3-misol. Berilgan funksiya hosilasini toping $f(x)=\dfrac{x}{x+1}$.

Yechimi: Bo'linmaning hosilasi formulasiga ko'ra

$$f'(x)=\frac{1(x+1)-x(1)}{(x+1)^2}=\frac{1}{(x+1)^2}.$$

4-misol. Quyida berilgan funksiyalarning hosilalarinini ikki xil usulda hisoblang:

$$1)\ g(x)=\frac{x^3+2x^2+3x^{-1}}{x},\ \ h(t)=\frac{t^2-1}{t-1}.$$

Yechimi:

1) Bo'linma hosilasidan foydalansak

$$g'(x)=\frac{(3x^2-4x-3x^{-2})x-(x^3+2x^2+3x^{-1})\cdot 1}{x^2}=$$
$$=\frac{2x^3+2x^2-6x^{-1}}{x^2}2x+2-6x^{-3}.$$

Yig'indining hosilasidan foydalansak,

$$g'(x)=(x^2+2x+3x^{-2})'=2x+2-6x^{-3}.$$

2) Bo'linma hosilasidan foydalanamiz,

$$h'(t)=\frac{2t(t-1)-(t^2-1)\cdot 1}{(t-1)^2}=\frac{t^2-2t+1}{(t-1)^2}=\frac{(t-1)^2}{(t-1)^2}=1.$$

Yig'indining hosilasidan foydalanamiz,

$$h'(t)=(t+1)'=1.$$

3B. Zanjir qoidasi. Oshkormas funksiyalarni differensiallash qoidasi

$y=f(u)$ va $u=g(x)$ funksiyalar berilgan bo'lsin. U holda $y=f(g(x))$ ga murakkab funksiya deb ataladi.

Agar $f(x)$ va $g(x)$ differensiallanuvchi funksiyalar bo'lsa, murakkab funksiyadan quyidagi formula yordamida hosila olish mumkin:

$$\frac{df(g(x))}{dx}=\frac{df(g)}{dg}\frac{dg(x)}{dx}.$$

$u=g(x)$ belgilashni kiritib yuqoridagini quyidagicha ham yozish mumkin:

$$\frac{dy}{dx}=f'(u)\frac{du}{dx} \quad yoki \quad \frac{dy}{dx}=\frac{dy}{du}\frac{du}{dx}$$

5-misol. Berilgan funksiya hosilasini toping $y=\sqrt{x^3+1}$.

Yechimi: Quyidagicha almashtirishlarni bajarib,

$$f(u)=\sqrt{u} \quad va \quad u=g(x)=x^3+1$$

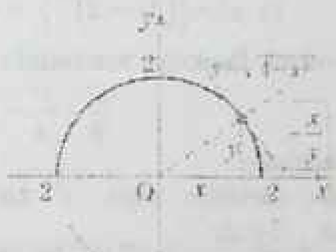
$f'(u)=\frac{1}{2}u^{-1/2}$ va $g'(x)=3x^2$ lardan va murakkab funksiya hosilasi formulasidan foydalangan holda

$$\frac{d}{dx}\sqrt{x^3+1}=\frac{1}{2}u^{-1/2}(3x^2)=\frac{1}{2}(x^3+1)^{-1/2}(3x^2)=\frac{3x^2}{2\sqrt{x^3+1}}$$ ni olamiz.

Funksiyalarning ikki xil ko'rinishi

Biz yuqorida oshkor $y=f(x)$ kabi funksiyalarni o'rgangan edik, masalan $y=x^2$.

Agar erkli va erksiz o'zgaruvchilar o'zaro oshkor bog'lanmagan bo'lsa, unga oshkormas funksiya deyiladi: $f(x,y)=0$, masalan $x^2+y^2=1$.



Hosilalarini hisoblashning ikki xil usuli

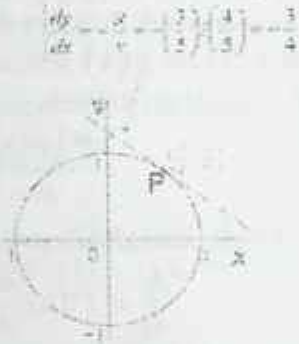
Misol uchun, aylana $x^2+y^2=4$ tenglama bilan berilgan bo'lsin.

(1) y ga nisbatan yechib differensiallaymiz

$$y=\pm\sqrt{4-x^2}\ ,\quad \frac{dy}{dx}=\left(\pm\frac{x}{\sqrt{4-x^2}}\right)=-\frac{x}{y}$$

(2) Zanjir qoidasini qo'llab har bir had hosilasini olamiz $x^2+y^2=1$.

Ikkala tarafidan ham differensial olib,



$$\frac{d}{dx}\left(x^2+y^2\right)=\frac{d}{dx}(1)$$

$$2x+\frac{d}{dx}\left(y^2\right)\frac{dy}{dx}=0$$

Zanjir qoidasini qo'llab topamiz

$$2x+\frac{d}{dx}\left(y^2\right)\frac{dy}{dx}=0,$$

$$2x+2y\frac{dy}{dx}=0,\ \frac{dy}{dx}=-\frac{x}{y}.$$

6-misol. Berilgan funksiyalar hosilalarini toping:

1) $y=\left(x^2+2x-1\right)^2$,

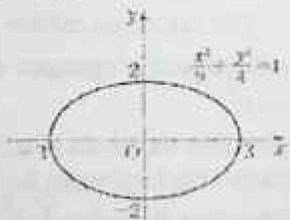
2) $y=\dfrac{1}{(3x-2)^2}$.

Yechimi: $y'=2\left(x^2+2x-1\right)(2x+2)=4\left(x^2+2x-1\right)(x+1)$,

1) $y'=\left\{(3x-2)^{-2}\right\}'=-2(3x-2)\cdot 3=-6(3x-2)$.

7-misol. Funksiya hosilasini toping:

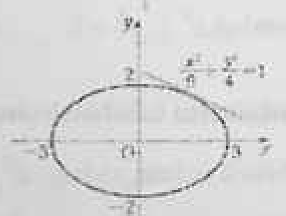
$$\frac{x^2}{9}+\frac{y^2}{4}=1$$



Yechimi:

Ikki tarafdan ham x bo'yicha hosila olib $\dfrac{2x}{9}+\dfrac{2y}{4}\dfrac{dy}{dx}=0$ ga ega bo'lamiz.

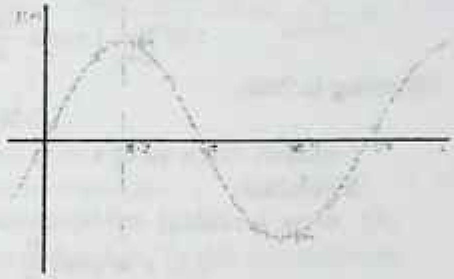
Yuqoridagidan $y\neq 0$ da $\dfrac{dy}{dx}=-\dfrac{4x}{9y}$ ga ega bo'lamiz.



2.4. Trigonometrik funksiyalarning hosilalari

4A. Asosiy trigonometrik funksiyalar hosilalari. $\frac{sinx}{x}$ ning limiti

Sinus funksiyasining hosilasi



Grafikdagi ko'k chiziq sinus funksiyani aks ettiradi. Chiziqlarning o'zgarishiga e'tiborimizni qaratsak. Grafikga berilgan nuqtada o'tqazilgan urinma hosilaning shu nuqtadagi ishorasini aniqlab beradi. $x=0$ da urinmaning burchak koeffisiyenti 1 ga teng, $x=\frac{\pi}{4}$ da esa kichikroq, $x=\frac{\pi}{2}$ da esa 0 ga teng.

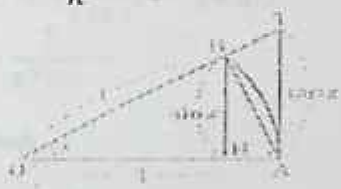
Shu tarzda davom ettirib funksiyaning hosilasini ko'rishimiz mumkin. Bu holda hosilani kosinus funksiyaga oxshatishimiz mumkin.

Endi esa hosilaga berilgan tarifga asosan sinus funksiyaning hosilasini hisoblaymiz. Hosila ta'rifi bo'yicha

$$f'(x)=\lim_{n\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{n\to 0}\frac{\sin(x+h)-sinx}{h}=$$

$$=\lim_{n\to 0}\frac{sinx\cdot cosh+cosx\cdot sinh-\sin x}{h}=$$

$$=\lim_{h\to 0}(\,cosx\cdot\frac{sinh}{h}-sinx\cdot\frac{1-cosh}{h}).$$



Demak, hosilani hisoblash sin x / x ning limitini, ya'ni aniqmaslikni hisoblashga kelar ekan.

Markaziy burchakka ega bo'lgan sektorni ko'rib chiqib $\triangle OAB$, sektor OAB, va $\triangle OAT$ yuzalarini solishtiramiz:

$$\frac{1}{2}1\cdot\sin x<(\pi\cdot 1^2)\,\frac{x}{2\pi}<\frac{1}{2}1\cdot tgx.$$

Bu yerdan $\therefore$ $\quad \sin x<x<tg\,x$

$\sin x>0\ (x>0)$ da

$$\therefore\quad 1<\frac{x}{sinx}<\frac{1}{cosx}$$

$$\therefore\quad 1>\frac{sinx}{x}>cosx$$

Agar oxirgi tengsizlikda $x\to 0$ da limitga o'tsak

$$\lim_{x\to 0}\frac{sinx}{x}=1$$

ni hosil qilamiz.

$$\frac{(1-cosh)(1+cosh\,)}{h(1+cosh)}=\frac{1-cos^2h}{h(1+cosh)}=\frac{sinh}{h}\frac{sinh}{(1+cosh)}.$$

Demak, sinus funksiyasining hosilasi uchun,

$$f'(x) = \lim_{h\to 0}\left(cosx\frac{sinh}{h} - sinx\frac{1-cosh}{h}\right) =$$
$$= \lim_{h\to 0}\left(cosx\frac{sinh}{h} - sinx\frac{sinh}{h}\frac{sinh}{(1+cosh)}\right).$$

Shuning uchun,

$$(sinx)' = cosx.$$

1-misol. cos x va tg x ning hosilasini toping.

Yechimi:

(1) o'ng tarafdagi uchburchakdan

$$cos\, x = sin(\frac{\pi}{2} - x)$$

Shuning uchun

$$(\cos x)' = (\sin(\frac{\pi}{2} - x))' = cos\left(\frac{\pi}{2} - x\right)\cdot\left(\frac{\pi}{2} - x\right)'$$
$$= \sin x \cdot (-1) = -\sin x$$

2) Bo'linmaning hosilasi formulasidan

$$(\operatorname{tg} x)' = (\frac{\sin x}{\cos x})' = \frac{\cos x \cdot \cos x - \sin x(-\sin x)}{cos^2x} = \frac{cos^2x + sin^2x}{cos^2x}$$
$$= \frac{1}{cos^2x} == sec^2x$$

Asosiy trigonometrik hosilalar

$$\frac{d}{dx}\sin x = \cos x, \qquad \frac{d}{dx}\cos x = -\sin x$$

(2) Boshqa standart munosabatlar

$$\frac{d}{dx}\operatorname{tg} x = sec^2x, \qquad \frac{d}{dx}\sec x = \sec x \operatorname{tg} x,$$
$$\frac{d}{dx}\operatorname{ctg} x = -cosec^2x, \qquad \frac{d}{dx}\operatorname{cosec} x = -\operatorname{cosec} x \operatorname{ctg} x$$

2-misol. Quyidagi funksiyalarning hosilalarini toping:

(1) $y = cos^2x$, *(2)* $y = x\, sinx + \cos x$.

Yechimi: 1) Darajaning hosilasi va murakkab funksiya hosilasi fomulasiga ko'ra, $y' = 2\cos x \cdot (\cos x)' = 2\cos\, x \cdot (-\sin x) = -2\sin x \cos x = -\sin 2x$

2) Ko'paytmaning hosilasi bo'yicha

$$y' = (x \cdot \sin x)' + (\cos x)' = (1 \cdot \sin x + x\cos x) - \sin x = x\cos x$$

3-misol. Quyidagi funksiyalarning hosilalarini toping:

(1) $y = \sin ax^2$, (2) $y = \frac{1}{\tan x}$, (3) $y = \cos(\frac{x}{2} + \frac{\pi}{6})$.

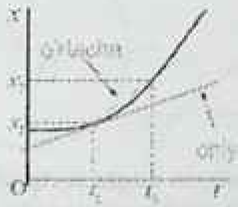
Yechimi: *(1)* $\frac{d}{dx}sin(ax^2) = \frac{d}{du}sin\,u\,\frac{d}{dx}(ax^2) = cos\,u \cdot (2ax) = 2axcos(ax^2)$

(2) $y' = \left(\frac{cos\,x}{sin\,x}\right)' = \frac{(-sin\,x)\,sin\,x - cos\,x(cosx)}{sin^2x} = -\frac{(sin^2x + cos^2x)}{sin^2x} = \frac{1}{sin^2x} = -\frac{1}{sin^2x}$

(3) $y' = \frac{d}{du}cos\,u\,\frac{d}{dx}\left(\frac{x}{2} + \frac{\pi}{6}\right) = -\frac{1}{2}sin(\frac{x}{2} + \frac{\pi}{6})$.

4B. Hosilalar va harakatlar

Vaziyat, tezlik va tezlanish. Oddiy tebranma harakat. Tezlik va tezlanish



P nuqta to'g'ri chiziq bo'ylab harakatlanadi. Uning vaziyati: Bu holda tezlik va tezlanishlarni quyidagicha aniqlaymiz:

t_1 va t_2 orasidagi o'rtacha tezlik

$$v = \frac{\Delta x}{\Delta t} = \frac{x_2 - x_1}{t_2 - t_1}$$

$T = t_1$ da bir oniy tezlik

$$v(t_1) = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t}, \qquad v = \frac{ds}{dt} = f'(t).$$

$T = t_1$ da oniy tezlanish $a = \frac{dv}{dt} = f''(t)$

4-misol. X o'qi bo'ylab harakatlanadigan nuqtaning vaziyati $s(t) = t^3 - 3t^2 - 9t + 10$ tenglama bilan berilgan.

(1) $t = 2$ nuqtada tezlik va tezlanishni toping;

(2) $-2 \le t \le 4$ vaqt davomida harakatni o'rganing.

Yechimi: (1) tezlik: $v = \frac{ds}{dt} = 3t^2 - 6t - 9 = 3(t+1)(t-3)$

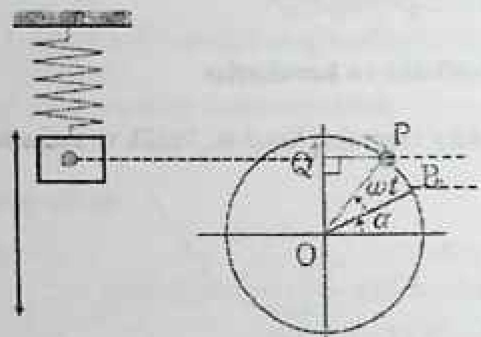
$$\therefore \quad v(2) = -9$$

Tezlanish: $a = \frac{dv}{dt} = 6(t-1) \qquad \therefore \quad a(2) = 6$

(2)

t	-2	$\cdots$	-1	$\cdots$	3	$\cdots$	4
v		$+$	0	$-$	0	$+$	
s	8	$\nearrow$	15	$\searrow$	-17	$\nearrow$	-10

Oddiy tebranma harakat



Prujinaga osilgan nuqta vertikal harakatlahmoqda. Uning holat, tezlik va tezlanishi quyidagicha aniqlanadi:

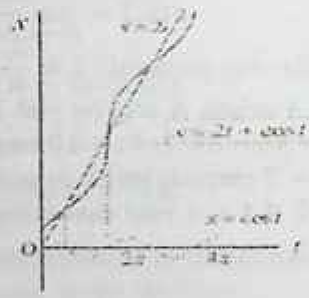
Vertikal holat: $y = Asin(\omega t + \alpha)$

Tezlik : $v = \frac{dy}{dt} = A\omega cos(\omega t + \alpha)$

Tezlanish: $a = \frac{dv}{dt} = \frac{d^2v}{dt^2} = -A\omega^2 si\,n(\omega t + \alpha)$.

5-misol. P nuqta x o'qi bo'yicha harakatlanmoqda. Uning vaziyati $x = 2t + \cos t$ kabi aniqlanadi. Nuqta maksimal tezligi va uning maksimal tezlikka ega bo'lgan vaqtini toping.

Yechimi.



Tezlik

$$v = \frac{dx}{dt} = 2 - sint$$

Maksimal tezlikka

$$sin\,t = -1$$

da erishadi, shuning uchun

$$t = \frac{3}{2}\pi + 2n\pi$$

Maksimal tezlik esa 3 ga teng.

2.5. Logarifmik va ko'rsatkichli funksiyalar hosilasi
5A. Logarifmik fumksiyaning hosilasi. Ko'rsatkichli funksiya

Ko'rsatkichli

$$y = a^x \quad (a > 0,\ a \neq 1)$$

funksiya berilgan bo'lsin. Unga teskari bo'lgan funksiani topamiz. U logarifmik funksiya deb atalib, quyidagicha aniqlanadi:

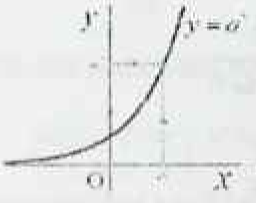
$$y = a^x$$

dan x ni topamiz

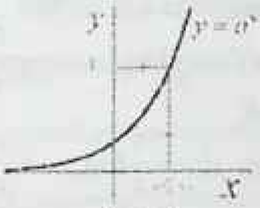
$$x = a^y$$

va x bilan y ning joyini almashtiramiz

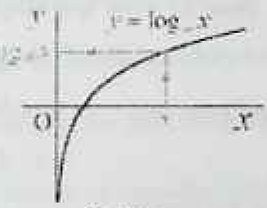
$$y = log_a x.$$



1 rasm



2 rasm



3 rasm

Logarifmik funksiyaning hosilasi

Ta'rifga ko'ra

$$(log_a x)' = \lim_{h\to 0}\frac{\log_a(x+h) - \log_a x}{h}$$

$$= \lim_{h\to 0}\frac{1}{h}\log_a\left(1+\frac{h}{x}\right) = \lim_{h\to 0}\left\{\frac{1}{x}\frac{x}{h}\log_a\left(1+\frac{h}{x}\right)\right\}.$$

Quyidagi belgilashni kiritamiz

$$\frac{h}{x} = k$$

$h \to 0$ da $k \to 0$, u holda

$$(\log_a x)' = \lim_{h\to 0}\left\{\frac{1}{x}\frac{1}{k}\log_a(1+k)\right\}$$

$$= \frac{1}{x}\lim_{h\to 0}\log_a(1+k)^{\frac{1}{k}} = \frac{1}{x}\log_a\left[\lim_{k\to 0}(1+k)^{\frac{1}{k}}\right]$$

Oxirgi limit ostidagi ifodaning qiymatlarini qaraymiz va jadval tuzamiz

h	$(1+h)^{\frac{1}{h}}$	h	$(1+h)^{\frac{1}{h}}$
0.1	2.59374……	−0.1	2.86797……
0.01	2.70481……	−0.01	2.73199……
0.001	2.71692……	−0.001	2.71964……
0.0001	2.71814……	−0.0001	2.71841……
0.00001	2.71826……	−0.00001	2.71829……

Demak ular biror songa yaqinlashar ekan. Unga e soni yoki Neper o'zgarmasi deb ataladi.

e soni

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = 2.718281828459045\ldots$$

Sunday qilib quyidagi tenglikka ega bo'ldik

$$(\log_a x)' = \frac{1}{x}\log_a e = \frac{1}{x\log_e a}$$

Logarifn asosini e ga tenglasak

$$(\log_e x)' = \frac{1}{x}\log_e e = \frac{1}{x}$$

ga ega bo'lamiz.

Asosi e ga ting bo'lgan logarifmga **natural logarifm** deb aytiladi.
Ya'ni:

$$\log_e x = \ln x$$

Eslatma.

$$\frac{d}{dx}(\log_a x) = \frac{1}{xlna},$$
$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

1-misol. Berilgan funksiyalarning hosilasini hisoblang

(1) $y = \ln 2x$,

(2) $y = \log_2(3x+2)$,

(3) $y = x\ln 3x$

Yechimi.

(1) $y' = \dfrac{1}{2x}(2x)' = \dfrac{2}{2x} = \dfrac{1}{x}$

(2) $y' = \dfrac{1}{(3x+2)\,ln\,2}(3x+2)' = \dfrac{3}{(3x+2)\,ln\,2}$

(3) $y' = x'ln3x + x(ln3x)' = ln3x + x\dfrac{3}{3x} = ln3x + 1$

Eslatma: e soni $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e = 2.7182\ldots$

2-misol. Berilgan funksiyalarni hosilasini hisoblang.

$$(1)\ y = (\ln x)^2$$

$$(2)\ y = x\ln x$$

$$(3)\ y = \log_{10} x$$

Yechimi.

$$(1)\ \frac{d}{dx}(\ln x)^2 = 2\ln x \cdot \frac{d}{dx}\ln x = \frac{2\ln x}{x}$$

$$(2)\ \frac{d}{dx}(x\ln x) = (x)' \cdot \ln x + x(\ln x)' = \ln x + x\frac{1}{x} = \ln x + 1$$

$$(3)\ \log_{10} x = \frac{\ln x}{\ln 10}$$

$$\frac{d}{dx}\log_{10} x = \frac{d}{dx}\frac{\ln x}{\ln 10} = \frac{1}{\ln 10}\frac{d}{dx}\ln x = \frac{1}{(\ln 10)x}$$

5B. Ko'rsatkichli fumksiyaning hosilasi.Teskari funksiyaning hosilasi

Bizga, f va g o'zaro teskari funksiyalar berilgan bo'lsin. U holda

$$y = f(x) \quad \rightarrow \quad x = g(y)$$

Bu tengliklarni har ikkala tomonini differensiallaymiz va quyidagilarti olamiz

$$1 = \frac{df(x)}{dy} = \frac{df(x)}{dx}\frac{dx}{dy} = \frac{df(x)}{dx}\frac{dg(y)}{dy}$$

$$1 = \frac{df(x)}{dx}\frac{dg(y)}{dy}$$

Demak

$$\frac{df(x)}{dx} = \frac{1}{\left(\frac{dg(y)}{dy}\right)}$$

yoki

$$\frac{dy}{dx} = \frac{1}{\left(\frac{dx}{dy}\right)}$$

Ko'rsatkichli funksiyaning hosilasi

Asosi e bo'lgan ko'rsatkichli funksiyani qaraymiz

$$y = f(x) = e^x$$

Unga teskari funksiya

$$x = g(y) = lny$$

Demak, yuqoridaginga ko'ra

$$\frac{dy}{dx} = \frac{df(x)}{dx} = \frac{1}{\left(\frac{dg(y)}{dy}\right)} = \frac{1}{\left(\frac{1}{y}\right)} = y$$

$$\frac{d}{dx}(e^x) = e^x$$

Endi $y = a^x$ ni qaraymiz.

Agar $a = e^x$ bo'lsa, $x = lna$ bo'ladi. Demak,

$$y = a^x = (e^{lna})^x = e^{xlna}$$

Qoidaga ko'ra

$$\frac{d}{dt}(a^x) = \frac{d}{dt}(e^{xlna}) = e^{xlna}\frac{d}{dt}(xlna) = a^x lna$$

$$\frac{d}{dt}(a^x) = a^x lna$$

3-misol. Berilgan funksiyalarni hosilasini hisoblang.

(1) $y = e^{2x}$ (2) $y = a^{-2x}$

Yechimi.

(1) Qoidaga ko'ra

$$y' = e^{2x} \cdot (2x)' = 2e^{2x}$$

(2) Qoidaga ko'ra

$$y' = (a^{-2x}\log a) \cdot (-2x)' = -2a^{-2x}\log a$$

4-misol. Berilgan funksiyalarni hosilasini hisoblang.

(1) $y = xa^x$ (2) $y = 2^{lnx}$ (3) $y = \frac{e^x - e^{-x}}{e^x + e^{-x}}$

Yechimi.

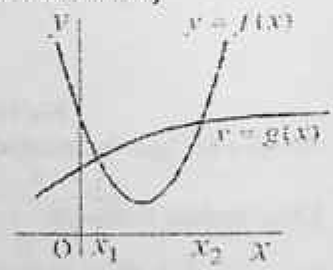
$$y' = (x)'a^x + x(a^x)' = a^x + x \cdot a^x \log a = a^x(1 + x\log a)$$

$$y' = 2^{\ln x}\ln 2 \cdot \left(\frac{1}{x}\right) = \frac{2^{\ln x}\ln 2}{x}$$

$$y' = \frac{(e^x + e^{-x})^2 - (e^x - e^{-x})^2}{(e^x + e^{-x})^2} = \frac{4}{(e^x + e^{-x})^2}.$$

2.6. Hosilaning tenglama va tengsizliklarga tadbiqi

6A.Grafiklar va tenglamalar (Parametrsiz hol)



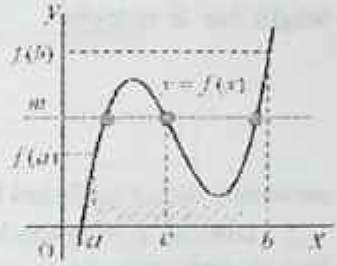
Ushbu $f(x)=g(x)$ tenlikning ildizlari $f(x)$ va $g(x)$ funksiyalarining kesishish nuqtalari koordinatalarini ko'rsatadi.

Xususiy hollar

Agar $f(x)=g(x)$ tenglama parametrli bo'lmasa, uning yechimini toppish uchun biz $y=f(x)-g(x)$ funksiya grafigining Ox o'qi bilan kesishish nuqtalari koordinatalarini qidiramiz.

Boshqacha qilib aytganda $f(x)-g(x)=0$ ni yechib nollarini topamiz.

O'rta qiymat haqidagi teorema



Aytaylik $y=f(x)$ tenglama bilan berilgan egri chiziq $[a,b]$ intervalda uzluksiz bo'lsin va m nuqta $f(a)$ va $f(b)$ lar orasida bo'lsin. U holda $[a,b]$ intervalda $f(c)=m$ tenklik bajariladigan shunday c nuqta albatta topiladi.

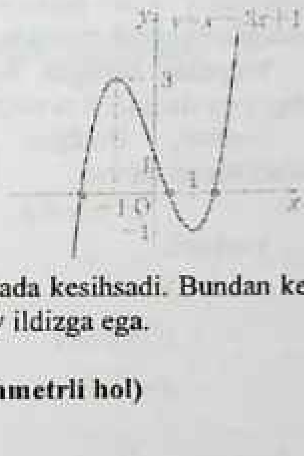
Izoh. Agar funksiya $[a,b]$da monoton o'ssa yoki kamaysa c son yagonadir.

1-misol. Berilgan tenglamaning haqiqiy sonlardagi yechimlari sonini toping:

$$x^3=3x-1.$$

Yechimi. Biz ushbu $y=x^3-3x+1$ tartibda funksiya yasab olib, undan $y'=3x^2-3=3(x+1)(x-1)$ ni olamiz va $y'=0$ tenglikni $x_1=1, x_2=1$ yechimlarni olamiz.

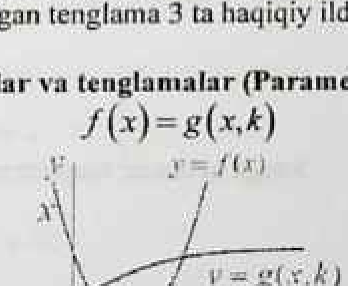
x	$\dots$	-1	$\dots$	$+1$	$\dots$
y'	$+$	0	$-$	0	$+$
y	$\nearrow$	3	$\searrow$	-1	$\nearrow$



Bu funksiyaning grafigi Ox o'qi bilan 3 ta nuqtada kesihsadi. Bundan kelib chiqadiki yuqorida berilgan tenglama 3 ta haqiqiy ildizga ega.

Grafiklar va tenglamalar (Parametrli hol)

$$f(x)=g(x,k)$$



Aytaylik tenglama quyidagi ko'rinishda bo'lsin

$$f(x) = k \cdot g(x) \Rightarrow -\frac{f(x)}{g(x)} = k, (g(x) \neq 0)$$

tenglikdan k ni topib

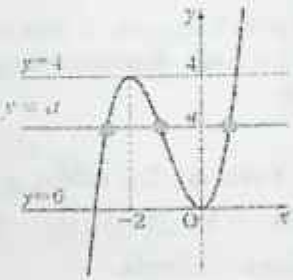
$$\left.\begin{aligned} y &= -\frac{f(x)}{g(x)} \\ y &= k \end{aligned}\right\}$$

sistemani yechib grafiklari kesishish nuqtalarini topish mumkin.

2-misol. a ning qanday qiymatlarida $x^3 + 3x^2 = a$ kubik tenglama 3 ta haqiqiy yechimga ega bo'ladi?

Yechimi. Ushbu $x^3 + 3x^2 = a$

x	......	-2	......	0	......
y'	$+$	0	$-$	0	$+$
y	↗	Max 4	↘	Min 0	↗

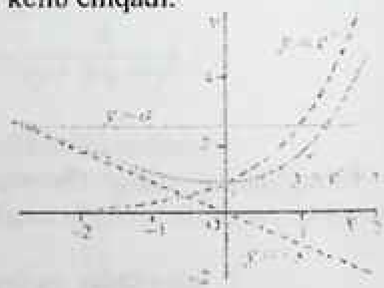


tenglikdagi $x^3 + 3x^2$ funksiya haqida quyidagilarni aytish mumkin.

Yuqorida berilgan funksiyaning grafigidan $0 < a < 4$ da funksiya grafigi $y=a$ chiziqni 3 ta nuqtada kesib o'tishi kelib chiqadi.

3-misol. Berilgan tenglamaning ildizlari sonini toping.

$$e^x = x + a$$



Yechimi.

$$y = e^x - x$$

$$y' = e^x - 1$$

$$x = 0 \;\; da \;\; y' = 0$$

G
a
a
a

x	...	0	...
y'	$-$	0	$+$
y	↘	1	↗

losaga kelamiz: tenglamaning ildizlari

4-misol. Aytaylik m haqiqiy son bo'lsin. m ning qiymatiga qarab $y = x^2 - x - 1$ va $y = 2x + m$ tenglamalarning kesishish nuqtalari sonini toping.

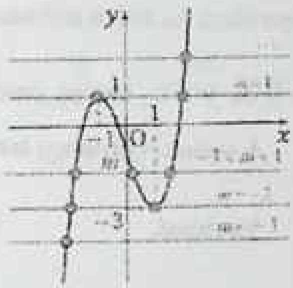
Yechimi.

$$x^3 - x - 1 = 3x + m$$
$$x^3 - 3x - 1 = m$$

Biz ikkita $y = x^3 - 3x - 1$ va $y = m$ funksiyalarni qaraymiz.

Yuqoridagidan: $y' = 3x^2 - 3 = 3(x-1)(x+1)$

x	...	-1	...	-1	...
y'	$+$	0	$-$	0	$+$
y	$\nearrow$	-1	$\searrow$	-3	$\nearrow$



Grafikka asosan, tenglamaning ildizlari soni:

$m < -3,\ \ m > 1$ da 1 ta

$m = -3;1$ da 2 ta

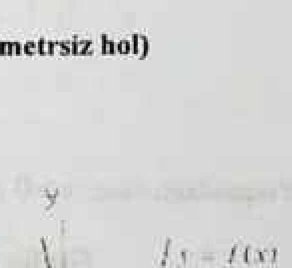
$-3 < m < 1$ da 3 ta yechimga ega.

6B. Grafik va tengsizliklar (Parametrsiz hol)

Ushbu

$$f(x) > g(x)$$

tengsizlikning yechimlari $y = f(x)$ funksiyaning grafigi $y = g(x)$ chiziqdan yuqorida bo'lgan x o'qi sohasida bo'ladi. $y = f(x) - g(x)$ almashtirish bajarib $y > 0$ qismini topamiz.



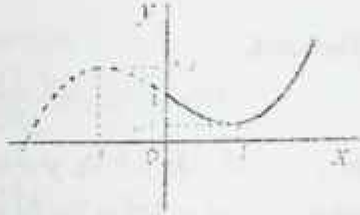
5-misol. Quyidagi tengsizlikni isbotlang.

$$x > 0 \ da \ x - 1 < \frac{1}{3}x^3$$

Yechimi.

$$y = \frac{1}{3}x^3 - x + 1$$
$$y' = x^2 - 1 = (x-1)(x+1)$$

x	0	...	+1	...
y'	—	—	0	+
y	1	↘	$\frac{1}{3}$	↗



Yuqoridagi funksiya minimum $\frac{1}{3}$ qiymatga $x=1$ nuqtada erishadi.

$x>0$ da $y>0$, demak barcha $x>0$ larda $x-1<\frac{1}{3}x^3$ bajariladi.

6-misol. Quyidagi tengsizlikni isbotlang.

$$x>0 da\ e^x>1+x$$

Yechimi.

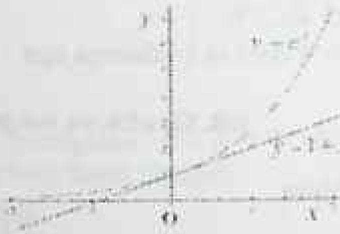
$$y=e^x-(1+x)$$

funksiyani qaraymiz

$$x>0 da\ \ y'=e^x-1>0$$

y monoton o'suvchiligidan $x=0$ da $y=0$.

x	0	...
y'	0	+
y	0	↗



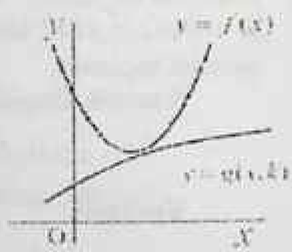
Yuqoridagilardan $x>0$ da $e^x>1+x$ bo'lishini topamiz.

Grafik va tengsizliklar (parametrlik hol)

k ning qanday qiymatida

$$f(x)>g(x,k)$$

tengsizlik bajarilish masalasi.



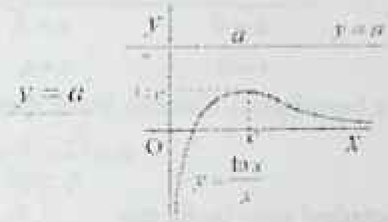
$$f(x)>kg(x)\Rightarrow\frac{f(x)}{g(x)}>k,\ (g(x))$$

$$\left.\begin{aligned} y&=\frac{f(x)}{g(x)} \\ y&=k \end{aligned}\right\}$$

sistemadan k ni $\dfrac{f(x)}{g(x)}>k$ ekanini topamiz.

7-misol. Quyidagi tengsizlik bajariladigan a ning qiymatlari to'plamini toping.

$$ax\geq \ln x \ (x>0)$$



Yechimi.

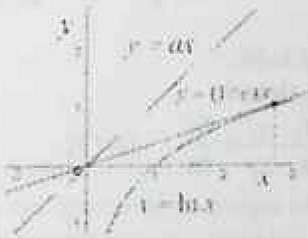
$$a\geq \frac{\ln x}{x}$$

$$y=\frac{\ln x}{x}$$

$$y'=\frac{\frac{1}{x}x-\ln x\cdot 1}{x^2}=\frac{1-\ln x}{x^2}$$

$$x=e \ da \ y'=0$$

x	$0<x<e$	e	$e<x$
y'	$+$	0	$-$
y	↗	$1/e$	↘



Ya'ni $x=e$ da funksiya eng kichik qiymatga erishadi.

Javob: $a\geq \dfrac{1}{e}$

8-misol. Quyidagi tengsizlikni isbotlang.

$$x\geq 0 \ da \ \ x^3-9x+27\geq 3x^2$$

Yechimi. Quyidagicha almashtirish bajaramiz

$$y=x^3-3x^2-9x+27$$

$$y'=3x^2-6x-9=3(x+1)(x+3)$$

$$x=-1;3 \ da \ y'=0$$

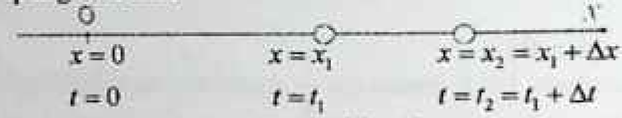
x	0	$\cdots$	3	$\cdots$
y'	$-$	$-$	0	$+$
y	27	↘	0	↗

Yuqoridagi jadvaldan $x^3-9x+27\geq 3x^2$ tengsizlik $x\geq 0$ da bajarilishini ko'rishimiz mumkin.

2.7. Fizikaga tadbiqlari

7A. Zarraning tezligi va tezlanishi. To‘g‘ri chiziqdagi harakat. Tezlik

Ox–o‘qidagi harakat



[t_1 , t_2] oraliqdagi o‘rtacha tezlik quyidagiga teng

$$\bar{v} = \frac{x_2 - x_1}{t_2 - t_1} = \frac{\Delta x}{\Delta t} \qquad [m/s]$$

t nuqtadagi oniy tezlik esa

$$v = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt} \qquad [m/s]$$

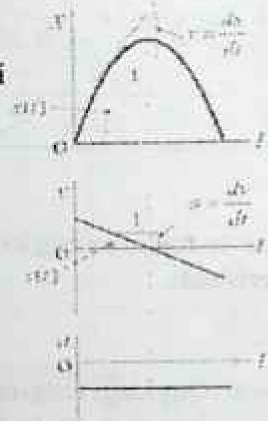
To‘g‘ri chiziqdagi harakat: Tezlanish

Yuqoridagi kabi:

O‘rtacha tezlanish

$$\bar{a} = \frac{v_2 - v_1}{t_2 - t_1} = \frac{\Delta v}{\Delta t} \qquad [m/s]$$

Joylashuvi



Oniy tezlanish

$$a = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = \frac{dv}{dt} \qquad [m/s]$$

Tezlik

Vektor va skalyar

Vektor – qiymati va yo‘nalishi bilan tavsiflanadigan, aniqlanadigan kattalik. Skalyar – yo‘nalishga ega bo‘lmagan va faqat son bilan ifodalanadigan kattalik.

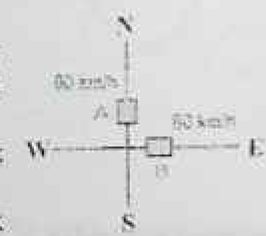
Tezlik va tezlik

Tezlik (ko‘chishning) - Obyektning qanchalik tez va qaysi yo‘nalishda harakatlanishini anglatuvchi vektor kattaligi.

Tezlik (yo‘lning) – Obyektning qanchalik tez harakatlanishini anglatuvchi skalyar kattalik.

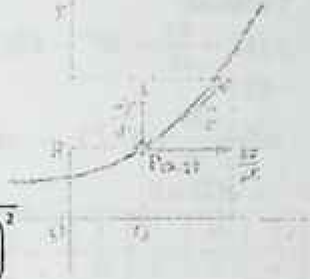
Rasmda A va B avtoulovlar mavjud bir xil tezlik bilan

turli tomonga harakatlanmoqda



Harakat tashkil etuvchilari

Q nuqta Ox o'qi bo'ylab, R nuqta esa Oy o'qi bo'ylab harakatlansin. Shuningdek Q va R lar P ning koordinatalari bo'lsin. U holda



Q nuqtaning tezligi : $\frac{dx}{dt}$,

R nuqtaning tezligi : $\frac{dy}{dt}$

P nuqtaning vector tezligi: $\bar{v}=\left(\frac{dx}{dt},\frac{dy}{dt}\right)$

P nuqtaning skalyartezligi : $v=\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}$

Tezlanish

P nuqtaning tezlanishi : $\bar{a}=\left(\frac{d^2x}{dt^2},\frac{d^2y}{dt^2}\right)$

Tezlanish kattaligi $a=\sqrt{\left(\frac{d^2x}{dt^2}\right)^2+\left(\frac{d^2y}{dt^2}\right)^2}$

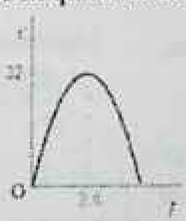
Doimiy tezlanish holati

1-misol. Boshlang'ich tezligi 25 m/s bo'lgan to'p yuqoriga tik otildi. Boshlang'ich nuqtasi sifatida $y-$o'qi bo'ylab yuqoriga yo'nalishini olamiz. Agar t vaqtni to'p qo'yib yuborilgan vaqtdan hisoblansa, u holda uning holati quyidagicha beriladi:

$$y(t)=y_0+v_0t-\frac{1}{2}gt^2$$

Bunda $y_0=0\ m,\ v_0=25\ m/s$ va $g=9{,}8\ m/s^2$. To'p eng baland nuqtaga yetgandagi vaqt va uning balandligi qancha?

Yechimi. Tezlik $v=\frac{dy}{dt}=v_0-gt$. Eng baland nuqtada tezlik nolga teng, bundan kelib chiqadi $v_0-gt_1=0$



$$t_1=\frac{v_0}{g}=\frac{25m/s}{9{,}8m/s^2}=2{,}6s$$

$$y(t_1)=0+25\times2{,}6-\frac{1}{2}\times9{,}8\times2{,}6^2=32\ m$$

2-misol. Zarrachaning $x-$ o'qidagi harakat tenglamasi quyidagicha berilgan

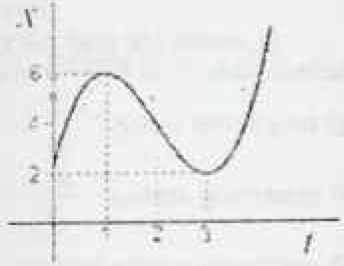
$$x(t)=t^3-6t^2+9t+2$$

(1) $t=2$ da tezlik va tezlanishni toping.
(2) Harakatni $0 \le t \le 4$ mobaynida o'rganing.

Yechimi.

(1) $v(t) = \frac{dx}{dt} = 3t^2 - 12t + 9 = 3(t-1)(t-3)$

(2) $a(t) = \frac{dv}{dt} = 6t - 12$ shuning uchun, $v(2) = -3$, $a(2) = 0$



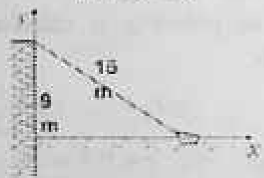
t	0	...	1	...	3	...	4
v	+	+	0	-	0	+	+
x	2	↗	6	↘	2	↗	66

7B. Bog'liq masalalar

3-misol. Bir kishi arqon yordamida qayiqni tepalik tepasidan tortib olmoqda. Odam suv sathidan 9 m balandlikda va arqonning o'zgarish tezligi $2\ m/s$. Arqonning uzunligi $15m$ bo'lganda qayiqning tezligi qancha?

Yechimi. Arqon uzunligi $l\ [m]$, qayiq va tepalik orasidagi masofa $u\ [m]$

$$u^2 + 9^2 = l^2$$



Bu tenglikni t bo'yicha differensiallasak

$$2u\frac{du}{dt} = 2l\frac{dl}{dt}$$

Shartga ko'ra,

$$\frac{dl}{dt} = -2\ m/s,\ l = 15\ m,\ u = \sqrt{15^2 - 9^2} = 12$$

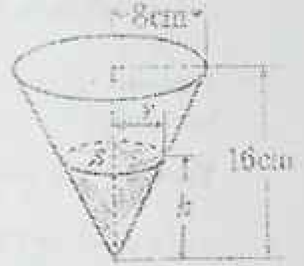
bundan $12 \times \frac{du}{dt} = 15 \times (-2)$ $\quad \frac{du}{dt} = -2{,}5\ m/s$.

4-misol. Balandligi $16\ sm$, asosining radiusi $8\ sm$ bo'lgan konus shaklidagi idishga $3\ sm^3/s$ hajmdagi suv quyilmoqda. $6\ sm$ chuqurlikda suv sathi balandligi o'zgarish tezligini toping.

Yechimi. Suv hajmi $V = \frac{1}{3}\pi r^2 h$

Uchburchaklar o'xshashligidan $8:r=16:h$, shuning uchun $r=\frac{1}{2}h$

$$V=\frac{1}{12}\pi h^3$$

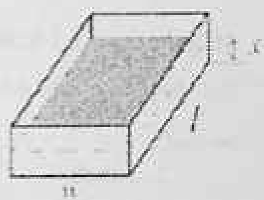


Differensiallasak t: $\frac{dV}{dt}=\frac{1}{4}\pi h^2\frac{dh}{dt}$

Natijada: $\frac{dV}{dt}=3sm^3/s$, $h=6sm$

Bundan esa $3=\frac{1}{4}\pi\times 6^2\times\frac{dh}{dt}$, $\frac{dh}{dt}=\frac{1}{3\pi}\ sm/s$.

5-misol. To'rtburchak shaklidagi suv idishi doimiy ravishda 20 $litr/s$ tezlikda to'ldirilmoqda. Idishning o'lchamlari $w=1\ m$ va $l=2\ m$. Idishdagi suv x balandligining o'zgarish tezligi qanday?



Yechimi. Yasalishiga ko'ra

$$V(t)=wlx(t)$$

Uni t bo'yicha differensiallaymiz $\frac{dV}{dt}=wl\frac{dx}{dt}$

yoki $$\frac{dx}{dt}=\frac{1}{wl}\left(\frac{dV}{dt}\right)$$

Berilganiga ko'ra: $\frac{dV}{dt}=20\ litr/s=20000\ sm^3/s$

Bundan quyidagi kelib chiqadi,

$$\frac{dx}{dt}=\frac{1}{wl}\left(\frac{dV}{dt}\right)=\frac{20000}{100\cdot 200}=1\ sm/s$$

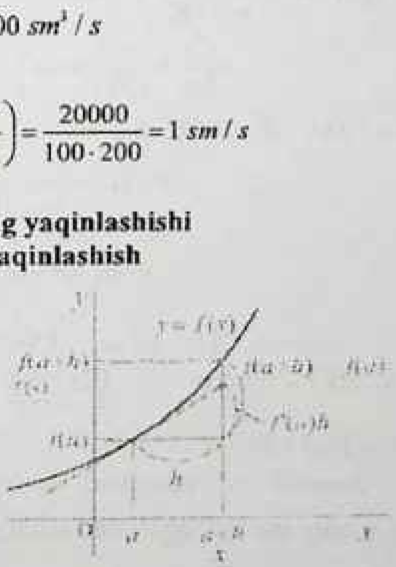
2.8. Funksiyaning yaqinlashishi
8A. Chiziqli yaqinlashish

Hosilaning ta'rifiga ko'ra

$$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}=f'(a)$$

Kichik $|h|$ lar uchun

$$\frac{f(a+h)-f(a)}{h}\approx f'(a)$$



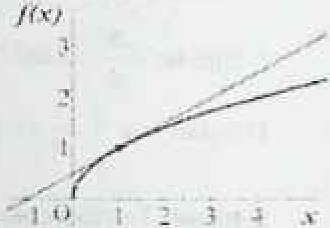
Bundan esa

$f(a+h) \approx f(a) + f'(a)h$

$a+h=x$ belgilashni qo'ysak,

$$f(x) \approx f(a) + f'(a)(x-a)$$

1-misol. $f(x)=\sqrt{x}$ ning $a=1$ bo'lgandagi taqribiy qiymatini hisoblang.



Yechimi. $f'(x)=\frac{1}{2}x^{-\frac{1}{2}}$ $\quad a=1$ da

$f(1)=\sqrt{1}=1$ va $f'(1)=\frac{1}{2}$

Shuning uchun

$\sqrt{x} \approx f(1)+f'(1)(x-1)=1+\frac{1}{2}(x-1)=\frac{1}{2}x+\frac{1}{2}$

Masalan: $x=1.1$ kalkulyatorda $\sqrt{1.1}=1.0488...$

Bu taqribiy hisoblashda $\sqrt{1.1}=\frac{1}{2}\times 1.1+\frac{1}{2}=1.05$. Bunda xatolik 1.1% bo'ladi.

2-misol. (1) h kichik bo'lganda, $\sin(a+h)$ ning chiziqli yaqinlashishini toping.

(2) $\sin 31^0$ ning taqribiy qiymatini toping.

Yechimi. (1) $f(x)=\sin x$ $\quad f'(x)=\cos x$. Formulaga ko'ra

$$\sin(a+h)=f(a+h)\approx f(a)+f'(a)h=\sin a+h\cos a$$

(2) $31^0=\frac{\pi}{6}+\frac{\pi}{180}$ bundan, $h=\frac{\pi}{180}$. Yuqoridagi tenglikka qo'ysak

$$\sin\left(\frac{\pi}{6}+\frac{\pi}{180}\right)\approx \sin\frac{\pi}{6}+\frac{\pi}{180}\cos\frac{\pi}{6}=\frac{1}{2}+\frac{\pi}{180}\cdot\frac{\sqrt{3}}{2}$$

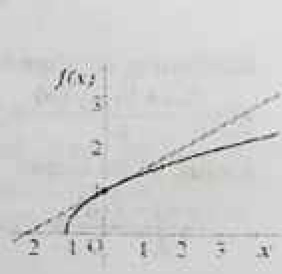
$\pi=3.142$, $\sqrt{3}=1.732$ deb olsak.

$$\sin 31^0 \approx \frac{1}{2}+\frac{3.142}{180}\cdot\frac{1.732}{2}\approx 0.5151$$

$|x|$ kichik bo'lgan holat

$a+h=x$ $\quad f(x)\approx f(a)+f'(a)(x-a)$

x kichik bo'lganda $\qquad$ Biz $a=0$ ni qo'yamiz



$f(x)\approx f(0)+f'(0)x$

3-misol. $|x|$ yetarlicha kichik bo'lganda, $\sqrt{1+x}$ ning taqribiy qiymatini toping.

Yechimi. $\left(\sqrt{1+x}\right)'=\frac{1}{2\sqrt{1+x}}$ shuning uchun

$f(x)\approx f(0)+f'(0)x=1+\frac{1}{2}x$

4-misol. $\cos 29^0$ ning taqribiy qiymatini ikkita usulda toping.

$f(x)=f(a+h)\approx f(a)+f'(a)h$ (1) formuladan foydalaning

$f(x)\approx f(0)+f'(0)x$ (2) formuladan foydalaning

Yechimi. $f(x)=\cos x$ ni (1) ga qo'yamiz, bundan $f'(x)=-\sin x$

$$\cos 29^0=\cos\left(\frac{\pi}{6}-\frac{\pi}{180}\right)\approx\cos\left(\frac{\pi}{6}\right)-\frac{\pi}{180}\sin\left(\frac{\pi}{6}\right)=$$

$$=\frac{\sqrt{3}}{2}-\frac{\pi}{180}\cdot\frac{1}{2}\approx\frac{1.732}{2}-\frac{3.142}{180}\cdot\frac{1}{2}=0.875$$

$g(x)=\cos\left(\frac{\pi}{6}+x\right)$ ni (2) ga qo'yamiz, bundan $g'(x)=-\sin\left(\frac{\pi}{6}+x\right)$

$$\cos 29^0=g\left(-\frac{\pi}{180}\right)\approx\cos\left(\frac{\pi}{6}\right)-\sin\left(\frac{\pi}{6}\right)\left(-\frac{\pi}{180}\right)=$$

$$=\frac{\sqrt{3}}{2}+\frac{\pi}{180}\cdot\frac{1}{2}\approx\frac{1.732}{2}+\frac{3.142}{180}\cdot\frac{1}{2}=0.875$$

8B. Teylor qatori

Aytaylik, $f(x)$ ning yoyilmasi quyidagicha bo'lsin:

$$f(x)=\sum_{n=0}^{\infty}a_n(x-c)^n=a_0+a_1(x-c)+a_2(x-c)^2+a_3(x-c)^3+...$$

Bu tenglikni ketma ket differensiallasak

$$f'(x)=a_1+2a_2(x-c)+3a_3(x-c)^2+4a_4(x-c)^3+...$$

$$f''(x)=2a_2+2\cdot 3a_3(x-c)+3\cdot 4a_4(x-c)^2+...$$

....

$x=c$ bo'lganda, bizda quyidagi hosil bo'ladi

$$f(c)=a_0,\ f'(c)=a_1,\ f''(c)=2a_2,\ ......\ f^{(k)}(c)=k!a_k,\ ......$$

ya'ni bunda koeffitsiyenlari quyidagicha yozish mumkin ekan:

$$a_k=\frac{f^{(k)}(c)}{k!}.$$

c nuqta atrofida $f(x)$ funksiyaning Teylor qatoriga yoyilmasi quyidagicha

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(c)}{n!}(x-c)^n=f(c)+f'(c)(x-c)+\frac{f''(c)}{2!}(x-c)^2+\frac{f'''(c)}{3!}(x-c)^3+...$$

$c=0$ uchun **Makloren qatori** hosil bo'ladi:

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\frac{f'''(0)}{3!}x^3+...$$

5-misol. $\sin x$ uchun $x=0$ da yettinchi tartibgacha taqribiy yoyilmasini toping.

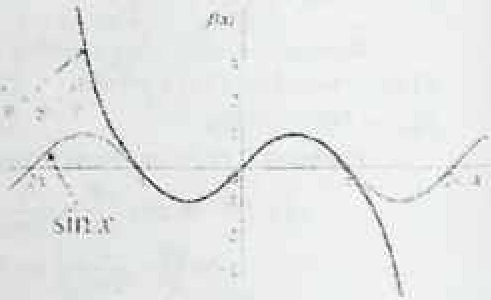
Yechimi. Biz yoyilmaga $f(x)=\sin x$ ni qo'yamiz.

$f'(x)=\cos x \quad f''(x)=-\sin x\ldots$

Umumiy holda

$$f^{(2n)}(x)=(-1)^n \sin x,$$
$$f^{(2n+1)}(x)=(-1)^n \cos x,$$

$x=0$ uchun:
$$f^{(2n)}(0)=0$$
$$f^{(2n+1)}(0)=(-1)^n$$



Bu qiymatlardan foydalanib quyidagiga ega bo'lamiz

$$\sin x = x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}.$$

6-misol. $f(x)=e^x$ uchun Makloren qatorini toping.

Yechimi. Bu funksiyaning barcha $n-$tartibli hosilalari $f^{(n)}(x)=e^x$ ga teng bo'ladi. Shunday qilib bu qiymatlar $x=0$ da quyidagiga teng bo'ladi

$$f(0)=f'(0)=f''(0)=\ldots=e^0=1.$$

Koeffitsiyentlari esa

$$a_k=\frac{f^{(k)}(0)}{k!}=\frac{1}{k!}$$

Shuning uchun

$$e^x=\sum_{k=1}^{\infty}\frac{x^k}{k!}=1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots$$

2.9. Boshlang'ich funksiya

9A Boshlang'ich funksiya

Biz funksiya berilganda uning hosilasini topishni o'rgandik. Masalan harakat tenglamasi berilganda tezlikni toppish mumkin. Endi teskari masalani qaraymiz. Hosila berilganda funksiyani toppish masalasini qaraylik:

Tezlik ma'lum bo'lsa $v(t)\left(=\frac{ds(t)}{dt}\right) \rightarrow s(t)$ bosib o'tilgan yo'l boshlang'ich bo'ladi.

$F(x)$ funksiya $f(x)$ ning boshlang'ich funksiyasi deyiladi agar $F'(x) = f(x)$ bo'lsa.

Misol. $F(x) = x^3$ funksiya $f(x) = 3x^2$ ning boshlang'ichidir, chunki

$$\frac{d}{dx}\left(x^3\right) = 3x^2$$

Aniqmas integral

$x^3, x^3 - 2, x^3 + 2$ lar $3x^2$ ning boshlang'ich funksiyasi.

Teorema. Agar $F'(x) = f(x)$ bo'lsa, har qanday boshlang'ich funksiya $F(x) + C$ (C:o'zgarmas) shaklga ega bo'ladi.

Bu $F(x) + C$ funksiya $f(x)$ ning aniqmas integrali deyiladi. Berilgan funksiyaning aniqmas integrali $\int f(x)dx$ kabi belgilanadi va teoremaga asosan, birorta $F(x)$ boshlang'ich funksiya bo'yicha quyidagi tenglik bilan aniqlanadi

$$\int f(x)\,dx = F(x) + C$$

Asosiy aniqmas integrallar

$$x^3 + C \quad \underset{Integral}{\overset{Hosila}{\rightleftarrows}} \quad 3x^2$$

Algebraik funksiyalar

<Hosila> <Integral>

$$\frac{d}{dx}x^{n+1} = (n+1)x^n \quad \rightarrow \quad \int x^n dx = \frac{x^{n+1}}{n+1} + C \qquad n \neq -1 \text{ uchun}$$

1-misol. $\int x^5 dx = \frac{1}{6}x^6 + C$

Eslatma. $n = -1$ uchun

$$\int x^{-1} dx = \frac{x^{-1+1}}{-1+1} + C = \frac{x^0}{0} + C \quad \text{ma'noga ega emas}$$

Grafik usuli

Differensial

Grafik: Og'shi:

$$y=f(x) \quad \rightarrow \quad \frac{df(x)}{dx}$$

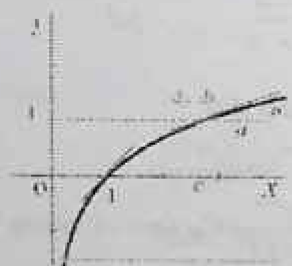
Integral

Og'ishi: Grafik:

$$\frac{df(x)}{dx} \quad \rightarrow \quad y=f(x)$$

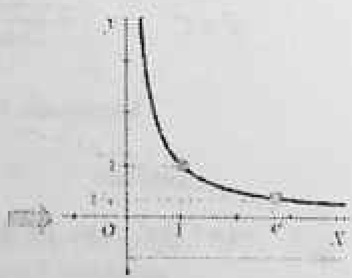
Differensialni grafik talqin qilish

Qanday qilib grafik orqali $y=\ln x$ dan $y=\frac{1}{x}$ hosilani topish mumkin.



Soha

$y=\ln x$

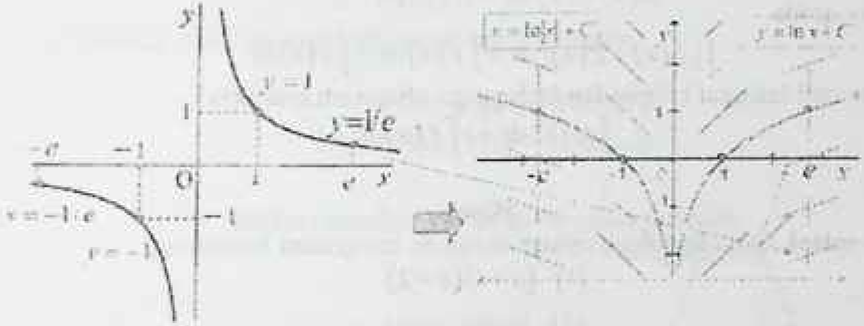


Soha

$y=\frac{d}{dx}\ln x=\frac{1}{x}$

Integralning grafik talqini

$y=\frac{1}{x}$ dan aniqmas integralni qanday topish mumkin



$$y=\frac{1}{x} \qquad\qquad y=\int\frac{1}{x}dx$$

Algebraik hosila

$x>0$ uchun ($y=\ln x$ aniqlanish sohasi)

$$\frac{d}{dx}\ln x=\frac{1}{x} \quad \Leftrightarrow \quad \int\frac{1}{x}dx=\ln x+C$$

$x<0$ uchun

Grafik ko'rinishidan, faraz qilaylik $y=\ln x$ $x=-u$ $(u>0)$ ni qo'yamiz. Differensiallash qoidasidan

$$\frac{dy}{dx}=\frac{d\left(\ln|x|\right)}{dx}=\frac{d\left(\ln u\right)}{du}\frac{du}{dx}=\frac{1}{u}\cdot(-1)=\frac{1}{x}$$

Barcha $x\neq 0$ uchun $\int\frac{1}{x}dx=\ln|x|+C$

Asosiy aniqmas integrallar – *davomi*

$$\frac{d}{dx}\sin x=\cos x \quad \Rightarrow \quad \int\cos x dx=\sin x+C$$

$$\frac{d}{dx}\cos x=-\sin x \quad \Rightarrow \quad \int\sin x dx=-\cos x+C$$

$$\frac{d}{dx}\tan x=\frac{1}{\cos^2 x} \quad \Rightarrow \quad \int\frac{1}{\cos^2 x}dx=\tan x+C$$

Ko'rsatkichli funksiyalar integrallari – *davomi*

Eksponensial funksiya integrali

$$\frac{d}{dx}e^x=e^x \quad \Rightarrow \quad \int e^x dx=e^x+C$$

$$\frac{d}{dx}a^x=a^x\ln a \quad \Rightarrow \quad \int a^x dx=\frac{a^x}{\ln a}+C$$

Integralning asosiy xossalari

Umumiy qoida

$$\int\{f(x)+g(x)\}dx=\int f(x)dx+\int g(x)dx$$

O'zgarmasni integral belgisidan tashqariga chiqarish mumkin

$$\int cf(x)dx=c\int f(x)dx$$

Misol

2-misol. Quyidagi funksiyalar aniqmas integralini hisoblang.

$$(1)\quad (x+1)(x-2)$$

$$(2)\quad \sin x+2\cos x$$

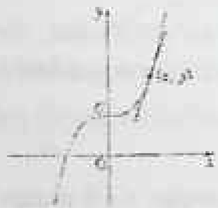
Yechimi.

$$(1)\quad \int(x+1)(x-2)dx\int\left(x^2-x-2\right)dx=\int x^2dx-\int xdx-2\int dx=\frac{x^3}{3}-\frac{x^2}{2}-2x+C$$

$$(2)\quad \int(\sin x+2\cos x)dx=\int\sin x+2\int\cos xdx=-\cos x+2\sin x+C$$

3-misol $f(x)=3x^2$ funksiyaning $(1,2)$ nuqtadan o'tuvchi boshlang'ich funksiyasini toping.

Yechimi.



$f'(x)=3x^2$

$f(x)=\int 3x^2dx=x^3+C$

Ushbu chiziq $(1,2)$ nuqtadan o'tganligi sababli,

$$f(1)=1+C=2\quad C=1$$

$$y=x^3+1$$

9B. O'rniga qo'yish usuli. Bo'laklab integrallash
O'rniga qo'yish usuli (1)

Integral ostidagi funksiya $f(u(x))u'(x)$ ko'rinishga ega bo'lganda, quyidagicha belgilashlar qo'llaniladi

Masalan

$$\left(x^2+1\right)^5x \qquad\qquad \sin^5x\cos x$$

$$u=x^2+1\quad u'=2x \qquad\qquad u=\sin x\quad u'=\cos x$$

$F'(u)=f(u)$ bo'lsin, bunda

$$F(u)=\int f(u)du \tag{1}$$

Differensiallash qoidasiga ko'ra $\frac{d}{dx}F(u(x))=F'(u)u'(x)=f(u)u'(x)$

$$F\left(u(x)\right)=\int f(u)u'(x)dx \qquad (2)$$

(1) va (2) integral orqali biz quyidagini yozamiz

O'rniga qo'yish usuli

$$\int f\left(u(x)\right)u'(x)dx=\int f(u)du$$

Differensiallar yordamida almashtirish

Differensial

$$u=u(x) \;\rightarrow\; \frac{du}{dx}=u'(x) \;\rightarrow\; du=u'(x)dx$$

$$du=\frac{du}{dx}dx$$

Masalan agar $u=x^3$, bundan $du=3x^2dx$

$$\int f(u)u'(x)dx=\int f(u)du$$

$$du=\frac{du}{dx}dx$$

4-misol. Quyidagi funksiyalarning aniqmas integralini toping.

(1) $\int\left(x^2+1\right)^5dx$ (2) $\int\sin^5x\cos xdx$

Yechimi. (1) Biz $x^2+1=u$ b=ni qo'yamiz, bundan

$$2x\frac{dx}{du}=1 \qquad xdx=\frac{1}{2}du$$

O'rniga qo'yamiz

$$\int\left(x^2+1\right)^5xdx=\int u^5\frac{1}{2}du=\frac{1}{12}u^6+C=\frac{1}{12}\left(x^2+1\right)^6+C$$

(2) Biz $\sin x=u$

$\cos x\dfrac{dx}{du}=1 \quad \cos xdx=\dfrac{1}{\cos x}du$ o'rniga qo'yamiz

$$\int\sin^5x\cos xdx=\int u^5du=\frac{1}{6}u^6+C=\frac{1}{6}\sin^6x+C$$

O'rniga qo'yish usuli (2)

$$\int f(x)dx=\int f\left(x(t)\right)x'(t)dt$$

Buni quyidagicha yozamiz

$$\int f(x)dx=\int f\left(x(t)\right)\frac{dx}{dt}dt$$

5-misol. Quyidagi funksiyaning aniqmas integralini toping

$$\int x\sqrt{1-x}dx$$

Yechimi. $\sqrt{1-x}=t$ deb belgilaymiz, bunda $x=1-t^2$ va $\dfrac{dx}{dt}=-2t$.

Shuning uchun

$$\int x\sqrt{1-x}dx=\int(1-t^2)t(-2t)dt=\int(1-t^2)t(-2t)dt=$$

$$=2\int(t^4-t^2)dt=2\left(\frac{t^5}{5}-\frac{t^3}{3}\right)+C=-\frac{2}{15}(2+3x)(1-x)\sqrt{1-x}+C$$

Bo'laklab integrallash

Biz ko'paytmaning hosilasini bilamiz

$$\{f(x)g(x)\}'=f'(x)g(x)+f(x)g'(x)$$

Bundan esa biz quyidagini hosil qilamiz

$$f(x)g'(x)=\{f(x)g(x)\}'-f'(x)g(x)$$

Bu tenglikni integrallab, biz quyidagi **bo'laklab integrallash** formulasini olamiz

$$\int f(x)g'(x)dx=f(x)g(x)-\int f'(x)g(x)dx$$

6-misol. Quyidagi integralni hisoblang.

$$\int x\cos xdx$$

Yechimi: $\int x\cos xdx=x\sin x-\int 1\cdot\sin xdx=x\sin x+\cos x+C$

7-misol. Quyidagilarni hisoblang.

(1) $\int\cos^2 x\sin xdx$ (2) $\int xe^{6x}dx$

Yechimi.

(1) $(\cos x)'=-\sin x$ bundan, $\cos x=u$ ni qo'yamiz $-\sin x=\dfrac{du}{dx}$

Shuning uchun $\int\cos^2 x\sin xdx=\int u^2(-du)=-\frac{1}{3}u^3+C=-\frac{1}{3}\cos^3 x+C$

(2) $$\int xe^{6x}dx=x\left(\frac{e^{6x}}{6}\right)-\int 1\cdot\left(\frac{e^{6x}}{6}\right)dx=\frac{x}{6}e^{6x}-\frac{x}{36}e^{6x}+C$$

2.10. Aniq integral

10A.Yuzani aniqlash va boshlang'ich funksiya. Egri chiziqli sohaning yuzasi

ABQP sohaning yuzasi $S(x)$ bo'lsin x - h ga orttirilsa, yuza esa quyidagicha ortadi

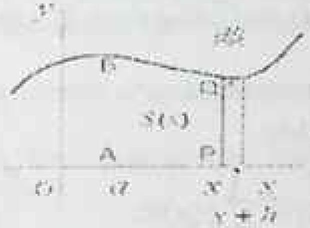
$$dS = S(x + h) - S(x) \approx hf(x)$$

$S(x)$ ning hosilasi esa

$$\frac{dS}{dx} = \lim_{h\to 0}\frac{S(x+h) - S(x)}{h} \approx f(x)$$

$S(x)$ funksiyaning yuzasi- $f(x)$ funksiyaning boshlang'ichlaridan biriga teng bo'ladi, ya'ni

$$S'(x) = f(x)$$



Aniq integral va yuza

$F(x)$ funksiya berilgan funksiyaning boshlang'ichi bo'lsin.

$$S(x) = F(x) + C$$

Berilganlarga ko'ra $x = a$ da yuza nolga teng $S(a) = 0$. U holda

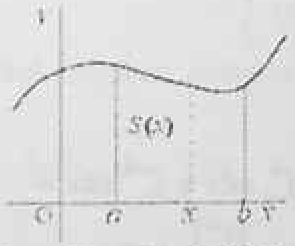
$$0 = F(a) + C$$

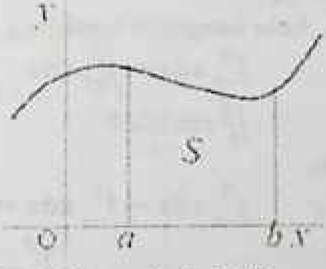
$$C = -F(a)$$

Demak, $S(x) = F(x) - F(a)$

Agar $x = b$ bo'lsa, biz to'liq sohaning yuzini hosil qilamiz

$$S = F(b) - F(a)$$





$F(b) - F(a)$ ga $f(x)$- ning [a,b] dagi aniq integrali deyiladi va

$$\int_a^b f(x)dx$$

yoki

$$F(b) - F(a) = [F(x)]_a^b$$

kabi belgilanadi.

Shuningdek

$$\int_a^b f(x)dx = F(b) - F(a) = [F(x)]_a^b$$

kabi yoziladi.

Aniq integralning xossalari

Aniq integralning chiziqlilik xossasi

(1) $\int_a^b kf(x)dx = k\int_a^b f(x)dx$

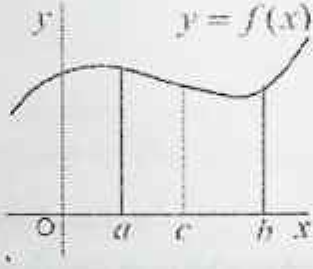
(2) $\int_a^b \{f(x) + g(x)\}dx = \int_a^b f(x)dx + \int_a^b g(x)dx$

(3) Teskari integrallash xossasi

$\int_a^b f(x)dx = -\int_b^a f(x)dx$

(4) Additivlik xossasi

$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$



1-misol. Aniq integralni hisoblang.

$$\int_{-2}^{4} (x^2 - 3x + 1)dx$$

Yechimi. $\int_{-2}^{4}(x^2 - 3x + 1)dx = \int_{-2}^{4} x^2dx - 3\int_{-2}^{4} xdx + \int_{-2}^{4} 1dx =$

$$= \left[\frac{x^3}{3}\right]_{-2}^{4} - 3\left[\frac{x^2}{2}\right]_{-2}^{4} + [x]_{-2}^{4} = \frac{4^3 - (-2)^3}{3} - 3\cdot\frac{4^2 - (-2)^2}{2}(4 - (-2))$$

$$= 12$$

2-misol. Aniq integralni hisoblang.

(1) $\int_{-1}^{3} xdx - \int_{-1}^{1} xdx$

(2) $\int_0^{\frac{\pi}{2}} sin2\theta d\theta$

Yechimi.

(1) $\int_{-1}^{3} xdx - \int_{-1}^{1} xdx = \int_{-1}^{3} xdx + \int_{1}^{-1} xdx =$

$$= \int_1^3 xdx = \left[\frac{x^2}{2}\right]_1^3 = 4$$

(2) $\int_0^{\frac{\pi}{2}} sin2\theta d\theta = \left[\frac{-cos2\theta}{2}\right]_0^{\frac{\pi}{2}} = \frac{-(-1)}{2} - \frac{(-1)}{2} = 1$

3-misol. Aniq integralni hisoblang.

$$\int_0^1 (x^2 - x)dx - \int_1^1 (x^2 - x)dx$$

Yechimi.

$$\int_0^1 (x^2 - x)dx - \int_1^1 (x^2 - x)dx = \int_0^3 (x^2 - x)dx == \left[\frac{x^3}{3}\right]_0^3 - \left[\frac{x^2}{2}\right]_0^3 = \frac{9}{2}$$

10B. Bo'laklab integrallash. O'zgaruvchilarni almashtirib integrallash (1)

Aniq integralni quyidagi usullarda o'zgaruvchini almashtirib integarallash mumkin.

(1)Funksiyani $x = g(t)$ kabi belgilaymiz. Agar

$a = g(\alpha)$ va $b = g(\beta)$

$$\int_a^b f(x)dx = \int_\alpha^\beta f(g(t))\, g'(t)dt$$

(2)Belgilab integrallash

Agar $\alpha = u(a)$ va $\beta = u(b)$ u holda

$$\int_a^b f(u(x))u'(x)dx = \int_\alpha^\beta f(u)\, du$$

4-misol. Quyidagi aniq integralni belgilash orqali yeching.

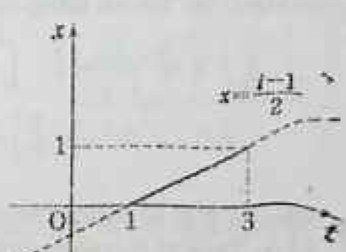
$$\int_0^1 (2x + 1)^3 dx$$

Yechimi. Belgilash kiritamiz $2x + 1 = t$

U holda

$$x = \frac{t-1}{2}, \quad dx = \frac{1}{2}dt$$

x	$0 \to 1$
t	$1 \to 3$



Demak,

$$\int_0^1 (2x+1)^3 dx = \int_1^3 t^3 \cdot \frac{1}{2} dt = \frac{1}{2}\int_1^3 t^3 dt = \frac{1}{2}\left[\frac{t^4}{4}\right]_1^3 = 10$$

5-misol. Quyidagi aniq integralni hisoblang

$$\int_0^2 x^2\sqrt{x^3+1}\,dx$$

Yechimi. Agar $u = x^3 + 1$ kabi belgilasak, bunda $du = 3x^2dx$

U holda

$$\int_0^2 x^2\sqrt{x^3+1}\,dx = \int_1^9 \sqrt{u}\left(\frac{1}{3}du\right) = \frac{1}{3}\int_1^9 u^{\frac{1}{2}}du = \frac{1}{3}\left[\frac{2}{3}u^{\frac{3}{2}}\right]_1^9 = \frac{52}{9}$$

Bo'laklab integrallash

Bizga quyidagi tenglik ma'lum

$$\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx$$

Aniq integral holida esa quyidagi hosil bo'ladi

$$\int_a^b f(x)g'(x)dx = [f(x)g(x)]_a^b - \int_a^b f'(x)g(x)dx$$

6-misol. Aniq integralni hisoblang

$$\int_{-3}^{-1}\frac{dx}{(2x+1)^3}$$

Yechimi. Belgilash kiritamiz $2x + 1 = t$. U holda

$x = \frac{t-1}{2}$, $dx = \frac{1}{2}dt$ va integralni chegaralari o'zgaradi

$$t(-3) = -5, \qquad t(-1) = -1.$$

Demak

$$\int_{-3}^{-1}\frac{dx}{(2x+1)^3} = \int_{-5}^{-1}\frac{dt}{2(t)^3} = \frac{1}{2}\left[\frac{t^{-2}}{-2}\right]_{-5}^{-1} = \frac{-6}{25}$$

7-misol. Quyidagi integralni hisoblang.

$$\int_0^2 x^2e^x\,dx$$

Yechimi. Bo'laklab integrallash formulasidan foydalansak,

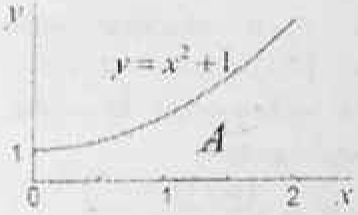
$$\int_0^2 x^2e^x dx = \left[x^2e^x\right]_0^2 - \int_0^2 2xe^x dx = 4e^2 - 2\left(\left[xe^x\right]_0^2 - \int_0^2 e^x dx\right)$$

$$= 4e^2 - 2\left(2e^2 - \left[e^x\right]_0^2\right) = 4e^2 - 4e^2 + 2\left(e^2 - 1\right) = 2\left(e^2 - 1\right)$$

2.11. To'g'ri to'rtburchaklar yordamida soha yuzini baholash

11A. To'g'ri to'rtburchak yordamida soha yuzini baholash

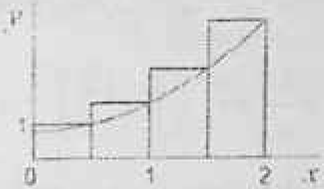
$y = x^2 + 1$ va $y = 0$ to'g'ri chiziqlar bilan chegaralangan A sohaning $[0,2]$ kesmadagi yuzini hisoblang. (Aytaylik, aniq integralni bilmaymiz).



Ushbu masalani yechish uchun berilgan [0,2] kesmadagi sohani ushbu tartibda 4 ta qismga bo'lamiz va quyidagi hisobni amalga oshiramiz.

$$A_R = \frac{1}{2}f\left(\frac{1}{2}\right) + \frac{1}{2}f(1) + \frac{1}{2}f\left(\frac{3}{2}\right) + \frac{1}{2}f(2) =$$

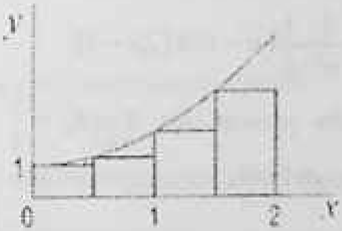
$$= \frac{1}{2}\left(\frac{5}{4}\right) + \frac{1}{2}(2) + \frac{1}{2}\left(\frac{13}{4}\right) + \frac{1}{2}(5) = 5.75$$



So'ngra berilgan A sohani yana ushbu tartibda 4 ta qismga ajratib quyidagi hisobni amalga oshiramiz.

$$A_L = \frac{1}{2}f(0) + \frac{1}{2}f\left(\frac{1}{2}\right) + \frac{1}{2}f\left(\frac{3}{2}\right) =$$

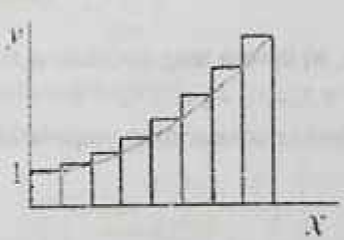
$$= \frac{1}{2}(1) + \frac{1}{2}\left(\frac{5}{4}\right) + \frac{1}{2}(2) + \frac{1}{2}\left(\frac{13}{4}\right) = 3.75$$



Demak, berilgan A sohaning yuzi $3.75 < A < 5.75$ oraliqda ekan.
Endi esa aynan shu hisobni berilgan sohani aynan 8 ta bo'lakka bo'lib amalga oshiramiz. U holda hisob-kitoblardan so'ng berilgan sohaning yuzi $4.1875 < A < 5.1875$ oraliqda ekanligini topamiz.
Aniq qiymat esa

$$A = \int_0^2 \left(x^2 + 1\right)dx = \left[\frac{x^3}{3} + x\right]_0^2 = \frac{14}{3} = 4.666$$

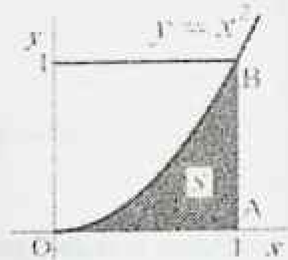
ga teng bo'lar ekan.



Teorema. Agar $f(x)$ $[a, b]$ kesmada uzluksiz bo'lsa, o'ng va chap chegara nuqtalar bo'yicha olingan yuzalar $N \to \infty$ da bir hil qiymatga yaqinlashadi, ya'ni

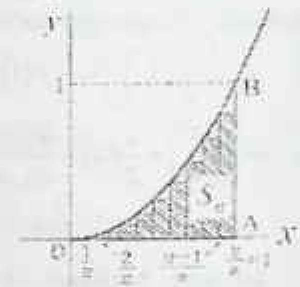
$$\lim_{N \to \infty} A_R = \lim_{N \to \infty} A_L = L.$$

1-misol. $y=x^2$ va $y=0$ chiziqlar bilan chegaralangan sohaning $[0,1]$ kesmadagi yuzini $n(n\to\infty)$ ta bo‘lakka bo‘lish orqali hisoblang. Quyidagi formuladan foydalaning.



$$1^2+2^2+3^2+...+n^2=\frac{n(n+1)(n+2)}{6}$$

Yechimi. $[0,1]$ kesmani n ta bo‘lakka bo‘lib, rasmda keltirilgan to‘rtburchaklar bo‘yicha yig‘indi tuzamiz



$$S_n=\frac{1}{n}\left\{0+\left(\frac{1}{n}\right)^2+\left(\frac{2}{n}\right)^2+...+\left(\frac{n+1}{n}\right)^2\right\}=$$

$$=\frac{1}{n^3}\left\{1^2+2^2+...+(n-1)^2\right\}=$$

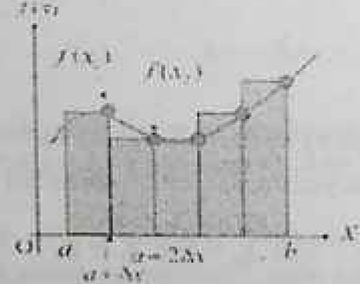
$$=\frac{1}{n^3}\cdot\frac{1}{6}(n-1)n(2n-1)$$

Bu yerdan esa, $\lim\limits_{n\to\infty}S_n=\frac{1}{3}$. Agar bu qiymatni integarallash yo‘li bilan taqqoslasak

$$S=\int_0^1 x^2dx=\left[\frac{x^3}{3}\right]_0^1=\frac{1}{3}.$$

Aniq integralning ta’rifi

[a, b] kesma teng ravishda n ta bo‘lakka ajratilgan bo‘lsin:
$a=x_0, x_1, x_2, ..., x_n=b$ va orttirma $\Delta x=(b-a)/n$ bo‘lsin. U holda o‘ng chetki nuqtalarining yaqinlashtirishdagi taqribiy qiymat



$$A_{Rn} = f(x_1)\Delta x + f(x_2)\Delta x + \ldots + f(x_n)\Delta x =$$
$$= \sum_{k=1}^{n} f(x_k)\Delta x$$
ga teng bo'lar ekan (f ning $[a,b]$ intervaldagi Riman yig'indisiga ko'ra).

Chap chetki nuqtalar bo'yicha olingan limit: $\int_a^b f(x)dx = \lim_{n\to\infty}\sum_{k=0}^{n-1} f(x_k)\Delta x$

O'ng chetki nuqtalar bo'yicha olingan limit: $\int_a^b f(x)dx = \lim_{n\to\infty}\sum_{k=1}^{n} f(x_k)\Delta x$

Agar bu limitlar o'zaro teng bo'lsa, unga f funksiyadan $[a,b]$ kesma bo'yicha olingan aniq integral deyiladi.

11B. Integral osti ifodaning ketma-ketlik qiymatini toppish

Agar biz $a=0, b=1$ qiymatlarni qo'ysak

$$\int_0^1 f(x)dx = \lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1} f\left(\frac{k}{n}\right) \quad \textit{Chap chetki nuqtalar bo'yicha}$$
$$= \int_0^1 f(x)dx = \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right) \quad \textit{O'ng chetki nuqtalar bo'yicha}$$

ekanligini topamiz. Bu munosabatlar ba'zida cheksiz qatorlarning qiymatini topishda ishlatiladi.

2-misol. Berilgan limitni toping

$$S = \lim_{n\to\infty}\left(\frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \ldots + \frac{1}{2n}\right).$$

Yechimi. Barcha qo'shiluvchilardan $\frac{1}{n}$ ni chiqarib olamiz

$$S = \lim_{n\to\infty}\frac{1}{n}\left(\frac{1}{1+\frac{1}{n}} + \frac{1}{1+\frac{2}{n}} + \frac{1}{1+\frac{3}{n}} + \ldots + \frac{1}{1+\frac{n}{n}}\right) = \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\left(\frac{1}{1+\frac{k}{n}}\right).$$

$\frac{k}{n}$ ni x bilan almashtirsak, bu limit $\frac{1}{1+x}$ funksiyaning integral yig'indisi ekanligini ko'ramiz. Demak integral ta'rifiga ko'ra

$S = \int_0^1 \frac{dx}{1+x} = \left[\ln(1+x)\right]\Big|_0^1 = \ln 2$ ekanini topamiz.

3-misol. Limitni hisoblang

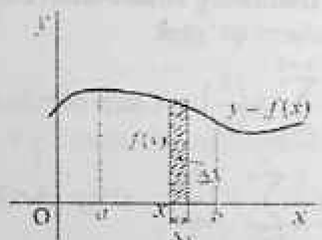
$$S=\lim_{n\to\infty}\frac{1}{n}\left(\sin\frac{\pi}{n}+\sin\frac{2\pi}{n}+\sin\frac{3\pi}{n}+...+\sin\frac{n\pi}{n}\right).$$

Yechimi. Yuqoridagi kabi

$$S=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\sin\left(\frac{k}{n}\pi\right)=\int_0^1\sin(\pi x)dx=\left[-\frac{\cos(\pi x)}{\pi}\right]\Bigg|_0^1=$$

$$=-\left(\frac{\cos\pi-1}{\pi}\right)=\frac{2}{\pi}.$$

2.12. Integrallarning tadbiqlari (1-qism)
12A. Tekislaikda yassi sohalar

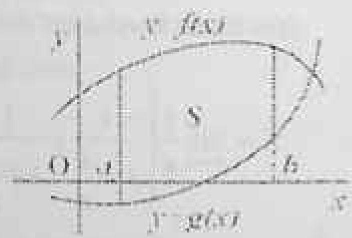


Strixlangan to'g'ri to'rtburchak yassi sohaning yuzi $\Delta S=f(x)dx$ bu erda, $f(x)>0$. U holda [a, b] kesma bo'yicha umumiy yuza uchun

$$S\approx\sum f(x)\Delta x \quad \rightarrow, \quad S=\int_a^b f(x)dx$$

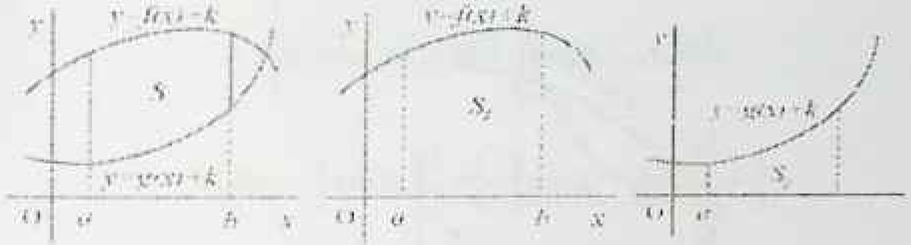
Agar [a, b] kesmada $f(x)\geq g(x)$ bo'lsa,

$$S=\int_a^b[f(x)-g(x)]dx$$



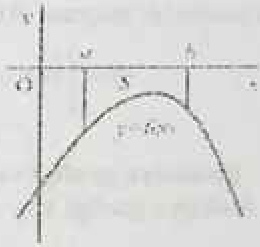
Isbot. $y=f(x)+k$, $y=g(x)+k$

$$S=S_1-S_2=\int_a^b[f(x)+k]dx-\int_a^b[g(x)+k]dx=\int_a^b[f(x)-g(x)]dx$$



Manfiy qiymatli funksiya bo'lgan hol

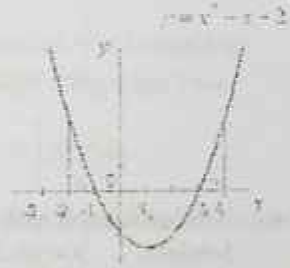
$[a,b]$ kesmada $f(x) < 0$ bo'lsa, yuqoridagi formulaga asosan



$$S = \int_a^b [0 - f(x)]dx = -\int_a^b f(x)dx$$

1-misol. $y = x^2 - x - 2$ va x o'qi bilan chegaralangan sohaning $x=-2$ va $x=3$ nuqtalar orasidagi qismining yuzini toping.

Yechimi. $y = x^2 - x - 2 = (x-2)(x+1)$



Sohalar S_1, S_2 va S_3 bo'lsin

$$S_1 = \int_{-2}^{-1}(x^2 - x - 2)dx = \left(\frac{x^3}{3} - \frac{x^2}{2} - 2x\right)\Bigg|_{-2}^{-1} = 1.833$$

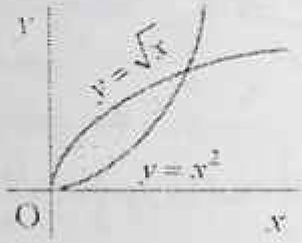
$$S_2 = -\int_{-1}^{2}(x^2 - x - 2)dx = 4.5$$

$$S_3 = -\int_{2}^{3}(x^2 - x - 2)dx = 1.833$$

O'z navbatida $S = S_1 + S_2 + S_3 \approx 1.833 + 4.5 - 1.833 = 8.166$

2-misol. $y = \sqrt{x}$ va $y = x^2$ funksiyalar bilan chegaralangan sohaning yuzini toping.

Yechimi. Chizmada soha ko'k shtrix chiziq bilan berilgan.Kesishish nuqtasini topamiz



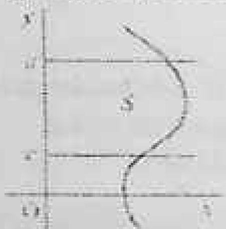
$$\sqrt{x}=x^2,\ x=x^4,\ \ x(x-1)(x^2+x+1)=0.$$

Bu yerdan kesishish nuqtasu (0,0), (1,1). Yuza esa

$$S=\int_0^1\left(\sqrt{x}-x^2\right)dx=\left(\frac{2}{3}x^{\frac{3}{2}}-\frac{x^3}{3}\right)\Bigg|_0^1=\frac{1}{3}$$

Funksiya grafigi va y o'qi bilan chegaralangan soha

$x=g(y)$ funksiya grafigi va y o'qi bilan chegaralangan soha yuzi



$$S=\int_c^d g(y)dy$$

formula bilan hisoblanadi.

3-misol. $y=\sin 2x$ va $y=\cos x$ funksiyalar bilan chegaralangan sohaning yuzini toping, bu yerda $0\le x\le\frac{\pi}{2}$.

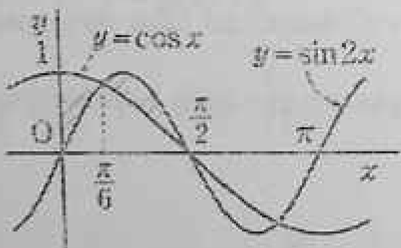
Yechimi. $\sin 2x=\cos x$ deb tenglashtiramiz

$$\cos x(2\sin x-1)=0$$

$$\therefore\ \cos x=0 \text{ yoki } \sin x=1/2$$

Bu yerdan ko'rsatilgan kesmadagi ildizlar $x=\pi/2,\ \pi/6$

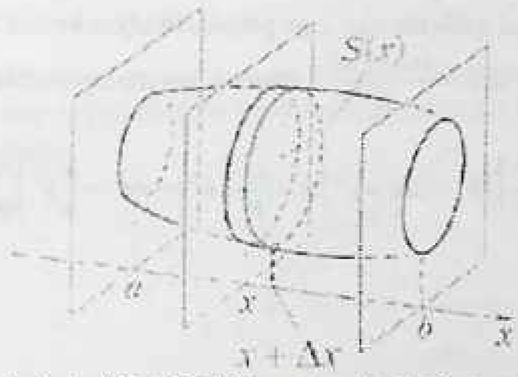
Demak, rasn



$$S=\int_0^{\frac{\pi}{6}}(\cos x-\sin 2x)dx+\int_{\pi/6}^{\pi/2}(\sin 2x-\cos x)dx=$$

$$=\left(\sin x-\frac{1}{2}\cos 2x\right)\Bigg|_0^{\frac{\pi}{6}}+\left(-\frac{1}{2}\cos 2x-\sin x\right)\Bigg|_{\frac{\pi}{6}}^{\frac{\pi}{2}}=\frac{1}{2}$$

12B. Aylanma jismlarning hajmlari

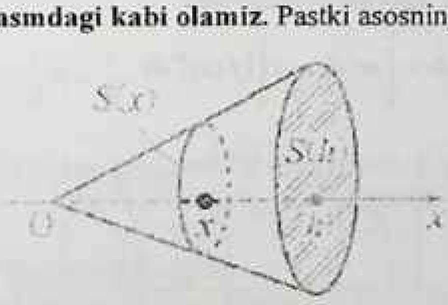


Silindrning hajmi $\Delta V\approx S(x)\Delta x$ ga teng bo'ladi. $x=a$ va $x=b$ orasidagi umumiy hajm esa, limit yordamida hisoblanib, quyidagi integralga teng

$$V\approx\sum S(x)\Delta x \qquad\rightarrow\qquad V=\int_a^b S(x)dx$$

4-misol. Asosi radiusi r va balandligi h konusning hajmini toping.

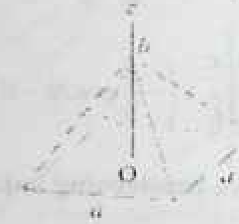
Yechimi. x **o'qini rasmdagi kabi olamiz.** Pastki asosning yuzi $S(h)=\pi r^2$



Konuslar o'xshasjligidan $S(x):S(h)=x^2:r^2$ yoki $S(x)=\frac{\pi r^2}{h^2}x^2$

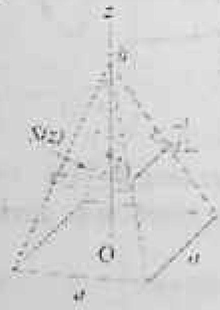
$$V=\int_0^h\left(\frac{\pi r^2}{h^2}x^2\right)dx=\frac{\pi r^2}{h^2}\left(\frac{x^3}{3}\right)\Bigg|_0^h=\frac{\pi}{3}r^2h$$

5-misol. Gorizontal kvadrat kesimga ega bo'lgan piramidaning hajmini toping. Pastki asosning tomoni a va balandligi h ga teng



Yechimi. z o'qini vertikal qilib olamiz. z ga perpendikulyar kesim uchun: $S(z):a^2=(h-z)^2:h^2$, $S(z)=\dfrac{(h-z)^2a^2}{h^2}$. Barcha parallelepipedlar bo'yicha yig'sak,

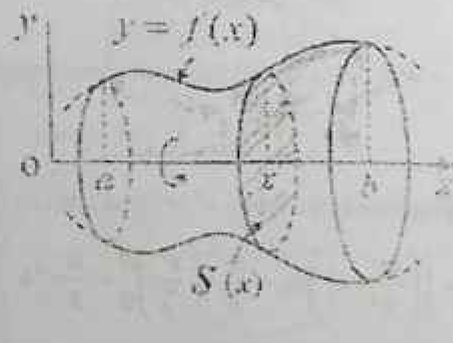
$$V=\int_0^h\frac{(h-z)^2a^2}{h^2}dz=\frac{a^2}{h^2}\int_0^h\left(h^2-2hz+z^2\right)dz=\frac{a^2}{h^2}\left(h^2z-hz^2+\frac{1}{3}z^3\right)\Big|_0^h=\frac{1}{3}a^2h$$



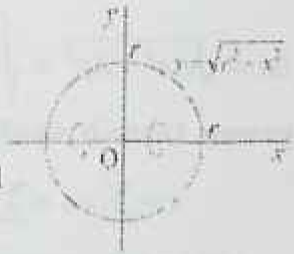
Umumiy holda aylanma jismning hajmi

Jism x o'qi atrofida aylanganida

$$V=\int_a^b S(x)dx=\int_a^b \pi r^2dx=\pi\int_a^b(f(x))^2dx$$



6-misol. r radiusli sharning hajmini toping



Yechimi. r radiusli aylana yuqori qismi tenglamasi

$$y=\sqrt{r^2-x^2}$$

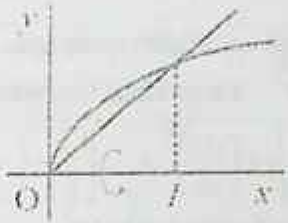
dan iborat. Bu yarim doirani aylantirib shar hosil qilamiz.

Bu sharning hajmi yuqoridagi formulaga asosan

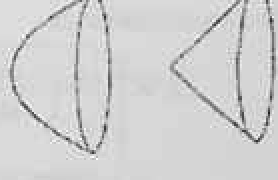
$$V=\int_{-r}^{r}\pi y^2dx=\int_{-r}^{r}\pi(r^2-x^2)dx=\pi\left[r^2x-\frac{x^3}{3}\right]_{-r}^{r}=\frac{4}{3}\pi r^3$$

7-misol. $f(x)=\sqrt{x}$ va $y=x$ chiziqlar bilan chegaralangan sohani Ox o'qi atrofida aylantirishdan hosil bo'lgan jismning hajmini toping.

Yechimi.



Bu hajmni ikkita aylanma jismlar hajmlari A dan B ni ayirishdan hosil qilish mumkin. Bu yerda A $f(x)=\sqrt{x}$ ni, B esa $y=x$ ni Ox o'qi atrofida aylantirishdan hosil bo'lgan jism.



A: $V_A=\pi\int_0^1(\sqrt{x})^2dx=\pi\left[\frac{1}{2}x^2\right]_0^1=\frac{1}{2}\pi$

B: $V_B=\pi\int_0^1(x)^2dx=\pi\left[\frac{1}{3}x^3\right]_0^1=\frac{1}{3}\pi$

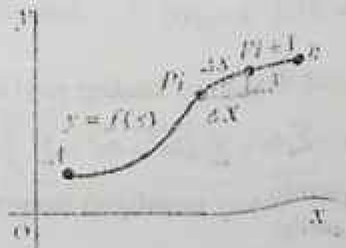
$V=V_A-V_B=\frac{1}{2}\pi-\frac{1}{3}\pi=\frac{1}{6}\pi$.

2.13. Integrallarning tadbiqlari (2-qism)

13A Egri chiziq yoyi uzunligi.

Teislikdagi egri chiziq

$y=f(x)$ funksiya berilgan bo'lsin. Uning grafigining A va B nuqtalar orasidagi yoyini qaraymiz. P_i va P_{i+1} nuqtalar orasidagi Δs yoyi uzunligi taqriban



$$\Delta s \approx \sqrt{\Delta x^2+\Delta y^2} \approx \sqrt{1+\left(\frac{\Delta y}{\Delta x}\right)^2}\Delta x$$

ga teng. Umumiy uzunligi esa

$$s \approx \sum \Delta s \approx \sum \sqrt{1+\left(\frac{\Delta y}{\Delta x}\right)^2}\Delta x$$

Bu yoyning aniq qiymati quyidagi integralga teng

$$s=\int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}dx=\int_a^b \sqrt{1+(f'(x))^2}dx$$

1-misol. $x^2+y^2=r^2$ aylana uzunligini hisoblang.

Yechimi. I chorakdagi uzunligini qaraymiz.

Bu erda $y=\sqrt{r^2-x^2}$. Yuqoridagi formulaga asosan

$$s_1=\int_0^r \sqrt{1+\left(\frac{dy}{dx}\right)^2}dx=\int_0^r \sqrt{1+\left(-\frac{x}{y}\right)^2}dx$$

Chunki oshkormas funksiya hosilasiga ko'ra

$$2x+2y\frac{dy}{dx}=0$$

$$\frac{dy}{dx}=-\frac{x}{y}$$

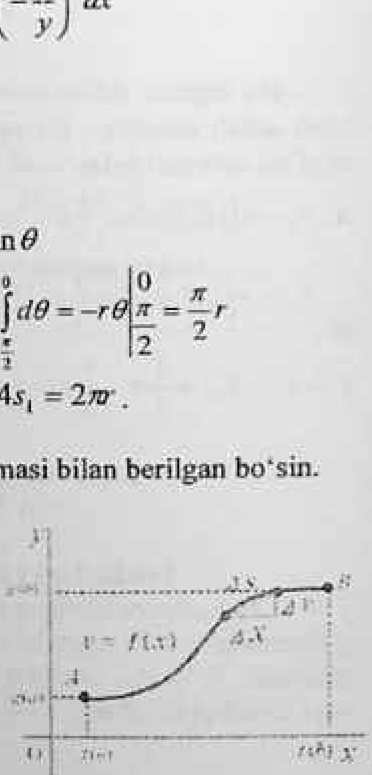
Qutb koordinatalarida $x=r\cos\theta,\ y=r\sin\theta$

$$s_1=\int_{\frac{\pi}{2}}^0 \sqrt{1+\left(-\frac{\cos\theta}{\sin\theta}\right)^2}(-r\sin\theta)d\theta=-r\int_{\frac{\pi}{2}}^0 d\theta=-r\theta\Big|_{\frac{\pi}{2}}^{0}=\frac{\pi}{2}r$$

Jami aylana uzunligi $s=4s_1=2\pi r$.

Tekislikdagi egri chiziq parametrik tenglamasi bilan berilgan bo'sin.

$x=f(t),\ y=g(t)$



Uning kichik bo'lakdagi yoyi uzunligi

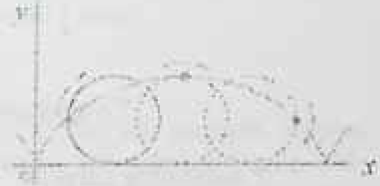
$$s=\sum\Delta s=\sum\sqrt{\Delta x^2+\Delta y^2}=\sum\sqrt{\left(\frac{\Delta x}{\Delta t}\right)^2+\left(\frac{\Delta y}{\Delta t}\right)^2}\Delta t$$

Umumiy yoy uzunligini integrallash yoli bilan topamiz

$$s=\int_a^b\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt=\int_a^b\sqrt{(f'(t))^2+(g'(t))^2}\,dt$$

2 misol Quyidagi sikloidaning uzunligini toping.

$x=2(t-\sin t),\quad y=2(1-\cos t),\ (0\le t\le 2\pi)$



[Eslatma] Sikloida bu g'ildirak to'g'ri chiziqli harakat qilganida g'ildirak chetidagi nuqtaning qoldirgan izi.

Yechimi.

$$\frac{dx}{dt}=2(1-\cos t),\quad \frac{dy}{dt}=2\sin t.$$

Yoyning umumiy uzunligi esa

$$s=\int_0^{2\pi}\sqrt{4(1-\cos t)^2+4\sin^2 t}\,dt=2\int_0^{2\pi}\sqrt{2(1-\cos t)}\,dt=4\int_0^{2\pi}\sqrt{\sin^2\frac{t}{2}}\,dt$$

$0\le t\le 2\pi$ kesmada $\sin\frac{t}{2}\ge 0$. Shuning uchun

$$s=4\int_0^{2\pi}\sin\frac{t}{2}\,dt=4\left[-2\cos\frac{t}{2}\right]_0^{2\pi}=16.$$

3-misol. $r=2\sin\theta\quad(0\le\theta\le\pi)$ chiziq haqida savolga javob bering.

(1) Chiziqni (x, y) to'g'ri chiziqli koordinatalar sistemasida tasvirlang.

(2) Chiziq uzunligini toping.

Yechimi.

(1) $x=r\cos\theta=2\sin\theta\cos\theta=\sin 2\theta$

$y=r\sin\theta=2\sin^2\theta=1-\cos 2\theta$

(2) Bu chiziq markazi (0,1) nuqtada bo'lgan aylana, chunki

$$x^2+(y-1)^2=\sin^2 2\theta+\cos^2\theta=1$$

$$s=\int_0^{\pi}\sqrt{\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2}\,d\theta=\int_0^{\pi}\sqrt{(2\cos 2\theta)^2+(2\sin 2\theta)^2}\,d\theta=2\int_0^{\pi}d\theta=2\pi$$

13B. Aniq integralning fizikaga tadbiqlari
Harakat, tezlik va tezlanish

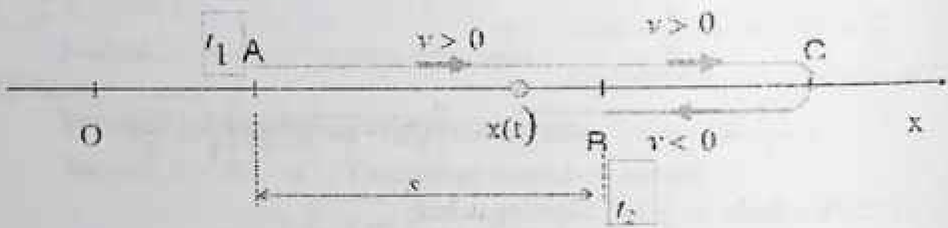
P nuqta x o'qi bo'ylab harakatlanmoqda

Harakat tenglamasi $x=x(t)$. U holda tezlik $v=\frac{dx}{dt}$, tezlanish esa $a=\frac{dv}{dt}$.
Tezlik yoki tezlanish ma'lum bo'lsa

$$x(t)=\int_{t_1}^{t} v\,dt + x_1 \qquad\qquad v(t)=\int_{t_1}^{t} a\,dt + v_1$$

To'g'ri chiziq bo'ylab bosib o'tilgan yol

P nuqtaning harakati



Yuqoridagi chizmaga asosan $t=t_1$ dan $t=t_2$ ga ko'chish

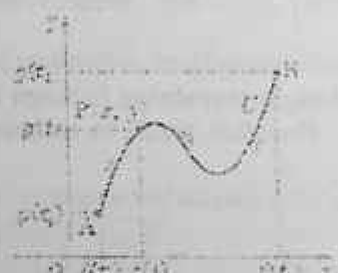
$$s=\int_{t_1}^{t_2} v(t)dt = x(t_2)-x(t_1)$$

t=a dan t=b gacha bosib o'tilgan yo'l

$$l=\int_{t_1}^{t_2} v(t)dt + \int_{t_2}^{t_3} (-v(t))dt = \int_{t_1}^{t_3} |v(t)|dt$$

Samolyotda bosib o'tilgan yo'l

P nuqta C chiziq bo'ylab harakatlanmoqda
$x=f(t)$, $y=g(t)$



Yoyning umumiy uzunligi

$$l=\int_{t_1}^{t_2}\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$$

Tezlik $v=\left(\frac{dx}{dt},\frac{dy}{dt}\right)$

Bosib o'tilgan yo'l $l=\int_{t_1}^{t_2}|v|dt=\int_{t_1}^{t_2}\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$

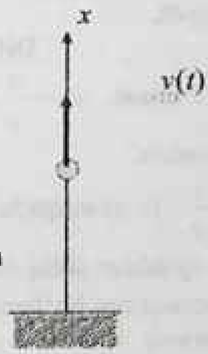
4 misol. Yerdan tepaga qarab 29,4 m/s boshlang'ich tezlikda otilgan sharni qaraymiz. x o'qini vertikal tarzda, boshlang'ich nuqtani yerda deb olamiz. Tezlanish 9,8 m/s^2. Quyidagi savollarga javob bering.

(1) Tezlikni vaqtning funksiyasi kabi ifodalang.

(2) Shar qachon yuqori nuqtaga erishadi?

(3) Shar qachon erga tushadi?
tushadi?

(4) Umumiy bosib o'tilgan yo'lni toping.



Yechimi. (1) Tezlik $v(t) = 29.4 - 9.8t.$

(2) Eng yuqori nuqtqda tezlik nolga teng, shuning uchun

$v(t)=29{,}4-9{,}8t=0$ dan $t=3$sek

(3) Harakat tenglamasi $x=29{,}4t-4.9t^2$, shuning uchun

$x=-4{,}9t(t-6)=0$ yoki $t=6$

(4) Umumiy bosib o'tilgan yo'l

$$l=\int_{t_1}^{t_2}|v|dt=\int_0^3(29.4-9{,}8t)dt+\int_3^6(-29{,}4+9{,}8t)dt=88{,}2m$$

5-misol. x o'qi bo'ylab harakatlanayotgan P nuqtaning tezligi $v(t)=t^2-4t+3$. $t=0$ va $t=6$ vaqtlar orasida bosib o'tilgan yo'lni toping.

Yechimi. $v(t)=(t-1)(t-3)=0$ dan topsak

$v(t)\ge 0$ agar $t\le 1$ va $t\ge 3$

$v(t)\le 0$ agar $1\le t\le 3$

O'z navbatida $v(t)=|t^2-4t+3|$

Bosib o'tilgan yo'l $l=\int_0^6|v|dt=\int_0^1(t^2-4t+3)dt-\int_1^3(t^2-4t+3)dt+\int_3^6(t^2-4t+3)dt=\frac{62}{3}.$

2.14. Differensial tenglamalar (1-qism)
14A. Asosiy Tushunchalar

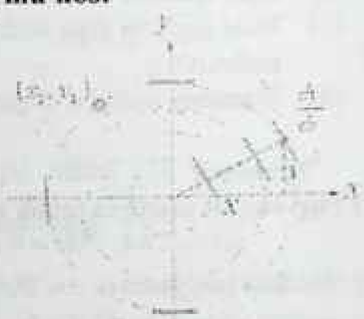
1-misol. Erkin tushayotgan sharning harakatini qaraymiz.

Nyutonning ikkinchi qonuniga asosan: $m\frac{d^2x}{dt^2}=-mg$

Bu tenglamada noma'lum funksiyaning hosilasi paydo bo'ldi.

Differensial tenglama- funksiya o'zi va uning har xil tartibli hosilalarini bog'laydi.

Differentsial tenglamaning ma'nosi



2-misol. $y'=-\frac{x}{y}$ differensial tenglamani o'rganamiz.

$y'=-\frac{x}{y}$ (x, y) nuqtada $\frac{dy}{dx}$ og'ishni beradi.

Agar og\ishlar silliq tutashtirilsa, $x^2+y^2=C^2$ tenglamaga ega bo'lamiz. (C: erkli constanta)

Chegaraviy shart $x=x_1$, $y=y_1$ bitta yechimni aniqlab beradi. (Xususiy yechim)

Differensial tenglamalar bo'yicha ba'zi atamalar

Birinchi tartibli differensial tenglama

Differensial tenglama faqat birichi tartibli hosilalarini o'z ichiga olsa. Masalan $y'=y-x$

Ikkinchi tartibli differensial tenglama

Differensial tenglama faqat ikkinchi tartibli hosilalarini o'z ichiga olsa. (va birinchi hosilalarni ham) $y''+y=1+y^2+x$

Chiziqli differensial tenglama

n-tartibli chiziqli differensial tenglama umumiy ko'rinishi

$y^{(n)}+P_n(x)y^{(n-1)}+...+P_2(x)y'+P_1(x)y=Q(x)$

Chiziqli bo'lmagan differensial tenglama

Chiziqli bo'lmagan differensial tenglama nochiziqli differensial tenglama deyiladi.

$yy'+5x=0$, $\qquad$ $\theta''+5\sin\theta=0$

Qanday qilib diffrensial tenglama yechiladi: Oddiy ish

Eng sodda differensial tenglamalar $y' = f(x)$.
Bu holda yechim $f(x)$ funksiyaning boshlang'ichidir. $y=\int f(x)dx$.

3-misol. Quyidagi savollarga javob bering.

(1) $y'=-7x$ ning umumiy yechimini toping.

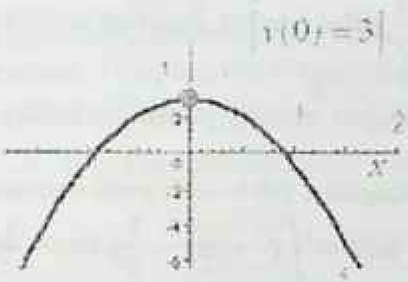
(2) $y(0)=3$ boshlang'ich shartni qanoatlantiradigan xususiy yechimini toping.

Yechimi.

(a) $y=\int(-7x)dx=-\frac{7}{2}x^2+C$

(b) $x=0,\ y=3$. chegaraviy shart.

$3=0+C,\ y=-\frac{7}{2}x^2+3$



4-misol. Quyidagi differensial tenglamalarning xususiy yechimini toping.

(1) $y'=5$, $x=0,\ y=2$ boshlang'ich shart

(2) $y'=\cos 3x$, $x=0,\ y=0$ boshlang'ich shart

Yechimi.

(1) Ikkala tomonini x bo'yicha integrallaymiz va $y=\int 5dx$, $y=5x+C$ ga ega bo'lamiz.
Boshlang'ich shartlarni qo'llasak. $2=5\times 0+C$, $y=5x+2$

(2) $y'=\cos 3x$, $y=\int\cos 3x dx+C$, $y=\frac{1}{3}\sin x+C$

Boshlang'ich shartlarni qo'llasak $x=0,\ y=0$

$$y=\frac{1}{3}\sin 3x$$

14B. Birinchi tartibli differensial tenglamning ba'zi turlari. O'zgaruvchilari ajraladigan differensial tenglamalar

O'zgaruvchilari ajraladigan differensial tenglama deb

$$\frac{dy}{dx}=f(x)g(y)$$

ko'rinishdagi tenglamaga aytiladi.Uni shakl almashtirib

$$\frac{dy}{g(y)}=f(x)dx$$

ga ega bo'lamiz. Uning ikkala tomonini integrallaymiz

$$\int\frac{dy}{g(y)}=\int f(x)dx$$

Integrallashdan so'ng, biz umumiy yechimga ega bo'lamiz (oshkormas ko'rinishda).

5-misol. $\frac{dy}{dx}=-\frac{x}{y}$ bilan bog'liq quyidagi savollarga javob bering.

(1) Umumiy yechimini toping;

(2) $x=0$, $y=1$dan o'tadigan xususiy yechimini toping.

Yechimi.

(1) tenglamani quyidagicha $ydy=-xdx$ yoza olamiz. Uni integrallasak

$$\therefore\int ydy=\int(-x)dx\therefore\frac{1}{2}y^2=-\frac{1}{2}x^2+C_1$$

Agar biz $C=2C_1$ni qo'ysak, bizda umumiy yechim paydo bo'ladi

$$x^2+y^2=C$$

(2) $x=0$, $y=1$ ni qo'ysak biz $C=2$ni topamiz.

Shuning uchun, $x^2+y^2=1$ bo'ladi.

6-misol. Quyidagi differensial tenglamani yeching $\frac{dy}{dx}=6y^2x$.

Boshlang'ich shart $x=1$, $y=\frac{1}{25}$

Yechimi. Tenglamani shaklini almashtirib,

$$\frac{dy}{y^2}=6xdx$$

ko'rinishda yozib olamiz.

Buni integrallaymiz, $\int\frac{dy}{y^2}=\int 6xdx$ va $-\frac{1}{y}=3x^2+C$ga ega bo'lamiz.

Boshlang'ich shartni qanoatlantirib, $C=-28$ ega bo'lamiz.

Xususiy yechim $-\frac{1}{y}=3x^2-28\therefore y=\frac{1}{28-3x^2}$ bo'ladi.

Bir jinsli differensial tenglama

Bir jinsli differensial tenglama

$$\frac{dy}{dx}=f\left(\frac{y}{x}\right)$$

ko'rinishdagi tenglamaga aytiladi.

Agar $\frac{y}{x}=u$ ni qo'ysak, $y=ux \therefore \frac{dy}{dx}=u+x\frac{du}{dx}$

o'zgartirish orqali

$$u+x\frac{du}{dx}=f(u)$$

ga ega bolamiz. U ozgaruvchisi ajraladigan tenglama, chunki

$$\frac{du}{f(u)-u}=\frac{dx}{x}\ .$$

Eslatma. Quyidagi oddiy differensial tenglama ham bir jinsli differensial tenglama deyiladi. Ularning ma'nolari butunlay boshqacha.

$$y^{(n)}+P_n(x)y^{(n-1)}+...+P_2(x)y'+P_1(x)y=0$$

7-misol. $y'=\frac{x^2+y^2}{2xy}$ ning umumiy yechimini toping.

Yechimi. Biz tenglamani $y'=\frac{1+(y/x)^2}{2(y/x)}$ ko'rinishda yoza olamiz.

$\frac{y}{x}=u$ deb belgilaymiz, $y=ux$ almashtirishdan so'ng

$$u+xu'=\frac{1+u^2}{2u} \therefore \int\frac{2u}{u^2-1}du=-\int\frac{1}{x}dx \therefore \ln\left|u^2-1\right|=-\ln|x|+C_1$$

ega bo'lamiz.

$\pm e^{C_1}=C$ deb olib,

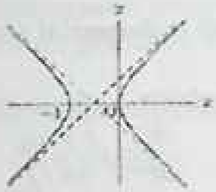
$$x(u^2-1)=C$$

$$x\left\{\left(\frac{y}{x}\right)^2-1\right\}=C$$

$$y^2-x^2=Cx.$$

ga ega bo'lamiz.

Agar C=1 bo'lsa



$x^2 - y^2 = x$

8-misol. $y' = \dfrac{x+y}{x}$ ning umumiy yechimini toping.

Yechimi. $\dfrac{y}{x} = u$ deb belgilaymiz. $y = ux$ almashtirishdan keyin,

$$u + xu' = 1 + u$$
$$x\frac{du}{dx} = 1$$
$$\int du = \int \frac{1}{x} dx$$

ga ega bo'lamiz.
Uni integrallab, $u = ln|x| + C$

$$\frac{y}{x} = ln|x| + C$$

ni hosil qilamiz.

2.15. Differensial tenglamalar(2-qism)

15A. Birinchi tartibli chiziqli differensial tenglama

$y' + P(x)y = Q(x)$ ko'rinishdagi tenglamaga birinchi tartibli chiziqli differensial tenglama deyiladi. Agar $Q(x) = 0$ bo'lsa bir jinsli birinchi tartibli chiziqli differensial tenglama hosil bo'ladi.
Birinchi tartibli chiziqli differensial tenglamani yechish yo'llarini qaraymiz:

1-ish. Bir jinsli differensial tenglama yechimi topiladi.
(Bir jinsli bo'lgani uchun differensial tenglama o'zgaruvchilari ajraladigan differensial tenglama bo'ladi, shu sababli biz oldingi tushunchalarni qo'llab yecha olamiz)

2-ish: Bir jinsli bo'lmagan differensial tenglama yechimi topiladi.
(Bir jinsli differensial tenglama yechimidan foydalanib)

Bir jinsli differensial tenglama yechimi

Birinchi tartiblibli chiziqli differensial tenglama

$$y' + P(x)y = 0$$

O'zgaruvchilarini ajratamiz

$$\frac{1}{y}dy=-P(x)dx$$

So'ngra bu tenglikni integrallaymiz

$$\int\frac{1}{y}dy=-\int P(x)dx \;\therefore\; \ln|y|=\int P(x)dx+C$$

Shundan

$$y=Ce^{-\int P(x)dx}$$

bo'ladi.

Bir jinsli bo'lmagan differensial tenglama yechimi

Bir jinsli bo'lmagan chiziqli differensial tenglama berilgan bo'lsin

$$y'+P(x)y=Q(x)$$

Yechimni quyidagi ko'rinishda faraz qilamiz

$$y=u(x)e^{-\int P(x)dx}$$

ya'ni bir jinsli tenglama $y=Ce^{-\int P(x)dx}$ yechimida C ni funksiya deb olamiz.

Uni berilgan tenglamaga qo'yamiz

$$\left[\frac{du}{dx}e^{-\int P(x)dx}-u(x)P(x)e^{-\int P(x)dx}\right]+P(x)u(x)e^{-\int P(x)dx}=Q(x)$$

$$\frac{du}{dx}=Q(x)e^{\int P(x)dx}$$

$$u(x)=\int\left(Q(x)e^{\int P(x)dx}\right)dx+C$$

U holda umumiy yechim

$$y=\left\{\int\left(Q(x)e^{\int P(x)dx}\right)dx+C\right\}e^{-\int P(x)dx}$$

kabi bo'ladi.

1-misol. $y'-2xy=2x$ ning yechimini toping.

Yechimi. Birinchi ish: $\frac{dy}{dx}-2xy=0$

$$\int\frac{dy}{y}=\int 2xdx$$

$$ln|y|=x^2+C_1$$

$$y=e^{x^2+C_1}=Ce^{x^2}$$

Ikkinchi ish: $y=u(x)e^{x^2}$ ni qo'yamiz.

$$\{u'e^{x^2}+2uxe^{x^2}\}-2xy=2x$$

$$u'e^{x^2}=2x$$

$$u = \int 2x\,e^{-x^2}dx + C = -e^{-x^2} + C$$

Uni olib borib qo'ysak

$$y = (-e^{-x^2} + C)e^{x^2} = Ce^{x^2} - 1$$

2-misol. Massasi m bo'lga jism ko'prikdan tashlab yuborildi. Havoning qarshiligi cv tezlikka to'g'ri proporsional. Nyuton qonuniga asosan harakat tenglamasi $m\frac{dv}{dt} = -cv + mg$ bo'yicha berilgan. v tezlikni t vaqt funksiyasi sifatida toping.

Yechimi. $\gamma = \frac{c}{m}$ ni berilgan tenglamaga qo'yib

$$\frac{dv}{dt} + \gamma v = g$$

ga ega bo'lamiz.

1-ish:

$$\frac{dv}{dt} + \gamma v = 0$$

$$\int \frac{dv}{v} = -\gamma \int dt$$

$$\ln|v| = -\gamma t + C_1$$

$$v = e^{-\gamma t}e^{C_1} = Ce^{-\gamma t}$$

2-ish. Yechimni $v = u(t)e^{-\gamma t}$ deb faraz qilamiz. Bu almashtirishdan biz

$$(u'e^{-\gamma t} - u\gamma e^{-\gamma t}) + \gamma ue^{-\gamma t} = g$$

$$u' = ge^{\gamma t}$$

$$u = g\int e^{\gamma t}dt$$

ga ega bo'lamiz.

Umumiy yechim esa

$$v = \left(g\int e^{\gamma t}dt\right)e^{-\gamma t} = \left(\frac{g}{\gamma}e^{\gamma t} + C\right)e^{-\gamma t} = \frac{g}{\gamma} + Ce^{-\gamma t}$$

$t = 0$ da $v = 0$ boshlang'ich shartini qo'yamiz.

$$0 = \frac{g}{\gamma} + Ce^{0} \therefore C = -\frac{g}{b_1}$$

Shunda yechim

$$v = \frac{g}{\gamma} - \frac{g}{\gamma}e^{-bt} = \frac{gm}{c}(1 - e^{-\frac{c}{m}t})$$

bo'ladi.

Ikkinchi tartibli differensial tenglama

O'zgarmas koeffitsentli ikkinchi tartibli differensial tenglama

Ikkinchi tartibli o'zgarmas koeffisiyentli bir jinsli bo'lmagan tenglamaning umumiy ko'rinishi

$$\frac{d^2y}{dx^2}+a\frac{dy}{dx}+by=f(x)$$

Uning umumiy yechimi

$$y=y_h+y_p$$

ko'rinishda bo'ladi. Bu yerda

y_h - Qo'shma bir jinsli differensial tenglamaning umumiy yechimi;

y_p - Differensial tenglamaning biror xususiy yechimi.

Differensial tenglamaga mos bir jinsli tenglama

$$\frac{d^2y}{dx^2}+a\frac{dy}{dx}+by=0$$

Faraz qilaylik uning yechimi $y=Ce^{\lambda x}$ko'rinishda.

Uni tenglamaga qo'ysak

$$C(\lambda^2+a\lambda+b)e^{\lambda x}=0$$

ga ega bo'lamiz. Bu yerdan

$$\therefore\ \lambda^2+a\lambda+b=0\ \therefore\ \lambda=\frac{a\pm\sqrt{b^2-4ac}}{2}(=\lambda_1,\lambda_2)$$

Bir jinsli tenglamaning umumiy yechimi

$$y=C_1e^{\lambda_1 x}+C_2e^{\lambda_2 x}$$

dan iborat bo'ladi.

Bir jinsli bo'lmagan differensial tenglamaning umumiy yechimi.

$$\frac{d^2y}{dx^2}+a\frac{dy}{dx}+by=f(x)$$

tenglamaning umumiy yechimi

$$y=C_1e^{\lambda_1 t}+C_2e^{\lambda_2 t}+y_p$$

bo'ladi. ($y=y_p$ xususiy yechim)

[Eslatma] D. T. aks ettiradigan holatlarni tushunish uchun, biz quyidagi uchta holatga mos keladigan natijalarni muhokama qilishimiz kerak

(1) λ_1 va λ_2 ikki turli haqiqiy ildiz;

(2) λ_1 va λ_2 ikki kompleks ildiz;

(3) λ_1 va λ_2 karrali ildiz.

3-misol. Tenglamaning umumiy yechimni toping. $\frac{d^2y}{dx^2}-4y=\sin 3x$

Yechimi. (1) Bir jinsli tenglamani qaraymiz

$$\frac{d^2y}{dx^2}-4y=0$$

Faraz qilaylik yechim $y=Ce^{\lambda x}$ ko'rinishida bo'lsin. Uni tenglamaga qo'ysak, u holda

$$C(\lambda^2-1)e^{\lambda x}=0$$
$$\lambda=2,-2$$

Yechim $y=C_1e^{2x}+C_2e^{-2x}$

(2) Xususiy yechimni $y_p=A\sin 3x$ ko'rinishida qidiramiz. Uni berilgan tenglamaga qo'yamiz, shunda

$$(-9-4)A\sin 3x=\sin 3x$$

Demak xususiy yechim

$$y_p=-\frac{1}{13}\sin 3x$$

bo'ladi.

(4)Umumiy yechim esa

$$y=C_1e^{\lambda_1 t}+C_2e^{\lambda_2 t}-\frac{1}{13}\sin 3x$$

dan iborat.

III KURS. Chiziqli algebra

3.1. Vektorlarning asosiy qoidalari

IA. Vektorning ta'rifi. Asosiy qoidalar. Vektorlarning koordinatalari

Son qiymati orqali ifodalanadigan miqdorga ckalyay miqdor deyiladi. Masalan: yuza S, harorat t, tezlik qiymati v, burchak θ va hokazolar skalyarga-misol bo'la oladi.

Vektor kattalik

Son qiymati va yo'nalishi bilan tavsiflanadigan каттaлик вектор катталик дейилади. Masalan: kuch, tezlik .

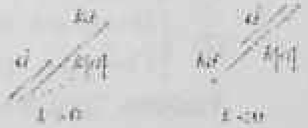
Vektor umumiy holda strelka bilan yoziladi. Yozilishi: $\mathbf{a}$, $\vec{a}$, $\overline{OP}$

Vektorning son qiymati $|\vec{a}|$ yoki a orqali belgilanadi.

Vektorlarning asosiy xossalari:

1.Tengligi. $\vec{a}=\vec{b}$ son qiymati va yo'nalishi bir xil bo'lgan vektorlar teng bo'ladi.

2. Vektorni songa ko'paytirish $\boxed{k\vec{a}}$ bu $k$ning ishorasi va moduliga bog'liq: k musbat bo'lsa yo'nalish o'zgarmaydi, manfiy bo'lsa qarama qarshiga o'zgaradi. k ning moduli cho'zilish koeffisientini anglatadi.



3. Qo'shish

Vektorlar yig'indisi geometrik jihatdan parallelogramning diagonaliга teng. Yoki vektorlarni uchburchak usulida ham qo'shish mumkin.

4. Ayirish

Vektorlarni ayirishni qo'shish yordamida hosil qilish mumkin.

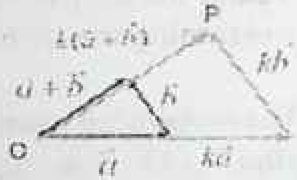
$\vec{b}+\overline{BA}=\vec{a}$ Shuning uchun $\overline{BA}=\vec{a}-\vec{b}$

Vektorlarning asosiy qonunlari

1. O‘rin almashtirish qonuni. $\bar{a}+\bar{b}=\bar{b}+\bar{a}$

2. Guruhlash qonuni $\left(\bar{a}+\bar{b}\right)+\bar{c}=\bar{a}+\left(\bar{b}+\bar{c}\right)$

3. Taqsimot qonuni $k\left(\bar{a}+\bar{b}\right)=k\bar{a}+k\bar{b}$



1-misol. $\bar{p}=3\bar{a}+2\bar{b}$ va $\bar{q}=-2\bar{a}+\bar{b}$ bo‘lsin. Quyidagi savollarga javob bering.

(1) $3(\bar{x}-\bar{q})=2\bar{p}+\bar{x}$ tenglamani qanoatlantiruvchi $\bar{x}$ ni toping.

(2) $\begin{cases} 2\bar{x}-3\bar{y}=\bar{p} & (i) \\ \bar{x}+\bar{y}=\bar{q} & (ii) \end{cases}$ ni qanoatlantiradiagan $\bar{x}$ va $\bar{y}$ ni toping.

Yechimi. Tenglikka $\bar{p}$ va $\bar{q}$ ni qo‘yib,

$$3\bar{x}-3\left(-2\bar{a}+\bar{b}\right)=2\left(3\bar{a}+2\bar{b}\right)+\bar{x}, \qquad \bar{x}=\frac{7}{2}\bar{b}$$

ga ega bo‘lamiz.

(1) (i)-2x(ii) dan $-5\bar{y}=\bar{p}-2\bar{q}$

$$\bar{y}=-\frac{1}{5}\bar{p}+\frac{2}{5}\bar{q}=\frac{1}{-5}\left(3\bar{a}+2\bar{b}\right)+\frac{2}{5}\left(-2\bar{a}+\bar{b}\right)=-\frac{7}{5}\bar{a}$$

ga ega bo‘lamiz. (ii) dan

$$\bar{x}=\bar{q}-\bar{y}=\left(-2\bar{a}+\bar{b}\right)-\left(-\frac{7}{5}\bar{a}\right)=-\frac{3}{5}\bar{a}+\bar{b}$$

ni hosil qilamiz.

2-misol. Quyidagi tenglamani qanoatlantiruvchi $\bar{x}$ va $\bar{y}$ ni toping.

$$\left.\begin{array}{ll} 3\bar{x}+2\bar{y}=\bar{a} & (i) \\ 4\bar{x}-3\bar{y}=\bar{b} & (ii) \end{array}\right\}$$

Yechimi. (i) ni 3 ga ko‘paytirib, tenglamadan $9\bar{x}+6\bar{y}=3\bar{a}$ ga ega bo‘lamiz.

(ii) ni 2 ga ko'paytirib, tenglamadan $8\vec{x}-6\vec{y}=2\vec{b}$ ga ega bo'lamiz:

Ularni qo'shib, $17\vec{x}=3\vec{a}+2\vec{b}$, yoki

$$\vec{x}=\frac{3}{17}\vec{a}+\frac{2}{17}\vec{b}$$

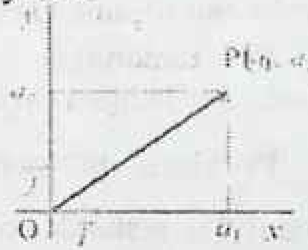
ga ega bo'lamiz. Birinchi tenglamadan

$$\vec{y}=-\frac{3}{2}\vec{x}+\frac{1}{2}\vec{a}=-\frac{3}{2}\left(\frac{3}{17}\vec{a}+\frac{2}{17}\vec{b}\right)+\frac{1}{2}\vec{a}=\frac{4}{17}\vec{a}-\frac{3}{17}\vec{b}$$

1B. Vektorlarning koordinatalari. Birlik vector

Uzunligi 1 ga teng bo'lgan vectorga birlik vektor deyiladi.

$$\vec{e}=\frac{\vec{a}}{|\vec{a}|} \qquad |\vec{e}|=1$$



$\vec{i}=(1,0)$ va $\vec{j}=(0,1)$ lar birlik vektor bo'lib, birlik bazis vektorlarni hosil qiladi.

$\vec{a}=a_1\vec{i}+a_2\vec{j}=(a_1,a_2)$ bo'lsa, (a_1,a_2) vektor koordinatalari bo'ladi.

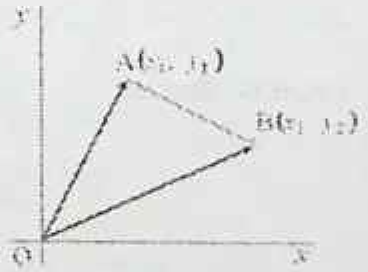
A va B nuqtalarni tutashtiruvchi vector uchun:

$\overline{AB}=\overline{OB}-\overline{OA}=(x_2,y_2)-(x_1,y_1)$

$=(x_2-x_1,y_2-y_1)$

AB vektor uzunligi: $|\overline{AB}|=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$

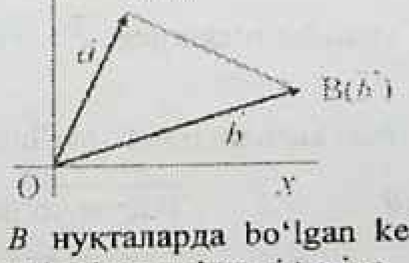
kabi topiladi.



Vektorning joylashuvi

Agar dastlab vektorning boshini tanlasak, vektor yordamida nuqta belgilangan bo'ladi.

A va B nuqtalarni tutashtiruvchi vektor: $\overline{AB}=\vec{b}-\vec{a}$



3-misol. Учи A va B нуқталарда bo'lgan kesmani $m:n$ nisbatda bo'luvchi C nuqta orqali hosil bo'lgan vektorni toping.

Yechimi. $\overrightarrow{AC}$ va $\overrightarrow{CB}$ bir xil yo'nalishga ega bo'lsin.

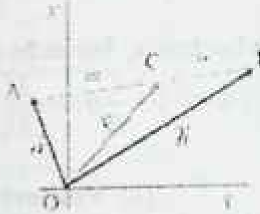
Berilganiga ko'ra son qiymatlari uchun: $\left|\overrightarrow{AC}\right| : \left|\overrightarrow{CB}\right| = m : n$

Shuning uchun,

$$n(\bar{x} - \bar{a}) = m(\bar{b} - \bar{x})$$

Bu yerdan

$$\therefore \quad \bar{x} = \frac{n\bar{a} + m\bar{b}}{(m+n)}$$



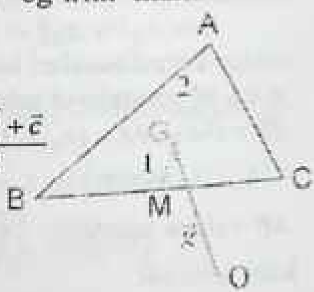
4-misol. ΔABC ning og'irlik markazi бўлган $\bar{g}$ vektorning joylashuvini toping. Oxirlari A, B va C bo'lgan vektorlar $\bar{a}, \bar{b}$ va $\bar{c}$ бўлсин.

BC tomonning o'rtasi bo'lgan M nuqta orqali, AM kesmani 2:1 nisbatda bo'ladigan berilgan o'girlik markazini toping.

Yechimi. BC tomon o'rtasi $\bar{m} = \dfrac{\bar{b} + \bar{c}}{2}$. G og'irlik markazi AM kesmani 2:1 nisbatda bo'lgani uchun

$$\bar{x} = \frac{\bar{a} + 2\bar{m}}{2+1} = \frac{\bar{a} + 2\left(\dfrac{\bar{b} + \bar{c}}{2}\right)}{3} = \frac{\bar{a} + \bar{b} + \bar{c}}{3}$$

ga ega bo'lamiz:



3.2. Skalyar ko'paytma

2A. Skalyar ko'paytma ta'rifi

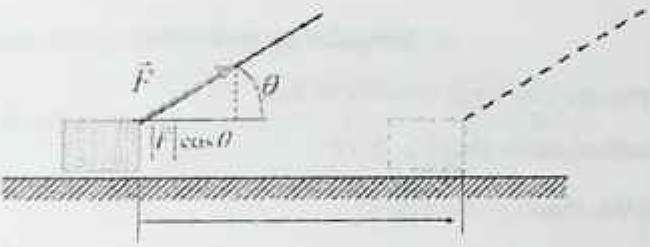
Asosiy qoidalar.

O'tgan mavzuda biz vektorlarni qo'shishni o'rgangan edik. Ushbu mavzuda esa biz amaliyotda ko'p foydalaniladigan vektorlarning ko'paytmasini o'rganamiz. Vektorlar ko'paytmasining ikkita turi mavjub bo'lib, ular skalyar va vektor ko'paytmalardir. Skalyar ko'paytma ichki ko'paytma ham deyiladi. 2A darsda biz skalyar ko'paytmaning asosiy xossalarini ko'rib chiqamiz.

Dastlab-misol sifatida, o'zgarmas $\overline{F}$ vektor holida bajarilgan ishni qaraymiz.

Misol. O'zgarmas kuch momenti. Ma'lumki, bajarilgan ish

$W = \left|\overline{F}\right| \cdot \left|\overline{a}\right| \cos\theta$ | Skalyar ko'paytma $\overline{F} \cdot \overline{a}$



$\vec{F}$ kuch bajargan ish $W=|\vec{F}|\cdot|\vec{a}|\cos\theta$ formula orqali topiladi. Bu erda θ – kuch va $\vec{a}$ vektor bo‘ylab ko‘chish orasidagi burchak, $|\vec{a}|$-ko‘chish. $\vec{F}$ kuchning gorizontal komponentasi $|\vec{F}|\cdot\cos\theta$ ekanidan ish bu $|\vec{F}|\cdot\cos\theta$ kuch va $\vec{a}$ ko‘chish orqali ifodalanadi, ya’ni $W=|\vec{F}|\cdot|\vec{a}|\cos\theta$.

$\vec{F}$ va $\vec{a}$ vektorlarning skalyar ko‘paytmasi $\vec{F}\cdot\vec{a}$ kabi belgilanadi.

Skalyar ko‘paytma ta’rifi.

$\vec{a}$ va $\vec{b}$ vektorlarning skalyar ko‘paytmasi deb

$$\vec{a}\cdot\vec{b}=|\vec{a}|\cdot|\vec{b}|\cos\theta$$

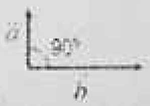
ga aytiladi (bu erda $0\le\theta\le\pi$).

Ikkita $\vec{a}=\{a_1,a_2,a_3\}$ va $\vec{b}=\{b_1,b_2,b_3\}$ vektorlarning skalyar ko‘paytmasi vektorlar orasiga $(\cdot)$ qo‘yish orqali bilan yoziladi. $\vec{a}\cdot\vec{b}=|\vec{a}|\cdot|\vec{b}|\cos\theta$, bu erda θ ikki vektor orasidagi burchak va $0\le\theta\le\pi$.

Ushbu-misolni o‘rganamiz.

1-misol. Quyidagi ko‘paytmalarni toping, $|\vec{a}|=2$ va $|\vec{b}|=3$

1)



2)

3)

Yechimi.

(1) $\vec{a}\cdot\vec{b}=3\cdot2\cos60^0=3$

(2) $\vec{a}\cdot\vec{b}=3\cdot2\cos90^0=0$

(3) $\vec{a}\cdot\vec{b}=3\cdot2\cos135^0=-3\sqrt{2}$

Xossalar va qoidalar

Skalyar ko'paytma quyidagi xossalarga ega.

1. Perpendikulyarlik sharti $\vec{a}\cdot\vec{b}=0$
2. Parallellik sharti $\vec{a}\cdot\vec{b}=\pm|\vec{a}|\cdot|\vec{b}|$
3. Birlik basis vektorlar uchun $\vec{i}\cdot\vec{j}=0$ $\quad \vec{i}\cdot\vec{i}=\vec{j}\cdot\vec{j}=1$

1. Ikkita vektor perpendikulyar bo'lganda $\vec{a}\cdot\vec{b}=0$ o'rinli, chunki $\theta=90^0$ va $\cos 90^0=0$.
2. Ikkita vektor parallel bulganda $\vec{a}\cdot\vec{b}=\pm|\vec{a}|\cdot|\vec{b}|$ munosabat o'rinli, chunki $\theta=0^0$ yoki $\theta=180^0 \Rightarrow \cos 0^0=1$ yoki $\cos 180^0=-1$.
3. Birlik vektorlar bazisi uchun $\vec{i}\cdot\vec{j}=0$ va $\vec{i}\cdot\vec{i}=\vec{j}\cdot\vec{j}=1$ o'rinli

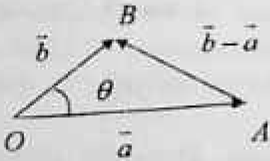
Muhim qoidalar

1. Kommutativlik qoidasi $\vec{a}\cdot\vec{b}=\vec{b}\cdot\vec{a}$
2. Distributivlik qonuni $\vec{a}\cdot(\vec{b}+\vec{c})=\vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{c}$
3. Skalyar songa ko'paytirish $k(\vec{a}\cdot\vec{b})=k\vec{b}\cdot\vec{a}=\vec{a}\cdot k\vec{b}$

Koordinatatalar orqali tasvirlash

Agar $\vec{a}=\{a_1,a_2\}$ va $\vec{b}=\{b_1,b_2\}$ bo'lsa u holda $\vec{a}\cdot\vec{b}=a_1\cdot b_1+a_2\cdot b_2$ bo'ladi.

Isbot. $\vec{a}$ va $\vec{b}$ vektorlardan qurilgan OAB uchburchakni qaraymiz



Kosinuslar teormasiga ko'ra

$$AB^2=OA^2+OB^2-2OAOB\cos\theta$$

$$|\vec{b}-\vec{a}|^2=|\vec{a}|^2+|\vec{b}|^2-2\vec{a}\cdot\vec{b}$$

Pifagor teoremasiga ko'ra

$$(b_1-a_1)^2+(b_2-a_2)^2=(a_1^2+a_2^2)+(b_1^2+b_2^2)-2\vec{a}\cdot\vec{b}$$

Bu tenglikni soddalashtirib biz quyidagiga ega bo'lamiz:

$$\vec{a}\cdot\vec{b}=a_1b_1+a_2b_2$$

2-misol. Quyidagi vektorlarning skalyar ko'paytmasini va ular orasidagi burchakni toping

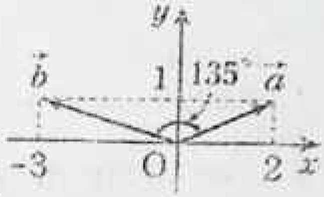
(1) $\vec{a}=\{2,1\}$, $\vec{b}=\{-3,1\}$ (2) $\vec{a}=\{-1,3\}$, $\vec{b}=\{6,2\}$

Yechimi. (1) Ichki ko'paytmani hisoblasak

$$\vec{a}\cdot\vec{b}=a_1b_1+a_2b_2=2\cdot(-3)+1\cdot1=-5$$

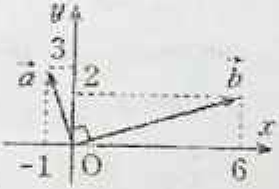
Faraz qilaylik vektorlar orasidagi burchak θ bo'lsin, u holda

$$\sqrt{2^2+1^2}\cdot\sqrt{(-3)^2+1^2}\cos\theta=-5$$



Shunday qilib $\cos\theta=\dfrac{-5}{\sqrt{5}\cdot\sqrt{10}}=-\dfrac{1}{\sqrt{2}}$, $\quad\theta=135^0$.

(2) $\vec{a}\cdot\vec{b}=a_1b_1+a_2b_2=-1\cdot6+3\cdot2=0$



Demak, bu vektorlar perpendikulyar, ya'ni $\theta=90^0$.

3-misol. $\vec{a}=\left(\sqrt{3},-1\right)$ vektorga perpendikulyar bo'lgan birlik vektorlarni toping.

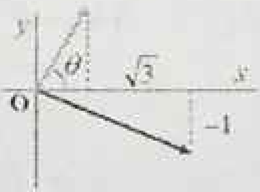
Yechimi.

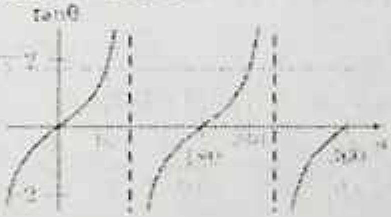
(1) Aytaylik $\vec{e}=(\cos\theta,\sin\theta)$ birlik vektor bo'lsin.

Shartga ko'ra $\vec{e}\cdot\vec{a}=\sqrt{3}\cos\theta-\sin\theta=0$

Bu yerdan $tg\theta=\sqrt{3}$,

$$\theta=60^0 \text{ yoki } \theta=240^0.$$





2B. Skalyar ko'paytmaning tadbiqi.

Uchburchakning yuzi

Keling, $\vec{a}$ va $\vec{b}$ vektorlar orqali aniqlangan $OAB\ \Delta$ ning yuzi uchun formulani keltirib chiqaraylik. Faraz qilaylik bu $\vec{a}$ va $\vec{b}$ vektorlar orasidagi burchak θ bo'lsin. B nuqtadan dan OA tomonga tushirilgan balandlik asosi H bo'lsin. Perpendikulyar kesma: BH $\quad BH=|\vec b|\sin\theta$

Demak, yuza $S=\frac{1}{2}|\vec a|\cdot BH=\frac{1}{2}|\vec a|\cdot|\vec b|\sin\theta=\frac{1}{2}|\vec a|\cdot|\vec b|\sqrt{1-\cos^2\theta}=\frac{1}{2}\sqrt{|\vec a|^2|\vec b|^2-(\vec a\cdot\vec b)^2}$

$$S=\frac{1}{2}\sqrt{|\vec a|^2\cdot|\vec b|^2-(\vec a\cdot\vec b)^2}.$$

4-misol.

1. Ikkita $\vec a=\{a_1,a_2\}$ va $\vec b=\{b_1,b_2\}$ vektordan qurilgan uchburchak yuzasini toping.
2. Uchta $A(2,1)$, $B(7,2)$, $C(4,5)$ nuqtalar berilgan. ABC Δ ning yuzini toping

Yechimi. 1. Vektorlarning modullari

$|\vec a|=\sqrt{a_1^2+a_2^2}$, $\quad |\vec b|=\sqrt{b_1^2+b_2^2}$

dan topamiz. Skalyar ko'paytmani $\vec a\cdot\vec b=a_1b_1+a_2b_2$ kabi hisoblaymiz.

U holda $S=\frac{1}{2}\sqrt{|\vec a|^2\cdot|\vec b|^2-(\vec a\cdot\vec b)^2}=\frac{1}{2}\sqrt{(a_1^2+a_2^2)(b_1^2+b_2^2)-(a_1b_1+a_2b_2)^2}$

$S=\frac{1}{2}\sqrt{|\vec a|^2\cdot|\vec b|^2-(\vec a\cdot\vec b)^2}=\frac{1}{2}\sqrt{(a_1b_2-a_2b_1)^2}=\frac{1}{2}|a_1b_2-a_2b_1|$

2. $\vec a=\overline{AB}=(7,2)-(2,1)=(5,1)$, $\qquad \vec b=\overline{AC}=(4,5)-(2,1)=(2,4)$

$$S=\frac{1}{2}|a_1b_2-a_2b_1|=\frac{1}{2}|5\cdot4-1\cdot2|=9$$

5-misol. Tekislikda uchta $A(4,1)$, $B(5,4)$, $C(2,3)$ nuqtalar berilgan. Quyidagi savollarga javob bering.

(1) $\angle BAC=\theta$ bo'lsin. $\cos\theta$ va $\sin\theta$ ni toping.

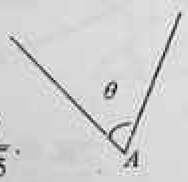
(2) ABC Δ-ning ıozini toping.

Yechmi. (1) $\vec a=\overline{AB}=(5,4)-(4,1)=(1,3)$, $\vec b=\overline{AC}=(2,3)-(4,1)=(-2,2)$

Demak, $\qquad |\overline{AB}|=\sqrt{1^2+3^2}=\sqrt{10}$, $\qquad |\overline{AC}|=\sqrt{(-2)^2+2^2}=2\sqrt{2}$

$\overline{AB}\cdot\overline{AC}=1\cdot(-2)+3\cdot2=4$

$\cos\theta=\dfrac{\overline{AB}\cdot\overline{AC}}{|\overline{AB}|\cdot|\overline{AC}|}=\dfrac{4}{\sqrt{10}\cdot2\sqrt{5}}=\dfrac{1}{\sqrt{5}}$

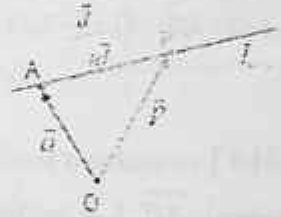


$\sin\theta>0$ ekanidan, $\sin\theta=\sqrt{1-\cos^2\theta}=\sqrt{1-\frac{1}{5}}=\frac{2}{\sqrt{5}}$.

(3) $S_{\Delta ABC}=\frac{1}{2}|\overline{AB}|\cdot|\overline{AC}|\cos\theta=\frac{1}{2}\cdot\sqrt{10}\cdot2\sqrt{2}\cdot\frac{1}{\sqrt{5}}=2$

3.3. Vektor tenglamalar
3A. To'g'ri chiziqning vektor tenglamasi

$A(\vec{a})$ nuqtadan o'tuvchi va $\vec{d}$ vektorga parallel to'g'ri chiziq tenglamasi quyidagicha



$$\boxed{\vec{p} = \vec{a} + t\vec{d}}$$

Bu yerda t : parameter, $\vec{d}$: yo'naltiruvchi vektor

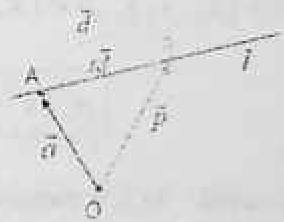
Koordinatalarda ifodalanishi

$\vec{a} = (x_0, y_0)$, $\vec{p} = (x, y)$ va $\vec{d} = (d_x, d_y)$ bo'lsa, u holda yuqoridagi tenglama quyidagicha bo'ladi

$$\boxed{(x, y) = (x_0, y_0) + t(d_x, d_y)}$$

Vektor tenglamadan:

$$\boxed{(x, y) = (x_0, y_0) + t(d_x, d_y)}$$



Parametrik tenglama hosil bo'ladi:

$$\left.\begin{aligned} x &= x_0 + td_x \\ y &= y_0 + td_y \end{aligned}\right\}$$

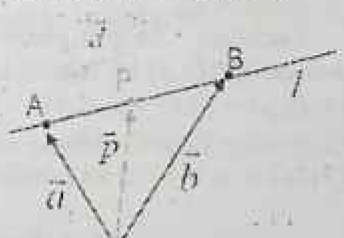
$A(\vec{a})$ va $B(\vec{b})$ nuqtalardan o'tuvchi to'g'ri chiziq tenglamasi

$$\vec{d} = \vec{b} - \vec{a}$$

$$\vec{p} = \vec{a} + t(\vec{b} - \vec{a})$$

yoki $\vec{p} = (1-t)\vec{a} + t\vec{b}$

kabi bo'ladi.

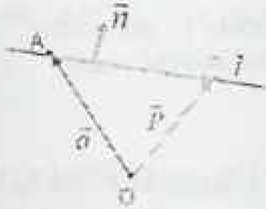


1-misol. A(1,2) va B(–2,5) nuqtalardan o'tuvchi to'g'ri chiziqning vektor tenglamasini toping.

Yechimi. A va B nuqtalarni tutashtiruvchi $\overline{AB}$ vektor $\overline{AB}=(-2-1,5-2)=(-3,3)$ bo'lgani uchun. $(x,y)=(1,2)+t(-3,3)$ bo'ladi.

To'g'ri chiziq va normal vektor

$A(\vec{a})$ nuqtadan o'tuvchi va $\vec{n}$ vektorga perpendikulyar to'g'ri chiziq tenglamasi $\overline{AP}\perp\vec{n}$ bo'lgani uchun



$$\boxed{\vec{n}\cdot(\vec{p}-\vec{a})=0}$$

bo'ladi. $\vec{n}$ ga normal vektor deyiladi.

Koordinatalarda ifodalanishi

Agar $\vec{a}=(x_0,y_0)$, $\vec{p}=(x,y)$ va $\vec{n}=(n_x,n_y)$ normal vector bo'lsa, u holda

$$\boxed{n_x(x-x_0)+n_y(y-y_0)=0}$$

bo'ladi.

2-misol. A(3,5) nuqtadan o'tuvchi va $\vec{d}=(-2,4)$ vektorga parallel bo'lgan to'g'ri chiziq haqidagi quyidagi savollarga javob bering.

(1) Vektor tenglamasini toping;

(2) Parametrik tenglamasini toping;

(3) Parametrdan qutulish orqali to'g'ri chiziq tenglamasini keltirib chiqaring.

Yechimi. To'g'ri chiziqda yotuvchi ixtiyoriy nuqtaning koordinatalari (x,y) bo'lsin. U holda

(1) (x,y) = (3,5) + t(–2,4);

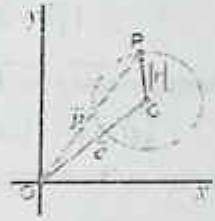
(2) x = 3 – 2t, y=5 + 4t;

(3) 2(3 – x) = y – 5 va bundan y = –2x+11.

3B. Aylananing vektor tenglamasi

Aylana radiusi r va markazi $C(\vec{c})$ nuqtada bo'lsin.

Aylananing ta'rifiga ko'ra $\left|\overline{CP}\right| = r$



Binobarin,

$$\boxed{\left|\vec{p} - \vec{c}\right| = r}$$

Ikki tomonini kvadratga oshiramiz:

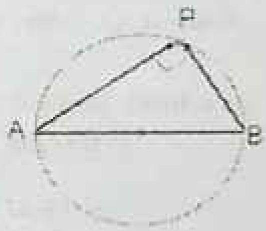
$$\left|\vec{p} - \vec{c}\right|^2 = r^2.$$

U holda, skalyar ko'paytmaning xossasiga ko'ra:

$$\boxed{(\vec{p} - \vec{c}) \cdot (\vec{p} - \vec{c}) = r^2}$$

Aylananing vektor tenglamasi (2)

Aylana diametri AB, $A(\vec{a})$ va $B(\vec{b})$ aylanada yotuvchi nuqta.



Aylana diametriga tortilgan ichki burchak 90 gradus bo'lganligi uchun,

$$\boxed{(\vec{p} - \vec{a}) \cdot (\vec{p} - \vec{b}) = 0}$$

Aylana radiusini r bo'lgani uchun

$$(\vec{a} - \vec{c}) \cdot (\vec{a} - \vec{c}) = \left|\vec{a} - \vec{c}\right|^2 = r^2$$

$$(\vec{p} - \vec{a}) \cdot (\vec{a} - \vec{c}) = \{\vec{p} - \vec{c} - (\vec{a} - \vec{c})\} \cdot (\vec{a} - \vec{c}) =$$

$$= (\vec{p} - \vec{c}) \cdot (\vec{a} - \vec{c}) - \left|\vec{a} - \vec{c}\right|^2 = (\vec{p} - \vec{c}) \cdot (\vec{a} - \vec{c}) - r^2 = 0$$

munosabatlar o'rinli ekanligidan,

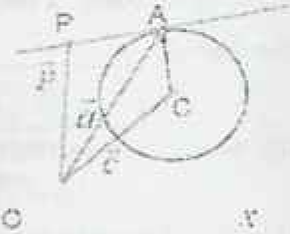
$$\boxed{(\vec{p} - \vec{c}) \cdot (\vec{a} - \vec{c}) = r^2}$$

kelib chiqadi.

3-misol. Aylanaga $A(\vec{a})$ nuqtada urinuvchi urinma to'g'ri chiziqning vektor tenglamasini toping. Bu aylananing radiusi r va markazi $C(\vec{c})$ nuqtada joylashgan.

Yechimi. AP va CA vektorlar o'zaro perpendikulyar bo'lganligi uchun

$$\boxed{(\vec{p}-\vec{a})\cdot(\vec{a}-\vec{c})=0}$$



Aylana radiusi r bo'lgani uchun

$$(\vec{a}-\vec{c})\cdot(\vec{a}-\vec{c})=|\vec{a}-\vec{c}|^2=r^2$$

O'rniga qo'yish orqali

$$(\vec{p}-\vec{a})\cdot(\vec{a}-\vec{c})=\{\vec{p}-\vec{c}-(\vec{a}-\vec{c})\}\cdot(\vec{a}-\vec{c})=$$

$$=(\vec{p}-\vec{c})\cdot(\vec{a}-\vec{c})-|\vec{a}-\vec{c}|^2=(\vec{p}-\vec{c})\cdot(\vec{a}-\vec{c})-r^2=0$$

munosabatga ega bo'lamiz va bundan

$$\boxed{(\vec{p}-\vec{c})\cdot(\vec{a}-\vec{c})=r^2}$$

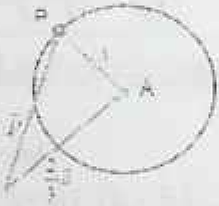
4-misol. Quyidagi vektor tenglama qanday shaklni ifodalaydi?

$$|3\vec{p}-2\vec{a}|=3$$

Yechimi. Berilgan tenglama quyidagicha yozilishi mumkin:

$$|3\vec{p}-2\vec{a}|=3\ \Rightarrow$$

$$\Rightarrow\left|\vec{p}-\frac{2}{3}\vec{a}\right|=1$$



Demak, vektor tenglama radiusi 1 ga teng bo'lgan va markazi $\frac{2}{3}\vec{a}$ da joylashgan aylanani ifodalar ekan.

3.4. Matrisaning asosiy qoyidalari

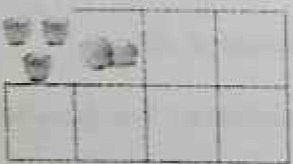
4A. Matrisaning ta'rifi.

Turli xil matrisalar

Ta'rif. Matrisa deb- satr va ustunlarida sonlar bo'lgan to'g'ri to'rtburchak shakidagi jadvalga aytiladi.

Misol.

narsalar qutisi



ustun

Matritsa

qator

$$\Longleftrightarrow\quad \begin{pmatrix} 3 & 2 & 2 & 1 \\ 2 & 5 & 3 & 3 \end{pmatrix}$$

$$A=\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{pmatrix}$$

Matritsaning turlari:

Kvadrat matrisa- satr va ustunlari soni teng bo'lgan matrisaga aytiladi.

$$\text{Masalan,}\begin{pmatrix} 12 & 5 & 7 \\ 5 & 5 & 6 \\ 1 & 3 & 2 \end{pmatrix}$$

Diagonal matrisa- kvadrat matrisaning diagonal elementlaridan boshqa barcha elementlari nolga teng bo'lgan matrisaga aytiladi. Masalan: $\begin{pmatrix} 12 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

Satr matrisa- 1 satrdan iborat bo'lgan matrisaga aytiladi. $\begin{pmatrix} 12 & 5 & 7 \end{pmatrix}$

Ustun matrisa – 1 ustundan iborat bo'lgan matrisaga aytiladi. $\begin{pmatrix} 12 \\ 5 \\ 1 \end{pmatrix}$

A matrisaga transponirlangan matrisa A^T - bunda i -satr elementlari j-ustun elementlari bilan almashib yozilgan matrisaga aytiladi.

$$b_{ij}=a_{ji} \qquad B=A^T \quad A=\begin{pmatrix} 12 & 5 \\ 5 & 4 \\ 1 & 3 \end{pmatrix} \qquad B=A^T=\begin{pmatrix} 12 & 5 & 1 \\ 5 & 4 & 3 \end{pmatrix}$$

Birlik matrisa- diagonal elementlari 1ga teng bo'lgan va boshqa elementlari nollardan tashkil topgan kvadrat matrisaga aytiladi va I harfi bilan belgilanadi. Masalan uchinchi tartibli birlik matrisa $I=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Simmetrik matrisa deb- transponirlanganda yana o'sha matrisaga aylanadigan kvadrat matrisaga aytiladi. $a_{ij}=a_{ji}$ $\begin{pmatrix} 12 & 5 & 7 \\ 5 & 4 & 3 \\ 7 & 3 & 1 \end{pmatrix}$

Qarama-qarshi simmetrik matrisa deb- matrisaning elementlarini transponirlaganda manfiy ishorali bo'lib qoladigan matrisaga aytiladi. $a_{ij}=-a_{ji}$ $\begin{pmatrix} 0 & 5 & -7 \\ -5 & 0 & 3 \\ 7 & 3 & 0 \end{pmatrix}$

Bir xil matrisalar - B va A matrisalarning satr va ustun elementlari bir xil bo'lsa,

$$A=\begin{pmatrix} 1 & 3 & 3 \\ 5 & 10 & 9 \end{pmatrix} \quad B=\begin{pmatrix} 1 & 3 & 3 \\ 5 & 10 & 9 \end{pmatrix}$$

Teng matrisalar- matrisaning barcha elementlari mos ravishda bir xil bo'lsa, ular teng deyiladi.

$$B=A \qquad a_{ij}=b_{ij}$$

1-misol. Quyidagi matrisada x, y, u, v larning qiymatlarini toping.

$$\begin{pmatrix} x+y & x-y \\ u-1 & 2v \end{pmatrix}=\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$$

Yechimi. Mos elementlarning tengligidan,
$x+y=1 \quad x-y=3 \quad u-1=2 \quad 2v=4$ ga ega bo'lamiz.

Bu tenglamalarni yechsak, $x=1 \quad y=-1 \quad u=3 \quad v=2$

2-misol.

1) 2x3 lik matrisaning elementlarini quyidagi formula yordamida toping. $a_{ij}=3i-2j$.

2) Quyidagi tenglikdan noma'lumlarni toping. $\begin{pmatrix} 5 & x \\ 2x & x-y \end{pmatrix}=\begin{pmatrix} x+y & z \\ z^2 & -1 \end{pmatrix}$

Yechimi.

1) $a_{11}=3\times1-2\times1=1 \quad a_{12}=3\times1-2\times2=-1 \quad a_{13}=3\times1-2\times3=-3$
$a_{21}=3\times2-2\times1=4 \quad a_{22}=3\times2-2\times2=2 \quad a_{23}=3\times2-2\times3=0$

Demak, $A=\begin{pmatrix} 1 & -1 & -3 \\ 4 & 2 & 0 \end{pmatrix}$

2) $5=x+y, \quad x=z, \quad 2x=z^2, \quad x-y=1$

Bu tenglamalar sistemasini yechsak, $x=3, y=2, z=2$.

4B. Matrisalarni qo'shish. Matrisalarni ko'paytirish. Skalyarga ko'paytirish

Matrisa quyidagi ikkita asosiy afzalliklarga ega:

1. Tenglamalarni va ma'lumotlarni ixcham shaklda yozish mumkin.
2. Tenglamalarni yechishning va ma'lumotlarni almashishning eng qulay usulidir.

Matrisalarni qo'shish va ayirish

Matrisa m ta satr va n ta ustundan iborat bo'lsa, u mxn tartibli deyiladi. Ikkita matrisani ularning tartiblari bir xil bo'lsagina qo'shish va ayirish mumkin va uni quyidagicha amalga oshirish mumkin.

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}+\begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{pmatrix}=\begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} & a_{13}+b_{13} \\ a_{21}+b_{21} & a_{22}+b_{22} & a_{23}+b_{23} \end{pmatrix}$$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}-\begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{pmatrix}=\begin{pmatrix} a_{11}-b_{11} & a_{12}-b_{12} & a_{13}-b_{13} \\ a_{21}-b_{21} & a_{22}-b_{22} & a_{23}-b_{23} \end{pmatrix}$$

Matrisani biror songa ko'paytirish uchun uning barcha elementlarini shu songa ko'paytirish kerak.

$A=\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ keyin $kA=\begin{pmatrix} ka_{11} & ka_{12} \\ ka_{21} & ka_{22} \end{pmatrix}$

Matrisalarni hisoblashda asosiy qoidalar

1. Kommutativlik xossasi $A+B=B+A$
2. Assotsiativlik xossasi $(A+B)+C=A+(B+C)$
3. Distributivlik xossasi $k(A+B)=kA+kB$
4. Skalyarga ko'paytirish haqidagi xossasi $(kl)A=k(lA)$

3-misol. Quyidagi matrisalar berilgan $A=\begin{pmatrix} 2 & -3 & 0 \\ 4 & 1 & 5 \end{pmatrix}$ va $B=\begin{pmatrix} -1 & 2 & 4 \\ 2 & 3 & -2 \end{pmatrix}$ Quyidagini toping $2(3A-2B)-3(A-B)$.

Yechimi.

$$2(3A-2B)-3(A-B)=6A-4B-3A+3B=3A-B$$

$$=3\begin{pmatrix} 2 & -3 & 0 \\ 4 & 1 & 5 \end{pmatrix}-\begin{pmatrix} -1 & 2 & 4 \\ 2 & 3 & -2 \end{pmatrix}=\begin{pmatrix} 6 & -9 & 0 \\ 12 & 3 & 15 \end{pmatrix}-\begin{pmatrix} -1 & 2 & 4 \\ 2 & 3 & -2 \end{pmatrix}$$

$$=\begin{pmatrix} 7 & -11 & -4 \\ 10 & 0 & 17 \end{pmatrix}$$

4-misol. Ushbu $A=\begin{pmatrix} 5 & -1 \\ 0 & 2 \end{pmatrix}$ va $B=\begin{pmatrix} 2 & 1 \\ 1 & -3 \end{pmatrix}$ matrisalar yordamida quyidagi tenglamaning noma'lum hadini toping $2A+X=B$.

$$X=B-2A=\begin{pmatrix} 2 & 1 \\ 1 & -3 \end{pmatrix}-2\begin{pmatrix} 5 & -1 \\ 0 & 2 \end{pmatrix}$$

$$=\begin{pmatrix} 2 & 1 \\ 1 & -3 \end{pmatrix}-\begin{pmatrix} 10 & -2 \\ 0 & 4 \end{pmatrix}=\begin{pmatrix} -8 & 3 \\ 1 & -7 \end{pmatrix}$$

3.5. Matrisalar ko'paytmasi
5A. Ustun-vektor va qator-vektor ko'paytmasi. Matrisalar ko'paytmasi. Ko'paytmaning asosiy qoidalari

$\vec{a}=(a_1,a_2)$ va $\vec{b}=(b_1,b_2)$ vektorlarning skalyar ko'paytmasi; $\vec{a}\cdot\vec{b}=a_1\cdot b_1+a_2\cdot b_2$ kabi edi. Ustun-vektor va qator-vektorlar mos ravishda quyidagicha belgilanadi:

$$(a_1\ a_2\ \cdots a_n)\ ,\qquad \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

Ustun-vektor va qator-vektorlarda elementlar soni bir xil bo'lsa ularni quyidagi qoidaga asosan ko'paytiramiz

$$(a_1\ a_2\ \cdots a_n)\cdot\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} = a_1\cdot b_1 + a_2\cdot b_2 + \cdots + a_n\cdot b_n$$

1-misol. Quyidagi ko'paytmalarni hisoblang:

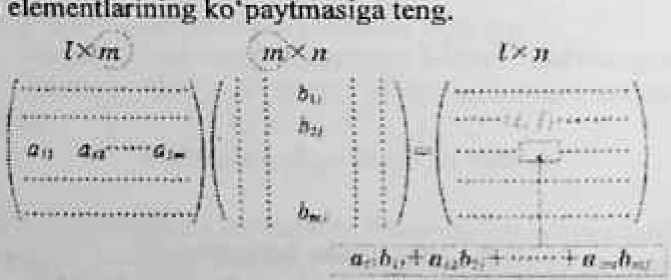
1. $(2\ 6\ 3)\cdot\begin{pmatrix} 8 \\ 1 \\ 4 \end{pmatrix}$ 2. $(2\ 6\ 3\ 1)\cdot\begin{pmatrix} 8 \\ 1 \\ 4 \end{pmatrix}$

Yechimi.

1. $(2\ 6\ 3)\cdot\begin{pmatrix} 8 \\ 1 \\ 4 \end{pmatrix} = 2\cdot 8 + 6\cdot 1 + 3\cdot 4 = 34$ 2. Bu ko'paytma mavjud emas, chunki ulardagi elementlar soni turlicha.

Ikki matrisaning ko'paytmasi

- A matrisani B matrisaga ko'paytirish mumkin bo'lishi uchun A $(l\times m)$ matrisaning ustunlar soni B $(m\times n)$ matrisaning qatorlar soniga teng bo'lishi kerak.
- Ko'paytmaning c_{ij} elementi A matrisaning i- qatori va B matrisaning j-ustuni elementlarining ko'paytmasiga teng.



2-misol. Quyidagi ko'paytmalarni hisoblang

1. $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x & y \\ z & u \end{pmatrix}$ 2. $\begin{pmatrix} 2 & 8 & 1 \\ 3 & 6 & 4 \end{pmatrix}\begin{pmatrix} 1 & 7 \\ 9 & -2 \\ 6 & 3 \end{pmatrix}$

Yechimi.

1. $\begin{pmatrix} ax+bz & ay+bu \\ cx+dz & cy+du \end{pmatrix}$ 2. $\begin{pmatrix} 2\cdot1+8\cdot9+1\cdot6 & 2\cdot7+8\cdot(-2)+1\cdot3 \\ 3\cdot1+6\cdot9+4\cdot6 & 3\cdot7+6\cdot(-2)+4\cdot3 \end{pmatrix}=\begin{pmatrix} 80 & 1 \\ 81 & 21 \end{pmatrix}$

Ko‘paytmaning asosiy qoidalari

Quyidagi qoidalar bajariladi.

1. Skalyarga ko‘paytirish uchun $(kA)B = A(kB) = k(AB)$
2. Assosiativlik $(AB)C = A(BC)$
3. Distributivlik $\begin{matrix} (A+B)C = AC+BC \\ A(B+C) = AB+AC \end{matrix}$

Lekin matrisalarni ko‘paytirish uchun kommutativlik bajarilmaydi.

$$AB \neq BA$$

$$l\times m \quad m\times n \quad m\times n \quad l\times m$$

Misol

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix}=\begin{pmatrix} 5 & 7 \\ 4 & 8 \end{pmatrix} \qquad \begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}=\begin{pmatrix} 7 & 5 \\ 6 & 6 \end{pmatrix}$$

Birlik matrisa, nol matrisa

Asosiy diagonalidagi elementlar birga, qolganlar elementlari nolga teng bo‘lgan matrisaga birlik matrisa deyiladi

$$I=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Hamma elementlari nolga teng bo‘lgan

$$O=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

matrisaga nol matrisa deb ataladi.

Xossalari

1. $AI = IA = A$
2. $AO = OA = O$

Nol emas matrisalarning ham ko‘paytmasi nol matrisa bo‘lishi mumkin

Misol $\begin{pmatrix} 2 & -2 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}=\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ $AB = O$

3-misol. Quyidagi ko‘paytmalarni hisoblang

(1) $\begin{pmatrix} 4 & 2 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 5 & -1 \end{pmatrix}$ (2) $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

Yechimi.

1. $\begin{pmatrix} 4 & 2 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 5 & -1 \end{pmatrix} = \begin{pmatrix} 4\cdot(-1)+2\cdot 5 & 4\cdot 0+2\cdot(-1) \\ 3\cdot(-1)+1\cdot 5 & 3\cdot 0+1\cdot(-1) \end{pmatrix} = \begin{pmatrix} 6 & -2 \\ 2 & -1 \end{pmatrix}$

2. $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1+0+0 & 1+1+0 & 1+1+1 \\ 0+0+0 & 0+1+0 & 0+1+1 \\ 0+0+0 & 0+0+0 & 0+0+1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$

5B. Determinant ta'rifi. Determinantni hisoblash

- Determinant deganda matrisa orqali hisoblanadigan aniq son tushuniladi.
- Chiziqli tenglamalar sistemasini o'rganishda, chiziqli akslantirishlarda va boshqa masalalarda bu tushuncha qo'llaniladi.
- Determinant faqat kvadrat matrisa uchun mavjud.

Belgilanishi

$$A=\begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow |A| \quad \begin{vmatrix} a & b \\ c & d \end{vmatrix} \quad \det A$$

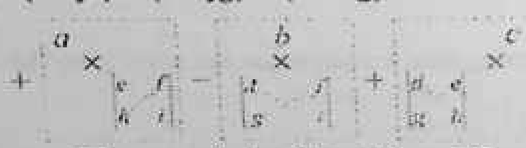
2×2 determinantni hisoblash. $A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$; Determinant $|A| = ad-bc$

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

Misol $\begin{vmatrix} 2 & 3 \\ 1 & 5 \end{vmatrix} = 2\cdot 5-3\cdot 1=7$

3 x 3 determinantni hisoblash. $A=\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$

Determinant $|A| = a(ei-fh)-b(di-fg)+c(dh-eg)$



Minor va algebraik to'ldiruvchi

$n\times n$ matrisa uchun $A=\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$

Minor. M_{ij} minor $(n-1)\times(n-1)$ determinant bo'lib, berilgan matrisaning i-nchi qator va j-inchi ustunini o'chirishdan hosil bo'ladi.

Algebraik to'ldiruvchi.

Algebraik to'ldiruvchi C_{ij} – bu ishorasi bor minor (бу ишораси i+j га боғлиқ бўлган минордир)

$C_{11}=M_{11}, C_{12}=-M_{12}, C_{13}=M_{13},\cdots$

$$C_{ij}=(-1)^{i+j}M_{ij}$$

Determinant $|A|=a_{11}C_{11}+a_{12}C_{12}+a_{13}C_{13}+\cdots$

4-misol. Matrisaning determinantini hisoblang:

$$A=\begin{pmatrix}2 & 1 & -1\\ 0 & 4 & 3\\ -5 & 0 & -2\end{pmatrix}$$

Yechimi.

$$|A|=\begin{vmatrix}2 & 1 & -1\\ 0 & 4 & 3\\ -5 & 0 & -2\end{vmatrix}=2\begin{vmatrix}4 & 3\\ 0 & -2\end{vmatrix}-\begin{vmatrix}0 & 3\\ -5 & -2\end{vmatrix}+(-1)\begin{vmatrix}0 & 4\\ -5 & 0\end{vmatrix}=2\cdot(-8)-1\cdot 15+(-1)\cdot 20=-51$$

5-misol. Quyidagi matrisalarning determinantini hisoblang:

1. $A=\begin{pmatrix}2 & 1\\ -1 & 3\end{pmatrix}$ 2. $A=\begin{pmatrix}2 & 1 & 3\\ 1 & 3 & 0\\ 0 & 0 & 1\end{pmatrix}$

Yechimi.

(1) $|A|=\begin{vmatrix}2 & 1\\ -1 & 3\end{vmatrix}=2\cdot 3-1\cdot(-1)=7$

(2) $|A|=\begin{vmatrix}2 & 1 & 3\\ 1 & 3 & 0\\ 0 & 0 & 1\end{vmatrix}=2\begin{vmatrix}3 & 0\\ 0 & 1\end{vmatrix}-1\begin{vmatrix}1 & 3\\ 0 & 1\end{vmatrix}+0\begin{vmatrix}0 & 4\\ -5 & 0\end{vmatrix}=2\cdot 3-1\cdot 1+0=5$

3.6. Teskari matrisa va birgalikdagi tenglamalar sistemasi

Berilgan a son uchun teskari son $x = \frac{1}{a} = a^{-1}$ kabi aniqlanar edi. Berilgan a sonning teskarisiga ko'paytmasi birga teng $xa = 1$.

Teskari matrisa tushinchasini kiritamiz. A kvadrat matrisa berilgan bo'lsin.

Agar AX=XA=I bo'lsa X matrisa A matrisa uchun teskari matrisa deb ataladi. A matrisaga teskari matrisani A^{-1} kadi belgilash qabul qilingan

(2x2) kvadrat matrisaga teskari matrisani hisoblash.

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ kvadrat matrisaga teskari matrisa quyidagicha aniqlanadi.

Berilgan matrisani determinantini hisoblaumiz

$$|A| = ad - bc \neq 0$$

Teskari matrisa

$$A^{-1} = \frac{1}{|A|}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Berilgan matrisaning determinanti |A|=ad-bc=0 bo'lsa, A^{-1} mavjud emas.

1-misol. $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ matrisaga teskari matrisani toping.

Yechimi. Faraz qilaylik, $X = \begin{pmatrix} x & z \\ y & w \end{pmatrix}$ matrisa A matrisaga teskari matrisa bo'lsin.

Demak AX=I shart bajarilishi shart, ya'ni

$$\begin{pmatrix} a & b \\ c & w \end{pmatrix}\begin{pmatrix} x & z \\ y & w \end{pmatrix} = \begin{pmatrix} ax + by & zaz + bw \\ cx + dy & cz + dw \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Demak

$$\begin{array}{ll} ax + by = 1 & az + dw = 0 \\ cx + dy = 0 & cz + dw = 1 \\ x(ad - bc) = d & z(ad - bc) = -b \\ y(ad - bc) = -c & w(ad - bc) = a \end{array}$$

(ad-bc)≠0 bo'lganda

$$x = \frac{d}{(ad - bc)}, \quad y = -\frac{c}{ad - bc}, \qquad z = -\frac{b}{ad - bc}, \quad w = \frac{a}{ad - bc}$$

(ad-bc)=0 bo'lganda esa

$a = b = c = d = 0$ bo'ladi, va AX=I shart bajarilmaydi.

2-misol. $A = \begin{pmatrix} 2 & 1 \\ 4 & 4 \end{pmatrix}$ matrisaga teskari matrisani toping.

Yechimi. A matrisani determinantini hisoblaymiz:

$$|A| = \begin{vmatrix} 2 & 1 \\ 4 & 4 \end{vmatrix} = 2 \cdot 4 - 1 \cdot 4 = 4$$

Teskari matrisani topamiz

$$A^{-1} = \frac{1}{4}\begin{vmatrix} 4 & -1 \\ -4 & 2 \end{vmatrix} = \begin{vmatrix} 1 & -1/4 \\ -1 & 1/2 \end{vmatrix}$$

3-misol. Matrisaga teskari matrisani toping

1) $A = \begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix}$ 2) $A = \begin{pmatrix} 2 & 4 \\ 3 & 6 \end{pmatrix}$

Yechimi.

1) Determinant

$$|A| = \begin{vmatrix} 3 & 4 \\ 2 & 3 \end{vmatrix} = 3 \times 3 - 4 \times 2 = 1 \neq 0$$

Demak

$$A^{-1} = \frac{1}{1}\begin{vmatrix} 3 & -4 \\ -2 & 3 \end{vmatrix} = \begin{vmatrix} 3 & -4 \\ -2 & 3 \end{vmatrix}.$$

2) Determinant

$$|A| = \begin{vmatrix} 2 & 4 \\ 3 & 6 \end{vmatrix} = 2 \times 6 - 4 \times 3 = 0$$

Demak matrisaga teskari matrisa mavjud emas.

Tenglamalar sistemasi

$$\left.\begin{array}{r} ax + by = p \\ cx + dy = q \end{array}\right\}$$

tenglamalar sistemasi berilgan bo'lsin.

Ya'ni qisqacha yozsak: $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} p \\ q \end{pmatrix}$

Ushbu belgilarni kiritamiz:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad X = \begin{pmatrix} x \\ y \end{pmatrix}, \quad P = \begin{pmatrix} p \\ q \end{pmatrix}$$

U holda berilgan sistemani ko'paytirish qoidasidan foydalanib, ushbu ekvivalent shaklda yozish mumkin:

$$AX = P$$

Agar A matrisa xosmas, ya'ni $ad - bc \neq 0$ bo'lsa, u holda

$$A^{-1}AX = A^{-1}P$$

$$X = A^{-1}P$$

ni topamiz. Bu esa berilgan tenglamalar sistemasi yechimining matrisali yozuvini beradi.

Agar $ad - bc = 0$ bo'lsa u holda berilgan sistema yechimga ega emas yoki cheksiz ko'p yechimga ega (misollarda ko'ramiz)

4-misol. Berilgan tenglamalar sistemasini yeching:

$$\left.\begin{array}{r} 3x + y = 3 \\ 9x + 4y = 6 \end{array}\right\}.$$

Yechimi. Berilgan sistemani quydagicha yozib olamiz:

$$\begin{pmatrix} 3 & 1 \\ 9 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \end{pmatrix}$$

Asosiy matrisaning determinant: $|A| = \begin{vmatrix} 3 & 1 \\ 9 & 4 \end{vmatrix} = 3$

Asosiy matrisaga teskari matrisa:

$$\begin{pmatrix} 3 & 1 \\ 9 & 4 \end{pmatrix}^{-1} = \frac{1}{3}\begin{pmatrix} 4 & -1 \\ -9 & 3 \end{pmatrix} = \begin{pmatrix} 4/3 & -1/3 \\ -3 & 1 \end{pmatrix}$$

Aniqlangan teskari matrisani P matrisaga ko'paytiramiz

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4/3 & -1/3 \\ -3 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ 6 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$$

Demak,

$$x = 2, \ \ y = -3$$

5-misol. Berilgan tenglamalar sistemasini yeching:

1) $\left.\begin{matrix} 2x - y = -5 \\ -6x + 3y = 15 \end{matrix}\right\}$

2) $\left.\begin{matrix} 2x + y = 2 \\ 4x + 2y = 3 \end{matrix}\right\}$

Yechimi. 1) Asosiy matrisaning determinanti

$$|A| = \begin{vmatrix} 2 & -1 \\ -6 & 3 \end{vmatrix} = 0$$

Agar 1) sistemaning ikkinchi tenglamasini har ikkala tomonini -3 ga bo'lamiz va quydagiga ega bo'lamiz

$$2x - y = -5$$

Demak ikkinchi tenglama birinchi tenglamaga teng. Bu yerda, berilgan sistema, $2x - y = -5$ tenglamani qanoatlantiruvchi, cheksiz ko'p (x, y) yechimga ega.

2) Sistemani asosiy matrisasi determinanti

$$|A| = \begin{vmatrix} 2 & 1 \\ 4 & 2 \end{vmatrix} = 0.$$

Sistemaning birinchi tenglamasini har ikkila tomonini 2 ga ko'paytirib $4x + 2y = 4$ ga kelamiz.

Bu esa ikkinchi tenglikga zid. Demak, sistema yechimga ega emas.

6-misol. Tenglamalar sistemasini yeching:

$$\left.\begin{matrix} 3x - y = 3 \\ -2x + 3y = 5 \end{matrix}\right\}$$

Yechimi. Berilgan sistemani matrisa ko'rinishida yo'zib olamiz:

$$\begin{pmatrix} 3 & -1 \\ -2 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$$

Sistemani asosiy matrisasi determinanti

$$|A| = \begin{vmatrix} 3 & -1 \\ -2 & 3 \end{vmatrix} = 7 \neq 0$$

Asosiy matrisaga teskari matrisa

$$\begin{pmatrix} 3 & -1 \\ -2 & 3 \end{pmatrix}^{-1} = \frac{1}{7}\begin{pmatrix} 3 & 1 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 3/7 & 1/7 \\ 2/7 & 3/7 \end{pmatrix}$$

Demak

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3/7 & 1/7 \\ 2/7 & 5/7 \end{pmatrix} \begin{pmatrix} 3 \\ 5 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$

3.7. Chiziqli akslantirish

7A. Chiziqli almashtirish. Geometrik almashtirishlardan nuqta koordinatalarini o'zgarishi.

Bu mavzu davomida chiziqli akislantirish ya'ni geometrik almashtirishlar bilan yaqindan tanishamiz. Quyidagi variantlar bilan tanishib chiqamiz.

Agar P(x,y) nuqta $Q(x', y')$ nuqtaga ko'chsa, bu hol chiziqli almashtirish deyiladi.

Ixtiyoriy almashtirish quyidagicha ifodalanadi:

$$f: (x, y) \rightarrow (x', y')$$

Chiziqli akislantirish quyidagicha bo'lib, uni matrisa yordamida yozib olamiz:

$$\left.\begin{aligned} x' = ax + by \\ y' = cx + dy \end{aligned}\right\} \rightarrow \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ matrisa, chiziqli almashtirishning asosiy matrisasidir. Berilgan nuqta koordinatasiga qaytish uchun, asosiy matrisaga teskari bo'lgan matrisaga, tenglikni har ikkala tomonini ko'paytiramiz.

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{ad - cb} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix}$$

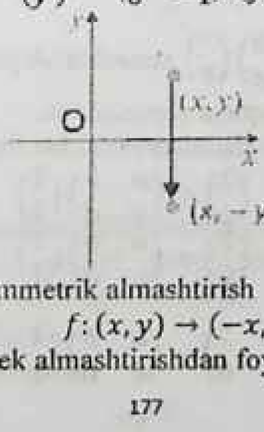
Bir necha xil chiziqli almashtirishlarni ko'rib chiqaylik:

1. Ox oqiga nisdatan simmetrik almashtirish

$$f: (x, y) \rightarrow (x, -y)$$

Bu yerda, quyidagidek almashtirishdan foydalanamiz

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

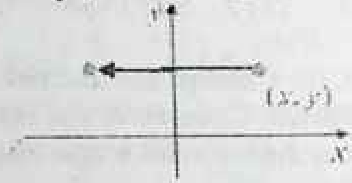


2. Oy oqiga nisdatan simmetrik almashtirish

$$f: (x, y) \rightarrow (-x, y)$$

Bu yerda, quyidagidek almashtirishdan foydalanamiz

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

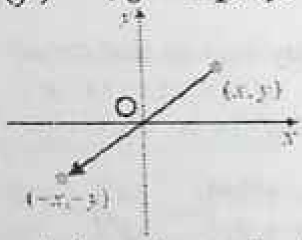


3. Koordinata boshiga nisdatan simmetrik almashtirish

$$f:(x,y) \to (-x,-y)$$

Bu almashtirish quyidagicha ifodalanadi

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

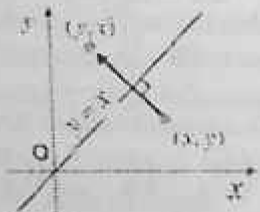


4. $x = y$ to'g'ri chiziqga nisdatan simmetrik almashtirish

$$f:(x,y) \to (y,x)$$

Bu almashtirish quyidagicha ifodalanadi

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$



1-misol. $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$ almashtirish yordamida (1,1) va (2,1) nuqtalarni (3,2) va (7,0) nuqtaga almashtirish.

Yechimi. Quydagicha almashtirishlardan foydalanamiz:

$$\begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 7 \\ 0 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Bu ifodalarni birlashtirib

$$\begin{pmatrix} 3 & 7 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$$

tenglikga ega bo'lamiz.
Dastlabki nuqtalar koordinatalaridan tuzilgan matrisa determinant noldan farqli, ya'ni

$$\begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} = 1 \times 1 - 2 \times 1 = -1 \neq 0$$

Demak, ushbu matrisaga teskari matrisa mavjud.

Teskari matrisa

$$\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}^{-1} = \frac{1}{(-1)}\begin{pmatrix} 1 & -2 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix}$$

Almashtirishimizni har ikki tomonini teskari matrisaga ko'paytiramiz:

$$\begin{pmatrix} 3 & 7 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix}$$

Demak

$$\therefore \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 3 & 7 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 4 & -1 \\ -2 & 4 \end{pmatrix}.$$

2-misol. $P(-2;2)$, $Q(0;2)$ va $R(2;-2)$ nuqtalar, ya'ni uchburchak uchlari berilgan. $x' = x + y$ va $y' = x - y$ almashtirish yordamida, qanday $P'Q'R'$ uchburchak hosil bo'ladi.

P' nuqta koordinatalarini hisoblaymiz:

$$\begin{pmatrix} x_P \\ y_P \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} -2 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ -4 \end{pmatrix}$$

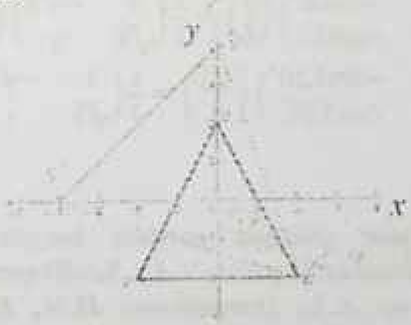
Q' nuqta koordinatalarini hisoblaymiz:

$$\begin{pmatrix} x_Q \\ y_Q \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \end{pmatrix}$$

R' nuqta koordinatalarini hisoblaymiz:

$$\begin{pmatrix} x_R \\ y_R \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 2 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \end{pmatrix}$$

Shunday qilib, quyidagi uchburchak hosil bo'ladi:



7B. Chiziqli akslantirishlarning qo'llanilishi. Burish

Bu mavzu davomida, chiziqli akslantirishlardan biri bo'lgan, burishni o'rganamiz. Koordinata bo'shiga nisbatan, soat strelkasiga qarshi, θ burchakga burishni o'rganamiz.

$P(x, y)$ nuqta koordinatalari

BURISH

$$x = r\cos\alpha$$
$$y = r\sin\alpha \quad (1)$$

$$x = r\cos\alpha$$
$$y = r\sin\alpha$$

berilgan bo'lsin. Burish natijasida burchak $\alpha + \theta$ ga teng bo'lganligi sababli, hosil bo'ladigan Q nuqta koordinatalari quyidagiga teng bo'ladi

$$x' = r\cos(\alpha+\theta) = r(\cos\alpha\cos\theta - \sin\alpha\cos\theta)$$
$$y' = r\sin(\alpha+\theta) = r(\sin\alpha\cos\theta + \cos\alpha\sin\theta)$$

Trigonometrik ayniyatlardan foydalanib, quyidagini olamiz

$$\left.\begin{aligned} x' &= x\cos\theta - y\sin\theta \\ y' &= x\sin\theta + y\cos\theta \end{aligned}\right\}$$

Matrisa ko'rinishida yozib olamiz

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \qquad (1)$$

3-misol. Berilgan nuqtani berilgan burchakga, soat strelkasiga qarshi, burishdan hosil bo'lgan nuqtaning koordinatasini to'ping.

1) $(2,4),\ 60°$ 2) $(\sqrt{3},1),\ 120°$

Yechimi. (1) burish formulasidan foydalanib, quyidagini yozamiz:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos60° & -\sin60° \\ \sin60° & \cos60° \end{pmatrix} \begin{pmatrix} 2 \\ 4 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} -2\sqrt{3} \\ \sqrt{3}+2 \end{pmatrix}.$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos120° & -\sin120° \\ \sin120° & \cos120° \end{pmatrix} \begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix} \begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix} = \begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix}.$$

Mualliflar tarjima qilishda yordam bergan Abdukarimov A., Mustapoqulov H.Y., Raxmatullayeva N.A., Xoldibayeva I., Dexqonov F., Esonov E.A., Qurbonov A.A., Quromboyev H.N., Egamberganova O.Y. "Oliy matematika" kafedrasi jamoasiga o'z minnatdorchiligini bildiradi.

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