INITIAL AND BOUNDARY CONDITIONS
THE HEAT EQUATION ON $[0, L]$
SOLUTIONS IN AN INFINITE MEDIUM
LAPLACE TRANSFORM TECHNIQUES HEAT

CHAPTER 17

The Heat Equation

17.1 Initial and Boundary Conditions

In Section 12.8, we used Gauss's divergence theorem to derive a partial differential equation modeling heat distribution, or diffusion. In the absence of sources or sinks within the medium, the one-dimensional heat equation is

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \tag{17.1}$$

in which k is a constant depending on the medium.

Equation (17.1) can be solved subject to a variety of boundary and initial conditions. For example,

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \text{ for } 0 < x < L,\ t > 0,$$

$$u(0, t) = T_1,\ u(L, t) = T_2 \text{ for } t \geq 0,$$

$$u(x, 0) = f(x) \text{ for } 0 \leq x \leq L$$

models the temperature distribution in a thin homogeneous bar of length L whose left end is kept at temperature T_1 and right end at temperature T_2, and having initial temperature $f(x)$ in the cross section at x.

The initial-boundary value problem

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \text{ for } 0 < x < L,\ t > 0,$$

$$\frac{\partial u}{\partial x}(0, t) = \frac{\partial u}{\partial x}(L, t) = 0 \text{ for } t \geq 0,$$

and

$$u(x, 0) = f(x) \text{ for } 0 \leq x \leq L$$

models the distribution in a bar of length L having no heat loss across its ends (*insulation conditions*) and initial temperature function f.

Other kinds of boundary conditions also can be specified. If the left end is kept at constant temperature T and the right end is insulated, then we would have

$$u(0, t) = T \text{ and } \frac{\partial u}{\partial x}(L, t) = 0 \text{ for } t > 0.$$

Free radiation or *convection* occurs when the bar loses energy by radiation from its ends into the surrounding medium, which is assumed to be maintained at constant temperature T. Now the boundary conditions have the form

$$\frac{\partial u}{\partial x}(0, t) = A[u(0, t) - T], \frac{\partial u}{\partial x}(L, t) = -A[u(L, t) - T] \text{ for } t \geq 0,$$

in which A is a positive constant. Notice that, if the bar is kept hotter than the surrounding medium, then the heat flow as measured by $\partial u/\partial x$ must be positive at one end and negative at the other.

Boundary conditions

$$u(0, t) = T_1, \frac{\partial u}{\partial x}(L, t) = -A[u(L, t) - T_2]$$

are used if the left end is kept at constant temperature T_1 while the right end radiates heat energy into a medium of constant temperature T_2.

As with the wave equation, we also consider the heat equation on the line or half-line, subject to various conditions.

SECTION 17.1 PROBLEMS

1. Formulate an initial-boundary value problem modeling heat conduction in a thin homogeneous bar of length L if the left end is kept at temperature zero and the right end is insulated. The initial temperature function is f.
2. Formulate an initial-boundary value problem modeling heat conduction in a thin homogeneous bar of length L if the left end is kept at temperature $\alpha(t)$ and the right end at temperature $\beta(t)$. The initial temperature function in the cross section at x is $f(x)$.
3. Formulate an initial-boundary value problem for the temperature distribution in a thin bar of length L if the left end is insulated and the right end is kept at temperature $\beta(t)$. The initial temperature function is f.

17.2 The Heat Equation on $[0, L]$

We will solve several initial-boundary value problems on an interval $[0, L]$.

17.2.1 Ends Kept at Temperature Zero

If the initial temperature in the cross section at x is $f(x)$ and the ends of the bar are kept at temperature zero, the problem for the temperature distribution function is

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \text{ for } 0 < x < L, t > 0,$$

$$u(0, t) = u(L, t) = 0 \text{ for } t \geq 0,$$

and

$$u(x, 0) = f(x) \text{ for } 0 \leq x \leq L.$$

Put $u(x,t) = X(x)T(t)$ into the heat equation to obtain

$$\frac{X''}{X} = \frac{T'}{kT} = -\lambda$$

in which λ is the separation constant. Now,

$$u(0,t) = X(0)T(t) = 0 = u(L,t) = X(L)T(t)$$

for all $t \geq 0$, so $X(0) = X(L) = 0$, and the problem for X is

$$X'' + \lambda X = 0;\ X(0) = X(L) = 0$$

with eigenvalues $\lambda_n = n^2\pi^2/L^2$ and eigenfunctions $\sin(n\pi x/L)$.

It is in the time dependence that the heat and wave equations differ. The equation for T is

$$T' + \frac{n^2\pi^2 k}{L^2}T = 0,$$

which is a first-order differential equation with solutions that are constant multiples of $e^{-n^2\pi^2 kt/L^2}$. For $n = 1, 2, \cdots$, we have functions

$$u_n(x,t) = c_n \sin\left(\frac{n\pi x}{L}\right) e^{-n^2\pi^2 kt/L^2},$$

which satisfy the heat equation and the boundary conditions $u(0,t) = u(L,t) = 0$. To satisfy the initial condition, we must (depending on f) use a superposition

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right) e^{-n^2\pi^2 kt/L^2}$$

and choose the coefficients so that

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right).$$

This is the Fourier sine expansion of f on $[0, L]$, so choose

$$c_n = \frac{2}{L}\int_0^L f(\xi)\sin\left(\frac{n\pi\xi}{L}\right) d\xi.$$

With this choice of the c_n's, the solution is

$$u(x,t) = \frac{2}{L}\sum_{n=1}^{\infty}\left(\int_0^L f(\xi)\sin(n\pi\xi/L)\,d\xi\right)\sin(n\pi x/L)e^{-n^2\pi^2 kt/L^2}. \tag{17.2}$$

EXAMPLE 17.1

Suppose the ends are kept at zero temperature and the initial temperature is $f(x) = A$, which is constant. Compute

$$c_n = \frac{2}{L}\int_0^L A\sin(n\pi\xi/L)\,d\xi = \frac{2A}{n\pi}[1 - (-1)^n].$$

The solution is

$$u(x,t) = \frac{2A}{\pi}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n}\sin\left(\frac{n\pi x}{L}\right)e^{-n^2\pi^2 kt/L^2}.$$

Since

$$1 - (-1)^n = \begin{cases} 2 & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even,} \end{cases}$$

we can omit the even values of n in the summation to write the solution

$$u(x, t) = \frac{4A}{\pi} \sum_{n=1}^{\infty} \frac{1}{2n-1} \sin\left(\frac{(2n-1)\pi x}{L}\right) e^{-(2n-1)^2\pi^2 kt/L^2}. \quad \blacklozenge$$

17.2.2 Insulated Ends

Suppose the bar has insulated ends, hence no energy is lost across the ends. The temperature distribution is modeled by the initial-boundary value problem

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \text{ for } 0 < x < L, t > 0,$$

$$\frac{\partial u}{\partial x}(0, t) = \frac{\partial u}{\partial x}(L, t) = 0 \text{ for } t \geq 0,$$

and

$$u(x, 0) = f(x) \text{ for } 0 \leq x \leq L.$$

As before, separation of variables yields

$$X'' + \lambda X = 0 \quad \text{and} \quad T' + \lambda k T = 0.$$

The insulation conditions give us

$$\frac{\partial u}{\partial x}(0, t) = X'(0)T(t) = \frac{\partial u}{\partial x}(L, t) = X'(L)T(t) = 0,$$

so $X'(0) = X'(L) = 0$, and the problem for X is

$$X'' + \lambda X = 0; \; X'(0) = X'(L) = 0.$$

Previously, we solved this problem for the eigenvalues $\lambda_n = n^2\pi^2/L^2$ and eigenfunctions $\cos(n\pi x/L)$ for $n = 0, 1, 2, \cdots$.

The equation for T is

$$T' + \frac{n^2\pi^2 k}{L^2} T = 0.$$

For $n = 0$, we get $T_0(t) =$ constant. For $n = 1, 2, \cdots$,

$$T_n(t) = e^{-n^2\pi^2 kt/L^2},$$

or constant multiples of this function. We now have a function

$$u_n(x, t) = c_n \cos\left(\frac{n\pi x}{L}\right) e^{-n^2\pi^2 kt/L^2},$$

which satisfies the heat equation and the insulation boundary conditions for $n = 0, 1, 2, \cdots$. To satisfy the initial condition, use the superposition

$$u(x, t) = \frac{1}{2}c_0 + \sum_{n=1}^{\infty} c_n \cos\left(\frac{n\pi x}{L}\right) e^{-n^2\pi^2 kt/L^2}.$$

Then

$$u(x, 0) = f(x) = \frac{1}{2}c_0 + \sum_{n=1}^{\infty} c_n \cos\left(\frac{n\pi x}{L}\right),$$

so choose the c_n's to be the Fourier cosine coefficients of f on $[0, L]$:

$$c_n = \frac{2}{L}\int_0^L f(\xi)\cos\left(\frac{n\pi\xi}{L}\right) d\xi.$$

EXAMPLE 17.2

Suppose the ends of the bar are insulated and the left half of the bar is initially at constant temperature A while the right half is initially at temperature zero. Then

$$f(x) = \begin{cases} A & \text{for } 0 \le x \le L/2 \\ 0 & \text{for } L/2 < x \le L. \end{cases}$$

Compute

$$c_0 = \frac{2}{L}\int_0^{L/2} A\,d\xi = A$$

and, for $n = 1, 2, \cdots$,

$$c_n = \frac{2}{L}\int_0^{L/2} A\cos\left(\frac{n\pi\xi}{L}\right)d\xi = \frac{2A}{n\pi}\sin(n\pi/2).$$

The solution is

$$u(x,t) = \frac{A}{2} + \frac{2A}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{n\pi}{2}\right)\cos\left(\frac{n\pi x}{L}\right)e^{-n^2\pi^2kt/L^2}.$$

Since $\sin(n\pi/2) = 0$ if n is even, we can retain only odd n in this summation to write this solution as

$$u(x,t) = \frac{A}{2} + \frac{2A}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\cos\left(\frac{(2n-1)\pi x}{L}\right)e^{-(2n-1)^2\pi^2kt/L^2}.$$ ♦

17.2.3 Radiating End

Suppose the left end of the bar is maintained at temperature zero, while the right end radiates energy into the surrounding medium, which is kept at temperature zero. If the initial temperature function is f, then the temperature distribution is modeled by the initial-boundary value problem

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \text{ for } 0 < x < L,\ t > 0,$$

$$u(0,t) = 0,\ \frac{\partial u}{\partial x}(L,t) = -Au(L,t) \text{ for } t > 0,$$

and

$$u(x,0) = f(x) \text{ for } 0 \le x \le L.$$

A is a positive constant called the *transfer coefficient*. Let $u(x,t) = X(x)T(t)$ to obtain

$$X'' + \lambda X = 0,\ T' + \lambda kT = 0.$$

Because $u(0,t) = X(0)T(t) = 0$, then $X(0) = 0$. From the radiation condition at the right end,

$$X'(L)T(t) = -AX(L)T(t) \text{ for } t > 0,$$

so

$$X'(L) + AX(L) = 0.$$

The problem for X is

$$X'' + \lambda X = 0;\ X(0) = 0,\ X'(L) + AX(L) = 0.$$

This is a regular Sturm-Liouville problem. To solve for the eigenvalues and eigenfunctions, consider cases on λ.

Case 1: $\lambda = 0$

We get $X(x) = cx + d$, so $X(0) = d = 0$. Then $X(x) = cx$, so

$$X'(L) + AX(L) = c + cL = 0$$

and this forces $c = 0$. This case yields only the trivial solution, so 0 is not an eigenvalue of this problem.

Case 2: $\lambda < 0$

Write $\lambda = -\alpha^2$ with $\alpha > 0$. Then $X'' - \alpha^2 X = 0$, with solutions

$$X(x) = ce^{\alpha x} + de^{-\alpha x}.$$

Now $X(0) = c + d = 0$ implies that $d = -c$, so

$$X(x) = c(e^{\alpha x} - e^{-\alpha x}).$$

From the radiation condition at L,

$$X'(L) = \alpha c(e^{\alpha L} + e^{-\alpha L}) = -AX(L) = -Ac(e^{\alpha L} - e^{-\alpha L}).$$

If $c \neq 0$, then

$$\alpha c(e^{\alpha L} + e^{-\alpha L}) > 0 \text{ and } - Ac\left(e^{\alpha L} - e^{-\alpha L}\right) < 0,$$

contradicting the preceding line. We conclude that $c = 0$, so this case also has only the trivial solution. This problem has no negative eigenvalue.

Case 3: $\lambda > 0$

Set $\lambda = \alpha^2$ with $\alpha > 0$. Now $X'' + \alpha^2 X = 0$, so

$$X(x) = c\cos(\alpha x) + d\sin(\alpha x).$$

Then $X(0) = c = 0$, so $X(x) = d\sin(\alpha x)$. Next,

$$X'(L) = \alpha d\cos(\alpha L) = -AX(L) = -Ad\sin(\alpha L).$$

If $d = 0$, we have only the trivial solution. If $d \neq 0$, then $\alpha\cos(\alpha L) = -A\sin(\alpha L)$, so

$$\tan(\alpha L) = -\frac{\alpha}{A}. \tag{17.3}$$

We have a nontrivial solution for X only if α is chosen to satisfy this transcendental equation, which we cannot solve algebraically for α. However, let $z = \alpha L$. Then equation (17.3) is $\tan(z) = -z/AL$. Part of the graphs of $y = \tan(z)$ and $y = -z/AL$ are shown in Figure 17.1, and they have infinitely many points of intersection for $z > 0$. The z-coordinates of these points are solutions of $\tan(z) = -z/AL$. Let these z-coordinates be $z_1, z_2, \cdots$ in increasing order. Since $\alpha = z/L$, the eigenvalues of this problem for X are

$$\lambda_n = \alpha_n^2 = \frac{z_n^2}{L^2}.$$

The eigenfunctions are functions $X_n(x) = \sin(\alpha_n x) = \sin(z_n x/L)$.

With these eigenvalues, the problem for T is

$$T' + \frac{z_n^2 k}{L^2} T = 0$$

with solutions that are constant multiples of

$$T_n(t) = e^{-z_n^2 kt/L^2}.$$

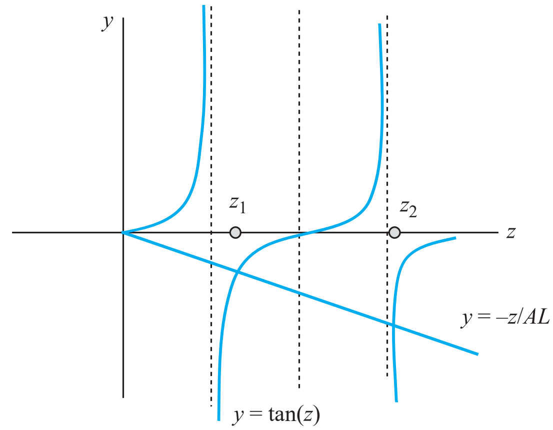

FIGURE 17.1 *Eigenvalues as the problem with radiating end.*

For each positive integer n, we now have a function

$$u_n(x,t) = X_n(x)T_n(t) = \sin\left(\frac{z_n x}{L}\right) e^{-z_n^2 kt/L^2}.$$

that satisfies the heat equation and the boundary conditions. To satisfy the initial condition, write the superposition

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{z_n x}{L}\right) e^{-z_n^2 kt/L^2}.$$

We must choose the c_n's to satisfy

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{z_n x}{L}\right).$$

This is not a Fourier sine series, but it is an eigenfunction expansion in the eigenfunctions of a Sturm-Liouville problem. The coefficients are

$$c_n = \frac{\int_0^L f(\xi)\sin(z_n \xi/L)\,d\xi}{\int_0^L \sin^2(z_n\xi/L)\,d\xi}.$$

The difficulty in computing these numbers is one we frequently encounter in such problems. We do not have an explicit formula for the z_n's. However, we can compute approximate values of the z_n's for a given L. For the first four values, with $L = 1$, we obtain the approximate values

$$z_1 \approx 2.0288,\ z_2 \approx 49132,\ z_3 \approx 7.9787,\ z_4 \approx 11.0855.$$

Using these numbers, we can perform numerical integrations to approximate

$$c_1 \approx 1.9207,\ c_2 \approx 2.6593,\ c_3 \approx 4.1457,\ c_4 \approx 5.6329.$$

With just the first four terms, we have an approximation

$$u(x,t) \approx 1.9027\sin(2.0288x)e^{-4.1160kt} + 2.6593\sin(4.9132x)e^{-24.1395kt}$$
$$+ 4.1457\sin(7.9787x)e^{-63.6597kt} + 5.6329\sin(11.0855x)e^{-1,228.883kt}.$$

Depending on k, these exponentials may be decaying so fast that these first four terms would be a good enough approximation for some applications.

17.2.4 Transformation of Problems

As with the wave equation, it may be impossible to separate variables in a diffusion problem, depending on the partial differential equation and/or the boundary conditions. Sometimes we can transform such a problem into one to which separation of variables applies.

EXAMPLE 17.3

A thin, homogeneous bar extends from $x = 0$ to $x = L$. The left end is maintained at constant temperature T_1 and the right end at constant temperature T_2. The initial temperature in the cross section at x is $f(x)$.

The initial-boundary value problem modeling this setting is

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \text{ for } 0 < x < L, t > 0,$$

$$u(0, t) = T_1, u(L, t) = T_2 \text{ for } t > 0,$$

and

$$u(x, 0) = f(x) \text{ for } 0 \leq x \leq L.$$

If we set $u(x, t) = X(x)T(t)$, we must have $X(0)T(t) = T_1$. If $T_1 = 0$, this is satisfied by requiring that $X(0) = 0$. If $T_1 \neq 0$, then $X(0) \neq 0$, and we must have $T(t) = T_1/X(0) =$ constant, which is not acceptable.

Perturb the temperature distribution function by setting

$$u(x, t) = U(x, t) + \psi(x).$$

The idea is to choose ψ to obtain a problem we know how to solve. Substitute U into the wave equation to get

$$\frac{\partial U}{\partial t} = k\left(\frac{\partial^2 U}{\partial x^2} + \psi''(x)\right).$$

This is the standard heat equation (17.1) if $\psi''(x) = 0$, hence, if $\psi(x) = cx + d$. Now

$$u(0, t) = T_1 = U(0, t) + \psi(0),$$

and this is the more friendly condition $U(0, t) = 0$ if $\psi(0) = T_1$. Thus, choose $d = T_1$ to have $\psi(x) = cx + T_1$. Next we need

$$u(L, t) = T_2 = U(L, t) + \psi(L),$$

and this is just $U(L, t) = 0$ if $\psi(L) = T_2$. We need $cL + T_1 = T_2$, so $c = (T_2 - T_1)/L$, and

$$\psi(x) = \frac{1}{L}(T_2 - T_1)x + T_1.$$

Finally,

$$u(x, 0) = f(x) = U(x, 0) + \psi(x),$$

so

$$U(x, 0) = f(x) - \psi(x) = f(x) - \frac{1}{L}(T_2 - T_1)x - T_1.$$

The problem for U is:

$$\frac{\partial U}{\partial t} = k\frac{\partial^2 U}{\partial x^2} \text{ for } 0 < x < L,\ t > 0,$$
$$U(0, t) = U(L, t) = 0 \text{ for } t > 0,$$

and

$$U(x, 0) = f(x) - \frac{1}{L}(T_2 - T_1)x - T_1 \text{ for } 0 \le x \le L.$$

We know the solution of this problem for U:

$$U(x, t) = \frac{2}{L}\sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right) e^{-n^2\pi^2 kt/L^2}$$

where

$$c_n = \frac{2}{L}\int_0^L \left[f(\xi) - \frac{1}{L}(T_2 - T_1)\xi - T_1 \right] \sin\left(\frac{n\pi\xi}{L}\right) d\xi.$$

The solution of the original problem is

$$u(x, t) = U(x, t) + \frac{1}{L}(T_2 - T_1)x + T_1.$$

As a specific example, suppose $T_1 = 1$, $T_2 = 2$ and $f(x) = 3/2$. Now

$$\psi(x) = \frac{1}{L}x + 1,$$

and

$$\begin{aligned} c_n &= \frac{2}{L}\int_0^L \left[\frac{3}{2} - \frac{1}{L}x - 1\right] \sin\left(\frac{n\pi\xi}{L}\right) d\xi \\ &= \frac{2}{L}\int_0^L \left(\frac{1}{2} - \frac{1}{L}x\right) \sin\left(\frac{n\pi\xi}{L}\right) d\xi \\ &= \frac{1 + (-1)^n}{n\pi}. \end{aligned}$$

The solution is

$$u(x, t) = \sum_{n=1}^{\infty} \left(\frac{1 + (-1)^n}{n\pi}\right) \sin\left(\frac{n\pi x}{L}\right) e^{-n^2\pi^2 kt/L^2} + \frac{1}{L}x + 1. \ \blacklozenge$$

Sometimes an initial-boundary value problem involving the heat equation can be transformed into a simpler problem by multiplying by an exponential function $e^{\alpha x + \beta t}$ and making appropriate choices for α and β. We will pursue this idea in the problems.

17.2.5 The Heat Equation with a Source Term

We will determine the solution of the initial-boundary value problem

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + F(x, t) \text{ for } 0 < x < L,\ t > 0,$$
$$u(0, t) = u(L, t) = 0 \text{ for } t > 0,$$

and

$$u(x, 0) = f(x) \text{ for } 0 \le x \le L.$$

The term $F(x, t)$, for example, could account for a source or loss of energy within the medium.

It is routine to try separation of variables and find that it does not apply in general to this heat equation. A different approach is needed.

As a starting point, we know that in the case that $F(x, t) = 0$ for all x and t the solution has the form

$$u(x, t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right) e^{-n^2\pi^2 kt/L^2}.$$

Taking a cue from this case, we will attempt a solution of the current problem of the form

$$u(x, t) = \sum_{n=1}^{\infty} T_n(t) \sin\left(\frac{n\pi x}{L}\right). \tag{17.4}$$

We must find the functions $T_n(t)$. If t is fixed, the left side of equation (17.4) is a function of x and the right side is its Fourier sine expansion on $[0, L]$, so the coefficients must be

$$T_n(t) = \frac{2}{L}\int_0^L u(\xi, t) \sin\left(\frac{n\pi\xi}{L}\right) d\xi. \tag{17.5}$$

Assume that for any $t \geq 0$, $F(x, t)$, thought of as a function of x, also can be expanded in a Fourier sine series on $[0, L]$:

$$F(x, t) = \sum_{n=1}^{\infty} B_n(t) \sin\left(\frac{n\pi\xi}{L}\right) \tag{17.6}$$

where

$$B_n(t) = \frac{2}{L}\int_0^L F(\xi, t) \sin\left(\frac{n\pi\xi}{L}\right) d\xi. \tag{17.7}$$

Differentiate equation (17.5) with respect to t to get

$$T_n'(t) = \frac{2}{L}\int_0^L \frac{\partial u}{\partial t}(\xi, t) \sin\left(\frac{n\pi\xi}{L}\right) d\xi. \tag{17.8}$$

Substitute for $\partial u/\partial t$ from the heat equation to get

$$T_n'(t) = \frac{2k}{L}\int_0^L \frac{\partial^2 u}{\partial x^2}(\xi, t) \sin\left(\frac{n\pi\xi}{L}\right) d\xi + \frac{2}{L}\int_0^L F(\xi, t) \sin\left(\frac{n\pi\xi}{L}\right) d\xi.$$

In view of equation (17.6), this equation becomes

$$T_n'(t) = \frac{2k}{L}\int_0^L \frac{\partial^2 u}{\partial x^2}(\xi, t) \sin\left(\frac{n\pi\xi}{L}\right) d\xi + B_n(t). \tag{17.9}$$

Integrate by parts twice, at the last step making use of the boundary conditions and equation (17.5). We get

$$\begin{aligned}
&\int_0^L \frac{\partial^2 u}{\partial x^2}(\xi, t) \sin\left(\frac{n\pi\xi}{L}\right) d\xi \\
&= \left[\frac{\partial u}{\partial x}(\xi, t)\sin\left(\frac{n\pi\xi}{L}\right)\right]_0^L - \int_0^L \frac{n\pi}{L}\frac{\partial u}{\partial x}(\xi, t)\cos\left(\frac{n\pi\xi}{L}\right) d\xi \\
&= -\frac{n\pi}{L}\left[u(\xi, t)\cos\left(\frac{n\pi\xi}{L}\right)\right]_0^L + \frac{n\pi}{L}\int_0^L -\frac{n\pi}{L} u(\xi, t)\sin\left(\frac{n\pi\xi}{L}\right) d\xi \\
&= -\frac{n^2\pi^2}{L^2}\int_0^L u(\xi, t)\sin\left(\frac{n\pi\xi}{L}\right) d\xi \\
&= -\frac{n^2\pi^2}{L^2}\frac{L}{2}T_n(t) = -\frac{n^2\pi^2}{2L}T_n(t).
\end{aligned}$$

Substitute this into equation (17.9) to get

$$T_n' = -\frac{n^2\pi^2 k}{L^2} T_n(t) + B_n(t).$$

For $n = 1, 2, \cdots$, this is a first-order differential equation for $T_n(t)$, which we write as

$$T_n'(t) + \frac{n^2\pi^2 k}{L^2} T_n(t) = B_n(t).$$

Next, use equation (17.5) to obtain the condition

$$T_n(0) = \frac{2}{L}\int_0^L u(\xi, 0)\sin\left(\frac{n\pi\xi}{L}\right) d\xi = \frac{2}{L}\int_0^L f(\xi)\sin\left(\frac{n\pi\xi}{L}\right) d\xi = b_n$$

which is the nth coefficient in the Fourier sine expansion of f on $[0, L]$. Solve the differential equation for $T_n(t)$ subject to this boundary condition to get

$$T_n(t) = \int_0^t e^{-n^2\pi^2 k(t-\tau)/L^2} B_n(\tau)\, d\tau + b_n e^{-n^2\pi^2 kt/L^2}.$$

Finally, substitute this into equation (17.4) to obtain the solution

$$u(x, t) = \sum_{n=1}^{\infty}\left(\int_0^t e^{-n^2\pi^2 k(t-\tau)/L^2} B_n(\tau)\, d\tau\right)\sin\left(\frac{n\pi x}{L}\right)$$
$$+ \frac{2}{L}\sum_{n=1}^{\infty}\left(\int_0^L f(\xi)\sin\left(\frac{n\pi\xi}{L}\right) d\xi\right)\sin\left(\frac{n\pi x}{L}\right) e^{-n^2\pi^2 kt/L^2}. \qquad (17.10)$$

Notice that the last term is the solution of the problem if $F(x, t) = 0$, while the first term is the effect of this source term on the solution.

EXAMPLE 17.4

We will solve the problem

$$\frac{\partial u}{\partial t} = 4\frac{\partial^2 u}{\partial x^2} + xt \text{ for } 0 < x < \pi,\ t > 0,$$
$$u(0, t) = u(\pi, t) = 0 \text{ for } t \geq 0,$$

and

$$u(x, 0) = f(x) = \begin{cases} 20 & \text{for } 0 \leq x \leq \pi/4 \\ 0 & \text{for } \pi/4 < x \leq \pi. \end{cases}$$

Since we have a formula (17.10) for the solution, we need only carry out the integrations. First compute

$$B_n(t) = \frac{2}{\pi}\int_0^\pi \xi t \sin(n\xi)\, d\xi = \frac{2(-1)^{n+1}}{n} t.$$

This enables us to evaluate

$$\int_0^t e^{-4n^2(t-\tau)} B_n(\tau)\, d\tau = \int_0^t \frac{2(-1)^{n+1}}{n}\tau e^{-4n^2(t-\tau)}\, d\tau$$
$$= \frac{1}{8}(-1)^{n+1}\frac{-1 + 4n^2 t + e^{-4n^2 t}}{n^5}.$$

Finally we need

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(\xi)\sin(n\xi)\,d\xi = \frac{40}{\pi}\int_0^{\pi/4}\sin(n\xi)\,d\xi = \frac{40}{\pi}\frac{1-\cos(n\pi/4)}{\pi}.$$

With all the components in place, the solution is

$$u(x,t) = \sum_{n=1}^{\infty}\left(\frac{1}{8}(-1)^{n+1}\frac{-1+4n^2t+e^{-4n^2t}}{n^5}\right)\sin(nx) + \sum_{n=1}^{\infty}\frac{40}{\pi}\frac{1-\cos(n\pi/4)}{n}\sin(nx)e^{-4n^2t}.$$

The second term on the right is the solution of the problem with the xt term omitted. Denote this "no-source" solution as

$$u_0(x,t) = \sum_{n=1}^{\infty}\frac{40}{\pi}\frac{1-\cos(n\pi/4)}{n}\sin(nx)e^{-4n^2t}.$$

Then the solution with the source term is

$$u(x,t) = u_0(x,t) + \sum_{n=1}^{\infty}\left(\frac{1}{8}(-1)^{n+1}\frac{-1+4n^2t+e^{-4n^2t}}{n^5}\right)\sin(nx).$$

Writing the solution in this way enables us to gauge the effect of the xt term on the solution. Figures 17.2 through 17.5 compare graphs of $u(x,t)$ and $u_0(x,t)$ at times $t = 0.3, 0.8, 1.2$ and 1.3. In each figure, $u_0(x,t)$ falls below $u(x,t)$. Both solutions decay to zero as t increases, but the effect of the xt term is to retard this decay. ♦

17.2.6 Effects of Boundary Conditions and Constants

We will investigate how constants and terms appearing in diffusion problems influence the behavior of solutions.

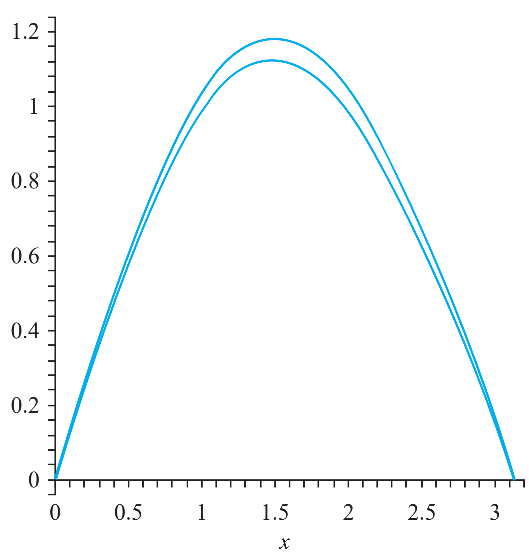

FIGURE 17.2 *$u(x,t)$ and $u_0(x,t)$ at $t = 0.3$.*

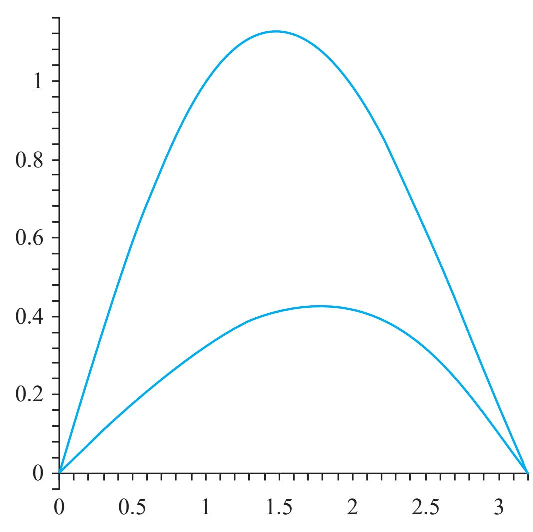

FIGURE 17.3 $t = 0.8$.

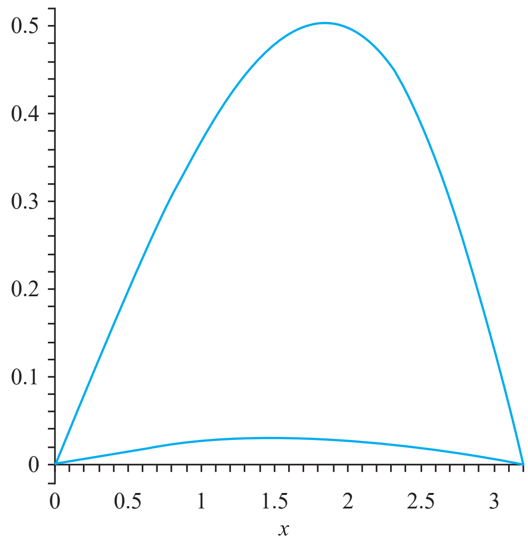

FIGURE 17.4 $t = 1.2$.

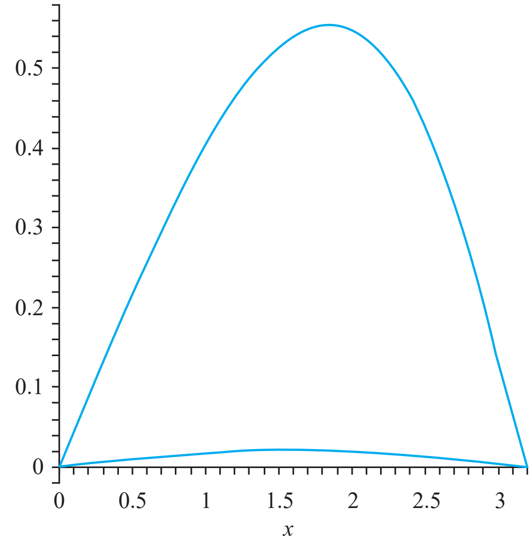

FIGURE 17.5 $t = 1.3$.

EXAMPLE 17.5

A homogeneous bar of length π has initial temperature function $f(x) = x^2 \cos(x/2)$ and ends maintained at temperature zero. The temperature distribution function satisfies

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \text{ for } 0 < x < \pi,\ t > 0,$$

$$u(0, t) = u(\pi, t) = 0 \text{ for } t > 0,$$

and

$$u(x, 0) = x^2 \cos(x/2) \text{ for } 0 \le x \le \pi.$$

The solution is

$$u(x, t) = \frac{2}{\pi}\sum_{n=1}^{\infty}\left(\int_0^{\pi} \xi^2 \cos(\xi/2)\sin(n\xi)\, d\xi\right)\sin(nx)e^{-n^2kt}$$

$$= \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{16\pi n(-1)^n - 64\pi n^3(-1)^n - 48n - 64n^3}{64n^6 - 48n^4 + 12n^2 - 1}\sin(nx)e^{-n^2kt}.$$

To gauge the effect of the diffusivity constant k on the solution, Figure 17.6 shows graphs of $y = u(x, t)$ for $t = 0.2$ and for $k = 0.3, 0.6, 1.1$, and 2.7. Figure 17.7 has the graphs for the same values of k, but $t = 1.2$. For each k, the temperature function decays with time, as we expect. However, for each time the temperature function has a smaller maximum as k increases. ♦

EXAMPLE 17.6

We will examine the effects on $u(x, t)$, depending on whether the ends of the bar are kept at temperature zero, or are insulated. Suppose the initial temperature function is $f(x) = x^2(\pi - x)$ and $L = \pi$ and $k = 1/4$. For the ends at temperature zero, we find that

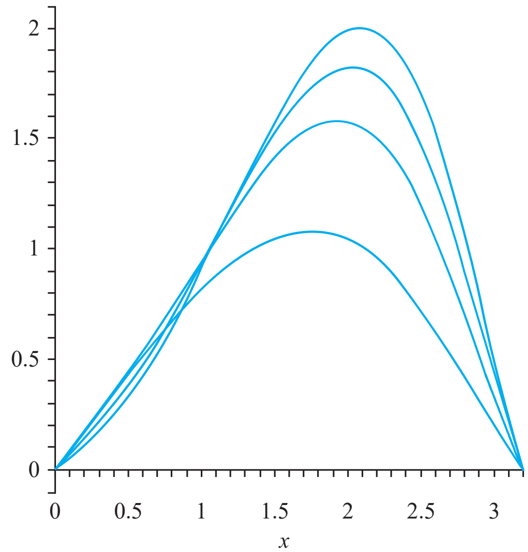

FIGURE 17.6 *The solution in Example 17.5 at* $t = 0.2$ *and* $k = 0.3, 0.6, 1.4,$ *and* 2.7.

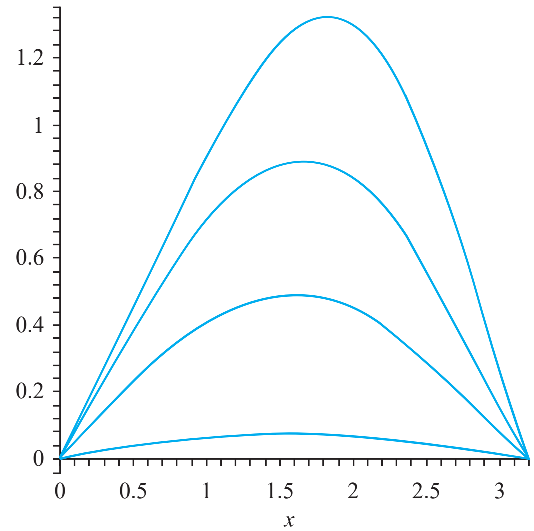

FIGURE 17.7 *The solution in Example 17.5 at* $t = 1.2$ *and* $k = 0.3, 0.6, 1.4,$ *and* 2.7.

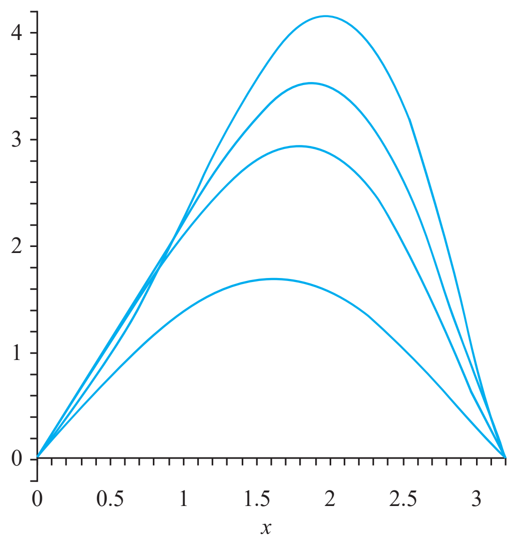

FIGURE 17.8 $u_1(x, t)$ *at times* $t = 0.4, 0.9, 1.5,$ *and* 3.6.

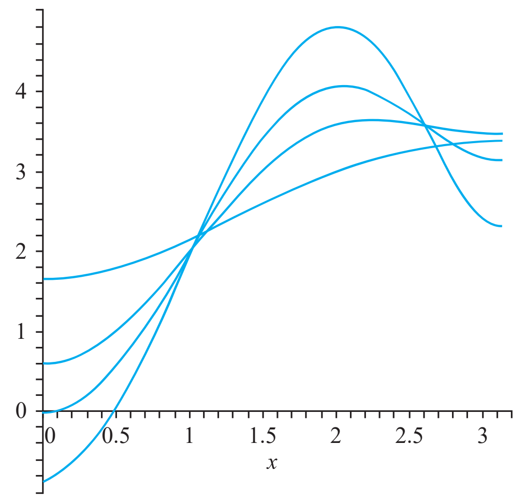

FIGURE 17.9 $u_2(x, t)$ *at times* $t = 0.4, 0.9, 1.5,$ *and* 3.6.

$$u_1(x, t) = \sum_{n=1}^{\infty} \left(\frac{8(-1)^{n+1} - 4}{n^3} \right) \sin(nx) e^{-n^2 t/4}.$$

If the ends are insulated the solution is

$$u_2(x, t) = \frac{1}{12}\pi^3 + \sum_{n=1}^{\infty} \left(\frac{n^2\pi^2(-1)^{n+1} + 6(-1)^n - 6}{n^4} \right) \cos(nx) e^{-n^2 t/4}.$$

Figure 17.8 shows $u_1(x, t)$, and Figure 17.9 shows $u_2(x, t)$, at times $t = 0.4, 0.9, 1.5$ and 3.6. Both solutions decrease as t increases. However, as $t \to \infty$, $u_2(x, t) \to \pi^3/12$, as suggested in Figure 17.9. ♦

SECTION 17.2 PROBLEMS

In each of Problems 1 through 7, write a solution of the initial-boundary value problem. Graph the twentieth partial sum of the series for $u(x, t)$ on the same set of axes for different values of the time.

1. $$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \text{ for } 0 < x < L,\ t > 0,$$
$$u(0, t) = u(L, t) = 0 \text{ for } t \geq 0,$$
$$u(x, 0) = x(L - x) \text{ for } 0 \leq x \leq L$$

2. $$\frac{\partial u}{\partial t} = 4\frac{\partial^2 u}{\partial x^2} \text{ for } 0 < x < L,\ t > 0,$$
$$u(0, t) = u(L, t) = 0 \text{ for } t \geq 0,$$
$$u(x, 0) = x^2(L - x) \text{ for } 0 \leq x \leq L$$

3. $$\frac{\partial u}{\partial t} = 3\frac{\partial^2 u}{\partial x^2} \text{ for } 0 < x < L,\ t > 0,$$
$$u(0, t) = u(L, t) = 0 \text{ for } t \geq 0,$$
$$u(x, 0) = L(1 - \cos(2\pi x/L)) \text{ for } 0 \leq x \leq L$$

4. $$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \text{ for } 0 < x < \pi,\ t > 0,$$
$$\frac{\partial u}{\partial x}(0, t) = \frac{\partial u}{\partial x}(\pi, t) = 0 \text{ for } t \geq 0,$$
$$u(x, 0) = \sin(x) \text{ for } 0 \leq x \leq \pi$$

5. $$\frac{\partial u}{\partial t} = 4\frac{\partial^2 u}{\partial x^2} \text{ for } 0 < x < 2\pi,\ t > 0,$$
$$\frac{\partial u}{\partial x}(0, t) = \frac{\partial u}{\partial x}(2\pi, t) = 0 \text{ for } t \geq 0,$$
$$u(x, 0) = x(2\pi - x) \text{ for } 0 \leq x \leq 2\pi$$

6. $$\frac{\partial u}{\partial t} = 4\frac{\partial^2 u}{\partial x^2} \text{ for } 0 < x < 3,\ t > 0,$$
$$\frac{\partial u}{\partial x}(0, t) = \frac{\partial u}{\partial x}(3, t) = 0 \text{ for } t \geq 0,$$
$$u(x, 0) = x^2 \text{ for } 0 \leq x \leq 3$$

7. $$\frac{\partial u}{\partial t} = 2\frac{\partial^2 u}{\partial x^2} \text{ for } 0 < x < 6,\ t > 0,$$
$$\frac{\partial u}{\partial x}(0, t) = \frac{\partial u}{\partial x}(6, t) = 0 \text{ for } t \geq 0,$$
$$u(x, 0) = e^{-x} \text{ for } 0 \leq x \leq 6$$

8. A thin, homogeneous bar of length L has insulated ends and initial temperature B, a positive constant. Find the temperature distribution in the bar.

9. A thin homogeneous bar of length L has initial temperature $f(x) = B$ where the right end $x = L$ is insulated, while the left end is kept at zero temperature. Find the temperature distribution in the bar.

10. A thin, homogeneous bar having thermal diffusivity of 9 and a length of 2 cm has insulated sides and its left end maintained at zero temperature, while its right end is perfectly insulated. The bar has an initial temperature $f(x) = x^2$ for $0 \leq x \leq 2$. Determine the temperature distribution $u(x, t)$ and $\lim_{t\to 0} u(x, t)$.

11. Show that the partial differential equation
$$\frac{\partial u}{\partial t} = k\left(\frac{\partial^2 u}{\partial x^2} + A\frac{\partial u}{\partial x} + Bu\right)$$
can be transformed into a standard heat equation for v by letting $u(x, t) = e^{\alpha x + \beta t}v(x, t)$ and choosing α and β appropriately.

12. Use the idea of Problem 11 to solve
$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + 4\frac{\partial u}{\partial x} + 2u \text{ for } 0 < x < \pi,\ t > 0,$$
$$u(0, t) = u(\pi, t) = 0 \text{ for } t \geq 0,$$
$$u(x, 0) = x(\pi - x) \text{ for } 0 \leq x \leq \pi.$$

13. Solve
$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + 6\frac{\partial u}{\partial x} \text{ for } 0 < x < 4,\ t > 0,$$
$$u(0, t) = u(4, t) = 0 \text{ for } t \geq 0,$$
$$u(x, 0) = 1 \text{ for } 0 \leq x \leq 4.$$
Graph the twentieth partial sum of the solution for selected times.

14. Solve
$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - 6\frac{\partial u}{\partial x} \text{ for } 0 < x < \pi,\ t > 0,$$
$$u(0, t) = u(\pi, t) = 0 \text{ for } t \geq 0,$$
$$u(x, 0) = x(\pi - x) \text{ for } 0 \leq x \leq \pi.$$
Graph the twentieth partial sum of the solution for selected times.

15. Solve
$$\frac{\partial u}{\partial t} = 16\frac{\partial^2 u}{\partial x^2} \text{ for } 0 < x < 1,\ t > 0,$$
$$u(0, t) = 2,\ u(1, t) = 5 \text{ for } t \geq 0,$$
$$u(x, 0) = x^2 \text{ for } 0 \leq x \leq 1.$$

16. Solve
$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \text{ for } 0 < x < L,\ t > 0,$$
$$u(0, t) = T,\ u(L, t) = 0 \text{ for } t \geq 0,$$
$$u(x, 0) = x(L - x) \text{ for } 0 \leq x \leq L.$$