SEVENTH EDITION

ADVANCED ENGINEERING MATHEMATICS

SOLUTION MANUAL

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Instructor's Solutions Manual

to Accompany

ADVANCED ENGINEERING MATHEMATICS

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Contents

Chapter 1

First-Order Differential Equations

1.1 Terminology and Separable Equations

1. For $x > 1$,
$$2\varphi\varphi' = 2\sqrt{x-1}\frac{1}{2\sqrt{x-1}} = 1,$$
so φ is a solution.

2. With $\varphi(x) = Ce^{-x}$,
$$\varphi' + \varphi = -Ce^{-x} + Ce^{-x} = 0,$$
so φ is a solution.

3. For $x > 0$, rewrite the equation as
$$2xy' + 2y = e^x.$$
With $y = \varphi(x) = \frac{1}{2}x^{-1}(C - e^x)$, compute
$$y' = \frac{1}{2}\left(-x^{-2}(C - e^x) - x^{-1}e^x\right).$$
Then
$$2xy' + 2y = x\left(-x^{-2}(C - e^x) - x^{-1}e^x\right) + x^{-1}(C - e^x) = e^x.$$
Therefore $\varphi(x)$ is a solution.

4. For $x \neq \pm\sqrt{2}$,
$$\varphi' = \frac{-2cx}{(x^2-2)^2} = \left(\frac{2x}{2-x^2}\right)\left(\frac{c}{x^2-2}\right) = \frac{2x\varphi}{2-x^2},$$
so φ is a solution.

5. On any interval not containing $x = 0$ we have
$$x\varphi' = x\left(\frac{1}{2} + \frac{3}{2x^2}\right) = x + \left(\frac{3}{2x} - \frac{x}{2}\right) = x - \left(\frac{x^2-3}{2x}\right) = x - \varphi,$$
so φ is a solution.

6. For all x,
$$\varphi' + \varphi = -Ce^{-x} + (1 + Ce^{-x}) = 1$$
so $\varphi(x) = 1 + Ce^{-x}$ is a solution.

7. Write
$$3\frac{dy}{dx} = \frac{4x}{y^2}$$
and separate variables:
$$3y^2\,dy = 4x\,dx.$$
Integrate to obtain
$$y^3 = 2x^2 + k,$$
which implicitly defines the general solution. We can also write
$$y = \left(2x^2 + k\right)^{1/3}.$$

8. Write the differential equation as
$$x\frac{dy}{dx} = -y$$
and separate the variables:
$$\frac{1}{y}\,dy = -\frac{1}{x}\,dx.$$
This separation requires that $x \neq 0$ and $y \neq 0$. Integration gives us $\ln|y| = -\ln|x| + c$. Then
$$\ln|y| + \ln|x| = c$$
so $\ln|xy| = c$. Then $xy = e^c = k$, in which k can be any positive constant. Notice now that $y = 0$ is also a solution of the original differential equation. Therefore, if we allow k to be any constant (positive, negative or zero), we can omit the absolute values and write the general solution in the implicit form $xy = k$.

9. Write the differential equation as
$$\begin{aligned}\frac{dy}{dx} &= \frac{\sin(x+y)}{\cos(y)}\\ &= \frac{\sin(x)\cos(y) + \cos(x)\sin(y)}{\cos(y)}\\ &= \sin(x) + \cos(x)\frac{\sin(y)}{\cos(y)}.\end{aligned}$$

There is no way to separate the variables in this equation, so the differential equation is not separable.

10. Since $e^{x+y} = e^x e^y$, we can write the differential equation as
$$e^x e^y \frac{dy}{dx} = 3x$$
or, in separated form,
$$e^y\, dy = 3xe^{-x}\, dx.$$
Integration gives us the implicitly defined general solution
$$e^y = -3e^{-x}(x+1) + c.$$

11. Write the differential equation as
$$x\frac{dy}{dx} = y(y-1).$$
This is separable. If $y \neq 0$ and $y \neq 1$, we can write
$$\frac{1}{x}\, dx = \frac{1}{y(y-1)}\, dy.$$
Use partial fractions to write this as
$$\frac{1}{x}\, dx = \frac{1}{y-1}\, dy - \frac{1}{y}\, dy.$$
Integrate to obtain
$$\ln|x| = \ln|y-1| - \ln|y| + c,$$
or
$$\ln|x| = \ln\left|\frac{y-1}{y}\right| + c.$$
This can be solved for x to obtain the general solution
$$y = \frac{1}{1-kx}.$$
The trivial solution $y(x) = 0$ is a singular solution, as is the constant solution $y(x) = 1$. We assumed that $y \neq 0, 1$ in the algebra of separating the variables.

12. This equation is not separable.

13. This equation is separable since we can write it as
$$\frac{\sin(y)}{\cos(y)}\, dy = \frac{1}{x}\, dx$$
if $\cos(y) \neq 0$ and $x \neq 0$. A routine integration gives the implicitly defined general solution $\sec(y) = kx$. Now $\cos(y) = 0$ if $y = (2n+1)\pi/2$ for n any integer. $y = (2n+1)\pi/2$ also satisfies the original differential equation and is a singular solution.

14. The differential equation itself assumes that $y \neq 0$ and $x \neq -1$. Write

$$\frac{x}{y}\frac{dy}{dx} = \frac{2y^2+1}{x+1},$$

which separates as

$$\frac{1}{y(2y^2+1)}\,dy = \frac{1}{x(x+1)}\,dx.$$

Use a partial fractions decomposition to write

$$\left(\frac{1}{y} - \frac{2y}{1+2y^2}\right)dy = \left(\frac{1}{x} - \frac{1}{1+x}\right)dx.$$

Integration this equation to obtain

$$\ln|y| - \frac{1}{2}\ln(1+2y^2) = \ln|x| - \ln|x+1| + c.$$

Then,

$$\ln\left(\frac{y}{\sqrt{1+2y^2}}\right) = \ln\left(\frac{x}{x+1}\right) + c,$$

in which we have taken the case that $y > 0$ and $x > 0$ to drop the absolute values. Finally, take the exponential of both sides of this equation to obtain the implicitly defined solution

$$\frac{y}{\sqrt{1+2y^2}} = k\left(\frac{x}{x+1}\right).$$

Since $y = 0$ satisfies the original differential equation, $y = 0$ is a singular solution.

15. This differential equation is not separable.

16. Substitute

$$\sin(x-y) = \sin(x)\cos(y) - \cos(x)\sin(y),$$
$$\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y),$$

and

$$\cos(2x) = \cos^2(x) - \sin^2(x)$$

into the differential equation to obtain the separated equation

$$(\cos(y) - \sin(y))\,dy = (\cos(x) - \sin(x))\,dx.$$

Upon integrating we obtain the implicitly defined solution

$$\cos(y) + \sin(y) = \cos(x) + \sin(x) + c.$$

17. If $y \neq -1$ and $x \neq 0$, we obtain the separated equation

$$\frac{y^2}{y+1}\,dy = \frac{1}{x}\,dx.$$

Write this as

$$\left(y - 1 + \frac{1}{1+y}\right)\,dy = \frac{1}{x}\,dx.$$

Integrate to obtain

$$\frac{1}{2}y^2 - y + \ln|1+y| = \ln|x| + c.$$

Now use the initial condition $y(3e^2) = 2$ to obtain

$$2 - 2 + \ln(3) = \ln(3) + 2 + c$$

so $c = -2$ and the solution is implicitly defined by

$$\frac{1}{2}y^2 - y + \ln(1+y) = \ln(x) - 2,$$

in which the absolute values have been removed because the initial condition puts the solution in a part of the $x, y-$ plane where $x > 0$ and $y > -1$.

18. Integrate

$$\frac{1}{y+2}\,dy = 3x^2\,dx$$

to obtain $\ln|2+y| = x^3 + c$. Substitute the initial condition to obtain $c = \ln(10) - 8$. The solution is defined by

$$\ln\left(\frac{2+y}{10}\right) = x^3 - 8.$$

19. Write $\ln(y^x) = x\ln(y)$ and separate the variables to write

$$\frac{\ln(y)}{y}\,dy = 3x\,dx.$$

Integrate to obtain $(\ln(y))^2 = 3x^2 + c$. Substitute the initial condition to obtain $c = -3$, so the solution is implicitly defined by $(\ln(y))^2 = 3x^2 - 3$.

20. Write $e^{x-y^2} = e^x e^{-y^2}$ and Separate the variables to obtain

$$2ye^{y^2}\,dy = e^x\,dx.$$

Integrate to get $e^{y^2} = e^x + c$. The condition $y(4) = -2$ requires that $c = 0$, so the solution is defined implicitly by $e^{y^2} = e^x$, or $x = y^2$. Since $y(4) = -2$, the explicit solution is $y = -\sqrt{x}$.

21. Separate the variables to obtain

$$y\cos(3y)\,dy = 2x\,dx,$$

with solution given implicitly by

$$\frac{1}{3}y\sin(3y) + \frac{1}{9}\cos(3y) = x^2 + c.$$

The initial condition requires that

$$\frac{\pi}{9}\sin(\pi) + \frac{1}{9}\cos(\pi) = \frac{4}{9} + c,$$

so $c = -5/9$. The solution is implicitly defined by

$$3y\sin(3y) + \cos(3y) = 9x^2 - 5.$$

22. By Newton's law of cooling the temperature function $T(t)$ satisfies $T'(t) = k(T-60)$, with k a constant of proportionality to be determined, and with $T(0) = 90$ and $T(10) = 88$. This is based on the object being placed in the environment at time zero. This differential equation is separable (as in the text) and we solve it subject to $T(0) = 90$ to obtain $T(t) = 60+30e^{kt}$. Now

$$T(10) = 88 = 60 + 30e^{10k}$$

gives us $e^{10k} = 14/15$. Then

$$k = \frac{1}{10}\ln\left(\frac{14}{15}\right) \approx -6.899287(10^{-3}).$$

Since $e^{10k} = 14/15$, we can write

$$T(t) = 60 + 30(e^{10k})^{t/10} = 60 + 30\left(\frac{14}{15}\right)^{t/10}.$$

Now

$$T(20) = 60 + 30\left(\frac{14}{15}\right)^2 \approx 86.13$$

degrees Fahrenheit. To reach 65 degrees, solve

$$65 = 60 + 30\left(\frac{14}{15}\right)^{t/10}$$

to obtain

$$t = \frac{10\ln(1/6)}{\ln(14/15)} \approx 259.7$$

minutes.

23. Suppose the thermometer was removed from the house at time $t = 0$, and let $t > 0$ denote the time in minutes since then. The house is kept at 70 degrees F. Let A denote the unknown outside ambient temperature, which is assumed constant. The temperature of the thermometer at time t is modeled by

$$T'(t) = k(T - A); T(0) = 70, T(5) = 60 \text{ and } T(15) = 50.4.$$

There are three conditions because we must find k and then A.

Separation of variables and the initial condition $T(0) = 70$ yield the expression $T(t) = A + (70 - A)e^{kt}$. The other two conditions now give us

$$T(5) = 60 = A + (70 - A)e^{5k} \text{ and } T(15) = 50.4 = A + (70 - A)e^{15k}.$$

Solve the first equation to obtain

$$e^{5k} = \frac{60 - A}{70 - A}.$$

Substitute this into the second equation to obtain

$$(7 - A)\left(\frac{60 - A}{70 - A}\right)^3 = 50.4 - A.$$

This yields the quadratic equation

$$10.4A^2 - 1156A + 30960 = 0$$

with roots $A = 45$ and 66.16. Clearly we require that $A < 50.4$, so $A = 45$ degrees Fahrenheit.

24. The amount $A(t)$ of radioactive material at time t is modeled by

$$A'(t) = kA; A(0) = e^3$$

together with the condition $A(\ln(2)) = e^3/2$, since we must also find k. Time is in weeks. Solve to obtain

$$A(t) = \left(\frac{1}{2}\right)^{t/\ln(2)} e^3$$

tons. Then $A(3) = e^3(1/2)^{3/\ln(2)} = 1$ ton.

25. Similar to Problem 24, we find that the amount of Uranium-235 at time t is

$$U(t) = 10\left(\frac{1}{2}\right)^{t/(4.5(10^9))},$$

with t in years. Then $U(10^9) = 10(1/2)^{1/4.5} \approx 8.57$ kg.

26. At any time t there will be $A(t) = 12e^{kt}$ gms, and $A(4) = 9.1$ requires that $e^{4k} = 9.1/12$, so

$$k = \frac{1}{4}\ln\left(\frac{9.1}{12}\right) \approx -0.06915805.$$

The half-life is the time t^* so that $A(t^*) = 6$, or $e^{kt^*} = 1/2$. This gives $t^* = -\ln(2)/k \approx 10.02$ minutes.

27. Compute

$$I'(x) = -\int_0^\infty \frac{2x}{t} e^{-(t^2+(x/t)^2)}\,dt.$$

Let $u = x/t$ to obtain

$$\begin{aligned} I'(x) &= 2\int_\infty^0 e^{-((x/u)^2+u^2)}\,du \\ &= -2\int_0^\infty e^{-(u^2+(x/u)^2)}\,du = -2I(x). \end{aligned}$$

This is the separable equation $I' = -2I$. Write this as

$$\frac{1}{I}\,dI = -2\,dx$$

and integrate to obtain $I(x) = ce^{-2x}$. Now

$$I(0) = \int_0^\infty e^{-t^2}\,dt = \frac{\sqrt{\pi}}{2},$$

a standard result often used in statistics. Then

$$I(x) = \frac{\sqrt{\pi}}{2}e^{-2x}.$$

Put $x = 3$ to obtain

$$\int_0^\infty e^{-t^2-(9/t^2)}\,dt = \frac{\sqrt{\pi}}{2}e^{-6}.$$

28. (a) For water h feet deep in the cylindrical hot tub, $V = 25\pi h$, so

$$25\pi\frac{dh}{dt} = -0.6\pi\left(\frac{5}{16}\right)^2\sqrt{64h},$$

with $h(0) = 4$. Thus

$$\frac{dh}{dt} = -\frac{3\sqrt{h}}{160}.$$

(b) The time it will take to drain the tank is

$$T = \int_4^0 \left(\frac{dt}{dh}\right) dh$$
$$= \int_4^0 -\frac{160}{3\sqrt{h}}\, dh = \frac{640}{3}$$

seconds.

(c) To drain the upper half will require

$$T_1 = \int_4^2 -\frac{160}{3\sqrt{h}}\, dh = \frac{320}{3}(2-\sqrt{2})$$

seconds, approximately 62.5 seconds. The lower half requires

$$T_2 = \int_2^0 -\frac{160}{3\sqrt{h}}\, dh = \frac{320}{3}\sqrt{2}$$

seconds, about 150.8 seconds.

29. Model the problem using Torricelli's law and the geometry of the hemispherical tank. Let $h(t)$ be the depth of the liquid at time t, $r(t)$ the radius of the top surface of the draining liquid, and $V(t)$ the volume in the container (See Figure 1.1). Then

$$\frac{dV}{dt} = -kA\sqrt{2gh} \text{ and } \frac{dV}{dt} = \pi r^2 \frac{dh}{dt}.$$

Here $r^2 + h^2 = 18^2$, since the radius of the tub is 18. We are given $k = 0.8$ and $A = \pi(1/4)^2 = \pi/16$ is the area of the drain hole. With $g = 32$ feet per second per second, we obtain the initial value problem

$$\pi(324 - h^2)\frac{dh}{dt} = 0.4\pi\sqrt{h}; h(0) = 18.$$

This is a separable differential equation with the general solution

$$1620\sqrt{h} - h^{5/2} = -t + k.$$

Then $h(0) = 18$ yields $k = 3888\sqrt{2}$, so

$$1620\sqrt{h} - h^{5/2} = 3888\sqrt{2} - t.$$

The hemisphere is emptied at the instant that $h = 0$, hence at $t = 3888\sqrt{2}$ seconds, about 91 minutes, 39 seconds.

30. From the geometry of the sphere (Figure 1.2), $dV/dt = -kA\sqrt{2gh}$ becomes

$$\pi(32A - (h-18)^2)\frac{dh}{dt} = -0.8\pi\left(\frac{1}{4}\right)^2\sqrt{64h},$$

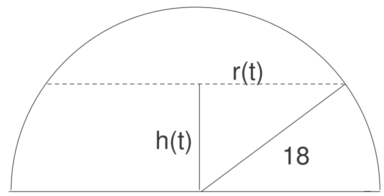

Figure 1.1: Problem 29, Section 1.1.

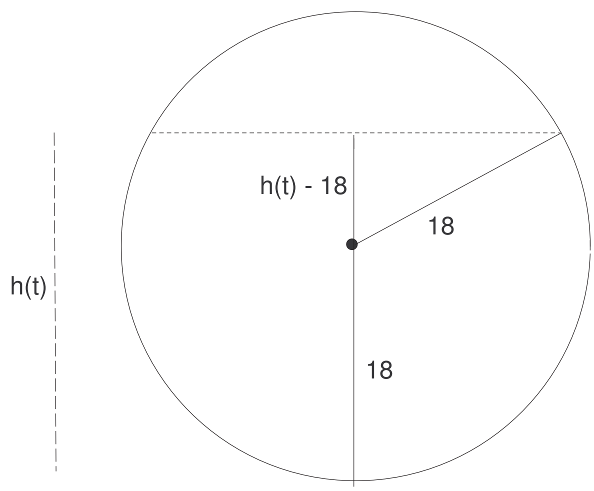

Figure 1.2: Problem 30, Section 1.1.

with $h(0) = 36$. Here $h(t)$ is the height of the upper surface of the fluid above the bottom of the sphere. This equation simplifies to

$$(36\sqrt{h} - h^{3/2})\,dh = -0.4\,dt,$$

a separated equation with general solution $h\sqrt{h}(60 - h) = -t + k$. Then $t = 0$ when $h = 36$ gives us $k = 5184$. The tank runs empty when $h = 0$, so $t = 5184$ seconds, about 86.4 minutes. This is the time it takes to drain this spherical tank.

31. (a) Let $r(t)$ be the radius of the exposed water surface and $h(t)$ the depth of the draining water at time t. Since cross sections of the cone are similar,

$$\pi r^2 \frac{dh}{dt} = -kA\sqrt{2gh},$$

with $h(0) = 9$. From similar triangles (Figure 1.3), $r/h = 4/9$, so $r = (4/9)h$. Substitute $k = 0.6$, $g = 32$ and $A = \pi(1/12)^2$ and simplify the resulting equation to obtain

$$h^{3/2}\frac{dh}{dt} = -27/160,$$

with $h(0) = 9$. This separable equation has the general solution given implicitly by

$$h^{5/2} = -\frac{27}{64}t + k.$$

Since $h(0) = 9$, then $k = 243$ and the tank empties out when $h = 0$, so

$$t = 243\left(\frac{64}{27}\right) = 576$$

seconds, about 9 minutes, 36 seconds.

(b) This problem is modeled like part (a), except now the cone is inverted. This changes the similar triangle proportionality (Figure 1.4) to

$$\frac{r}{9-h} = \frac{4}{9}.$$

Then $r = (4/9)(9 - h)$. The separable differential equation becomes

$$\frac{(9-h)^2}{\sqrt{h}}\,dh = -\frac{27}{160},$$

with $h(0) = 9$. This initial value problem has the solution

$$162\sqrt{h} - 12h^{3/2} + \frac{2}{5}h^{5/2} = -\frac{27}{160}t + \frac{1296}{5}.$$

The tank runs dry at $h = 0$, which occurs when

$$t = \frac{160}{27}\left(\frac{1296}{5}\right) = 1536$$

seconds, about 25 minutes, 36 seconds.

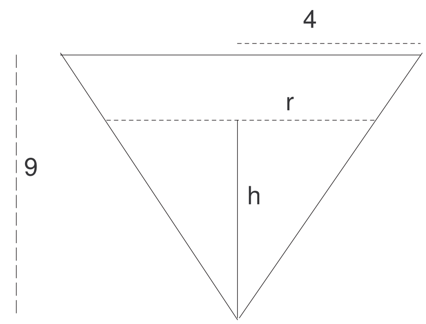

Figure 1.3: Problem 31(a), Section 1.1.

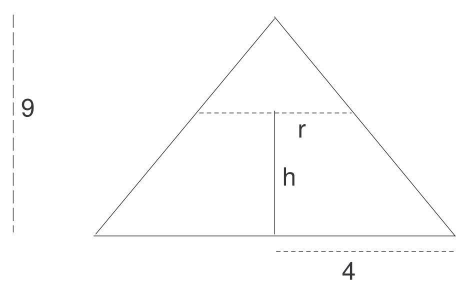

Figure 1.4: Problem 31(b), Section 1.1.

32. From the geometry of the cone and Torricelli's law,

$$\frac{dV}{dt} = \pi\left(\frac{16}{81}\right)h^2\frac{dh}{dt} = -\frac{(0.6)(8\pi)}{144}\sqrt{h-2}$$

when the drain hole is two feet above the vertex. With the drain hole at the bottom of the tank we get

$$\frac{dV}{dt} = \pi\left(\frac{16}{81}\right)^2 h^2\frac{dh}{dt} = -\frac{(0.6)(8\pi)}{144}\sqrt{h}.$$

If we know the rates of change of depth of the water in these two instances, then we can locate the drain hole height above the bottom of the tank, knowing the hole size, since

$$\pi\left(\frac{16}{81}\right)h^2\left(\frac{dh}{dt}\right)_1 = -kA\sqrt{2g(h-h_0)}$$

divided by

$$\pi\left(\frac{16}{81}\right)^2 h^2\left(\frac{dh}{dt}\right)_2 = -kA\sqrt{2gh}$$

yields

$$\frac{h-h_0}{\sqrt{h}} = \frac{(dh/dt)_1}{(dh/dt)_2} = r,$$

a known constant. We can therefore solve for h_0, the location of the hole above the bottom of the tank.

33. Begin with the logistic equation

$$P'(t) = aP(t) - bP(t)^2$$

in which a and b are positive constants. Then

$$\frac{dP}{dt} = (a-bP)P.$$

This is separable and we can write

$$\frac{1}{(a-bP)P}\,dP = dt.$$

Use a partial fractions decomposition to write

$$\left(\frac{1}{a}\frac{1}{P} + \frac{b}{a}\frac{1}{a-bP}\right)dP = dt.$$

Integrate to obtain

$$\frac{1}{a}\ln(P) - \frac{1}{a}\ln(a-bP) = t + c.$$

Here we assume that $P(t) > 0$ and $a - bP(t) > 0$. Write this equation as

$$\ln\left(\frac{P}{a - bP}\right) = at + k,$$

with $k = ac$ still a constant to be determined. Then

$$\frac{P}{a - bP} = e^{at+k} = e^k e^{at} = Ke^{at},$$

where $K = e^k$ is the constant to be determined. Now $P(0) = p_0$, so

$$K = \frac{p_0}{a - bp_0}.$$

Then

$$\frac{P}{a - bP} = \frac{p_0}{a - bp_0}e^{at}.$$

It is a straightforward algebraic manipulation to solve for P and obtain

$$P(t) = \frac{ap_0}{a - bp_0 + bp_0e^{at}}e^{at}.$$

Notice that $P(t)$ is a strictly increasing function. Further, by multiplying numerator and denominator by e^{-at}, and using the fact that $a > 0$, we have

$$\begin{aligned}\lim_{t\to\infty} P(t) &= \lim_{t\to\infty} \frac{ap_0}{(a - bp_0)e^{-at} + bp_0} \\ &= \frac{ap_0}{bp_0} = \frac{a}{b}.\end{aligned}$$

34. With a and b taking on the given values, and $p_0 = 3,929,214$, the population in 1790, we obtain the logistic model for the United States population growth:

$$P(t) = \frac{123,141.5668}{0.03071576577 + 0.0006242342282e^{0.03134t}}e^{0.03134t}.$$

Table 1.1 shows compares the population figures given by $P(t)$ with the actual numbers, together with the percent error (positive if $P(t)$ exceeds the actual population, negative if $P(t)$ is an underestimate).

An exponential model can also be constructed as $Q(t) = Ae^{kt}$. Then

$$A = Q(0) = 3,929,214,$$

the initial (1790) population. To find k, use the fact

$$Q(10) = 5308483 = 3929214e^{10k}$$

year	population	$P(t)$	percent error	$Q(t)$	percent error
1790	3,929,214	3,929,214	0	3,929,214	0
1800	5,308,483	5,336,313	0.52	5,308,483	0
1810	7,239,881	7,228,471	-0.16	7,179,158	-0.94
1820	9,638,453	9,757,448	1.23	9,689,468	0.53
1830	12,886,020	13,110,174	1.90	13,090,754	1.75
1840	17,069,453	17,507,365	2.57	17,685,992	3.61
1850	23,191,876	23,193,639	0.008	23,894,292	3.03
1860	31,443,321	30,414,301	-3.27	32,281,888	2.67
1870	38,558,371	39,374,437	2.12	43,613,774	13.11
1880	50,189,209	50,180,383	-0.018	58,923,484	17.40
1890	62,979,766	62,772,907	-0.33	79,073,491	26.40
1900	76,212,168	76,873,907	0.87	107,551,857	41.12
1910	92,228,496	91,976,297	-0.27	145,303,703	57.55
1920	106,021,537	107,398,941	1.30	196,312,254	85.16
1930	123,202,624	122,401,360	-0.65		
1940	132,164,569	136,320,577	3.15		
1950	151,325,798	148,679,224	-1.75		
1960	179,323,175	159,231,097	-11.2		
1970	203,302,031	167,943,428	-17.39		
1980	226,547,042	174,940,040	-22.78		

Table 1.1: Census and model data for Problems 33 and 34

to solve for k, obtaining

$$k = \frac{1}{10} \ln \left(\frac{5308483}{3929214} \right) \approx 0.03008667012.$$

Thus the exponential model determined using these two data points (1790 and 1800) is

$$Q(t) = 3929214 e^{0.03008667012t}.$$

Population figures predicted by this model are also included in Table 1.1, along with percentage errors. Notice that the logistic model remains quite accurate until 1960, at which time the error increases dramatically for the next three years. The exponential model becomes increasingly inaccurate by 1870, after which the error rapidly becomes so large that it is not worth computing further. Exponential models do not work well over time with complex populations, such as fish in the ocean or countries throughout the world.

1.2 Linear Equations

1. With $p(x) = -3/x$, an integrating factor is

$$e^{\int p(x)\,dx} = e^{-3\ln(x)} = x^{-3}.$$

Multiply the differential equation by x^{-3} to obtain

$$\frac{d}{dx}(yx^{-3}) = \frac{2}{x}.$$

A routine integration gives us $yx^{-3} = 2\ln(x) + c$, or

$$y = cx^3 + 2x^3\ln|x|$$

for $x \neq 0$.

2. $e^{\int dx} = e^x$ is an integrating factor. Multiply the differential equation by e^x to obtain

$$y'e^x + ye^x = (ye^x)' = \frac{1}{2}\left(e^{2x} - 1\right).$$

Integrate to obtain

$$ye^x = \frac{1}{4}e^{2x} - \frac{1}{2}x + c.$$

Then

$$y = \frac{1}{4}e^x - \frac{1}{2}xe^{-x} + ce^{-x}.$$

3. $e^{\int 2\,dx} = e^{2x}$ is an integrating factor. Multiply the differential equation by e^{2x} to obtain

$$y'e^{2x} + 2y = (ye^{2x})' = xe^{2x}.$$

Integrate to obtain

$$ye^{2x} = \int xe^{2x}\,dx = \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + c.$$

The general solution is

$$y = \frac{1}{2}x - \frac{1}{4} + ce^{-2x}.$$

4. An integrating factor is

$$e^{\int \sec(x)\,dx} = e^{\ln|\sec(x)+\tan(x)|} = \sec(x) + \tan(x).$$

Multiply the differential equation by $\sec(x) + \tan(x)$ to obtain

$$\begin{aligned} &y'(\sec(x) + \tan(x)) + (\sec(x)\tan(x) + \sec^2(x))y \\ &= (y(\sec(x) + \tan(x)))' = \cos(x)(\sec(x) + \tan(x)) \\ &= 1 + \sin(x). \end{aligned}$$

Integrate this equation to obtain

$$y(\sec(x) + \tan(x)) = x - \cos(x) + k.$$

Multiply both sides of this equation by

$$\frac{1}{\sec(x) + \tan(x)} = \frac{\cos(x)}{1 + \sin(x)}$$

to obtain

$$\begin{aligned} y &= (x - \cos(x) + k)\left(\frac{\cos(x)}{1 + \sin(x)}\right) \\ &= \frac{x\cos(x) - \cos^2(x) + k\cos(x)}{1 + \sin(x)}. \end{aligned}$$

5. An integrating factor is $e^{\int -2\,dx} = e^{-2x}$. Multiply the differential equation by e^{-2x} to obtain

$$y'e^{-2x} - 2ye^{-2x} = (ye^{-2x})' = -8x^2e^{-2x}.$$

Integrate to obtain

$$ye^{-2x} = \int -8x^2e^{-2x}\,dx = 4x^2e^{-2x} + 4xe^{-2x} + 2e^{-2x} + c.$$

The general solution is

$$y = 4x^2 + 4x + 2 + ce^{2x}.$$

6. $e^{\int 3\,dx} = e^{3x}$ is an integrating factor. Multiply the differential equation by e^{3x} to obtain

$$y'e^{3x} + 3ye^{3x} = (ye^{3x})' = 5e^{5x} - 6e^{3x}.$$

Integrate to obtain the general solution

$$ye^{3x} = e^{5x} - 2e^{3x} + c.$$

The general solution is

$$y = e^{2x} - 2 + ce^{-3x}.$$

Now we need

$$y(0) = 1 - 2 + c = 2,$$

so $c = 3$. The initial value problem has the solution

$$y = e^{2x} - 2 + 3e^{-3x}.$$

7. Notice that, if we multiply the differential equation by $x-2$, we obtain

$$y'(x-2)+y=((x-2)y)'=3x(x-2).$$

Integrate to obtain

$$(x-2)y=x^3-3x^2+c.$$

The general solution is

$$y=\frac{1}{x-2}(x^3-3x^2+c).$$

Now

$$y(3)=27-27+c=4$$

so the initial value problem has the solution

$$y=\frac{x^3-3x^2+4}{x-2}=x^2-x-2.$$

8. Multiply the differential equation by the integrating factor e^{-x} to obtain

$$(ye^{-x})'=2e^{3x}.$$

Integrate to obtain

$$ye^{-x}=\frac{2}{3}e^{3x}+c.$$

The general solution is

$$y=\frac{2}{3}e^{4x}+ce^{x}.$$

Then

$$y(0)=-3=\frac{2}{3}+c$$

so $c=-11/3$ and the initial value problem has the solution

$$y=\frac{2}{3}e^{4x}-\frac{11}{3}e^{x}$$

9. An integrating factor is

$$e^{\int (2/(x+1))\,dx}=e^{2\ln|x+1|}=e^{\ln((x+1)^2)}=(x+1)^2.$$

Multiply the differential equation by $(x+1)^2$ to obtain

$$(x+1)^2y'+2(x+1)y=((x+1)^2y)'=3(x+1)^2.$$

Integrate to obtain

$$(x+1)^2y=(x+1)^3+c.$$

Then

$$y=(x+1)+\frac{c}{(x+1)^2}.$$

Now

$$y(0) = 5 = 1 + c$$

so $c = 4$ and the solution of the initial value problem is

$$y = x + 1 + \frac{4}{(x+1)^2}.$$

10. An integrating factor is

$$e^{\int (5/9x)\,dx} = e^{(5/9)\ln(x)} = e^{\ln(x^{5/9})} = x^{5/9}.$$

Multiply the differential equation by $x^{5/9}$ to obtain

$$(yx^{5/9})' = 3x^{32/9} + x^{14/9}.$$

Integrate to obtain

$$yx^{5/9} = \frac{27}{41}x^{41/9} + \frac{9}{23}x^{23/9} + c.$$

Then

$$y = \frac{27}{41}x^4 + \frac{9}{23}x^2 + cx^{-5/9}.$$

We need

$$y(-1) = 4 = \frac{27}{41} + \frac{9}{23} - c,$$

so $c = -2782/943$. The solution is

$$y = \frac{27}{41}x^4 + \frac{9}{23}x^2 - \frac{2782}{943}x^{-5/9}$$

11. Let (x, y) be a point on the curve. The tangent line at (x, y) must pass through $(0, 2x^2)$, hence must have slope $(y - 2x^2)/x$. But this slope is y', so we have the differential equation

$$y' = \frac{y - 2x^2}{x}.$$

This is the linear differential equation

$$y' - \frac{1}{x}y = -2x,$$

which has the general solution $y = -2x^2 + cx$.

12. If $A(t)$ is the amount of salt in the tank at time $t \geq 0$, then

$$\begin{aligned}\frac{dA}{dt} &= \text{ rate salt is added } - \text{ rate salt is removed}\\ &= 6 - 2\left(\frac{A(t)}{50+t}\right),\end{aligned}$$

and the initial condition is $A(0) = 28$.

This differential equation is linear:

$$A' + \frac{2}{50+t}A = 6,$$

with integrating factor $(50+t)^2$. The general solution is

$$A(t) = 2(50+t) + \frac{C}{(50+t)^2},$$

The initial condition gives us $C = -180,000$, so

$$A(t) = 2(50+t) - \frac{180000}{(50+t)^2}.$$

The tank contains 100 gallons when $t = 50$ and $A(50) = 176$ pounds of salt.

13. If $A_1(t)$ and $A_2(t)$ are the amounts of salt in tanks one and two, respectively, at time t, we have

$$A_1'(t) = \frac{5}{2} - \frac{5A_1(t)}{100}; A_1(0) = 20$$

and

$$A_2'(t) = \frac{5A_1(t)}{100} - \frac{5A_2(t)}{150}; A_2(0) = 90.$$

Solve the first initial value problem to obtain

$$A_1(t) = 50 - 30e^{-t/20}.$$

Substitute this into the problem for $A_2(t)$ to obtain

$$A_2' + \frac{1}{30}A_2 = \frac{5}{2} - \frac{3}{2}e^{-t/20}; A_2(0) = 90.$$

Solve this to obtain

$$A_2(t) = 75 + 90e^{-t/20} - 75e^{-t/30}.$$

Tank 2 has its minimum when $A_2'(t) = 0$, hence when

$$2.5e^{-t/30} - 4.5e^{-t/20} = 0.$$

Then $e^{t/60} = 9/5$, or $t = 60\ln(9/5)$. Then

$$A_2(t)_{\text{min}} = A_2(60\ln(9/5)) = \frac{5450}{81}$$

pounds.

1.3 Exact Equations

In the following we assume that the differential equation has the form $M(x, y) + N(x, y)y' = 0$, or, in differential form, $M\,dx + N\,dy = 0$.

1. Since
$$\frac{\partial M}{\partial y} = 4y + e^{xy} + xye^{xy} = \frac{\partial N}{\partial x}$$
for all x and y, the equation is exact in the entire plane. One way to find a potential function is to integrate
$$\frac{\partial \varphi}{\partial x} = M(x, y) = 2y^2 + ye^{xy}$$
with respect to x to obtain
$$\varphi(x, y) = 2xy^2 + e^{xy} + \alpha(y).$$
Then we need
$$\frac{\partial \varphi}{\partial y} = 4xy + xe^{xy} + \alpha'(y) = N(x, y) = 4xy + xe^{xy} + 2y.$$
This requires that $\alpha'(y) = 2y$ so we may choose $\alpha(y) = y^2$. A potential function has the form
$$\varphi(x, y) = 2xy^2 + e^{xy} + y^2.$$
The general solution is implicitly defined by
$$\varphi(x, y) = 2xy^2 + e^{xy} + y^2 = c.$$
We could have also started by integrating $\partial N/\partial y = 4xy + xe^{xy} + 2y$ with respect to y.

2. Since $\partial M/\partial y = 4x = \partial N/\partial x$ for all x and y, the equation is exact in the plane. We can find a potential function by integrating
$$\frac{\partial \varphi}{\partial y} = 2x^2 + 3y^2$$
with respect to y to obtain
$$\varphi(x, y) = 2x^2y + y^3 + \beta(x).$$
Then
$$\frac{\partial \varphi}{\partial x} = 4xy + \beta'(x) = 4xy + 2x,$$
so $\beta'(x) = 2x$ and we can choose $\beta(x) = x^2$. A potential function is
$$\varphi(x, y) = 2x^2y + y^3 + x^2$$
and the general solution is defined implicitly by
$$2x^2y + y^3 + x^2 = c.$$

3. $\partial M/\partial y = 4 + 2x^2$ and $\partial N/\partial x = 4x$, so this equation is not exact.

4. $$\frac{\partial M}{\partial y} = -2\sin(x+y) - 2x\cos(x+y) = \frac{\partial N}{\partial x}$$

   so the equation is exact over the plane. Routine integrations yield the potential function is $\varphi(x,y) = 2x\cos(x+y)$ and the general solution is implicitly defined by $2x\cos(x+y) = c$.

5. $\partial M/\partial y = 1 = \partial N/\partial x$, so the equation is exact for all (x,y) with $x \neq 0$, where the equation is not defined. Integrate $\partial\varphi/\partial x = M$ or $\partial\varphi/\partial y = N$ to obtain the potential function

   $$\varphi(x,y) = \ln|x| + xy + y^3.$$

   The general solution is defined implicitly by

   $$\varphi(x,y) = \ln|x| + xy + y^3 = c$$

   for $x \neq 0$.

6. For the equation to be exact, we need

   $$\frac{\partial M}{\partial y} = \alpha x y^{\alpha-1} = \frac{\partial N}{\partial x} = -2xy^{\alpha-1}.$$

   This holds if $\alpha = -2$. By integrating, we find the potential function $\varphi(x,y) = x^3 + x^2/2y^2$, so the general solution is defined implicitly by

   $$x^3 + \frac{x^2}{2y^2} = c.$$

7. For exactness we need

   $$\frac{\partial M}{\partial y} = 6xy^2 - 3 = \frac{\partial N}{\partial x} = -3 - 2\alpha x y^2$$

   and this requires that $\alpha = -3$. By integration, we find a potential function $\varphi(x,y) = x^2y^3 - 3xy - 3y^2$. The general solution is implicitly defined by

   $$x^2y^3 - 3xy - 3y^2 = c.$$

8. Compute

   $$\frac{\partial M}{\partial y} = 2 - 2y\sec^2(xy^2) - 2xy^3\sec^2(xy^2)\tan(xy^2)$$

   and

   $$\frac{\partial N}{\partial x} = 2 - 2y\sec^2(xy^2) - 2xy^3\sec^2(xy^2)\tan(xy^2).$$

Since these partial derivatives are equal for all x and y for which the functions are defined, the differential equation is exact for such x and y. To find a potential function, we can start by integrating $\partial\varphi/\partial x = 2y - y^2\sec^2(xy^2)$ with respect to x to obtain

$$\varphi(x,y) = 2xy - \tan(xy^2) + \alpha(y).$$

Now we need

$$\begin{aligned}\frac{\partial\varphi}{\partial y} &= 2x - 2xy\sec^2(xy^2)\\ &= 2x - 2xy\sec^2(xy^2) + \alpha'(y).\end{aligned}$$

This requires that $\alpha'(y) = 0$ and we may choose $\alpha(y) = 0$. A potential function is

$$\varphi(x,y) = 2xy - \tan(xy^2).$$

The general solution is implicitly defined by

$$2xy - \tan(xy^2) = c.$$

For the initial condition we need $y = 2$ when $x = 1$, which requires that

$$2(2) - \tan(4) = c.$$

The unique solution of the initial value problem is implicitly defined by

$$2xy - \tan(xy^2) = 4 - \tan(4).$$

9. Since $\partial M/\partial y = 12y^3 = \partial N\partial x$, the differential equation is exact for all x and y. Straightforward integrations yield the potential function

$$\varphi(x,y) = 3xy^4 - x.$$

The general solution is implicitly defined by

$$3xy^4 - x = c.$$

For the initial condition, we need $y = 2$ when $x = 1$, so

$$3(1)(2^4) - 1 = 47 = c.$$

The initial value problem has the unique solution implicitly defined by

$$3xy^4 - x = 47.$$

10. Compute

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{1}{x}e^{y/x} - \frac{1}{x}e^{y/x} - \frac{y}{x^2}e^{y/x}\\ &= -\frac{y}{x^2}e^{y/x} = \frac{\partial N}{\partial x},\end{aligned}$$

so the differential equation is exact for all $x \neq 0$ and all y. For a potential function, begin with

$$\frac{\partial \varphi}{\partial y} = e^{y/x}$$

and integrate with respect to y to obtain

$$\varphi(x, y) = xe^{y/x} + \beta(x).$$

Then

$$\frac{\partial \varphi}{\partial x} = 1 + e^{y/x} - \frac{y}{x}e^{y/x} = e^{y/x} - \frac{y}{x}e^{y/x} + \beta'(x).$$

This requires that $\beta'(x) = 1$ so choose $\beta(x) = x$. Then

$$\varphi(x, y) = xe^{y/x} + x.$$

The general solution is implicitly defined by

$$xe^{y/x} + x = c.$$

For the initial value problem, we need to choose c so that

$$e^{-5} + 1 = c.$$

The solution of the initial value problem is implicitly defined by

$$xe^{y/x} + x = 1 + e^{-5}.$$

11. Compute

$$\frac{\partial M}{\partial y} = -2x\sin(2y - x) - 2\cos(2y - x) = \frac{\partial N}{\partial x},$$

so the differential equation is exactly. For a potential function, integrate

$$\frac{\partial \varphi}{\partial y} = -2x\cos(2y - x)$$

with respect to y to get

$$\varphi(x, y) = -x\sin(2y - x) + \alpha(x).$$

Then we must have

$$\begin{aligned}\frac{\partial \varphi}{\partial x} &= x\cos(2y - x) - \sin(2y - x)\\ &= -\sin(2y - x) + x\cos(2y - x) + \alpha'(x).\end{aligned}$$

Then $\alpha'(x) = 0$ and we may choose $\alpha(x) = 0$ to obtain

$$\varphi(x, y) = -x\sin(2y - x).$$

The general solution has the form

$$-x\sin(2y-x)=c.$$

For $y(\pi/12)=\pi/8$, we need

$$-\frac{\pi}{12}\sin\left(\frac{\pi}{4}-\frac{\pi}{12}\right)=-\frac{\pi}{12}\sin(\pi/6)=-\frac{\pi}{24}=c.$$

The solution of the initial value problem is implicitly defined by

$$x\sin(2y-x)=\frac{\pi}{24}.$$

12. The equation is exact over the entire plane because

$$\frac{\partial M}{\partial y}=e^y=\frac{\partial N}{\partial x}.$$

Integrate

$$\frac{\partial\varphi}{\partial x}=e^y$$

with respect to x to get

$$\varphi(x,y)=xe^y+\alpha(y).$$

Then we need

$$\frac{\partial\varphi}{\partial y}=xe^y+\alpha'(y)=xe^y-1.$$

Then $\alpha'(y)=-1$ and we can take $\alpha(y)=-y$. Then

$$\varphi(x,y)=xe^y-y.$$

The general solution is implicitly defined by

$$xe^y-y=c.$$

For the initial condition, we need $y=0$ when $x=5$, so choose $c=5$ to obtain the implicitly defined solution

$$xe^y-y=5.$$

13. $\varphi+c$ is also a potential function if φ is because

$$\frac{\partial(\varphi+c)}{\partial x}=\frac{\partial\varphi}{\partial x}$$

and

$$\frac{\partial(\varphi+c)}{\partial y}=\frac{\partial\varphi}{\partial y}$$

Any function defined implicitly by $\varphi(x,y)=k$ is also defined by $\varphi(x,y)+c=k$, because, if k can assume any real value, so can $k-c$ for any c.

14. (a)
$$\frac{\partial M}{\partial y} = 1 \text{ and } \frac{\partial N}{\partial x} = -1$$
so this differential equation is not exact over any rectangle in the plane.

(b) Multiply the differential equation by x^{-2} to obtain
$$yx^{-2} - x^{-1}y' = 0.$$
This is exact over any rectangle not containing $x = 0$, because
$$\frac{\partial M^*}{\partial y} = x^{-2} = \frac{\partial N^*}{\partial x}.$$
This equation has potential function $\varphi(x, y) = -yx^{-1}$, so the general solution is defined implicitly by
$$-yx^{-1} = c.$$

(c) If we multiply the differential equation by y^{-2} we obtain
$$y^{-1} - xy^{-2}y' = 0.$$
This is exact on any region not containing $y = 0$ because
$$\frac{\partial M^{**}}{\partial y} = -y^{-2} = \frac{\partial N^{**}}{\partial x}.$$
This has potential function $\varphi(x, y) = xy^{-1}$, so the differential equation has the general solution
$$xy^{-1} = c.$$

(d) Multiply the differential equation by xy^{-2} to obtain
$$xy^{-2} - x^2y^{-3}y' = 0.$$
Now
$$\frac{\partial M^{***}}{\partial y} = -2xy^{-3} = \frac{\partial N^{***}}{\partial x}$$
so this differential equation is exact. Integrate $\partial\varphi/\partial x = xy^{-2}$ with respect to x to obtain
$$\varphi(x, y) = \frac{1}{2}x^2y^{-2} + \beta(y).$$
Then
$$\frac{\partial\varphi}{\partial y} = -x^2y^{-3} + \beta'(y) = -x^2y^{-3}$$
so choose $\beta(y) = 0$. The general solution in this case is given implicitly by
$$x^2y^{-2} = c.$$

(e) As a linear equation, we have

$$y' - \frac{1}{x}y = 0,$$

or $xy' - y = (x^{-1}y)' = 0$. This has general solution defined implicitly by $x^{-1}y = c$.

(f) The general solutions obtained in (b) through (e) are the same. For example, in (b) we obtained $-yx^{-1} = c$. Since c is an arbitrary constant, this can be written $y = kx$. In (d) we obtained $x^2y^{-2} = c$. This can be written $y^2 = Cx^2$, or $y = kx$.

15. Multiply the differential equation by $\mu(x, y) = x^a y^b$ to obtain

$$x^{a+1}y^{b+1} + x^a y^{b-3/2} + x^{a+2}y^b y' = 0.$$

For this to be exact, we need

$$\begin{aligned}\frac{\partial M}{\partial y} &= (b+1)x^{a+1}y^b + \left(b - \frac{3}{2}\right)x^a y^{b-5/2} \\ &= \frac{\partial N}{\partial x} = (a+2)x^{a+1}y^b.\end{aligned}$$

Divide this by $x^a y^b$ to require that

$$(b+1)x + \left(b - \frac{3}{2}\right)y^{-5/2} = (a+2)x.$$

This will be true for all x and y if we let $b = 3/2$, and then choose a so that $(b+1)x = (a+2)x$, so $b + 1 = a + 2$. Therefore

$$a = \frac{1}{2} \text{ and } b = \frac{3}{2}.$$

Multiply the original differential equation by $\mu(x, y) = x^{1/2}y^{3/2}$ to obtain

$$x^{3/2}y^{5/2} + x^{1/2} + x^{5/2}y^{3/2}y' = 0.$$

Integrate $\partial\varphi/\partial y = x^{5/2}y^{3/2}$ to obtain

$$\varphi(x, y) = \frac{2}{5}x^{5/2}y^{5/2} + \beta(x).$$

Then we need

$$\frac{\partial \varphi}{\partial x} = x^{3/2}y^{5/2} + \beta'(x) = x^{3/2}y^{5/2} + x^{1/2}.$$

Then $\beta(x) = 2x^{3/2}/3$ and a potential function is

$$\varphi(x, y) = \frac{2}{5}x^{5/2}y^{5/2} + \frac{2}{3}x^{3/2}.$$

The general solution of the original differential equation is

$$\varphi(x,y) = \frac{2}{5}x^{5/2}y^{5/2} + \frac{2}{3}x^{3/2} = c.$$

The differential equation multiplied by the integrating factor has the same solutions as the original differential equation because the integrating factor is assumed to be nonzero. Thus we must exclude $x = 0$ and $y = 0$, where $\mu = 0$.

16. Multiply the differential equation by $x^a y^b$:

$$2x^a y^{b+2} - 9x^{a+1}y^{b+1} + (3x^{a+1}y^{b+1} - 6x^{a+2}y^b)y' = 0.$$

For this to be exact, we must have

$$\begin{aligned}\frac{\partial M}{\partial y} &= (b+2)2x^a y^{b+1} - 9(b+1)x^{a+1}y^b \\ &= \frac{\partial N}{\partial x} = 3(a+1)x^a y^{b+1} - 6(a+2)x^{a+1}y^b.\end{aligned}$$

Divide by $x^a y^b$ to obtain, after some rearrangement,

$$(2(b+2) - 3(a+1))y = ((9(b+1) - 6(a+2))x.$$

Since x and y are independent, this equation can hold only if the coefficients of x and y are zero, giving us two equations for a and b:

$$-3a + 2b = -1, -6a + 9b = 3.$$

Then $a = b = 1$, so $\mu(x,y) = xy$ is an integrating factor. Multiply the differential equation by xy:

$$2xy^3 - 9x^2y^2 + (3x^2y^2 - 6x^3y)y' = 0.$$

It is routine to check that this equation is exact. For a potential function, integrate

$$\frac{\partial \varphi}{\partial x} = 2xy^3 - 9x^2y^2$$

with respect to x to get

$$\varphi(x,y) = x^2y^3 - 3x^3y^2 + \beta(y).$$

Then

$$\frac{\partial \varphi}{\partial y} = 3x^2y^2 - 6x^3y + \beta'(y).$$

We may choose $\beta(y) = 0$, so $\varphi(x,y) = x^2y^3 - 3x^3y^2$. The general solution is implicitly defined by

$$x^2y^3 - 3x^3y^2 = c.$$

1.4 Homogeneous, Bernoulli and Riccati Equations

1. This is a Riccati equation with solution $S(x) = x$ (by inspection). Put $y = x + 1/z$ and substitute to obtain

$$2 - \frac{z'}{z^2} = \frac{1}{x^2}\left(x + \frac{1}{z}\right)^2 - \frac{1}{x}\left(x + \frac{1}{z}\right) + 1.$$

Simplify this to obtain

$$z' + \frac{1}{x}z = -\frac{1}{x^2}.$$

This linear differential equation can be written $(xz)' = -1/x$ and has the solution

$$z = -\frac{\ln(x)}{x} + \frac{c}{x}.$$

Then

$$y = x + \frac{x}{c - \ln(x)}$$

for $x > 0$.

2. This is a Bernoulli equation with $\alpha = -4/3$. Put $v = y^{7/3}$, or $y = v^{3/7}$. Substitute this into the differential equation to get

$$\frac{3}{7}v^{-4/7}v' + \frac{1}{x}v^{3/7} = \frac{2}{x^3}v^{-4/7}.$$

This simplifies to the linear equation

$$v' + \frac{7}{3x}v = \frac{14}{3x^2}.$$

This has integrating factor $x^{7/3}$ and can be written

$$(vx^{7/3})' = \frac{14}{3}x^{1/3}.$$

Integration yields

$$vx^{7/3} = \frac{7}{2}x^{4/3} + c.$$

Since $v = y^{7/3}$, we obtain

$$2y^{7/3}x^{7/3} - 7x^{4/3} = k.$$

This implicitly defined the general solution.

3. This is a Bernoulli equation with $\alpha = 2$ and we obtain the general solution

$$y = \frac{1}{1 + ce^{x^2/2}}.$$

4. This equation is homogeneous. With $y = xu$, we obtain

$$u + xu' = u + \frac{1}{u}.$$

Then

$$x\frac{du}{dx} = \frac{1}{u},$$

a separable equation. Write

$$u\,du = \frac{1}{x}\,dx.$$

Integrate to obtain

$$u^2 = 2\ln|x| + c.$$

Then

$$\frac{y^2}{x^2} = 2\ln|x| + c$$

implicitly defines the general solution of the original differential equation.

5. This differential equation is homogeneous, and $y = xu$ yields the general solution implicitly defined by

$$y\ln|y| - x = cy.$$

6. The differential equation is Riccati and we see one solution $S(x) = 4$. We obtain the general solution

$$y = 4 + \frac{6x^3}{c - x^3}.$$

7. This equation is exact, with general solution defined by

$$xy - x^2 - y^2 = c.$$

8. The differential equation is homogeneous, and $y = xu$ yields the general solution defined by

$$\sec\left(\frac{y}{x}\right) + \tan\left(\frac{y}{x}\right) = cx.$$

9. The differential equation is Bernoulli, with $\alpha = -3/4$. The general solution is given by

$$5x^{7/4}y^{7/4} + 7x^{-5/4} = c.$$

10. The differential equation is homogeneous and $y = xu$ yields

$$\frac{2\sqrt{3}}{\sqrt{3}}\arctan\left(\frac{2y - x}{\sqrt{3}x}\right) = \ln|x| + c.$$

11. The equation is Bernoulli with $\alpha = 2$. We obtain
$$y = 2 + \frac{2}{cx^2 - 1}.$$

12. The equation is homogeneous and $y = xu$ yields
$$\frac{1}{2}\frac{x^2}{y^2} = \ln|x| + c.$$

13. The equation is Riccati with one solution $S(x) = e^x$. The general solution is
$$y = \frac{2e^x}{ce^{2x} - 1}.$$

14. The equation is Bernoulli with $\alpha = 2$ and general solution
$$y = \frac{2}{3 + cx^2}.$$

15. For the first part,
$$F\left(\frac{ax + by + c}{dx + py + r}\right) = F\left(\frac{a + b(y/x) + c/x}{d + p(y/x) + r/x}\right) = f\left(\frac{y}{x}\right)$$
if and only if $c = r = 0$.

Now suppose $x = X + h$ and $y = Y + k$. Then
$$\frac{dY}{dX} = \frac{dY}{dx}\frac{dx}{dX} = \frac{dy}{dx}$$
so
$$\begin{aligned}\frac{dY}{dX} &= F\left(\frac{a(X + h) + b(Y + k) + c}{d(X + h) + p(Y + k) + r}\right)\\ &= F\left(\frac{aX + bY + c + ah + bk + c}{dX + pY + r + dh + pk + r}\right)\end{aligned}$$

This equation is homogeneous exactly when
$$ah + bk = -c \text{ and } dh + pk = -r.$$

This two by two system has a solution when the determinant of the coefficients is nonzero: $ap - bd \neq 0$.

16. Here $a = 0, b = 1, c = -3$ and $d = p = 1, r = -1$. Solve
$$k = 3, h + k = 1$$
to obtain $k = 3$ and $h = -2$. Thus let $x = X - 2, y = Y + 3$ to obtain
$$\frac{dY}{dX} = \frac{Y}{X + Y},$$

a homogeneous equation. Letting $U = Y/X$ we obtain, after some manipulation,

$$\frac{1+U}{U}\,dU = \frac{1}{X}\,dX,$$

a separable equation with general solution

$$U\ln|U| - 1 = -U\ln|X| + KU,$$

in which K is the arbitrary constant. In terms of x and y,

$$(y-3)\ln|y-3| - (x+2) = K(y-3).$$

17. Set $x = X + 2, y = Y - 3$ to obtain

$$\frac{dY}{dX} = \frac{3X - Y}{X + Y}.$$

This homogeneous equation has general solution (in terms of x and y)

$$3(x-2)^2 - 2(x-2)(y+3) - (y+3)^2 = K.$$

18. With $x = X - 5$ and $y = Y - 1$ we obtain

$$(x+5)^2 + 4(x+5)(y+1) - (y+1)^2 = K.$$

19. with $x = X + 2$ and $y = Y - 1$ we obtain

$$(2x + y - 3)^2 = K(y - x + 3).$$

1.5 Additional Applications

1. Once released, the only force acting on the ballast bag is due to gravity. If $y(t)$ is the distance from the bag to the ground at time t, then $y'' = -g = -32$, with $y(0) = 4$. With two integrations, we obtain

$$y'(t) = 4 - 32t \text{ and } y(t) = 342 + 4t - 16t^2.$$

The maximum height is reached when $y'(t) = 0$, or $t = 1/8$ second. This maximum height is $y(1/8) = 342.25$ feet. The bag remains aloft until $y(t) = 0$, or $-16t^2 + 4t + 342 = 0$. This occurs at $t = 19/4$ seconds, and the bag hits the ground with speed $|y'(19/4)| = 148$ feet per second.

2. With a gradient of $7/24$ the plane is inclined at an angle θ for which $\sin(\theta) = 7/25$ and $\cos(\theta) = 24/25$. The velocity of the box satisfies

$$\frac{48}{32}\frac{dv}{dt} = -48\left(\frac{24}{25}\right)\left(\frac{1}{3}\right) + 48\left(\frac{7}{25}\right) - \frac{3}{2}v; v(0) = 16.$$

Solve this initial value problem to obtain

$$v(t) = \frac{432}{25}e^{-t} - \frac{32}{25}$$

feet per second. This velocity reaches zero when $t_s = \ln(27/2)$ seconds. The box will travel a distance of

$$\begin{aligned} s(t_s) &= \int_0^{t_s} v(\xi)\,d\xi = \frac{432}{25}(1 - e^{-t_s}) - \frac{32}{25}t_s \\ &= \frac{432}{25}\left(1 - \frac{2}{27}\right) - \frac{32}{25}\ln\left(\frac{27}{2}\right) \approx 12.7 \end{aligned}$$

feet.

3. Until the parachute is opened at $t = 4$ seconds, the velocity $v(t)$ satisfies the initial value problem

$$\left(\frac{192}{32}\right)\frac{dv}{dt} = 192 - 6v; v(0) = 0.$$

This has solution $v(t) = 32(1 - e^{-t})$ for $0 \le t \le 4$. When the parachute opens at $t = 4$, the skydiver has a velocity of $v(4) = 32(1 - e^{-4})$ feet per second. Velocity with the open parachute satisfies the initial value problem

$$\left(\frac{192}{32}\right)\frac{dv}{dt} = 192 - 3v^2, v(4) = 32(1 - e^{-4}) \text{ for } t \ge 4.$$

This differential equation is separable and can be integrated using partial fractions:

$$\int \left[\frac{1}{v+8} - \frac{1}{v-8}\right] dv = -\int 8t\,dt.$$

This yields

$$\ln\left(\frac{v+8}{v-8}\right) = -8t + \ln\left(\frac{5 - 4e^{-4}}{3 - 4e^{-4}}\right) + 32.$$

Solve for $v(t)$ to obtain

$$v(t) = \frac{8(1 + ke^{-8(t-4)})}{1 - ke^{-8(t-4)}} \text{ for } t \ge 4.$$

We find using the initial condition that

$$k = \frac{3 - 4e^{-4}}{5 - 4e^{-4}}.$$

Terminal velocity is $\lim_{t\to\infty} v(t) = 8$ feet per second. The distance fallen is

$$s(t) = \int_0^t v(\xi)\,d\xi = 32(t - 1 + e^{-t})$$

for $0 \le t \le 4$, while

$$s(t) = 32(3 + e^{-4}) + 8(t - 4) + 2\ln(1 - ke^{-8(t-4)}) - 2\ln\left(\frac{2}{5 - 4e^{-4}}\right)$$

for $t \ge 4$.

4. When fully submerged the buoyant force will be $F_B = (1)(2)(3)(62.5) = 375$ pounds upward. The mass is $m = 384/32 = 12$ slugs. The velocity $v(t)$ of the sinking box satisfies

$$12\frac{dv}{dt} = 384 - 375 - \frac{1}{2}v; v(0) = 0.$$

This linear problem has the solution

$$v(t) = 18(1 - e^{-t/24}).$$

In t seconds the box has sunk $s(t) = 18(t + 24e^{-t/24} - 24)$ feet. From $v(t)$ we find the terminal velocity

$$\lim_{t\to\infty} v(t) = 18$$

feet per second. To answer the question about velocity when the box reaches the bottom $s = 100$, we would normally solve $s(t) = 100$ and substitute this t into the velocity. This would require a numerical solution, which can be done. However, there is another approach we can also use. Find t_* so that $v(t_*) = 10$ feet per second, and calculate $s(t_*)$ to see how far the box has fallen. With this approach we solve $18(1 - e^{-t/24}) = 10$ to obtain $t_* = 24\ln(9/4)$ seconds. Now compute

$$s(t_*) = 432\ln(9/4) - 240 \approx 110.3$$

feet. Therefore at the bottom $s = 100$, the box has not yet reached a velocity of 10 feet per second.

5. If the box loses 32 pounds of material on impact with the bottom, then $m = 11$ slugs. Now

$$11\frac{dv}{dt} = -352 + 375 - \frac{1}{2}v; v(0) = 0$$

in which we have taken up as the positive direction. This gives us

$$v(t) = 46(1 - e^{-t/22})$$

so the distance traveled up from the bottom is

$$s(t) = 46(t + 22e^{-t/22} - 22)$$

feet. Solve $s(t) = 100$ numerically to obtain $t \approx 10.56$ seconds. The surfacing velocity is approximately $v(10.56) \approx 17.5$ feet per second.

6. The statement of gravitational attraction inside the Earth gives $v'(t) = -kr$, where r is the distance to the Earth's center. When $r = R$, the acceleration is g, so $k = -g/R$ and $v'(t) = -gr/R$. Use the chain rule to write
$$\frac{dv}{dt} = \frac{dv}{dr}\frac{dr}{dt} = v\frac{dv}{dr}.$$
This gives us the separable equation
$$v\frac{dv}{dr} = -\frac{gr}{R},$$
with the condition $v(R) = 0$. Integrate to obtain
$$v^2 = gR - \frac{gr^2}{R}.$$
Put $r = 0$ to get the speed at the center of the Earth. This is $v = \sqrt{gR} = \sqrt{24} \approx 4.9$ miles per second.

7. Let θ be the angle the chord makes with the vertical. Then
$$m\frac{dv}{dt} = mg\cos(\theta); v(0) = 0.$$
This gives us $s(t) = \frac{1}{2}gt^2\cos(\theta)$, so the time of descent is
$$t = \left(\frac{2s}{g\cos(\theta)}\right)^{1/2},$$
where s is the length of the chord. By the law of cosines, the length of this chord satisfies
$$s^2 = 2R^2 - 2R^2\cos(\pi - 2\theta) = 2R^2(1 + \cos(2\theta)) = 4R^2\cos^2(\theta).$$
Therefore
$$t = 2\sqrt{\frac{R}{g}},$$
and this is independent of θ.

8. The loop currents in Figure 1.13 satisfy the equations
$$10i_1 + 15(i_1 - i_2) = 10$$
$$15(i_2 - i_1) + 30i_2 = 0$$
so
$$i_1 = \frac{1}{2} \text{ amp and } i_2 = \frac{1}{6} \text{ amp}.$$

9. The capacitor charge is modeled by

$$250(10^3)i + \frac{1}{2(10^{-6})}q = 80; q(0) = 0.$$

Put $i = q'$ to obtain, after some simplification,

$$q' + 2q = 32(10^{-5}),$$

a linear equation with solution $q(t) = 16(10^{-5})(1 - e^{-2t})$. The capacitor voltage is

$$E_C = \frac{1}{C}q = 80(1 - e^{-2t}).$$

The voltage reaches 76 volts when $t = (1/2)\ln(20)$, which is approximately 1.498 seconds after the switch is closed. Calculate the current at this time by

$$\frac{1}{2}\ln(20)i = q'(\ln(20)/2) = 32(10^{-5})e^{-\ln(20)} = 16 \text{ micro amps.}$$

10. The loop currents satisfy

$$\begin{aligned} 5(i_1' - i_2') + 10i_2 &= 6, \\ -5i_1' + 5i_2' + 30i_2 + 10(q_2 - q_3) &= 0, \\ -10q_2 + 10q_3 + 15i_3 + \frac{5}{2}q_3 &= 0. \end{aligned}$$

Since $q_1(0+) = q_2(0+) = q_3(0+) = 0$, then from the third equation we have $i_3(0+) = 0$. Add the three equations to obtain

$$10i_1(0+) + 30i_2(0+) = 6.$$

From the upper node between loops 1 and 2, we conclude that $i_1(0+) = i_2(0+)$. Therefore

$$i_1(0+) = i_2(0+) = \frac{3}{20} \text{ amps.}$$

11. (a) Calculate

$$i'(t) = \frac{E}{R}e^{-Rt/L} > 0,$$

implying that the current increases with time.

(b) Note that $(1 - e^{-1}) = 0.63+$, so the inductive time constant is $t_0 = L/R$.

(c) For $i(0) \neq 0$, the time to reach 63 percent of E/R is

$$t_0 = \frac{L}{R}\ln\left(\frac{e(E - Ri(0))}{E}\right),$$

which decreases with $i(0)$.

12. (a) For

$$q' + \frac{1}{RC}q = \frac{E}{R}; q(0) = q_0,$$

the differential equation is linear with integrating factor $e^{t/RC}$. The differential equation becomes

$$(qe^{t/RC})' = \frac{E}{R}e^{t/RC}$$

so

$$q(t) = EC + ke^{-t/RC}.$$

$q(0) = q_0$ gives $k = q_0 - EC$, so

$$q(t) = EC + (q_0 - EC)e^{-T/RC}.$$

(b) $\lim_{t\to\infty} q(t) = EC$, and this independent of q_0.

(c) If $q_0 > EC$, $q_{\text{max}} = q(0) = q_0$, there is no minimum in this case but $q(t)$ decreases toward EC. If $q_0 = EC$, then $q(t) = EC$ for all t. If $q_0 < EC$, $q_{\text{min}} = q(0) = q_0$ and there is no maximum in this case, but $q(t)$ increases toward EC.

(d) To reach 99 percent of the steady-state value, solve

$$EC + (q_0 - EC)e^{-t/RC} = EC(1 \pm 0.01),$$

so

$$t = RC \ln\left(\frac{q_0 - EC}{0.1EC}\right).$$

13. The differential equation of the given family is

$$\frac{dy}{dx} = \frac{4x}{3}.$$

Orthogonal trajectories satisfy

$$\frac{dy}{dx} = -\frac{3}{4x}$$

and are given by

$$y = -\frac{3}{4}\ln|x| + c.$$

14. Differentiate $x + 2y = k$ implicitly to obtain the differential equation $y' = -1/2$ of this family. The orthogonal trajectories satisfy $y' = 2$, and are the graphs of $y = 2x + c$.

15. The differential equation of the family is

$$y' = 2kx = \frac{2x(y-1)}{x^2} = \frac{2(y-1)}{x}.$$

Orthogonal trajectories satisfy $y' = x/2(y-1)$ and are the graphs of the family of ellipses

$$(y-1)^2 + \frac{1}{2}x^2 = c.$$

16. The differential equation of the given family is $dy/dx = -x/2y$. The orthogonal trajectories satisfy $dy/dx = 2y/x$ and are given by $y = cx^2$, a family of parabolas.

17. The differential equation of the given family is found by solving for k and differentiating to obtain $k = \ln(y)/x$, so

$$\frac{dy}{dx} = \frac{y\ln(y)}{x}.$$

Orthogonal trajectories satisfy

$$\frac{dy}{dx} = -\frac{x}{y\ln(y)}.$$

This is separable with solutions

$$y^2(\ln(y^2) - 1) = c - 2x^2.$$

18. At time $t = 0$, assume that the dog is at the origin of an x, y - system and the man is located at $(A, 0)$ on the x - axis. The man moves directly upward into the first quadrant and at time t is at (A, vt). The position of the dog at time $t > 0$ is (x, y) and the dog runs with speed $2v$, always directly toward his master. At time $t > 0$, the man is at (A, vt), the dot is at (x, y), and the tangent to the dog's path joins these two points. Thus

$$\frac{dy}{dx} = \frac{vt - y}{A - x}$$

for $x < A$. To eliminate t from this equation use the fact that during the time the man has moved vt units upward, the dog has run $2vt$ units along his path. Thus

$$2vt = \int_0^x \left[1 + \left(\frac{dy}{d\xi}\right)^2\right]^{1/2} d\xi.$$

Use this integral to eliminate the vt term in the original differential equation to obtain

$$2(A - x)y'(x) = \int_0^x \left[1 + \left(\frac{dy}{d\xi}\right)^2\right]^{1/2} d\xi - 2y.$$

Differentiate this equation to obtain

$$2(A-x)y'' - 2y' = (1+(y')^2)^{1/2} - 2y',$$

or

$$2(A-x)y'' = (1+(y')^2)^{1/2},$$

subject to $y(0) = y'(0) = 0$. Let $u = y'$ to obtain the separable equation

$$\frac{1}{\sqrt{1+u^2}}\,du = \frac{1}{2(A-x)}\,dx.$$

This has the solution

$$\ln(u+\sqrt{1+u^2}) = -\frac{1}{2}\ln(A-x) + c.$$

Using $y'(0) = u(0) = 0$ gives us

$$u + \sqrt{1+u^2} = \frac{\sqrt{A}}{\sqrt{A-x}},$$

or, equivalently,

$$y' + \sqrt{(1+(y')^2)} = \frac{\sqrt{A}}{\sqrt{A-x}}; y(0) = 0.$$

From the equation for y'', we obtain

$$\sqrt{1+(y')^2} = 2(A-x)y'',$$

so

$$y' + 2(A-x)y'' = \frac{\sqrt{A}}{\sqrt{A-x}}; y(0) = y'(0) = 0$$

for $x < A$. Let $w = y'$ to obtain the linear first order equation

$$w' + \frac{1}{2(A-x)}w = \frac{\sqrt{A}}{2(A-x)^{3/2}}.$$

An integrating factor is $1/\sqrt{A-x}$ and we can write

$$\frac{d}{dx}\left[\frac{w}{\sqrt{A-x}}\right] = \frac{\sqrt{A}}{2(A-x)^2}.$$

The solution, subject to $w(0) = 0$, is

$$w(x) = \frac{A}{\sqrt{2}}\frac{1}{\sqrt{A-x}} - \frac{1}{2\sqrt{A}}\sqrt{A-x} = \frac{dy}{dx}.$$

Integrate one last time to obtain

$$y(x) = -\sqrt{A}\sqrt{A-x} + \frac{1}{3\sqrt{A}}(A-x)^{1/2} + \frac{2}{3}A,$$

in which we have used $y(0) = 0$ to evaluate the constant of integration. The dog catches the man at $x = A$, so they meet at $(A, 2A/3)$. Since this is also (A, vt) when they meet, we conclude that $vt = 2A/3$, so they meet at time

$$t = \frac{2A}{3v}.$$

19. (a) Clearly each bug follows the same curve of pursuit relative to the corner from which it started. Place a polar coordinate system as suggested and determine the pursuit curve for the bug starting at $\theta = 0, r = a/\sqrt{2}$. At any time $t > 0$, the bug will be at $(f(\theta), \theta)$ and its target will be at $(f(\theta), \theta + \pi/2)$, and

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{f'(\theta)\sin(\theta) + f(\theta)\cos(\theta)}{f'(\theta)\cos(\theta) - f(\theta)\sin(\theta)}.$$

On the other hand, the tangent direction must be from $(f(\theta), \theta)$ to $(f(\theta), \theta + \pi/2)$, so

$$\begin{aligned} \frac{dy}{dx} &= \frac{f(\theta)\sin(\theta + \pi/2) - f(\theta)\sin(\theta)}{f(\theta)\cos(\theta + \pi/2) - f(\theta)\cos(\theta)} \\ &= \frac{\cos(\theta) - \sin(\theta)}{-\sin(\theta) - \cos(\theta)} \\ &= \frac{\sin(\theta) - \cos(\theta)}{\sin(\theta) + \cos(\theta)}. \end{aligned}$$

Equate these two expressions for dy/dx and simplify to obtain

$$f'(\theta) + f(\theta) = 0$$

with $f(0) = a/\sqrt{2}$. Then

$$r = f(\theta) = \frac{a}{\sqrt{2}} e^{-\theta}$$

is the polar coordinate equation of the pursuit curve.

(b) The distance traveled by each bug is

$$\begin{aligned} D &= \int_0^\infty \sqrt{(r')^2 + r^2}\, d\theta \\ &= \int_0^\infty \left[\left(\frac{a}{\sqrt{2}} e^{-\theta} \right)^2 + \left(\frac{-a}{\sqrt{2}} e^{-\theta} \right)^2 \right]^{1/2} d\theta \\ &= a \int_0^\infty e^{-\theta}\, d\theta = a. \end{aligned}$$

(c) Since $r = f(\theta) = ae^{-\theta}/\sqrt{2} > 0$ for all θ, no bug reaches its quarry. The distance between pursuer and quarry is $ae^{-\theta}$.

20. (a) Assume the disk rotates counterclockwise with angular velocity ω radians per second and the bug steps on the rotating disk at point $(a, 0)$. By the chain rule,

$$\frac{dr}{dt} = \frac{dr}{d\theta}\frac{d\theta}{dt},$$

so

$$\frac{dr}{d\theta} = -\frac{v}{\omega}.$$

Then

$$r = c - \frac{\theta v}{\omega}, r(0) = a$$

gives us

$$r(\theta) = a - \frac{\theta v}{\omega}.$$

This is a spiral.

(b) To reach the center, solve $r = 0 = a - \theta v/\omega$ to get $\theta = a\omega/v$ radians, or $\theta = a\omega/2\pi v$ revolutions.

(c) The distance traveled is

$$\begin{aligned} s &= \int_0^{a\omega/v} \sqrt{r^2 + (r')^2}\, d\theta \\ &= \int_0^{a\omega/v} \sqrt{\left(a - \frac{v\theta}{\omega}\right)^2 + \left(\frac{v}{\omega}\right)^2}\, d\theta. \end{aligned}$$

To evaluate this integral let $\theta = -z + a\omega/v$, so

$$\begin{aligned} s &= \frac{v}{\omega}\int_0^{a\omega/v} \sqrt{1 + z^2}\, dz \\ &= \frac{1}{2}\left[\frac{a\omega}{v^2}\sqrt{a\omega^2 + v^2} + \ln\left(\frac{a\omega + \sqrt{\omega^2 + v^2}}{v}\right)\right]. \end{aligned}$$

21. Let $x(t)$ denote the length of chain hanging down from the table at time t, and note that once the chain starts moving, all 24 feet move with velocity v. The motion is modeled by

$$\rho x = \frac{24\rho}{g}\frac{dv}{dt} = \frac{3\rho}{4} v \frac{dv}{dx},$$

with $v(6) = 0$. Thus $x^2 = \frac{3}{4}v^2 + c$ and $v(6) = 0$ gives $c = 36$, so

$$v^2 = \frac{4}{3}(x^2 - 36).$$

When the end leaves the table, $x = 24$ so $v = 12\sqrt{5} \approx 26.84$ feet per second. The time is

$$\begin{aligned} t_f &= \int_6^{24} \frac{1}{v(x)}\, dx = \int_6^{24} \frac{\sqrt{3}}{2\sqrt{x^2-36}}\, dx \\ &= \frac{\sqrt{3}}{2} \ln(6+\sqrt{35}) \approx 2.15 \end{aligned}$$

seconds.

22. The force pulling the chain off the table is due to the four feet of chain hanging between the table and the floor. Let $x(t)$ denote the distance the free end of the chain on the table has moved. The motion is modeled by

$$4\rho = \frac{d}{dt}\left[(22-x)\frac{\rho}{g}v\right];\, v = 0 \text{ when } x = 0.$$

Rewrite this as

$$128 + v^2 = (22-x)v\frac{dv}{dx},$$

a separable differential equation which we solve to get

$$c - \ln|22-x| = \frac{1}{2}\ln(128+v^2)$$

Since $v = 0$ when $x = 0$, then $c = \ln(176\sqrt{2})$. The end of the chain leaves the table when $x = 18$, so at this time

$$v = \sqrt{3744} \approx 61.19 \text{ feet per second.}$$

1.6 Existence and Uniqueness Questions

1. Both $f(x,y) = \sin(xy)$ and $\partial f/\partial y = x\cos(xy)$ are continuous (for all (x,y)).

2. $f(x,y) = \ln|x-y|$ and

$$\frac{\partial f}{\partial y} = -\frac{1}{x-y}$$

are continuous on a sufficiently small rectangle about $(3,\pi)$, for example, on a square centered at $(3,\pi)$ and having side length $1/100$.

3. Both $f(x,y) = x^2 - y^2 + 8x/y$ and

$$\frac{\partial f}{\partial y} = -2y - \frac{8x}{y^2}$$

are continuous on a sufficiently small rectangle centered at $(3,-1)$, for example, on the square of side length 1.

4. Both $f(x,y) = \cos(e^{xy})$ and $\partial f/\partial y = -xe^{xy}\sin(e^{xy})$ are continuous over the entire plane.

5. By taking $|y'| = y'$, we get $y' = 2y$ and the initial value problem has the solution $y(x) = y_0 e^{2(x-x_0)}$. However, if we take $|y'| = -y'$, then the initial value problem has the solution $y(x) = y_0 e^{-2(x-x_0)}$.

   In this problem we have $|y'| = 2y = f(x,y)$, so we actually have $y' = \pm 2y$, and $f(x,y) = \pm 2y$. This is not even a function, so the terms of Theorem 1.2 do not apply and the theorem offers no conclusion.

6. (a) Since both $f(x,y) = 2-y$ and $\partial f/\partial y = -1$ are continuous everywhere, the initial value problem has a unique solution. In this case the solution is easy to find: $y = 2 - e^{-x}$. This is the answer to (b).

   (c)

$$
\begin{aligned}
&y_0 = 1, y_1 = 1 + \int_0^x dt = 1 + x,\\
&y_2 = 1 + \int_0^x (1-t)\,dt = 1 + x - \frac{x^2}{2},\\
&y_3 = 1 + \int_0^x \left(1 - t + \frac{t^2}{2}\right) dt = 1 + x - \frac{x^2}{2} + \frac{x^3}{3!},\\
&y_4 = 1 + \int_0^x \left(1 - t + \frac{t^2}{2} - \frac{t^3}{3!}\right) dt = 1 + x - \frac{x^2}{2} + \frac{x^3}{3!} - \frac{x^4}{4!},\\
&y_5 = 1 + \int_0^x y_4(t)\,dt = 1 + x - \frac{x^2}{2} + \frac{x^3}{3!} - \frac{x^4}{4!} + \frac{x^5}{5!},\\
&y_6 = 1 + \int_0^x y_5(t)\,dt = 1 + x - \frac{x^2}{2} + \frac{x^3}{3!} - \frac{x^4}{4!} + \frac{x^5}{5!} - \frac{x^6}{6!}.
\end{aligned}
$$

   Based on these computations, we conjecture that

$$y_n(x) = 1 + x - \frac{x^2}{2} + \frac{x^3}{3!} - \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots + (-1)^{n+1}\frac{x^n}{n!}$$

   (d)

$$
\begin{aligned}
2 - e^{-x} &= 2 - \left(1 + x - \frac{x^2}{2} + \frac{x^3}{3!} - \frac{x^4}{4!} + \cdots + (-1)^n \frac{x^n}{n!}\right)\\
&= 1 + x - \frac{x^2}{2!} + \frac{x^3}{3!} - \cdots + (-1)^{n+1}\frac{x^n}{n!} + \cdots
\end{aligned}
$$

   Since

$$2 - e^{-x} = 2 - \lim_{n\to\infty} \sum_{k=0}^{n} (-1)^k \frac{x^k}{k!} = \lim_{n\to\infty} y_n(x),$$

   the Picard iterates converge to the unique solution of the initial value problem.

7. (a) Since both $f(x,y) = 4 + y$ and $\partial f/\partial y = 1$ are continuous everywhere, the initial value problem has a unique solution.

(b) This linear differential equation is easily solved to yield $y = -4 + 7e^x$ as the unique solution of the initial value problem.

(c)

$$
\begin{aligned}
y_0 &= 3, y_1 = 3 + \int_0^x 7\,dt = 3 + 7x,\\
y_2 &= 3 + \int_0^x (7 + 7t)\,dt = 3 + 7x + 7\frac{x^2}{2},\\
y_3 &= 3 + \int_0^x \left(7 + 7t + 7\frac{t^2}{2}\right)\,dt = 3 + 7x + 7\frac{x^2}{2} + 7\frac{x^3}{3!},\\
y_4 &= 3 + \int_0^x \left(7 + 7t + 7\frac{t^2}{2} + 7\frac{t^3}{3!}\right)\,dt = 3 + 7x + 7\frac{x^2}{2} + 7\frac{x^3}{3!} + 7\frac{x^4}{4!},\\
y_5 &= 3 + \int_0^x y_4(t)\,dt = 3 + 7x + 7\frac{x^2}{2} + 7\frac{x^3}{3!} + 7\frac{x^4}{4!} + 7\frac{x^5}{5!},\\
y_6 &= 3 + \int_0^x y_5(t)\,dt = 3 + 7x + 7\frac{x^2}{2} + 7\frac{x^3}{3!} + 7\frac{x^5}{5!} + 7\frac{x^6}{6!}.
\end{aligned}
$$

(d) We conjecture that

$$y_n(x) = 3 + 7x + 7\frac{x^2}{2} + 7\frac{x^3}{3!} + \cdots + 7\frac{x^n}{n!}.$$

Note that

$$y_n(x) = -4 + 7\sum_{k=0}^{n} \frac{x^k}{k!}$$

and that

$$\lim_{n\to\infty} y_n(x) = -4 + 7\sum_{k=0}^{\infty} \frac{x^k}{k!} = -4 + 7e^x.$$

Thus the Picard iterates converge to the solution.

8. (a) Both $f(x,y) = 2x^2$ and $\partial f/\partial y = 0$ are continuous everywhere, so the initial value problem has a unique solution.

(b) The solution is

$$y = \frac{2}{3}x^3 + \frac{7}{3}.$$

(c)

$$y_0 = 3, y_1 = 3 + \int_1^x 2t^2\,dt = \frac{2}{3}x^3 + \frac{7}{3}.$$

Because $f(x,y)$ is independent of y, $y_n(x) = y_1(x)$ for all n.

(d) The sequence of Picard iterates is a constant sequence. We can write

$$y = \frac{2}{3}x^3 + \frac{7}{3} = 3 + 2(x-1) + 2(x-1)^2 + \frac{2}{3}(x-1)^3$$

and this is the Taylor expansion of the solution about 1. For $n \geq 3$ the nth partial sum of this finite series is the solution. Certainly $y_n \to y$ as $n \to \infty$.

9. (a) $f(x,y) = \cos(x)$ and $\partial f/\partial y = 0$ are continuous for all (x,y), so the problem has a unique solution.

(b) The solution is $y = 1 + \sin(x)$.

(c)

$$y_0 = 1, y_1 = 1 + \int_\pi^x \cos(t)\,dt = 1 + \sin(x).$$

In this example, $y_n = y_1$ for $n = 2, 3, \cdots$.

(d) For $n \geq 1$,

$$y = 1 + \sin(x) = 1 + \sum_{k=0}^{\infty} \frac{(-1)^{2k+1}x^{2k+1}}{(2k+1)!}.$$

The nth partial sum T_n of this Taylor series does not agree with the nth Picard iterate $y_n(x)$. However,

$$\lim_{n\to\infty} T_n(x) = \lim_{n\to\infty} y_n(x) = 1 + \sin(x),$$

so both sequences converge to the unique solution.

Chapter 2

Linear Second-Order Equations

2.1 Theory of the Linear Second-Order Equation

In Problems 1 - 5, verification that the given functions are solutions of the differential equation is a straightforward differentiation, which we omit.

1. The general solution is $y(x) = c_1 \sin(6x) + c_2 \cos(6x)$. For the initial conditions, we need $y(0) = c_2 = -5$ and $y'(0) = 6c_1 = 2$. Then $c_1 = 1/3$ and the solution of the initial value problem is
$$y(x) = \frac{1}{3}\sin(6x) - 5\cos(6x).$$

2. The general solution is $y(x) = c_1 e^{4x} + c_2 e^{-4x}$. For the initial conditions, compute
$$y(0) = c_1 + c_2 = 12 \text{ and } y'(0) = 4c_1 - 4c_2 = 3.$$
Solve these algebraic equations to obtain $c_1 = 51/8$ and $c_2 = 45/8$. The solution of the initial value problem is
$$y(x) = \frac{51}{8}e^{4x} + \frac{45}{8}e^{-4x}.$$

3. The general solution is $y(x) = c_1 e^{-2x} + c_2 e^{-x}$. For the initial conditions, we have
$$y(0) = c_1 + c_2 = -3 \text{ and } y'(0) = -2c_1 - c_2 = -1.$$
Solve these to obtain $c_1 = 4$, $c_2 = -7$. The solution of the initial value problem is
$$y(x) = 4e^{-2x} - 7e^{-x}.$$

4. The general solution is $y(x) = c_1 e^{3x}\cos(2x) + c_2 e^{3x}\sin(2x)$. Compute

$$\begin{aligned} y'(x) &= 3c_1 e^{3x}\cos(2x) - 2c_1 e^{3x}\sin(2x) \\ &\quad + 3c_2 e^{3x}\sin(2x) + 2c_2 e^{3x}\cos(2x). \end{aligned}$$

From the initial conditions,

$$y(0) = c_1 = -1 \text{ and } y'(0) = 3c_1 + 2c_2 = 1.$$

Then $c_2 = 2$ and the solution of the initial value problem is

$$y(x) = -e^{3x}\cos(2x) + 2e^{3x}\sin(2x).$$

5. The general solution is $y(x) = c_1 e^x\cos(x) + c_2 e^x\sin(x)$. Then $y(0) = c_1 = 6$. We find that $y'(0) = c_1 + c_2 = 1$, so $c_2 = -5$. The initial value problem has solution

$$y(x) = 6e^x\cos(x) - 5e^x\sin(x).$$

6. The general solution is

$$y(x) = c_1\sin(6x) + c_2\cos(6x) + \frac{1}{36}(x-1).$$

7. The general solution is

$$y(x) = c_1 e^{4x} + c_2 e^{-4x} - \frac{1}{4}x^2 + \frac{1}{2}.$$

8. The general solution is

$$y(x) = c_1 e^{-2x} + c_2 e^{-x} + \frac{15}{2}.$$

9. The general solution is

$$y(x) = c_1 e^{3x}\cos(2x) + c_2 e^{3x}\sin(2x) - 8e^x.$$

10. The general solution is

$$y(x) = c_1 e^x\cos(x) + c_2 e^x\sin(x) - \frac{5}{2}x^2 - 5x - 4.$$

11. For conclusion (1), begin with the hint to the problem to write

$$\begin{aligned} y_1'' + py_1' + qy_1 &= 0, \\ y_2'' + py_2' + qy_2 &= 0. \end{aligned}$$

Multiply the first equation by y_2 and the second by $-y_1$ and add the resulting equations to obtain

$$y_1''y_2 - y_2''y_1 + p(y_1'y_2 - y_2'y_1) = 0.$$

Since $W = y_1 y_2 - y_2 y_1$, then

$$W' = y_1 y_2'' - y_1'' y_2,$$

so

$$W' + pW = y_1 y_2'' - y_1'' y_2 + p(y_1 y_2' - y_1' y_2) = 0.$$

Therefore the Wronskian satisfies the linear differential equation $W' + pW = 0$. This has integrating factor $e^{\int p(x)\,dx}$ and can be written

$$\left(We^{\int p(x)\,dx}\right)' = 0.$$

Upon integrating we obtain the general solution

$$W = ce^{-\int p(x)\,dx}.$$

If $c = 0$, then this Wronskian is zero for all x in I. If $c \neq 0$, then $W \neq 0$ for x in I because the exponential function does not vanish for any x.

Now turn to conclusion (2). Suppose first that $y_2(x) \neq 0$ on I. By the quotient rule for differentiation it is routine to verify that

$$y_2^2 \frac{d}{dx}\left(\frac{y_1}{y_2}\right) = -W(x).$$

If $W(x)$ vanishes, then the derivative of y_1/y_2 is identically zero on I, so y_1/y_2 is constant, hence y_1 is a constant multiple of y_1, making the two functions linearly dependent. Conversely, if the two functions are linearly independent, then one is a constant multiple of the other, say $y_1 = cy_2$, and then $W(x) = 0$.

If there are points in I at which $y_2(x) = 0$, then we have to use this argument on the open intervals between these points and then make use of the continuity of y_2 on the entire interval. This is a technical argument we will not pursue here.

12.

$$W(x) = \begin{vmatrix} x^2 & x^3 \\ 2x & 3x^2 \end{vmatrix} = x^4.$$

Then $W(0) = 0$, while $W(x) \neq 0$ if $x \neq 0$. However, the theorem only applies to solutions of a linear second-order differential equation on an interval containing the point at which the Wronskian is evaluated. x^2 and x^3 are not solutions of such a second-order linear equation on an open interval containing 0.

13. It is routine to verify by substitution that x and x^2 are solutions of the given differential equation. The Wronskian is

$$W(x) = \begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix} = -x^2,$$

which vanishes at $x = 0$, but at no other points. However, the theorem only applies to solutions of linear second order differential equations. To write the given differential equation in standard linear form, we must write

$$y'' - \frac{2}{x}y' + \frac{2}{x^2}y = 0,$$

which is not defined at $x = 0$. Thus the theorem does not apply.

14. If y_1 and y_2 have relative extrema at some point x_0 within the interval,

$$y_1'(x_0) = y_2'(x_0) = 0.$$

Then

$$W(x_0) = \begin{vmatrix} y_1(x_0) & y_2(x_0) \\ 0 & 0 \end{vmatrix} = 0.$$

Therefore y_1 and y_2 are linearly dependent.

15. Suppose $\varphi'(x_0) = 0$. Then φ is the unique solution of the initial value problem

$$y'' + py' + qy = 0; y(x_0) = y'(x_0) = 0$$

on I. But the functions that is identically zero on I is also a solution of this problem. Therefore $\varphi(x) = 0$ for all x in I.

2.2 The Constant Coefficient Case

1. The characteristic equation is $\lambda^2 - \lambda - 6 = 0$, with roots $-2, 3$. The general solution is

$$y = c_1e^{-2x} + c_2e^{3x}.$$

2. The characteristic equation is $\lambda^2 - 2\lambda + 10 = 0$, with roots $1 \pm 3i$. The general solution is

$$y = c_1e^x \cos(3x) + c_2e^x \sin(3x).$$

3. The characteristic equation is $\lambda^2 + 6\lambda + 9 = 0$, with repeated root -3. The general solution is

$$y = c_1e^{-3x} + c_2xe^{-3x}.$$

4. The characteristic equation is $\lambda^2 - 3\lambda = 0$, with roots $0, 3$. The general solution is

$$y = c_1 + c_2e^{3x}.$$

5. The characteristic equation is $\lambda^2 + 10\lambda + 26 = 0$, with roots $-5 \pm i$. The general solution is

$$y = c_1e^{-5x} \cos(x) + c_2e^{-5x} \sin(x).$$

6. The characteristic equation is $\lambda^2 + 6\lambda - 40 = 0$, with roots $-10, 4$. The general solution is
$$y = c_1 e^{-10x} + c_2 e^{4x}.$$

7. The characteristic equation is $\lambda^2+3\lambda+18 = 0$, with roots $-3/2 \pm 3\sqrt{7}i/2$. The general solution is
$$y = e^{-3x/2}\left[c_1 \cos\left(\frac{3\sqrt{7}x}{2}\right) + c_2 \sin\left(\frac{3\sqrt{7}x}{2}\right)\right].$$

8. The characteristic equation is $\lambda^2 + 16\lambda + 64 = 0$, with repeated root -8. The general solution is
$$y = e^{-8x}(c_1 + c_2 x).$$

9. The characteristic equation is $\lambda^2 - 14\lambda + 49 = 0$, with repeated root 7. The general solution is
$$y = e^{7x}(c_1 + c_2 x).$$

10. The characteristic equation is $\lambda^2 - 6\lambda + 7 = 0$, with roots $3 \pm \sqrt{2}i$. The general solution is
$$y = e^{3x}[c_1 \cos(\sqrt{2}x) + c_2 \sin(\sqrt{2}x)].$$

In each of Problems 11 through 20, the solution is obtained by finding the general solution of the differential equation and then solving for the constants to satisfy the initial conditions. We provide the details only for Problems 11 and 12, the other problems proceeding similarly.

11. The characteristic equation is $\lambda^2 + 3\lambda = 0$, with roots $0, -3$. The general solution of the differential equation is $y = c_1 + c_2 e^{-3x}$. To find a solution satisfying the initial conditions, we need
$$y(0) = c_1 + c_2 = 3 \text{ and } y'(0) = -3c_2 = 6.$$
Then $c_1 = 5$ and $c_2 = -2$, so the solution of the initial value problem is $y = 5 - 2e^{-3x}$.

12. The characteristic equation is $\lambda^2 + 2\lambda - 3 = 0$, with roots $1, -3$. The general solution of the differential equation is
$$y(x) = c_1 e^x + c_2 e^{-3x}.$$
Now we need
$$y(0) = c_1 + c_2 = 6 \text{ and } y'(0) = c_1 - 3c_2 = -2.$$
Then $c_1 = 4$ and $c_2 = 2$, so the solution of the initial value problem is
$$y(x) = 4e^x + 2e^{-3x}.$$

13. $y = 0$ for all x

14. $y = e^{2x}(3 - x)$

15.
$$y = \frac{1}{7}[9e^{3(x-2)} + 5e^{-4(x-2)}]$$

16.
$$y = \frac{\sqrt{6}}{4}e^{x}\left[e^{\sqrt{6}x} - e^{-\sqrt{6}x}\right]$$

17. $y = e^{x-1}(29 - 17x)$

18.
$$y = -4(5 - \sqrt{23})e^{5(x-2)/7}\sin\left(\frac{\sqrt{23}}{2}(x-2)\right)$$

19.
$$y = e^{(x+2)/2}\left[\cos\left(\frac{\sqrt{15}}{2}(x+2)\right) + \frac{5}{\sqrt{15}}\sin\left(\frac{\sqrt{15}}{2}(x+2)\right)\right]$$

20.
$$y = ae^{(-1+\sqrt{5})x/2} + be^{(-1-\sqrt{5})x/2},$$

where
$$a = \frac{(9 + 7\sqrt{5})}{2\sqrt{5}}e^{-2+\sqrt{5}}$$

and
$$b = \frac{(7\sqrt{5} - 9)}{2\sqrt{5}}e^{-2-\sqrt{5}}$$

21. (a) The characteristic equation is $\lambda^2 - 2\alpha\lambda + \alpha^2 = 0$, with repeated roots $\lambda = \alpha$. The general solution is
$$y(x) = \varphi(x) = (c_1 + c_2x)e^{\alpha x}.$$

(b) The characteristic equation is $\lambda^2 - 2\alpha\lambda + (\alpha^2 - \epsilon^2) = 0$, with roots $\alpha \pm \epsilon$. The general solution is
$$y_\epsilon(x) = \varphi_\epsilon(x) = e^{\alpha x}(c_1e^{\epsilon x} + c_2e^{-\epsilon x}).$$

(c) In general,
$$\lim_{\epsilon\to 0} y_\epsilon(x) = e^{\alpha x}(c_1 + c_2) \neq y(x).$$

22. (a) We find

$$y = \psi(x) = e^{\alpha x}(c + (d - ac)x).$$

(b) We obtain

$$y_\epsilon = \psi_\epsilon(x) = \frac{1}{2\epsilon}e^{\alpha x}\left[(d - ac + \epsilon c)e^{\epsilon x} + (ac - d + \epsilon c)e^{-\epsilon x}\right].$$

(c) Using l'Hospital's rule, take the limit

$$\lim_{\epsilon \to 0}\psi_\epsilon(x) =$$
$$\frac{1}{2}e^{\alpha x}\lim_{\epsilon \to 0}\left[(d - ac + \epsilon c)xe^{\epsilon x} - (ac - d + \epsilon c)xe^{-\epsilon x} + ce^(\epsilon x + ce^{-\epsilon x})\right]$$
$$= e^{\alpha x}(c + (d - ac)x) = \psi(x).$$

23. The characteristic equation has roots

$$\lambda_1 = \frac{1}{2}(-a + \sqrt{a^2 - 4b}), \lambda_2 = \frac{1}{2}(-a - \sqrt{a^2 - 4b}).$$

As we have seen, there are three cases.

If $a^2 = 4b$, then

$$y = e^{-ax/2}(c_1 + c_2 x) \to 0 \text{ as } x \to \infty,$$

because $a > 0$.

If $a^2 > 4b$, then $a^2 - 4b < a^2$ and λ_1 and λ_2 are both negative, so

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \to 0 \text{ as } x \to \infty.$$

Finally, if $a^2 < 4b$, then the general solution has the form

$$y(x) = e^{-ax/2}(c_1\cos(\beta x) + c_2\sin(\beta x)),$$

where $\beta = \sqrt{4b - a^2}/2$. Because $a > 0$, this solution also has limit zero as $x \to \infty$.

24. We will use the fact that, for any positive integer n,

$$i^{2n} = (i^2)^n = (-1)^n \text{ and } i^{2n+1} = i^{2n}i = (-1)^n i.$$

Now suppose a is real and split the exponential series into two series, one for even values of the summation index, and the other for odd values:

$$\begin{aligned}
e^{ia} &= \sum_{n=0}^{\infty} \frac{1}{n!} i^n a^n \\
&= \sum_{n=0}^{\infty} \frac{1}{(2n)!} i^{2n} a^{2n} + \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} i^{2n+1} a^{2n+1} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{2n!} a^{2n} + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} i a^{2n+1} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} a^{2n} + i \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} a^{2n+1} \\
&= \cos(a) + i \sin(a).
\end{aligned}$$

2.3 The Nonhomogeneous Equation

1. Two independent solutions of $y'' + y = 0$ are $y_1 = \cos(x)$ and $y_2 = \sin(x)$. The Wronskian is

$$W(x) = \begin{vmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{vmatrix} = 1.$$

To use variation of parameters, seek a particular solution of the differential equation of the form

$$y = u_1 y_1 + u_2 y_2.$$

Let $f(x) = \tan(x)$. We found that we can choose

$$\begin{aligned}
u_1(x) &= -\int \frac{y_2(x) f(x)}{W(x)} \, dx = -\int \tan(x) \sin(x) \, dx \\
&= -\int \frac{\sin^2(x)}{\cos(x)} \, dx \\
&= -\int \frac{1 - \cos^2(x)}{\cos(x)} \, dx \\
&= \int \cos(x) \, dx - \int \sec(x) \, dx \\
&= \sin(x) - \ln|\sec(x) + \tan(x)|
\end{aligned}$$

and

$$\begin{aligned}
u_2(x) &= \int \frac{y_1(x) f(x)}{W(x)} \, dx = \int \cos(x) \tan(x) \, dx \\
&= \int \sin(x) \, dx = -\cos(x).
\end{aligned}$$

The general solution can be written

$$\begin{aligned}
y &= c_1\cos(x) + c_2\sin(x) + \sin(x)\cos(x) \\
&\quad - \cos(x)\ln|\sec(x)+\tan(x)| - \sin(x)\cos(x) \\
&= c_1\cos(x) + c_2\sin(x) - \cos(x)\ln|\sec(x)+\tan(x)|
\end{aligned}$$

2. Two independent solutions of the associated homogeneous equation are $y_1(x) = e^{3x}$ and $y_2(x) = e^x$. These have Wronskian $W(x) = -2e^{4x}$. Then

$$\begin{aligned}
u_1(x) &= -\int \frac{2e^x\cos(x+3)}{-2e^{4x}}\,dx \\
&= \int e^{-3x}\cos(x+3)\,dx \\
&= -\frac{3}{10}e^{-3x}\cos(x+3) + \frac{1}{10}e^{-3x}\sin(x+3)
\end{aligned}$$

and

$$\begin{aligned}
v(x) &= \int \frac{2e^{3x}\cos(x+3)}{-2e^{4x}}\,dx \\
&= \int e^{-x}\cos(x+3)\,dx \\
&= \frac{1}{2}e^{-x}\cos(x+3) - \frac{1}{2}e^{-x}\sin(x+3).
\end{aligned}$$

The general solution is

$$\begin{aligned}
y(x) &= c_1e^{3x} + c_2e^x \\
&\quad - \frac{3}{10}\cos(x+3) + \frac{1}{10}\sin(x+3) \\
&\quad + \frac{1}{2}\cos(x+3) - \frac{1}{2}\sin(x+3).
\end{aligned}$$

This can be written

$$\begin{aligned}
y(x) &= c_1e^{3x} + c_2e^x \\
&\quad + \frac{1}{5}\cos(x+3) - \frac{2}{5}\sin(x+3).
\end{aligned}$$

For Problems 3 through 6 we will omit some of the details and give an outline of the solution.

3. $y_1 = \cos(3x)$ and $y_2 = \sin(3x)$ are linearly independent solutions of the associated homogeneous equation. Their Wronskian is $W = 3$. With $f(x) = 12\sec(3x)$, carry out the integrations in the equations for u_1 and u_2 to obtain the general solution

$$y(x) = c_1\cos(3x) + c_2\sin(3x) + 4x\sin(3x) + \frac{4}{3}\cos(3x)\ln|\cos(3x)|.$$

4. $y_1 = e^{3x}$ and $y_2 = e^{-x}$, with Wronskian $-4e^{-2x}$. With $f(x) = 2\sin^2(x) = 1 - \cos(2x)$, obtain u_1 and u_2 to write the general solution

$$y(x) = c_1 e^{3x} + c_2 e^{-x} - \frac{1}{3} + \frac{7}{65}\cos(2x) + \frac{4}{65}\sin(2x).$$

5. $y_1 = e^{x}$ and $y_2 = e^{2x}$, with Wronskian $W = e^{3x}$. With $f(x) = \cos(e^{-x})$, carry out the integrations to obtain u_1 and u_2 to write the general solution

$$y(x) = c_1 e^{x} + c_2 e^{2x} - e^{2x}\cos(e^{-x})$$

6. $y_1 = e^{3x}$ and $y_2 = e^{2x}$, with Wronskian $W = -e^{5x}$. Use the identity $8\sin^2(4x) = 4\cos(8x) - 4$ to help find u_1 and u_2 and write the general solution

$$y = c_1 e^{3x} + c_2 e^{2x} + \frac{2}{3} + \frac{58}{1241}\cos(8x) + \frac{40}{1241}\sin(8x).$$

In Problems 7 - 16 we use the method of undetermined coefficients in writing the general solution. For Problems 7 and 8 all the details are included, while for Problems 9 through 16 the important details of the solution are outlined.

7. Two independent solutions of the associated homogeneous equation are $y_1 = e^{2x}$ and $y_2 = e^{-x}$. Since $2x^2 + 5$ is a second degree polynomial, we attempt such a polynomial as a particular solution:

$$y_p(x) = Ax^2 + Bx + C.$$

Substitute this into the (nonhomogeneous) differential equation to obtain

$$2A - (2Ax + B) - 2(Ax^2 + Bx + C) = 2x^2 + 5.$$

Then

$$\begin{aligned} 2A - B - 2C &= 5, \\ -2A - 2B &= 0, \\ -2A &= 2. \end{aligned}$$

Then $A = -1, B = 1$ and $C = -4$. The general solution is

$$y = c_1 e^{2x} + c_2 e^{-x} - x^2 + x - 4.$$

8. We find $y_1 = e^{3x}$ and $y_2 = e^{-2x}$. Since $f(x) = 8e^{2x}$, which is not a constant multiple of y_1 or y_2, try $y_p(x) = Ae^{2x}$ to obtain

$$y = c_1 e^{3x} + c_2 e^{-2x} - 2e^{2x}.$$

9. $y_1 = e^x \cos(3x)$ and $y_2 = e^x \sin(3x)$. With $f(x)$ a second degree polynomial, try $y_p(x) = Ax^2 + Bx + C$ to obtain
$$y = e^x[c_1 \cos(3x) + c_2 \sin(3x)] + 2x^2 + x - 1.$$

10. $y_1 = e^{2x} \cos(x)$ and $y_2 = e^{2x} \sin(x)$. With $f(x) = 21e^{2x}$, try $y_p(x) = Ae^{2x}$ to obtain
$$y = e^{2x}[c_1 \cos(x) + c_2 \sin(x)] + 21e^{2x}.$$

11. $y_1 = e^{2x}$ and $y_2 = e^{4x}$. With $f(x) = 3e^x$, try $y_p(x) = Ae^x$, noting that e^x is not a solution of the associated homogeneous equation. Obtain the general solution
$$y = c_1e^{2x} + c_2e^{4x} + e^x.$$

12. $y_1 = e^{-3x}$ and $y_2 = xe^{-3x}$. Because $f(x) = 9\cos(3x)$, try $y_p(x) = A\cos(x)+B\sin(x)$, obtaining both a $\cos(3x)$ and a $\sin(3x)$ term, to obtain
$$y = e^{-3x}[c_1 + c_2x] + \frac{1}{2}\sin(3x).$$
Although the general solution does not contain a $\cos(3x)$ term, this does not automatically follow and in general both the sine and cosine term must be included in our attempt at $y_p(x)$.

13. $y_1 = e^x$ and $y_2 = e^{2x}$. With $f(x) = 10\sin(x)$, try $y_p(x) = A\cos(x) + B\sin(x)$ to obtain
$$y = c_1e^x + c_2e^{2x} + 3\cos(x) + \sin(x).$$

14. $y_1 = 1$ and $y_2 = e^{-4x}$. With $f(x) = 8x^2 + 2e^{3x}$, try $y_p(x) = Ax^2 + Bx + C + De^{3x}$, since e^{3x} is not a solution of the homogeneous equation. This gives us the general solution
$$y = c_1 + c_2e^{-4x} - \frac{2}{3}x^3 - \frac{1}{2}x^2 - \frac{1}{4}x - \frac{2}{3}e^{3x}.$$

15. $y_1 = e^{2x}\cos(3x)$ and $y_2 = e^{2x}\sin(3x)$. Since neither e^{2x} nor e^{3x} is a solution of the homogeneous equation, try $y_p(x) = Ae^{2x} + Be^{3x}$ to obtain the general solution
$$y = e^{2x}[c_1 \cos(3x) + c_2 \sin(3x)] + \frac{1}{3}e^{2x} - \frac{1}{2}e^{3x}.$$

16. $y_1 = e^x$ and $y_2 = xe^x$. Because $f(x)$ is a first degree polynomial plus a $\sin(3x)$ term, try
$$y_p(x) = Ax + B + C\sin(3x) + D\cos(3x)$$

to obtain the general solution

$$y = e^x[c_1 + c_2x] + 3x + 6 + \frac{3}{2}\cos(3x) - 2\sin(3x).$$

Notice that the solution contains both a $\sin(3x)$ term and a $\cos(3x)$ term, even though $f(x)$ has just a $\sin(3x)$ term.

In Problems 17 through 24, we first find the general solution of the differential equation, then solve for the constants to satisfy the initial conditions. Problems 17 through 22 are well suited to the method of undetermined coefficients, while Problems 23 and 24 can be solved fairly directly by variation of parameters.

17. $y_1 = e^{2x}$ and $y_2 = e^{-2x}$. Since e^{2x} is a solution of the homogeneous equation, try $y_p(x) = Axe^{2x} + Bx + C$ to obtain the general solution

$$y = c_1e^{2x} + c_2e^{-2x} - \frac{7}{4}xe^{2x} - \frac{1}{4}x.$$

Now

$$y(0) = c_1 + c_2 = 1 \text{ and } y'(0) = 2c_1 - 2c_2 - \frac{7}{4} = 3.$$

Then $c_1 = 7/4$ and $c_2 = -3/4$. The solution of the initial value problem is

$$y = -\frac{7}{4}e^{2x} - \frac{3}{4}e^{-2x} - \frac{7}{4}xe^{2x} - \frac{1}{4}x.$$

18. Two independent solutions of the homogeneous equation are $y_1 = 1$ and $y_2 = e^{-4x}$. For a particular solution we might try $A + B\cos(x) + C\sin(x)$, but A is a solution of the homogeneous equation, so try $y_p(x) = Ax + B\cos(x) + C\sin(x)$. The general solution is

$$y(x) = c_1 + c_2e^{-4x} - 2\cos(x) + 8\sin(x) + 2x.$$

Now

$$y(0) = c_1 + c_2 - 2 = 3 \text{ and } y'(0) = -4c_2 + 8 + 2 = 2.$$

These lead to the solution of the initial value problem:

$$y = 3 + 2e^{-4x} - 2\cos(x) + 8\sin(x) + 2x.$$

19. We find the general solution

$$y(x) = c_1e^{-2x} + c_2e^{-6x} + \frac{1}{5}e^{-x} + \frac{7}{12}.$$

Solve for the constants to obtain the solution

$$y(x) = \frac{3}{8}e^{-2x} - \frac{19}{120}e^{-6x} + \frac{1}{5}e^{-x} + \frac{7}{12}$$

20. The general solution is

$$y(x) = c_1 + c_2 e^{3x} - \frac{1}{5} e^{2x}(\cos(x) + 3\sin(x)).$$

The solution of the initial value problem is

$$y = \frac{1}{5} + e^{3x} - \frac{1}{5} e^{2x}[\cos(x) + 3\sin(x)].$$

21. The general solution is

$$y(x) = c_1 e^{4x} + 2e^{-2x} - 2e^{-x} - e^{2x}.$$

The initial value problem has the solution

$$y = 2e^{4x} + 2e^{-2x} - 2e^{-x} - e^{2x}.$$

22. The general solution is

$$y = e^{x/2}\left[c_1 \cos\left(\frac{\sqrt{3}}{2}x\right) + c_2 \sin\left(\frac{\sqrt{3}}{2}x\right)\right] + 1$$

To make it easier to fit the initial conditions specified at $x = 1$, we can also write this general solution as

$$y = e^{x/2}\left[d_1 \cos\left(\frac{\sqrt{3}}{2}(x-1)\right) + d_2 \sin\left(\frac{\sqrt{3}}{2}(x-1)\right)\right] + 1.$$

Now

$$y(1) = e^{1/2} d_1 + 1 = 4 \text{ and } y'(1) = \frac{1}{2} e^{1/2} d_1 + \frac{\sqrt{3}}{2} e^{1/2} d_2 = -2.$$

Solve these to get $d_1 = 3e^{-1/2}$ and $d_2 = -7e^{-1/2}/\sqrt{3}$. The solution of the initial value problem is

$$y = e^{(x-1)/2}\left[3\cos\left(\frac{\sqrt{3}}{2}(x-1)\right) - \frac{7}{\sqrt{3}} \sin\left(\frac{\sqrt{3}}{2}(x-1)\right)\right] + 1.$$

23. We find the general solution

$$y(x) = c_1 e^x + c_2 e^{-x} - \sin^2(x) - 2.$$

The initial value problem has the solution

$$y = 4e^{-x} - \sin^2(x) - 2.$$

24. The general solution is

$$y(x) = c_1 \cos(x) + c_2 \sin(x) - \cos(x) \ln|\sec(x) + \tan(x)|.$$

The solution of the initial value problem is

$$y = 4\cos(x) + 4\sin(x) - \cos(x) \ln|\sec(x) + \tan(x)|.$$

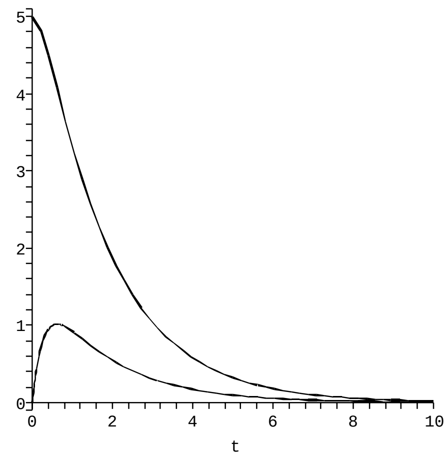

Figure 2.1: Solutions to Problem 1, Section 2.4.

2.4 Spring Motion

1. The solution with initial conditions $y(0) = 5, y'(0) = 0$ is

$$y_1(t) = 5e^{-2t}[\cosh(\sqrt{2}t) + \sqrt{2}\sinh(\sqrt{2}t)].$$

With initial conditions $y(0) = 0, y'(0) = 5$, we obtain

$$y_2(t) = \frac{5}{\sqrt{2}}e^{-2t}\sinh(\sqrt{2}t).$$

Graphs of these solutions are shown in Figure 2.1.

2. With $y(0) = 5$ and $y' = 0$, $y_1(t) = 5e^{-2t}(1 + 2t)$; with $y(0) = 0$ and $y'(0) = 5$, $y_2(t) = 5te^{-2t}$. Graphs are given in Figure 2.2.

3. With $y(0) = 5$ and $y' = 0$,

$$y_1(t) = \frac{5}{2}e^{-t}[2\cos(2t) + \sin(2t)].$$

With $y(0) = 0$ and $y'(0) = 5$, $y_2(t) = \frac{5}{2}e^{-t}\sin(2t)$. Graphs are given in Figure 2.3.

4. The solution is

$$y(t) = Ae^{-t}[\cosh(\sqrt{2}t) + \sqrt{(2)}\sinh(\sqrt{2}t)].$$

Graphs for $A = 1, 3, 6, 10, -4$ and -7 are given in Figure 2.4.

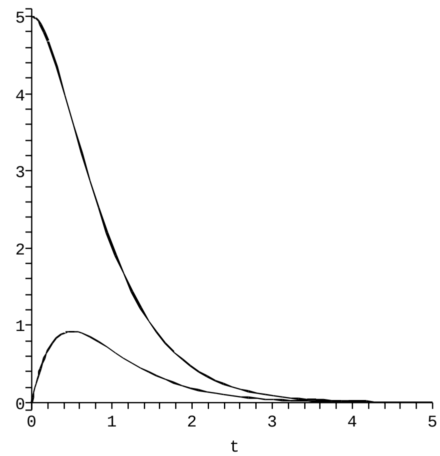

Figure 2.2: Solutions to Problem 2, Section 2.4.

5. The solution is
$$y(t) = \frac{A}{\sqrt{2}} e^{-2t} \sinh(\sqrt{(2)}t)$$
and is graphed for $A = 1, 3, 6, 10, -4$ and -7 in Figure 2.5.

6. The solution is $y(t) = Ae^{-2t}(1 + 2t)$ and is graphed for $A = 1, 3, 6, 10, -4, -7$ in Figure 2.6.

7. The solution is $y(t) = Ate^{-2t}$, graphed for $A = 1, 3, 6, 10, -4$ and -7 in Figure 2.7.

8. The solution is
$$y(t) = \frac{A}{2} e^{-t}[2\cos(2t) + \sin(2t)],$$
graphed in Figure 2.8 for $A = 1, 3, 6, 10, -4$ and -7.

9. The solution is
$$y(t) = \frac{A}{2} e^{-t} \sin(2t)$$
and is graphed for $A = 1, 3, 6, 10, -4$ and -7 in Figure 2.9.

10. From Newton's second law of motion,
$$y'' = \text{ sum of the external forces } = -29y - 10y'$$
so the motion is described by the solution of
$$y'' + 10y' + 29y = 0; y(0) = 3, y'(0) = -1.$$

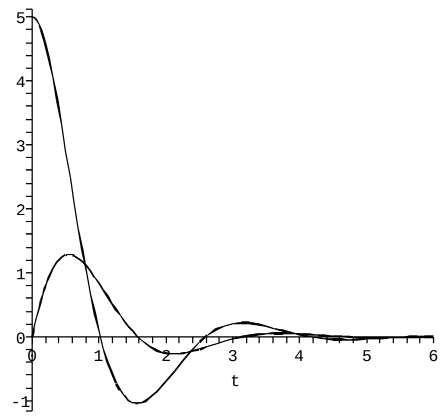

Figure 2.3: Solutions to Problem 3, Section 2.4.

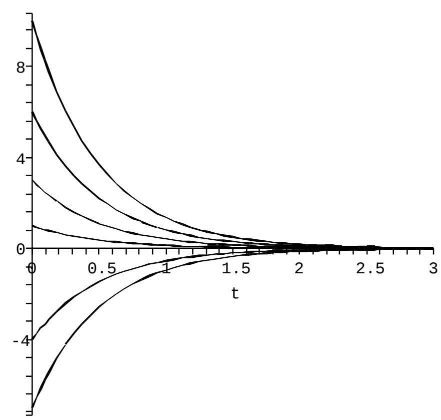

Figure 2.4: Solutions to Problem 4, Section 2.4.

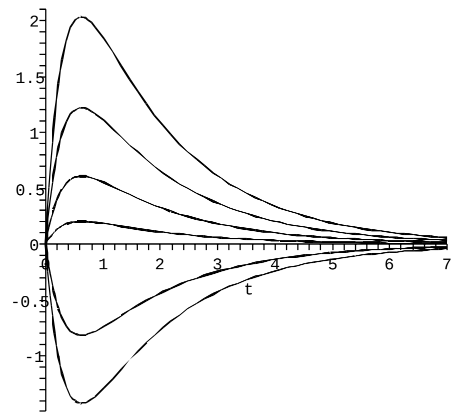

Figure 2.5: Solutions to Problem 5, Section 2.4.

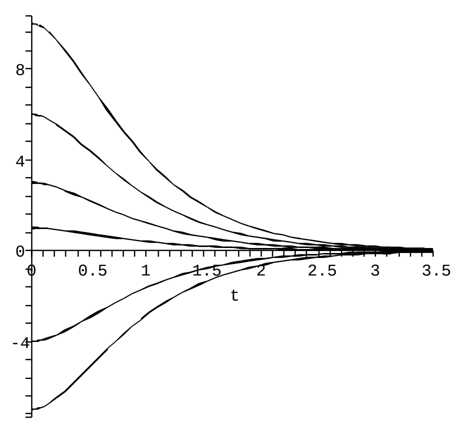

Figure 2.6: Solutions to Problem 6, Section 2.4.

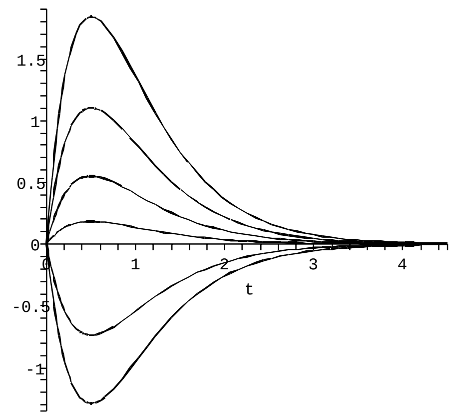

Figure 2.7: Solutions to Problem 7, Section 2.4.

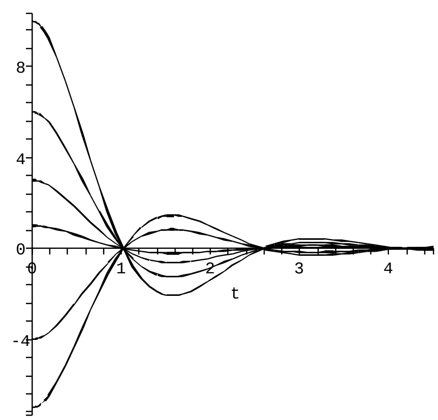

Figure 2.8: Solutions to Problem 8, Section 2.4.

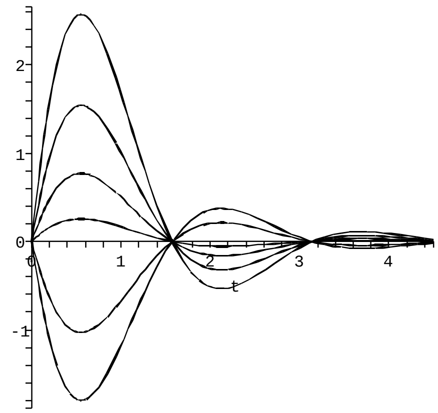

Figure 2.9: Solutions to Problem 9, Section 2.4.

The solution in this underdamped problem is

$$y(t) = e^{-5t}[3\cos(2t) + 7\sin(2t)].$$

If the condition on $y'(0)$ is $y'(0) = A$, this solution is

$$y(t) = e^{-5t}\left[3\cos(2t) + \left(\frac{A+15}{2}\right)\sin(2t)\right].$$

Graphs of this solution are shown in Figure 2.10 for $A = -1, -2, -4, 7, -12$ cm/sec (recall that down is the positive direction).

11. For overdamped motion the displacement is given by

$$y(t) = e^{-\alpha t}(A + Be^{\beta t}),$$

where α is the smaller of the roots of the characteristic equation and is positive, and β equals the larger root minus the smaller root. The factor $A + Be^{\beta t}$ can be zero at most once and only for some $t > 0$ if $-A/B > 1$. The values of A and B are determined by the initial conditions. In fact, if $y_0 = y(0)$ and $v_0 = y'(0)$, we have

$$A + B = y_0 \text{ and } -\alpha(A+B) + \beta B = v_0.$$

We find from these that

$$-\frac{A}{B} = 1 - \frac{\beta y_0}{v_0 + \alpha y_0}.$$

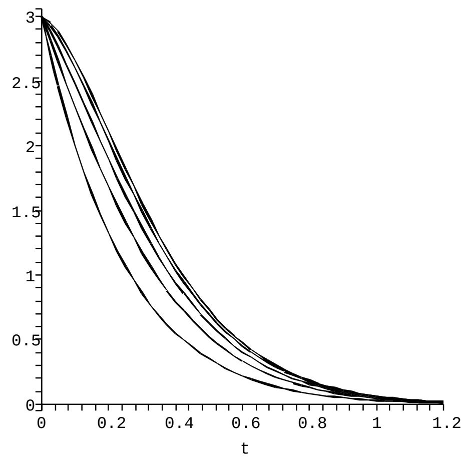

Figure 2.10: Solutions to Problem 10, Section 2.4.

No condition on only y_0 will ensure that $-A/B \leq 1$. If we also specify that $v_0 > -\alpha y_0$, we ensure that the overdamped bob will never pass through the equilibrium point.

12. For critically damped motion the displacement has the form

$$y(t) = e^{-\alpha t}(A + Bt),$$

with $\alpha > 0$ and A and B determined by the initial conditions. From the linear factor, the bob can pass through the equilibrium at most once, and will do this for some $t > 0$ if and only if $B \neq 0$ and $AB < 0$. Now note that $y_0 = A$ and $v_0 = y'(0) = -\alpha A + B$. Thus to ensure that the bob never passes through equilibrium we need $AB > 0$, which becomes the condition $(v_0 + \alpha y_0)y_0 > 0$. No condition on y_0 alone can ensure this. We would also need to specify $v_0 > -\alpha y_0$, and this will ensure that the critically damped bob never passes through the equilibrium point.

13. For underdamped motion, the solution has the appearance

$$y(t) = e^{-ct/2m}[c_1 \cos(\sqrt{4km - c^2}t/2m) + c_2 \sin(\sqrt{4km - c^2}t/2m)]$$

having frequency

$$\omega = \frac{\sqrt{4km - c^2}}{2m}.$$

Thus increasing c decreases the frequency of the the motion, and decreasing c increases the frequency.

14. For critical damping,

$$y(t) = e^{-ct/2m}(A + Bt).$$

For the maximum displacement at time t^* we need $y'(t^*) = 0$. This gives us

$$t^* = \frac{2mB - cA}{Bc}.$$

Now $y(0) = A$ and $y'(0) = B - Ac/2m$. Since we are given that $y(0) = y'(0) \neq 0$, we find that

$$t^* = \frac{4m^2}{2mc + c^2}$$

and this is independent of $y(0)$. The maximum displacement is

$$y(t^*) = \frac{y(0)}{c}(2m + c)e^{-2m/(2m+c)}.$$

15. The general solution of the overdamped problem

$$y'' + 6y' + 2y = 4\cos(3t)$$

is

$$\begin{aligned} y(t) = e^{-3t}&[c_1\cosh(\sqrt{7}t) + c_2\sinh(\sqrt{7}t)] \\ &- \frac{28}{373}\cos(3t) + \frac{72}{373}\sin(3t). \end{aligned}$$

(a) The initial conditions $y(0) = 6, y'(0) = 0$ give us

$$c_1 = \frac{2266}{373} \text{ and } c_2 = \frac{6582}{373\sqrt{7}}.$$

Now the solution is

$$y_a(t) = \frac{1}{373}[e^{-3t}[2266\cosh(\sqrt{7}t) + \frac{6582}{\sqrt{7}}\sinh(\sqrt{7}t)] - 28\cos(3t) + 72\sin(3t)].$$

(b) The initial conditions $y(0) = 0, y'(0) = 6$ give us $c_1 = 28/373$ and $c_2 = 2106/373$ and the unique solution

$$y_b(t) = \frac{1}{373}[e^{-3t}[29\cosh(\sqrt{7}t) + \tfrac{2106}{\sqrt{7}}\sinh(\sqrt{7}t)] - 28\cos(3t) + 72\sin(3t)].$$

These solutions are graphed in Figure 2.11.

16. The general solution of the critically damped problem

$$y'' + 4y' + 4y = 4\cos(3t)$$

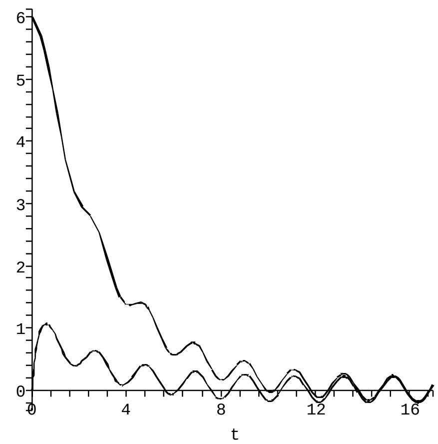

Figure 2.11: Solutions to Problem 15, Section 2.4.

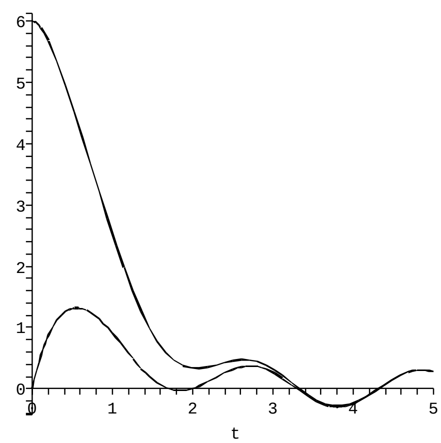

Figure 2.12: Solutions to Problem 16, Section 2.4.

is

$$y(t) = e^{-2t}[c_1 + c_2 t] - \frac{20}{169}\cos(3t) + \frac{48}{169}\sin(3t).$$

(a) The initial conditions $y(0) = 6, y'(0) = 0$ give us the unique solution

$$y_a(t) = \frac{1}{169}[e^{-2t}[1034 + 1924t] - 20\cos(3t) + 48\sin(3t)].$$

(b) The initial conditions $y(0) = 0, y'(0) = 6$ give us the unique solution

$$y_b(t) = \frac{1}{169}[e^{-2t}[20 + 910t] - 20\cos(3t) + 48\sin(3t)].$$

These solutions are graphed in Figure 2.12.

17. The general solution of the underdamped problem

$$y''(t) + y' + 3y = 4\cos(3t)$$

is

$$y(t) = e^{-t/2}\left[c_1\cos\left(\frac{\sqrt{11}t}{2}\right) + c_2\sin\left(\frac{\sqrt{11}t}{2}\right)\right] - \frac{24}{45}\cos(3t) + \frac{12}{45}\sin(3t).$$

(a) The initial conditions $y(0) = 6, y'(0) = 0$ yield the unique solution

$$y_a(t) = \frac{1}{15}\left[e^{-t/2}\left[98\cos\left(\frac{\sqrt{11}t}{2}\right) + \frac{74}{\sqrt{11}}\sin\left(\frac{\sqrt{11}t}{2}\right)\right] - 8\cos(3t) + 4\sin(3t)\right].$$

(b) The initial conditions $y(0) = 0, y'(0) = 6$ yield the unique solution

$$y_b(t) = \frac{1}{15}\left[e^{-t/2}\left[8\cos\left(\frac{\sqrt{11}t}{2}\right) + \frac{164}{\sqrt{11}}\sin\left(\frac{\sqrt{11}t}{2}\right)\right] - 8\cos(3t) + 4\sin(3t)\right].$$

These solutions are graphed in Figure 2.13.

2.5 Euler's Equation

In Problems 1 - 3, details are given with the solution. Solutions for Problems 4 through 10, just the general solution is given. All solutions are for $x > 0$.

1. Let $x = e^t$ to obtain

$$Y'' + Y' - 6Y = 0$$

which we can read directly from the original differential equation without further calculation. Then

$$Y(t) = c_1 e^{2t} + c_2 e^{-3t}.$$

In terms of x,

$$y(x) = c_1 e^{2\ln(x)} + c_2 e^{-3\ln(x)} = c_1 x^2 + c_2 x^{-3}.$$

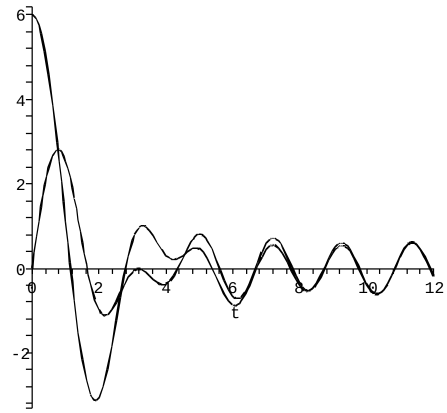

Figure 2.13: Solutions to Problem 17, Section 2.4.

2. The differential equation transforms to

$$Y'' + 2Y' + Y = 0,$$

with general solution

$$Y(t) = c_1 e^{-t} + c_2 t e^{-t}.$$

Then

$$y(x) = c_1 x^{-1} + c_2 x^{-1} \ln(x) = \frac{1}{x}(c_1 + c_2 \ln(x)).$$

3. Solve

$$Y'' + 4Y = 0$$

to obtain

$$Y(t) = c_1 \cos(2t) + c_2 \sin(2t).$$

Then

$$y(x) = c_1 \cos(2\ln(x)) + c_2 \sin(2\ln(x)).$$

4. $y(x) = c_1 x^2 + c_2 x^{-2}$
5. $y(x) = c_1 x^4 + c_2 x^{-4}$
6. $y(x) = x^{-2}(c_2 \cos(3\ln(x)) + c_2 \sin(3\ln(x))$
7. $y(x) = c_1 x^{-2} + c_2 x^{-3}$

8. $y(x) = x^2(c_1 \cos(7\ln(x)) + c_2 \sin(7\ln(x))$

9. $y(x) = x^{-12}(c_1 + c_2 \ln(x))$

10. $y(x) = c_1 x^7 + c_2 x^5$

11. The general solution of the differential equation is
$$y(x) = c_1 x^3 + c_2 x^{-7}.$$
We need
$$y(2) = 1 = c_1 2^3 + c_2 2^{-7} \text{ and } y'(2) = 0 = 3c_1 2^2 - 7c_2 2^{-8}.$$
Solve for c_1 and c_2 to obtain the solution of the initial value problem
$$y(x) = \frac{7}{10}\left(\frac{x}{2}\right)^3 + \frac{3}{10}\left(\frac{x}{2}\right)^{-7}$$

12. The solution of the initial value problem is
$$y(x) = -3 + 2x^2$$

13. $y(x) = x^2(4 - 3\ln(x))$

14. $y(x) = -4x^{-12}(1 + 12\ln(x))$

15. $y(x) = 3x^6 - 2x^4$

16.
$$y(x) = \frac{11}{4}x^2 + \frac{17}{4}x^{-2}$$

17. The transformation $x = e^t$ transforms the Euler equation $x^2y'' + axy' + by = 0$ into
$$Y'' + (a-1)Y' + bY = 0,$$
with characteristic equation
$$\lambda^2 + (a-1)\lambda + b = 0,$$
with roots λ_1 and λ_2. If we substitute $y = x^r$ directly into Euler's equation, we obtain
$$r(r-1)x^r + arx^r + bx^r = 0,$$
or, after dividing by x^r,
$$r^2 + (a-1)r + b = 0.$$
This equation for r is the same as the quadratic equation for λ, so its roots are $r_1 = \lambda_1$ and $r_2 = \lambda_2$. Therefore both the transformation method, and direct substitution of $y = x^r$ into Euler's equation, lead to the same solutions.

18. If $x < 0$, use the transformation $x = -e^t$, so $t = \ln(-x) = \ln|x|$. Note that
$$\frac{dt}{dx} = \frac{1}{-x}(-1) = \frac{1}{x},$$
just as in the case that $x > 0$. With $y(x) = y(-e^t) = Y(t)$, proceeding as in the text with chain rule derivatives. First
$$y'(x) = \frac{dY}{dt}\frac{dt}{dx} = \frac{1}{x}Y'(t)$$
and, similarly,
$$\begin{aligned} y''(x) &= \frac{d}{dx}\left(\frac{1}{x}Y'(t)\right) \\ &= -\frac{1}{x^2}Y'(t) + \frac{1}{x}\frac{dt}{dx}Y''(t) \\ &= -\frac{1}{x^2}Y'(t) + \frac{1}{x^2}Y''(t). \end{aligned}$$
Then,
$$x^2y''(x) = Y''(t) - Y'(t)$$
just as in the case that x is positive. Therefore Euler's equation transforms to
$$Y'' + (A-1)Y' + BY = 0,$$
and in effect we obtain the solution of Euler's equation for negative x by replacing x with $|x|$. For example, suppose we want to solve
$$x^2y'' + xy' + y = 0$$
for $x < 0$. We know that, for $x > 0$, this Euler equation transforms to
$$Y'' + Y = 0,$$
so $Y(t) = c_1\cos(t) + c_2\sin(t)$ and
$$y(x) = c_1\cos(\ln(x)) + c_2\sin(\ln(x))$$
for $x > 0$. For $x < 0$, the solution is
$$y(x) = c_1\cos(\ln(|x|)) + c_2\sin(\ln(|x|)).$$

Chapter 3

The Laplace Transform

3.1 Definition and Notation

1. From entry (9) of the table,

$$F(s) = \frac{3(s^2 - 4)}{(s^2 + 4)^2}$$

2. From entry (10),

$$G(s) = \frac{8}{(s + 4)^2 + 64}$$

3. From entries (2) and (6) and the linearity of the transform,

$$H(s) = \frac{14}{s^2} - \frac{7}{s^2 + 49}$$

4. From (7) applied twice, and the linearity of the transform,

$$W(s) = \frac{s}{s^2 + 9} - \frac{s}{s^2 + 49}$$

5. From entries (4) and (6) and the linearity of the transform,

$$K(s) = \frac{-10}{(s + 4)^3} + \frac{3}{s^2 + 9}$$

6. From entry (12) of the table, $r(t) = \frac{7}{3}\sinh(3t)$.

7. From (7) of the table, $q(t) = \cos(8t)$.

8. From entries (6) and (7),

$$g(t) = \tfrac{5}{\sqrt{12}}\sin(\sqrt{12}t) - 4\cos(\sqrt{8}t).$$

9. From entries (3) and (4),
$$p(t) = e^{-42t} - \tfrac{1}{6}t^3 e^{-3t}.$$

10. From entry (8),
$$f(t) = -\tfrac{5}{2}t\sin(t).$$

11. From the definition,
$$F(s) = \lim_{R\to\infty}\int_0^R e^{-st}f(t)\,dt.$$
For each R, let N be the largest integer so that $(N+1)T \le R$ and use the additivity of the integral to write
$$\int_0^R e^{-st}f(t)\,dt = \sum_{n=0}^{N}\int_{nT}^{(n+1)T} e^{-st}f(t)\,dt + \int_{(N+1)T}^{R} e^{-st}f(t)\,dt.$$
Assuming that $F(s)$ exists, then by choosing R sufficiently large,
$$\int_{(N+1)T}^{R} e^{-st}f(t)\,dt$$
can be made as small as we like. Also, as $R\to\infty$, $N\to\infty$. Therefore
$$\int_0^\infty e^{-st}f(t)\,dt = \sum_{n=0}^{\infty}\int_{nT}^{(n+1)T} e^{-st}f(t)\,dt.$$

12. Use the periodicity of f and make the change of variables $u = t - nT$ to write
$$\begin{aligned}\int_{nT}^{(n+1)T} e^{-st}f(t)\,dt &= \int_0^T e^{-s(u+nT)}f(u+nT)\,du\\ &= e^{-snT}\int_0^T e^{-su}f(u)\,du,\end{aligned}$$
since $f(u+nT) = f(u)$.

13. By the results of Problems 11 and 12,
$$\begin{aligned}\mathcal{L}[f](s) &= \sum_{n=0}^{\infty}\int_{nT}^{(n+1)T} e^{-st}f(t)\,dt\\ &= \sum_{n=0}^{\infty} e^{-snT}\int_0^T e^{-st}f(t)\,dt\\ &= \left[\sum_{n=0}^{\infty} e^{-snT}\right]\int_0^T e^{-st}f(t)\,dt,\end{aligned}$$
since the summation is independent of the integral.

14. For $s > 0$, $0 < e^{-st} < 1$, so by the geometric series,

$$\sum_{n=0}^{\infty} e^{-nst} = \sum_{n=0}^{\infty} (e^{-sT})^n = \frac{1}{1 - e^{-sT}}.$$

Then, by the result of Problem 13,

$$\mathcal{L}[f](s) = \frac{1}{1 - e^{-sT}} \int_0^T e^{-st} f(t)\, dt.$$

15. Since f has period $T = 6$ and

$$\int_0^6 e^{-st} f(t)\, dt = \int_0^3 5e^{-st}\, dt + \int_3^6 e^{-st} \cdot 0\, dt = \frac{5}{s}(1 - e^{-3s}),$$

then

$$\begin{aligned}\mathcal{L}[f](s) &= \frac{5}{s} \frac{1 - e^{-3s}}{1 - e^{-6s}} \\ &= \frac{5}{s} \frac{1 - e^{-3s}}{(1 - e^{-3s})(1 + e^{-3s})} \\ &= \frac{5}{s(1 + e^{-3s})}.\end{aligned}$$

16. f has period π/ω. Further,

$$\begin{aligned}\int_0^T e^{-st} f(t)\, dt &= \int_0^{\pi/\omega} e^{-st} E \sin(\omega t)\, dt \\ &= \frac{E\omega}{s^2 + \omega^2}(1 + e^{-\pi s/\omega}).\end{aligned}$$

Therefore

$$\mathcal{L}[f](s) = \frac{E\omega}{s^2 + \omega^2} \left(\frac{1 + e^{-\pi s/\omega}}{1 - e^{-\pi s/\omega}} \right).$$

This can also be written as

$$\mathcal{L}[f](s) = \frac{E\omega}{s^2 + \omega^2} \left(\frac{e^{\pi s/2\omega} + e^{-\pi s/2\omega}}{e^{\pi s/2\omega} - e^{-\pi s/2\omega}} \right),$$

which can in turn be stated in terms of the hyperbolic cotangent function as

$$\mathcal{L}[f](s) = \frac{E\omega}{s^2 + \omega^2} \coth\left(\frac{\pi s}{2\omega} \right).$$

17. f has period $T = 25$, and, from the graph,

$$f(t) = \begin{cases} 0 & \text{for } 0 < t \le 5, \\ 5 & \text{for } 5 < t \le 10, \\ 0 & \text{for } 10 < t \le 25. \end{cases}$$

Now

$$\int_0^{25} e^{-st} f(t)\,dt = \int_5^{10} 5e^{-st}\,dt = \frac{5}{s}e^{-5s}(1-e^{-5s}).$$

Then

$$\mathcal{L}[f](s) = \frac{5e^{-5s}(1-e^{-5s})}{s(1-e^{-25s})}.$$

18. $f(t) = t/3$ for $0 \le t < 6$ and f has period 6, so compute

$$\int_0^6 \frac{1}{3}te^{-st}\,dt = \frac{1}{3s^2}(1-6se^{-6s}-e^{-6s})$$

to obtain

$$\mathcal{L}[f](s) = \frac{1}{3s^2}\frac{1-6se^{-6s}-e^{-6s}}{1-e^{-6s}}.$$

19. f has period $2\pi/\omega$, and

$$f(t) = \begin{cases} E\sin(\omega t) & \text{for } 0 \le t < \pi/\omega, \\ 0 & \text{for } \pi/\omega \le t < 2\pi/\omega. \end{cases}$$

Compute

$$\begin{aligned}\int_0^{2\pi/\omega} f(t)e^{-st}\,dt &= \int_0^{\pi/\omega} E\sin(\omega t)e^{-st}\,dt \\ &= \frac{E\omega}{s^2+\omega^2}(1+e^{-\pi s/\omega}).\end{aligned}$$

Then

$$\begin{aligned}\mathcal{L}[f](s) &= \frac{E\omega}{s^2+\omega^2}\left(\frac{1+e^{-\pi s/\omega}}{1-e^{-2\pi s/\omega}}\right) \\ &= \frac{E\omega}{s^2+\omega^2}\frac{1}{1-e^{-\pi s/\omega}}.\end{aligned}$$

20.

$$f(t) = \begin{cases} 3t/2 & \text{for } 0 < t < 2, \\ 0 & \text{for } 2 \le 2 \le 8. \end{cases}$$

Here $T = 8$ and

$$\begin{aligned}\mathcal{L}[f](s) &= \frac{1}{1-e^{-8s}}\int_0^8 e^{-st}f(t)\,dt \\ &= \frac{3}{2s^2}\frac{1-2se^{-2s}-e^{-2s}}{1-e^{-8s}}.\end{aligned}$$

21. We have

$$f(t) = \begin{cases} h & \text{for } 0 < t \le a, \\ 0 & \text{for } a < t \le 2a \end{cases}$$

and $T = 2a$. Now

$$\int_0^{2a} e^{-st} f(t)\, dt = \int_0^a he^{-st}\, dt = \frac{h}{s}(1 - e^{-as}).$$

Then

$$\mathcal{L}[f](s) = \frac{h}{s}\frac{1 - e^{-as}}{1 - e^{-2as}} = \frac{h}{s}\frac{1}{1 + e^{-as}}.$$

22. $T = 2a$ and

$$f(t) = \begin{cases} ht/a & \text{for } 0 \le t < a, \\ -\frac{h}{a}(t - 2a) & \text{for } a < t \le 2a. \end{cases}$$

Now

$$\begin{aligned} \int_0^{2a} e^{-st} f(t)\, dt &= \frac{h}{a}\int_0^a te^{-st}\, dt - \frac{h}{a}\int_a^{2a} (t - 2a)e^{-st}\, dt \\ &= \frac{h}{as^2}(1 - e^{-as})^2. \end{aligned}$$

Then

$$\begin{aligned} \mathcal{L}[f](s) &= \frac{h}{as^2}\frac{(1 - e^{-as})^2}{1 - e^{-2as}} \\ &= \frac{h}{as^2}\frac{1 - e^{-as}}{1 + e^{-as}}. \end{aligned}$$

This can also be written in terms of the hyperbolic tangent function:

$$\mathcal{L}[f](s) = \frac{h}{as^2}\tanh\left(\frac{as}{2}\right).$$

3.2 Solution of Initial Value Problems

In many of these problems we use a partial fractions decomposition to write $Y(s)$ as a sum of terms whose inverse Laplace transforms can be computed fairly easily (for example, directly from a table). Partial fractions decompositions are reviewed at the end of this chapter.

1. Transform the differential equation to obtain

$$sY(s) - y(0) + 4Y(s) = \frac{1}{s}.$$

Set $y(0) = -3$ to obtain

$$sY + 3 + 4Y = \frac{1}{s},$$

or

$$Y(s) = \frac{1}{s+4}\left[\frac{1}{s} - 3\right] = \frac{-3s+1}{s(s+4)}.$$

Use a partial fractions decomposition to write this as

$$Y(s) = -\frac{13}{4}\left(\frac{1}{s+4}\right) + \frac{1}{4}\left(\frac{1}{s}\right).$$

The purpose of this decomposition is that we can easily compute the inverse transform of each term on the right, obtaining the solution of the initial value problem:

$$\begin{aligned} y(t) &= -\frac{13}{4}\mathcal{L}^{-1}\left(\frac{1}{s+4}\right) + \frac{1}{4}\mathcal{L}^{-1}\left(\frac{1}{s}\right) \\ &= -\frac{13}{4}e^{-4t} + \frac{1}{4}. \end{aligned}$$

2. Take the transform of the differential equation to obtain

$$sY(s) - y(0) - 9Y(s) = \frac{5}{s}.$$

Substitute $y(0) = 5$ and solve for Y to obtain

$$\begin{aligned} Y(s) &= \frac{1}{s-9}\left(\frac{1}{s^2+5}\right) \\ &= \frac{406}{81}\left(\frac{1}{s-9}\right) - \frac{1}{9}\left(\frac{1}{s^2}\right) - \frac{1}{81}\left(\frac{1}{s}\right), \end{aligned}$$

For the last part of this equation we have again used a partial fractions decomposition. Finally, take the inverse transform of $Y(s)$ to obtain the solution

$$y(t) = \frac{406}{81}e^{9t} - \frac{1}{9}t - \frac{1}{81}$$

3. Take the transform of the differential equation, insert the initial condition and solve for Y to obtain

$$\begin{aligned} Y(s) &= \frac{1}{s+4}\left(\frac{s}{s^2+1}\right) \\ &= -\frac{4}{17}\left(\frac{1}{s+4}\right) + \frac{1}{17}\frac{4s+1}{s^2+1}. \end{aligned}$$

The solution is

$$y(t) = -\frac{4}{17}e^{-4t} + \frac{4}{17}\cos(t) + \frac{1}{17}\sin(t)$$

4. Take the transform of the differential equation to obtain
$$sY - y(0) + 2Y = \frac{1}{s+1}.$$
Insert $y(0) = 1$ and solve for Y to obtain
$$Y(s) = \frac{1}{s+2}\left(\frac{1}{s+1} + 1\right) = \frac{1}{s+1}$$
The solution is
$$y(t) = e^{-t}$$

5. Transform the differential equation to obtain (with $y(0) = 4$),
$$sY - 4 - 2Y = \frac{1}{s} - \frac{1}{s^2}.$$
Then
$$\begin{aligned} Y(s) &= \frac{1}{s-2}\left(\frac{1}{s} - \frac{1}{s^2} + 4\right) \\ &= \frac{1}{2s^2} - \frac{1}{4s} + \frac{17}{4}\left(\frac{1}{s-2}\right) \end{aligned}$$
The solution is
$$y(t) = \frac{1}{2}t - \frac{1}{4} + \frac{17}{4}e^{2t}$$

6. Transform the differential equation, this time using the $n = 2$ case of the operational formula, to obtain
$$s^2Y - sy(0) - y'(0) + Y = \frac{1}{s},$$
or
$$s^2Y - 6s + Y = \frac{1}{s}.$$
Then
$$Y(s) = \frac{1}{s^2+1}\left(\frac{1}{s} + 6s\right) = \frac{1}{s} + 5\left(\frac{s}{s^2+1}\right)$$
The solution is
$$y(t) = 1 + 5\cos(t)$$

7. Transform the differential equation to obtain
$$\begin{aligned} Y(s) &= \frac{1}{(s-2)^2}\left[\frac{s}{s^2+1} + s - 1 + 4\right] \\ &= -\frac{13}{5}\left(\frac{1}{(s-2)^2}\right) + \frac{22}{25}\frac{1}{s-2} \\ &\quad + \frac{3}{25}\frac{s}{s^2+1} - \frac{4}{25}\left(\frac{1}{s^2+1}\right) \end{aligned}$$

The solution is

$$y(t) = -\frac{13}{5}te^{2t} + \frac{22}{25}e^{2t} + \frac{3}{25}\cos(t) - \frac{4}{25}\sin(t)$$

8. Transform the differential equation to obtain

$$\begin{aligned} Y(s) &= \frac{1}{s^2+9}\left(\frac{2}{s^3}\right) \\ &= \frac{2}{9}\left(\frac{1}{s^3}\right) - \frac{2}{81}\left(\frac{1}{s}\right) + \frac{2}{81}\left(\frac{s}{s^2+9}\right) \end{aligned}$$

The solution is

$$y(t) = \frac{1}{9}t^2 - \frac{2}{81} + \frac{2}{81}\cos(3t)$$

9. Transforming the differential equation, we have

$$\begin{aligned} Y(s) &= \frac{1}{s^2+16}\left[\frac{1}{s} + \frac{1}{s^2} - 2s + 1\right] \\ &= \frac{1}{16}\left(\frac{1}{s^2}\right) + \frac{1}{16}\left(\frac{1}{s}\right) - \frac{33}{16}\left(\frac{s}{s^2+16}\right) + \frac{15}{64}\left(\frac{1}{s^2+16}\right) \end{aligned}$$

The solution is

$$y(t) = \frac{1}{16}(t+1) - \frac{33}{16}\cos(4t) + \frac{15}{64}\sin(4t)$$

10. Transforming the differential equation, we obtain

$$\begin{aligned} Y(s) &= \frac{1}{(s-2)(s-3)}\left[\frac{1}{s+1} - 10\right] \\ &= \frac{29}{3}\left(\frac{1}{s-2}\right) - \frac{39}{4}\left(\frac{1}{s-3}\right) + \frac{1}{12}\left(\frac{1}{s+1}\right) \end{aligned}$$

The solution is

$$y(t) = \frac{29}{3}e^{2t} - \frac{39}{4}e^{3t} + \frac{1}{12}e^{-t}$$

11. Begin with the definition of the Laplace transform and integrate by parts to obtain

$$\begin{aligned} \mathcal{L}[f'(t)](s) &= \int_0^\infty e^{-st}f'(t)\,dt \\ &= e^{-st}f(t)\Big]_0^\infty - \int_0^\infty -se^{-st}f(t)\,dt \\ &= -f(0) + s\int_0^\infty e^{-st}f(t)\,dt \\ &= sF(s) - f(0). \end{aligned}$$

12. Begin with the definition of the Laplace transform, applied to $f''(t)$ and integrate by parts, then use the fact that

$$\mathcal{L}[f'(t)](s) = sF(s) - f(0)$$

to obtain:

$$\begin{aligned}\mathcal{L}[f''(t)](s) &= \int_0^\infty e^{-st} f''(t)\,dt \\ &= e^{-st} f'(t)\Big]_0^\infty - \int_0^\infty -se^{-st} f'(t)\,dt \\ &= -f'(0) + s\mathcal{L}[f'(t)](s) \\ &= -f'(0) + s(sF(s) - f(0)) \\ &= s^2F(s) - sf(0) - f'(0).\end{aligned}$$

3.3 Shifting and the Heaviside Function

In the following, if we shift $f(t)$ by a, replacing t with $t-a$, we may write

$$[f(t)]_{t \to t-a}.$$

In the same spirit, if we want to replace s with $s-a$ in the transform of f, write

$$\mathcal{L}[f(t)]_{s \to s-a}.$$

This notation is sometimes useful in applying a shifting theorem.

1.

$$\begin{aligned}\mathcal{L}[(t^3 - 3t + 2)e^{-2t}] &= \mathcal{L}[t^3 - 3t + 2]_{s \to s+2} \\ &= \left[\frac{6}{s^4} - \frac{3}{s^2} + \frac{2}{s}\right]_{s \to s+2} \\ &= \frac{6}{(s+2)^4} - \frac{3}{(s+2)^2} + \frac{2}{s+2}\end{aligned}$$

2. Use the facts that

$$\mathcal{L}[t](s) = \frac{1}{s^2} \text{ and } \mathcal{L}[-2](s) = -\frac{1}{s}$$

to write

$$\begin{aligned}\mathcal{L}[te^{-3t} - 2e^{-3t}](s) &= \mathcal{L}[te^{-3t}](s) - \mathcal{L}[te^{-3t}](s) \\ &= \left(\frac{1}{s^2}\right)_{s \to s+3} - \left(\frac{2}{s}\right)_{s \to s+3} \\ &= \frac{1}{(s+3)^2} - \frac{2}{s+3}\end{aligned}$$

3. Write

$$\begin{aligned}f(t) &= [1 - H(t-7)] + H(t-7)\cos(t)\\ &= [1 - H(t-7)] + H(t-7)\cos((t-7)+7)\\ &= [1 - H(t-7)] + \cos(7)H(t-7)\cos(t-7) - \sin(7)H(t-7)\sin(t-7).\end{aligned}$$

Then

$$\mathcal{L}[f(t)](s) = \frac{1}{s}(1 - e^{-7s}) + \frac{s}{s^2+1}\cos(7)e^{-7s} - \frac{1}{s^2+1}\sin(7)e^{-7s}$$

4.

$$\begin{aligned}\mathcal{L}[f](s) &= \left(\frac{1}{s^2}\right)_{s\to s+4} - \left(\frac{s}{s^2+1}\right)_{s\to s+4}\\ &= \frac{1}{(s+4)^2} - \frac{s+4}{(s+4)^2+1}.\end{aligned}$$

5. Write

$$\begin{aligned}f(t) &= t + (1-4t)H(t-3) = t + (1 - 4(t-3) + 3)H(t-3)\\ &= t - 11H(t-3) - 4(t-3)H(t-3).\end{aligned}$$

Then

$$\mathcal{L}[f(t)](s) = \frac{1}{s^2} - \frac{11}{s}e^{-3s} - \frac{4}{s^2}e^{-3s}.$$

6. Write

$$f(t) = [2(t-\pi) + 2\pi + \sin(t-\pi)][1 - H(t-\pi)],$$

to obtain

$$\mathcal{L}[f(t)](s) = \frac{2}{s^2} - \frac{1}{s^2+1} - \frac{2}{s^2}e^{-\pi s} - \frac{2\pi}{s}e^{-\pi s} - \frac{1}{s^2+1}e^{-\pi s}.$$

7. Replace s with $s+1$ in the transform of $1 - t^2 + \sin(t)$ to obtain

$$\mathcal{L}[f](s) = \frac{1}{s+1} - \frac{2}{(s+1)^3} + \frac{1}{(s+1)^2+1}$$

8. First write $f(t)$ in terms of the Heaviside function:

$$\begin{aligned}f(t) &= t^2 + (1 - t - 4t^2)H(t-2)\\ &= t^2 + [1 - (t-2) - 2 - 4((t-2)+2)^2]H(t-2)\\ &= t^2 - [17 + 17(t-2) + 4(t-2)^2]H(t-2).\end{aligned}$$

Then

$$\mathcal{L}[f(t)](s) = \frac{2}{s^3} - \left(\frac{17}{s} + \frac{17}{s^2} + \frac{8}{s^3}\right)e^{-2s}.$$

9. First write

$$f(t) = (1 - H(t - 2\pi))\cos(t) + H(t - 2\pi)(2 - \sin(t))$$

to obtain

$$\mathcal{L}[f](s) = \frac{s}{s^2+1} + \left(\frac{2}{s} - \frac{s}{s^2+1} - \frac{1}{s^2+1}\right)e^{-2\pi s}$$

10. Write

$$\begin{aligned} f(t) &= -4(1 - H(t-1)) + e^{-t}H(t-3) \\ &= -4 + 4H(t-1) + e^{-3}e^{-(t-3)}H(t-3) \end{aligned}$$

so

$$\mathcal{L}[f(t)](s) = -\frac{4}{s} + \frac{4}{s}e^{-s} + \frac{e^{-3}}{s+1}e^{-3s}.$$

11. Since

$$\mathcal{L}[t\cos(3t)](s) = \frac{s^2-9}{(s^2+9)^2},$$

we obtain the transform of $te^{-t}\cos(3t)$ by replacing s with $s+1$:

$$\mathcal{L}[f](s) = \frac{(s+1)^2-9}{((s+1)^2+9)^2}.$$

12. The transform of $1 - \cosh(t)$ is

$$\frac{1}{s} - \frac{s}{s^2-1},$$

so the transform of $e^t(1-\cosh(t))$ is obtained by replacing s with $s-1$ to obtain

$$\begin{aligned} \mathcal{L}[f](s) &= \frac{1}{s-1} - \frac{s-1}{(s-1)^2-1} \\ &= \frac{1}{s-1} - \frac{s-1}{s(s-2)}. \end{aligned}$$

13. First write

$$\begin{aligned} f(t) &= (1 - H(t-16))(t-2) - H(t-16) \\ &= t - 2 + H(t-16)(2 - t - 1) = t - 2 + (1-t)H(t-16). \end{aligned}$$

Then

$$F(s) = \frac{1}{s^2} - \frac{2}{s} + \left(\frac{1}{s} - \frac{1}{s^2}\right)e^{-16s}$$

14. Write
$$f(t) = (1 - \cos(2t)) - (1 - \cos(2t))H(t - 3\pi).$$
Then
$$F(s) = (\frac{1}{s} - \frac{s}{s^2+4})(1 - e^{-3\pi s}).$$

15. Replace s with $s + 5$ in the transform of $t^4 + 2t^2 + t$ to obtain
$$F(s) = \frac{24}{(s+5)^5} + \frac{4}{(s+5)^3} + \frac{1}{(s+5)^2}.$$

16. Write
$$F(s) = \frac{1}{(s+2)^2+8}.$$
This is the transform of $(1/2\sqrt{2})\sin(2\sqrt{2}t)$ with s replaced by $s+2$. Therefore
$$f(t) = \frac{1}{2\sqrt{2}}e^{-2t}\sin(2\sqrt{2}t).$$

17. Write
$$F(s) = \frac{1}{(s-2)^2+1},$$
which we recognize as the transform of $\sin(t)$ with s replaced by $s - 2$. Therefore
$$f(t) = e^{-2t}\sin(t).$$

18.
$$\begin{aligned}\mathcal{L}^{-1}\left[\frac{e^{-5s}}{s^3}\right](t) &= \mathcal{L}^{-1}\left[\frac{1}{s^3}\right]_{t\to t-5} \\ &= \frac{1}{2}(t-5)^2H(t-5).\end{aligned}$$

19. Since $3/(s^2 + 9)$ is the transform of $\sin(3t)$, then
$$f(t) = \frac{1}{3}\sin(3(t-2))H(t-2).$$

20. Since the transform of $3e^{-2t}$ is $3/(s + 2)$, then
$$f(t) = 3e^{-2(t-4)}H(t-4).$$

21. Since
$$F(s) = \frac{1}{(s+3)^2-2},$$
then
$$f(t) = \frac{1}{\sqrt{2}}\sinh(\sqrt{2}t)e^{-3t}.$$

22. Since
$$F(s) = \frac{s-4}{(s-4)^2-6},$$
then
$$f(t) = e^{4t}\cosh(\sqrt{6}t).$$

23. Write
$$F(s) = \frac{(s+3)-1}{(s+3)^2-8}$$
to obtain
$$f(t) = e^{-3t}\cosh(2\sqrt{2}t) - \frac{1}{2\sqrt{2}}e^{-3t}\sinh(2\sqrt{2}t).$$

24. Put $a = 1$ and $F(s) = 1/(s-5)$ in the second shifting theorem. Then $f(t) = e^{5t}$ in this formula, yielding
$$\mathcal{L}^{-1}\left[\frac{1}{s-5}e^{-s}\right](t) = e^{5(t-1)}H(t-1).$$

25. Write
$$\frac{1}{s(s^2+16)} = \frac{1}{16}\frac{1}{s} - \frac{1}{16}\frac{s}{s^2+16}$$
to obtain
$$f(t) = \frac{1}{16}(1-\cos(4(t-21)))H(t-21).$$

26. By the first shifting theorem
$$\mathcal{L}\left[e^{-2t}\int_0^t e^{2w}\cos(3w)\,dw\right] = F(s+2),$$
where
$$F(s) = \mathcal{L}\left[\int_0^t e^{2w}\cos(3w)\,dw\right].$$
Now
$$\frac{d}{dt}\int_0^t e^{2w}\cos(3w)\,dw = e^{2t}\cos(3t).$$
By the operational rule for the Laplace transform, applied in the case of a first derivative, we can write
$$\mathcal{L}\left[\frac{d}{dt}\int_0^t e^{2w}\cos(3w)\,dw\right] = \mathcal{L}\left[e^{2t}\cos(3t)\right]$$
$$= s\mathcal{L}\left[\int_0^t e^{2w}\cos(3w)\,dw\right] = sF(s).$$

Therefore

$$F(s) = \frac{1}{s}\mathcal{L}\left[e^{2t}\cos(3t)\right] = \frac{1}{s}\frac{s-2}{(s-2)^2+9}.$$

Therefore

$$\mathcal{L}\left[e^{-2t}\int_0^t e^{2w}\cos(3w)\,dw\right] = \frac{s}{(s+2)(s^2+9)}.$$

27. The initial value problem

$$y'' + 4y = 3H(t-4); y(0) = 1, y'(0) = 0$$

transforms to

$$(s^2+4)Y(s) = \frac{3}{s}e^{-4s} + s.$$

Then

$$Y(s) = \frac{3}{4}\left[\frac{1}{s} - \frac{s}{s^2+4}\right]e^{-4s} + \frac{s}{s^2+4}.$$

Inverting this gives the solution

$$y(t) = \cos(2t) + \frac{3}{4}(1-\cos(2(t-4)))H(t-4).$$

28. The problem is

$$y'' - 2y' - 3y = 12H(t-4); y(0) = 1, y'(0) = 0.$$

Transform this and solve for $Y(s)$ to obtain

$$Y(s) = \frac{1}{4}\left[\frac{1}{s-3} + \frac{3}{s+1}\right] + \left[\frac{1}{s-3} + \frac{3}{s+1} - \frac{4}{s}\right]e^{-4s}.$$

Invert this to obtain the solution

$$y(t) = \frac{1}{4}(e^{3t} + 3e^{-t}) + (e^{3(t-4)} + 3e^{-(t-4)} - 4)H(t-4).$$

29. The problem is

$$y^{(3)} - 8y' = 2H(t-6); y(0) = y'(0) = y''(0) = 0.$$

Transform this problem and solve for $Y(s)$ to obtain

$$Y(s) = \left[-\frac{1}{4s} + \frac{1}{12}\frac{1}{s-2} + \frac{1}{6}\frac{s}{s^2+2s+4}\right]e^{-6s}.$$

Invert this to obtain

$$y(t) = \left[-\frac{1}{4} + \frac{1}{12}e^{-2(t-6)} + \frac{1}{6}e^{-(t-6)}\cos(\sqrt{3}(t-6))\right]H(t-6).$$

30. The problem is

$$y'' + 5y' + 6y = -2(1 - H(t-3)); y(0) = y'(0) = 0.$$

Transform this problem and solve for $Y(s)$ to obtain

$$Y(s) = \left[\frac{1}{s+2} - \frac{2}{3}\frac{1}{s+3} - \frac{1}{3s}\right] H(t-3).$$

Invert this to obtain the solution

$$y(t) = e^{-2t} - \frac{2}{3}e^{-3t} - \frac{1}{3} - \left[e^{-2(t-3)} - \frac{2}{3}e^{-3(t-3)}\right] H(t-3).$$

31. The problem is

$$y^{(3)} - y'' + 4y' - 4y = 1 + H(t-5); y(0) = y'(0) = 0, y''(0) = 1.$$

Transform this and solve for $Y(s)$ to obtain

$$Y(s) = \left[-\frac{1}{4s} + \frac{2}{5}\frac{1}{s-1} - \frac{3}{20}\frac{s}{s^2+4} - \frac{2}{5}\frac{1}{s^2+4}\right](1 - e^{-5s}).$$

Invert this to obtain

$$\begin{aligned} y(t) = &-\frac{1}{4} + \frac{2}{5}e^t - \frac{3}{20}\cos(2t) - \frac{1}{5}\sin(2t) \\ &- \left[-\frac{1}{4} + \frac{2}{5}e^{t-5} - \frac{3}{20}\cos(2(t-5)) - \frac{1}{5}\sin(2(t-5))\right] H(t-5). \end{aligned}$$

32. The initial value problem is

$$y'' - 4y' + 4y = t + 2H(t-3); y(0) = -2, y'(0) = 1.$$

Transform this and solve for $Y(s)$ to obtain

$$\begin{aligned} Y(s) = &\frac{1}{4s} + \frac{1}{4s^2} - \frac{9}{4}\frac{1}{s-2} - \frac{43}{4}\frac{1}{s-2)^2} \\ &+ \left[\frac{1}{2s} - \frac{1}{2}\frac{1}{s-2} + \frac{1}{(s-2)^2}\right] e^{-3s}. \end{aligned}$$

Invert this to obtain

$$\begin{aligned} y(t) = &\frac{1}{4} + \frac{1}{4}t - \frac{9}{4}e^{2t} - \frac{43}{4}te^{2t} \\ &+ \left[\frac{1}{2} - \frac{1}{2}e^{2(t-3)} + (t-3)e^{2(t-3)}\right] H(t-3). \end{aligned}$$

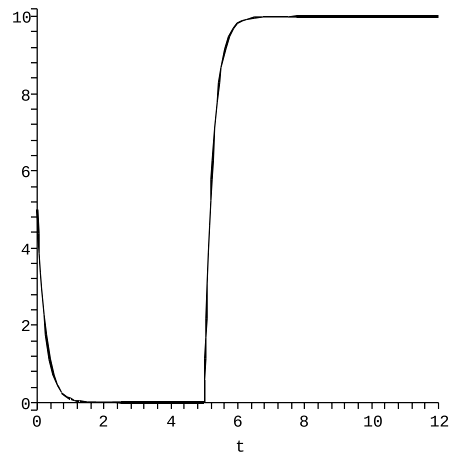

Figure 3.1: Output voltage in Problem 27, Section 3.3.

33. Assume that the switch is held in position for B seconds, then switched to position A and left there. The charge q on the capacitor is modeled by the initial value problem

$$250,000q' + 10^6 q = 10H(t-5); q(0) = C, E(0) = 5(10^{-6}).$$

Transform this problem and solve for $Q(s)$ to obtain

$$Q(s) = \frac{5(10^{-6})}{s+4} + 10^{-6}\left[\frac{1}{s} - \frac{1}{s+4}\right]e^{-5s}.$$

Invert this to obtain

$$E_{\text{out}} = \frac{q(t)}{C} = 10^6 q(t) = 5e^{-4t} + 10(1 - e^{-4(t-5)})H(t-5).$$

This output function is graphed in Figure 3.1.

34. The current is modeled by the initial value problem

$$Li' + Ri = 2H(t-5); i(0) = 0.$$

Transform this problem and solve for $I(s)$ to obtain

$$I(s) = \frac{2}{s(Ls+4)}e^{-5s} = \frac{2}{R}\left[\frac{1}{s} - \frac{1}{s+R/L}\right]e^{-5s}.$$

Invert this to obtain the current function

$$i(t) = \frac{2}{R}(1 - e^{-R(t-5)/L})H(t-5).$$

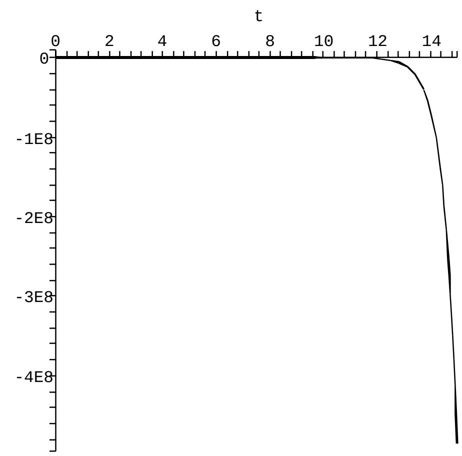

Figure 3.2: Current function in Problem 28, Section 3.3.

This function is graphed in Figure 3.2.

35. The current is modeled by

$$Li' + Ri = k(1 - H(t-5)); i(0) = 0.$$

Transform this problem and solve for $I(s)$ to obtain

$$I(s) = \frac{k}{s(Ls+R)}(1-e^{-5s}) = \frac{k}{R}\left[\frac{1}{s} - \frac{1}{s+R/L}\right](1-e^{-5s}).$$

Invert this to obtain

$$i(t) = \frac{k}{R}(1-e^{-Rt/L}) - \frac{k}{R}(1-e^{-R(t-5)/L})H(t-5).$$

36. The hint does most of the work. If we write $(s-a_j)p(s)/q(s)$ in the suggested way, then

$$\begin{aligned}\lim_{s\to a_j}(s-a)j)\frac{p(s)}{q(s)} &= \lim_{s\to a_j}\frac{p(s)}{(q(s)-q(a_j))/(s-a_j)} \\ &= \frac{p(a_j)}{q'(a_j)}.\end{aligned}$$

Upon summing over the zeros a_j of $q(s)$, we obtain Heaviside's formula. The reader familiar with singularities in complex analysis will recognize the limit formula just proved as the residue of $p(s)/q(s)$ at the simple pole a_j.

3.4 Convolution

1. Let

$$F(s) = \frac{1}{s^2+4} \text{ and } G(s) = \frac{1}{s^2-4}.$$

Then

$$\mathcal{L}^{-1}[F(s)] = \frac{1}{2}\sin(2t) \text{ and } \mathcal{L}^{-1}[G(s)] = \frac{1}{2}\sinh(2t).$$

By the convolution theorem,

$$\begin{aligned}
&\mathcal{L}^{-1}\left[\left(\frac{1}{s^2+4}\right)\left(\frac{1}{s^2-4}\right)\right] \\
&= \frac{1}{4}\sin(2t) * \sinh(2t) \\
&= \frac{1}{4}\int_0^t \sin(2(t-\tau))\sinh(2\tau)\,d\tau \\
&= \frac{1}{16}\left[\sin(2(t-\tau))\cosh(2\tau) + \cos(2(t-\tau))\sinh(2\tau)\right]_0^t \\
&= \frac{1}{16}[\sinh(2t) - \sin(2t)].
\end{aligned}$$

2. Choose

$$F(s) = \frac{s}{s^2+16} \text{ and } G(s) = \frac{e^{-2s}}{s}.$$

Then

$$\begin{aligned}
\mathcal{L}^{-1}[F(s)G(s)] &= \cos(4t) * H(t-2) \\
&= \int_0^t \cos(4(t-\tau))H(\tau-2)\,d\tau \\
&= \begin{cases} 0 & \text{if } t < 2, \\ \int_2^t \cos(4(t-\tau)\,d\tau & \text{if } t \geq 2. \end{cases}
\end{aligned}$$

The last integration gives us

$$\mathcal{L}^{-1}\left[\frac{e^{-2s}}{s^2+16}\right] = \frac{1}{4}\sin(4(t-2))H(t-2).$$

3. There are two cases. First suppose that $a^2 \neq b^2$. Then

$$\begin{aligned}
\mathcal{L}^{-1}\left[\frac{s}{(s^2+a^2)}\frac{1}{(s^2+b^2)}\right] &= \cos(at) * \frac{\sin(bt)}{b} \\
&= \frac{1}{b}\int_0^t \cos(a(t-\tau)\sin(b\tau)\,d\tau \\
&= \frac{1}{2b}\int_0^t [\sin((b-a)\tau+at)+\sin((b+a)\tau-at)]\,d\tau \\
&= \frac{1}{2b}\left[-\frac{\cos((b-a)\tau+at)}{b-a}-\frac{\cos((b+a)\tau-at)}{b+a}\right]_0^t \\
&= \frac{1}{2b}\left[-\frac{\cos(bt)}{b-a}-\frac{cos(bt)}{b+a}+\frac{\cos(at)}{b-a}+\frac{\cos(at)}{b+a}\right] \\
&= \frac{\cos(at)-\cos(bt)}{(b-a)(b+a)}.
\end{aligned}$$

If $b^2 = a^2$,

$$\begin{aligned}
\mathcal{L}^{-1}\left[\frac{s}{(s^2+a^2)}\frac{1}{(s^2+a^2)}\right] &= \cos(at) * \frac{\sin(at)}{a} \\
&= \frac{1}{a}\int_0^t \cos(a(t-\tau))\sin(a\tau)\,d\tau \\
&= \frac{1}{2a}\int_0^t (\sin(at)+\sin(2a\tau-at))\,d\tau \\
&= \frac{1}{2a}\left[\tau\sin(at)-\frac{\cos(a(2\tau-t))}{2a}\right]_0^t \\
&= \frac{t\sin(at)}{2a}.
\end{aligned}$$

4. First write

$$\frac{s}{(s-3)(s^2+5)} = \frac{1}{s-3}\left(\frac{s}{s^2+5}\right).$$

Then

$$\begin{aligned}
\mathcal{L}^{-1}\left[\frac{s}{(s-3)(s^2+5)}\right] &= e^{3t} * \cos(\sqrt{5}t) \\
&= \int_0^t \cos(\sqrt{5}\tau)e^{3(t-\tau)}\,d\tau \\
&= e^{3t}\int_0^t \cos(\sqrt{5}\tau)e^{-3\tau}\,d\tau \\
&= e^{3t}\left[\frac{e^{-3\tau}}{14}-3\cos(\sqrt{5}\tau)+\sqrt{5}\sin(\sqrt{5}\tau)\right]_0^t \\
&= \frac{3}{14}e^{3t}-\frac{3}{14}\cos(\sqrt{5}t)+\frac{\sqrt{5}}{14}\sin(\sqrt{5}t).
\end{aligned}$$

5. First observe that

$$\mathcal{L}^{-1}\left[\frac{1}{s(s^2+a^2)}\right] = \frac{1-\cos(at)}{a^2} \text{ and } \mathcal{L}^{1}\left[\frac{1}{s^2+a^2}\right] = \frac{\sin(at)}{a}.$$

Then

$$\begin{aligned}
\mathcal{L}^{-1}\left[\frac{1}{s(s^2+a^2)^2}\right] &= \frac{1}{a^2}[1-\cos(at)] * \sin(at) \\
&= \frac{1}{a^3}\int_0^t [1-\cos(a(t-\tau))]\sin(a\tau)\,d\tau \\
&= \frac{1}{a^3}\left[-\frac{\cos(a\tau)}{a} - \frac{\tau\sin(at)}{2} + \frac{\cos(2a\tau-at)}{4a}\right]_0^t \\
&= \frac{1}{a^4}[1-\cos(at)] - \frac{t}{2a^3}\sin(at).
\end{aligned}$$

6.

$$\begin{aligned}
\mathcal{L}^{-1}\left[\frac{1}{s^4}\frac{1}{(s-5)}\right] &= e^{5t} * \frac{t^3}{6} \\
&= \frac{1}{6}\int_0^t e^{5(t-\tau)}\tau^3\,d\tau = \frac{1}{6}e^{5t}\int_0^t \tau^3 e^{-5\tau}\,d\tau \\
&= \frac{1}{625}e^{5t} - \frac{1}{30}t^3 - \frac{1}{50}t^2 - \frac{1}{125}t - \frac{1}{625}.
\end{aligned}$$

7.

$$\begin{aligned}
\mathcal{L}^{-1}\left[\frac{1}{(s+2)}\frac{e^{-4s}}{s}\right] &= e^{-2t} * H(t-4) \\
&= \int_4^t e^{-2(t-\tau)} = \frac{1}{2}(1-e^{-2(t-4)})
\end{aligned}$$

if $t > 4$, and zero if $t \leq 4$. Therefore

$$\mathcal{L}^{-1}\left[\frac{1}{(s+2)}\frac{e^{-4s}}{s}\right] = \frac{1}{2}(1-e^{-2(t-4)})H(t-4).$$

8.

$$\begin{aligned}
\mathcal{L}^{-1}\left[\frac{2}{s^2}\frac{1}{(s^2+5)}\right] &= t^2 * \frac{\sin(\sqrt{5}t)}{\sqrt{5}} \\
&= \frac{1}{\sqrt{5}}\int_0^t \tau^2\sin(\sqrt{5}(t-\tau))\,d\tau \\
&= \frac{1}{5}t - \frac{2}{25} + \frac{2}{25}\cos(\sqrt{5}t).
\end{aligned}$$

9. Take the transform of the initial value problem to obtain

$$Y(s) = \frac{F(s)}{s^2 - 5s + 6} = \left[\frac{1}{s-3} - \frac{1}{s-2}\right] F(s).$$

By the convolution theorem,

$$y(t) = e^{3t} * f(t) - e^{2t} * f(t).$$

10. Taking the transform of the initial value problem, we obtain

$$\begin{aligned} Y(s) &= \frac{F(s)}{(s+6)(s+4)} + \frac{s}{(s+6)(s+4)} \\ &= \frac{1}{2}F(s)\left[\frac{1}{s+4} - \frac{1}{s+6}\right] + \left[\frac{3}{s+6} - \frac{2}{s+4}\right]. \end{aligned}$$

Then

$$y(t) = \frac{1}{2}e^{-4t} * f(t) - \frac{1}{2}e^{-6t} * f(t) + 3e^{-6t} - 2e^{-4t}.$$

In Problems 11 - 16, we give the solution of the initial value problem without the details of taking the transform of the differential equation.

11. We obtain

$$y(t) = \frac{1}{4}e^{6t} * f(t) - \frac{1}{4}e^{2t} * f(t) + 2e^{6t} - 5e^{2t}.$$

12.

$$y(t) = \frac{1}{6}e^{5t} * f(t) - \frac{1}{6}e^{-t} * f(t) + \frac{1}{2}e^{5t} + \frac{3}{2}e^{-t}$$

13.

$$y(t) = \frac{1}{3}\sin(3t) * f(t) - \cos(3t) + \frac{1}{3}\sin(3t)$$

14.

$$y(t) = \frac{1}{k}\sinh(kt) * f(t) - 2\cosh(kt) - \frac{4}{k}\sinh(kt)$$

15.

$$y(t) = \frac{1}{4}e^{2t} * f(t) + \frac{1}{12}e^{-2t} * f(t) - \frac{1}{3}e^{t} * f(t) - \frac{1}{4}e^{2t} - \frac{1}{12}e^{-2t} + \frac{4}{3}e^{t}$$

16.

$$y(t) = \frac{1}{42}e^{3t} * f(t) - \frac{1}{42}e^{-3t} * f(t) - \frac{\sqrt{2}}{28}e^{\sqrt{2}t} * f(t) - \frac{\sqrt{2}}{28}e^{-\sqrt{2}t} * f(t)$$

17. The integral equation can be expressed as

$$f(t) = -1 + f(t) * e^{-3t}.$$

Take the transform of this to obtain

$$F(s) = -\frac{1}{s} + \frac{F(s)}{s+3}.$$

Then

$$F(s) = -\frac{s+3}{s(s+2)} = \frac{1}{2(s+2)} - \frac{3}{2s}.$$

Inverting this leads to the solution

$$f(t) = \frac{1}{2}e^{-2t} - \frac{3}{2}.$$

18. The equation is $f(t) = -t + f(t) * \sin(t)$. Take the transform to obtain

$$F(s) = -\frac{(s^2+1)}{s^4} = -\frac{1}{s^2} - \frac{1}{s^4}.$$

Then

$$f(t) = -t - \frac{1}{6}t^3.$$

19. The equation is $f(t) = e^{-t} + f(t) * 1$. Transform this and solve for $F(s)$ to obtain

$$F(s) = \frac{s}{(s+1)(s-1)} = \frac{1}{2}\left[\frac{1}{s+1} + \frac{1}{s-1}\right].$$

Now invert to obtain

$$f(t) = \frac{1}{2}e^{-t} + \frac{1}{2}e^{t} = \cosh(t).$$

20. Write the equation as $f(t) = -1 + t - 2f(t) * \sin(t)$. Take the transform and solve for $F(s)$ to obtain

$$\begin{aligned} F(s) &= \frac{(1-s)(s^2+1)}{s^2(s^2+3)} \\ &= \frac{1}{3s^2} - \frac{1}{3s} - \frac{2}{3}\left(\frac{s}{s^2+3}\right) + \frac{2}{3}\left(\frac{1}{s^2+3}\right). \end{aligned}$$

Invert this to obtain

$$f(t) = \frac{1}{3}t - \frac{1}{3} - \frac{2}{3}\cos(\sqrt{3}t) + \frac{2\sqrt{3}}{9}\sin(\sqrt{3}t).$$

21. The equation is $f(t) = 3 + f(t) * \cos(2t)$. From this we obtain

$$F(s) = \frac{3(s^2+4)}{s(s^2-s+4)} = \frac{3}{s} + \frac{3}{s^2-s+4}.$$

Invert this to obtain

$$f(t) = 3 + \frac{2\sqrt{15}}{5} e^{t/2} \sin\left(\frac{\sqrt{15}}{2} t\right).$$

22. The equation can be written

$$f(t) = \cos(t) + \int_0^t f(\tau) e^{-2(t-\tau)}\, d\tau = \cos(t) + f(t) * e^{-2t}.$$

Transform this and solve the resulting equation for $F(s)$ to obtain

$$\begin{aligned} F(s) &= \frac{s(s+2)}{(s+1)(s^2+1)} \\ &= -\frac{1}{2(s+1)} + \frac{3}{2}\left(\frac{s}{s^2+1}\right) + \frac{1}{2}\left(\frac{1}{s^2+1}\right). \end{aligned}$$

Invert this to obtain

$$f(t) = -\frac{1}{2} e^{-2t} + \frac{3}{2}\cos(t) + \frac{1}{2}\sin(t).$$

23. We want $r(t)$ if $f(t) = A =$ constant and $m(t) = e^{-kt}$. Begin with

$$R(s) = \frac{F(s) - f(0)M(s)}{sM(s)}.$$

For this problem,

$$F(s) = \frac{A}{s} \text{ and } M(s) = \frac{1}{s+k}.$$

Then

$$R(s) = \frac{\frac{A}{s} - \frac{A}{s+k}}{\frac{s}{s+k}} = \frac{Ak}{s^2}.$$

Therefore $r(t) = Akt$. This function has a straight line graph, shown in Figure 3.3 for $A = 3, k = 1/5$.

24. We have $f(t) = A + Bt$ and $m(t) = e^{-kt}$. Then

$$F(s) = \frac{A}{s} + \frac{B}{s^2} \text{ and } M(t) = \frac{1}{s+k}.$$

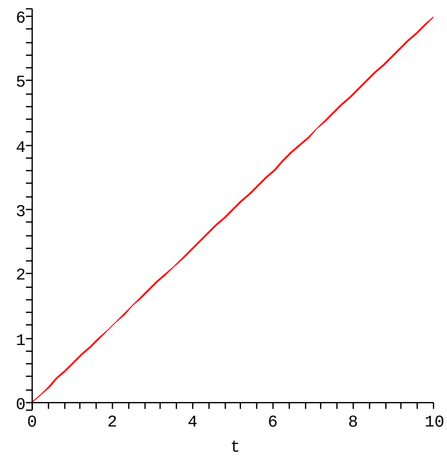

Figure 3.3: Replacement function in Problem 23, Section 3.4.

Then

$$\begin{aligned} R(s) &= \frac{\frac{A}{s}+\frac{B}{s^2}-\frac{A}{s+k}}{\frac{s}{s+k}} \\ &= \frac{Ak+B}{s^2+\frac{Bk}{s^3}}. \end{aligned}$$

Then

$$r(t) = (Ak+B)t + \frac{1}{2}Bkt^2.$$

This is graphed in Figure 3.4 for $A = 2, B = 1, k = 1/5$.

25. Now $f(t) = A + Bt + Ct^2$ and $m(t) = e^{-kt}$, so

$$F(s) = \frac{A}{s} + \frac{B}{s^2} + \frac{2C}{s^3} \text{ and } M(s) = \frac{1}{s+k}.$$

Then, by a routine algebraic calculation,

$$\begin{aligned} R(s) &= \frac{\frac{A}{s}+\frac{B}{s^2}+\frac{2C}{s^3}-A\left(\frac{1}{s+k}\right)}{\frac{s}{s+k}} \\ &= \frac{Ak+B}{s^2}+\frac{2C+Bk}{s^3}+\frac{2Ck}{s^4}. \end{aligned}$$

Then

$$r(t) = (Ak+B)t + \left(\frac{1}{2}Bk + C\right)t^2 + \frac{1}{3}Ckt^3.$$

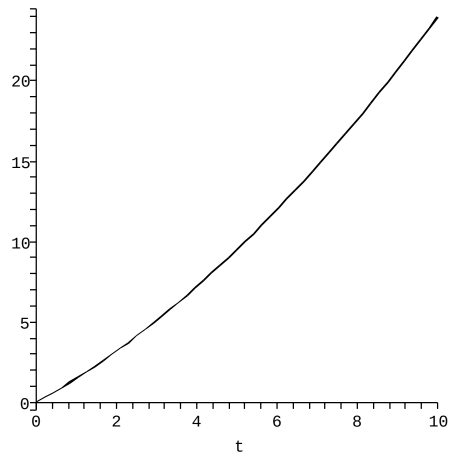

Figure 3.4: Replacement function in 24, Section 3.4.

Figure 3.5 shows a graph of this replacement function for $A = 1$, $B = 4$, $k = 1/5$ and $C = 2$.

26. Begin by writing

$$F(s)G(s) = F(s)\int_0^\infty e^{-st}g(t)\,dt = \int_0^\infty F(s)e^{-s\tau}g(\tau)\,d\tau.$$

Now recall that

$$e^{-s\tau}F(s) = \mathcal{L}[H(t-\tau)f(t-\tau)](s).$$

Substitute this into the integral for $F(s)G(s)$:

$$F(s)G(s) = \int_0^\infty \mathcal{L}[H(t-\tau)f(t-\tau)](s)g(\tau)\,d\tau.$$

From the definition of the Laplace transform,

$$\mathcal{L}[H(t-\tau)f(t-\tau)](s) = \int_0^\infty e^{-st}H(t-\tau)f(t-\tau),dt.$$

Substitute this into the previous equation to obtain

$$\begin{aligned} F(s)G(s) &= \int_0^\infty \left[\int_0^\infty e^{-st}H(t-\tau)f(t-\tau)\,dt\right]g(\tau)\,d\tau \\ &= \int_0^\infty \int_0^\infty e^{-st}g(\tau)H(t-\tau)f(t-\tau)\,dt\,d\tau. \end{aligned}$$

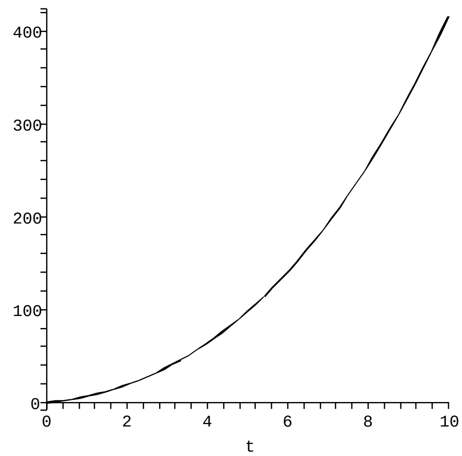

Figure 3.5: Replacement function in Problem 25, Section 3.4.

Since $H(t-\tau) = 0$ if $t < \tau$ and $H(t-\tau) = 1$ if $t \geq \tau$, then the last equation becomes

$$F(s)G(s) = \int_0^\infty \int_\tau^\infty e^{-st} g(\tau) f(t-\tau)\, dt\, d\tau.$$

This integral is over the triangular wedge in the t, τ - plane defined by $0 \leq \tau \leq t < \infty$. Reverse the order of integration to write

$$\begin{aligned} F(s)G(s) &= \int_0^\infty \int_0^t e^{-st} g(\tau) f(t-\tau)\, d\tau\, dt \\ &= \int_0^\infty \left[\int_0^t g(\tau) f(t-\tau)\, d\tau \right] dt \\ &= \int_0^\infty e^{-st} (f * g)(t)\, dt \\ &= \mathcal{L}[f * g](s). \end{aligned}$$

3.5 Impulses and the Dirac Delta Function

In Problem 1 we include details of the solution. These are similar in Problems 2 - 5, and only the solutions are given for these problems.

1. Transform the initial value problem to obtain

$$(s^2 + 5s + 6)Y(s) = 3e^{-2s} - 4e^{-5s}.$$

Then

$$Y(s) = 3\left[\frac{1}{s+2} - \frac{1}{s+3}\right]e^{-2s} - 4\left[\frac{1}{s+2} - \frac{1}{s+3}\right]e^{-5s}.$$

Invert this to obtain

$$y(t) = 3[e^{-2(t-2)} - e^{-3(t-2)}]H(t-2) - 4[e^{-2(t-5)} - e^{-3(t-5)}]H(t-5).$$

2.
$$y(t) = \frac{4}{3}e^{2(t-3)}\sin(3(t-3))H(t-3)$$

3.
$$y(t) = 6(e^{-2t} - e^{-t} + te^{-t})$$

4.
$$y(t) = 3\cos(4t) + 3\sin(4(t-5\pi/8))H(t-5\pi/8)$$

5.
$$\varphi(t) = (B+9)e^{-2t} - (B+6)e^{-3t}; \varphi(0) = 3, \varphi'(0) = B$$

The Dirac delta function $\delta(t-t_0)$ applied at time t_0 imparts a unit velocity to the unit mass.

6. The motion is modeled by the initial value problem

$$my'' + ky = 0; y(0) = 0, y'(0) = v_0.$$

Taking the Laplace transform of this problem, we obtain

$$Y(s) = \frac{mv_0}{ms^2 + k},$$

and inverting this yields

$$y(t) = v_0\sqrt{\frac{m}{k}}\sin\left(\sqrt{\frac{k}{m}}t\right).$$

The initial momentum is mv_0.

7. The motion is modeled by the problem

$$my'' + ky = mv_0\delta(t); y(0) = y'(0) = 0.$$

We find that

$$Y(s) = \frac{mv_0}{ms^2 + k},$$

so

$$y(t) = v_0\sqrt{\frac{m}{k}}\sin\left(\sqrt{\frac{k}{m}}t\right).$$

8. $F(x) = kx$ gives us $k = 2(8/3)(12) = 9$ pounds per foot, and $m = 2/32 = 1/16$ slugs. The motion is modeled by

$$\frac{1}{16}y'' + 9y = \frac{1}{4}\delta(t); y(0) = y'(0) = 0.$$

Transform to obtain

$$Y(s) = \frac{4}{s^2 + 144},$$

so

$$y(t) = \frac{1}{3}\sin(12t).$$

The initial velocity is $y'(0) = 4$ feet per second. The frequency is $6/\pi$ hertz and the amplitude is $1/3$ feet, or 4 inches.

9. Begin by writing, for $\epsilon > 0$,

$$\int_0^\infty f(t)\delta_\epsilon(t-a)\,dt = \int_0^\infty \frac{1}{\epsilon}[H(t-a) - H(t-a-\epsilon)]f(t)\,dt$$
$$= \frac{1}{\epsilon}\int_a^{a+\epsilon} f(t)\,dt.$$

By the mean value theorem for integrals, there is some t_ϵ between a and $a+\epsilon$ such that

$$\int_a^{a+\epsilon} f(t)\,dt = \epsilon f(t_\epsilon).$$

Then

$$\int_0^\infty f(t)\delta_\epsilon(t-a)\,dt = f(t_\epsilon).$$

As $\epsilon \to 0+$, $a + \epsilon \to a$, so $t_\epsilon \to a$ and, by continuity, $f(t_\epsilon) \to f(a)$. Then

$$\lim_{\epsilon\to 0+}\int_0^\infty f(t)\delta_\epsilon(t-a)\,dt = \int_0^\infty f(t)\lim_{\epsilon\to 0+}\delta_\epsilon(t-a)\,dt$$
$$= \int_0^\infty f(t)\delta(t-a)\,dt$$
$$= \lim_{\epsilon\to 0+} f(t_\epsilon) = f(a).$$

3.6 Solution of Systems

1. Take the Laplace transform of the system:

$$sX - 2sY = \frac{1}{s},$$
$$sX - X + Y = 0.$$

Solve these for X and Y:

$$X(s) = \frac{1}{s^2(2s-1)} = -\frac{1}{s^2} - \frac{2}{s} + \frac{4}{2s-1},$$
$$Y(s) = \frac{1-s}{s^2(2s-1)} = -\frac{1}{s^2} - \frac{1}{s} + \frac{2}{2s-1}.$$

Apply the inverse transform to obtain the solution

$$x(t) = -t - 2 + 2e^{t/2},$$
$$y(t) = -t - 1 + e^{t/2}.$$

2. Take the transform of the system to obtain

$$2sX + (2s-3)Y = 0,$$
$$sX + sY = \frac{1}{s^2}.$$

Solve for X and Y:

$$X(s) = \frac{-2s+3}{3s^2} = -\frac{2}{3}\frac{1}{s^2} + \frac{1}{s^2},$$
$$Y(s) = \frac{2}{3}\frac{1}{s^2}.$$

Then

$$x(t) = -\frac{2}{3}t + \frac{1}{2}t^2 \text{ and } y(t) = \frac{2}{3}t.$$

3. After taking the transform of the system, we have

$$sX + (2s-1)Y = \frac{1}{s} \text{ and } sX + Y = 0.$$

Then

$$X(s) = \frac{-1}{s^2(4s-3)} = \frac{1}{3}\frac{1}{s^2} + \frac{4}{9}\frac{1}{s} - \frac{16}{9}\frac{1}{4s-3},$$
$$Y(s) = \frac{2}{s(4s-3)} = -\frac{2}{3}\frac{1}{s} + \frac{8}{3}\frac{1}{4s-3}.$$

The solution is

$$x(t) = \frac{1}{3}t + \frac{4}{9} - \frac{4}{9}e^{3t/4} \text{ and } y(t) = -\frac{2}{3} + \frac{2}{3}e^{3t/4}.$$

4. The transform of the system yields

$$(s-1)X + sY = \frac{s}{s^2+4} \text{ and } sX + 2sY = 0.$$

Then

$$X(s) = \frac{2s^2}{(s^2+4)(s^2-3s)} = \frac{1}{2}\frac{1}{s-1} - \frac{1}{2}\frac{s}{s^2+4} + \frac{1}{s^2+4},$$
$$Y(s) = \frac{-s^2}{(s^2+4)(s^2-3s)} = -\frac{1}{4}\frac{1}{s-1} + \frac{1}{4}\frac{s}{s^2+4} - \frac{1}{2}\frac{1}{s^2+4}.$$

The solution is

$$x(t) = \frac{1}{2}e^t - \frac{1}{2}\cos(2t) + \frac{1}{2}\sin(2t),$$
$$y(t) = -\frac{1}{4}e^t + \frac{1}{4}\cos(2t) - \frac{1}{4}\sin(2t).$$

5. Take the Laplace transform:

$$3sX - Y = \frac{2}{s^2} \text{ and } sX + (s-1)Y = 0.$$

Then

$$X(s) = \frac{2(s-1)}{s^3(3s-2)} = \frac{1}{s^3} + \frac{1}{2}\frac{1}{s^2} + \frac{3}{4}\frac{1}{s} - \frac{9}{4}\frac{1}{3s-2},$$
$$Y(s) = \frac{-2}{s^2(3s-2)} = \frac{1}{s^2} + \frac{3}{2}\frac{1}{s} - \frac{9}{2}\frac{1}{3s-2}.$$

The solution is

$$x(t) = \frac{1}{2}t^2 + \frac{1}{2}t + \frac{3}{4} - \frac{3}{4}e^{2t/3},$$
$$y(t) = t + \frac{3}{2} - \frac{3}{2}e^{2t/3}.$$

6. The transform of the system yields

$$sX + (4s-1)Y = 0, sX + 2Y = \frac{1}{s+1}.$$

Then

$$X(s) = \frac{-4s+1}{(s+1)(4s^2-3s)} = \frac{5}{7}\frac{1}{s+1} - \frac{1}{3}\frac{1}{s} - \frac{32}{21}\frac{1}{4s-3},$$
$$Y(s) = \frac{s}{(s+1)(4s^2-3s)} = -\frac{1}{7}\frac{1}{s+1} + \frac{4}{7}\frac{1}{4s-3}.$$

The system's solution is

$$x(t) = \frac{5}{7}e^{-t} - \frac{1}{3} - \frac{8}{21}e^{3t/4},$$
$$y(t) = -\frac{1}{7}e^{-t} + \frac{1}{7}e^{3t/4}.$$

7. From the system, obtain

$$(s+2)X - sY = 0 \text{ and } (s+1)X + Y = \frac{2}{s^3}.$$

Then

$$X(s) = \frac{2}{s^2(s^2+2s+2)} = \frac{1}{s^2} - \frac{1}{s} + \frac{s+1}{(s+1)^2+1},$$
$$Y(s) = \frac{2(s+2)}{s^3(s^2+2s+2)} = \frac{2}{s^3} - \frac{1}{s^2} + \frac{1}{(s+1)^2+1}.$$

Invert these to obtain

$$x(t) = t - 1 + e^{-t}\cos(t) \text{ and } y(t) = t^2 - t + e^{-t}\sin(t).$$

8. The system yields

$$(s+4)X - Y = 0 \text{ and } sX + sY = \frac{1}{s^2}.$$

Then

$$X(s) = \frac{1}{s^2(s^2+5s)} = \frac{1}{125}\frac{1}{s} - \frac{1}{25}\frac{1}{s^2} + \frac{1}{5}\frac{1}{s^3} - \frac{1}{125}\frac{1}{s+5},$$
$$Y(s) = \frac{s+4}{s^2(s^2+5s)} = -\frac{1}{125}\frac{1}{s} + \frac{1}{25}\frac{1}{s^2} + \frac{4}{5}\frac{1}{s^3} + \frac{1}{125}\frac{1}{s+5}.$$

Invert these to obtain the solution

$$x(t) = \frac{1}{125} - \frac{1}{25}t + \frac{1}{10}t^2 - \frac{1}{125}e^{-5t},$$
$$y(t) = -\frac{1}{125} + \frac{1}{25}t + \frac{1}{5}t^2 + \frac{1}{125}e^{-5t}.$$

9. The transform of the system yields

$$(s+1)X + (s-1)Y = 0 \text{ and } (s+1)X + 2sY = \frac{1}{s}.$$

Then

$$X(s) = \frac{1-s}{s(s+1)^2} = \frac{1}{s} - \frac{1}{s+1} - \frac{2}{(s+1)^2},$$
$$Y(s) = \frac{1}{s(s+1)} = \frac{1}{s} - \frac{1}{s+1}.$$

The solution is

$$x(t) = 1 - e^{-t} - 2te^{-t} \text{ and } y(t) = 1 - e^{-t}.$$

10. The transform of the system gives us

$$(s-1)X + 2sY = 0 \text{ and } 4sX + (3s+1)Y = -\frac{6}{s}.$$

Solve for X and Y to get

$$\begin{aligned}
X(s) &= \frac{-12}{5s^2+2s} = -\frac{6}{s} + \frac{30}{5s+2}, \\
Y(s) &= \frac{6(s-1)}{s(5s^2+2s)} = -\frac{21}{2}\frac{1}{s} + \frac{3}{s^2} + \frac{105}{2}\frac{1}{5s+2}.
\end{aligned}$$

Apply the inverse transform to these equations to obtain the solution

$$x(t) = -6 + 6e^{-2t/5}, y(t) = -\frac{21}{2} + 3t + \frac{21}{2}e^{-2t/5}.$$

11. The transform of the system yields

$$\begin{aligned}
sY_1 - 2sY_2 + 3Y_3 &= 0, \\
Y_1 - 4sY_2 + 3Y_3 &= \frac{1}{s^2}, \\
Y_1 - 2sY_2 + 3sY_3 &= -\frac{1}{s}.
\end{aligned}$$

Then

$$\begin{aligned}
Y_1(s) &= \frac{1+s-s^2}{s^2(s^2-1)} = -\frac{1}{s^2} - \frac{1}{s} + \frac{1}{2}\frac{1}{s-1} + \frac{1}{2}\frac{1}{s+1}, \\
Y_2(s) &= -\frac{s+1}{2s^3} = -\frac{1}{2}\frac{1}{s^2} - \frac{1}{2}\frac{1}{s^3}, \\
Y_3(s) &= \frac{-2s^2+1}{3s^2(s^2-1)} = -\frac{1}{3}\frac{1}{s^2} - \frac{1}{6}\frac{1}{s-1} + \frac{1}{6}\frac{1}{s+1}.
\end{aligned}$$

Invert these to obtain the solution

$$\begin{aligned}
y_1(t) &= -t - 1 + \frac{1}{2}e^t + \frac{1}{2}e^{-t}, \\
y_2(t) &= -\frac{1}{2}t - \frac{1}{4}t^2, \\
y_3(t) &= -\frac{1}{3}t - \frac{1}{6}e^t + \frac{1}{6}e^{-t}.
\end{aligned}$$

12. The loop currents satisfy the 2×2 system

$$\begin{aligned}
2i_1 + 5(i_1 - i_2)' + 3i_1 &= E(t) = 2H(t-4) - H(t-5), \\
i_2 + 4i_2 + 5(i_2 - i_1)' &= 0.
\end{aligned}$$

Simplify these and take the Laplace transform to obtain

$$\begin{aligned} 5(s+1)I_1 - 5sI_2 &= \frac{2}{s}e^{-4s} - \frac{1}{s}e^{-5s}, \\ -5sI_1 + 5(s+1)I2 &= 0. \end{aligned}$$

Solve for I_1 and I_2 to get

$$\begin{aligned} I_1(s) &= \frac{2}{5}\left[\frac{1}{s} - \frac{2}{2s+1}\right]e^{-4s} - \frac{1}{5}\left[\frac{1}{s} - \frac{2}{2s+1}\right]e^{-5s}, \\ I_2(s) &= -\frac{2}{5(2s+1)}e^{-4s} + \frac{1}{5(2s+1)}e^{-5s}. \end{aligned}$$

Apply the inverse transform to solve for the loop currents:

$$\begin{aligned} i_1(t) &= \frac{2}{5}(1 - e^{-(t-4)})H(t-4) - \frac{1}{5}(1 - e^{-(t-5)})H(t-5), \\ i_2(t) &= -\frac{2}{5}e^{-(t-4)}H(t-4) + \frac{1}{5}e^{-(t-5)}H(t-5). \end{aligned}$$

13. The loop currents satisfy

$$\begin{aligned} 5i_1' + 5i_1 - 5i_2' &= 1 - H(t-4)\sin(2(t-4)), \\ -5i_1' + 5i_2' + 5i_2 &= 0. \end{aligned}$$

Simplify these equations and solve take the Laplace transform to obtain

$$\begin{aligned} I_1(s) &= \frac{s+1}{5(2s+1)}\left[\frac{1}{s} - \frac{2e^{-4s}}{s^2+4}\right] \\ &= \frac{1}{5}\left[\frac{1}{s} - \frac{1}{2s+1}\right] - \frac{2}{85}\left[\frac{2}{2s+1} - \frac{s}{s^2+4} + \frac{9}{s^2+4}\right]e^{-4s}, \\ I_2(s) &= \frac{1}{5(2s+1)}\left[1 - \frac{2se^{-4s}}{s^2+4}\right] \\ &= \frac{1}{5(2s+1)} + \frac{2}{85}\left[\frac{2}{2s+1} - \frac{s}{s^2+4} - \frac{8}{s^2+4}\right]e^{-4s}. \end{aligned}$$

Apply the inverse transform to obtain the currents:

$$\begin{aligned} i_1(t) &= \frac{1}{5}\left[1 - \frac{1}{2}e^{-t/2}\right] \\ &\quad - \frac{2}{85}\left[e^{-(t-4)/2} - \cos(2(t-4)) + \frac{9}{2}\sin(2(t-4))\right]H(t-4), \\ i_2(t) &= \frac{1}{10}e^{-t/2} \\ &\quad + \frac{2}{85}\left[e^{-(t-4)/2} - \cos(2(t-4)) - 4\sin(2(t-4))\right]H(t-4). \end{aligned}$$

14. Let x_1 and x_2 be the downward displacements of the masses m_1 and m_2, respectively. By Newton's second law of motion and from the equilibrium of the system, the equations of motion are

$$m_1 x_1'' = -k_1 x_1 + k_2(x_2 - x_1) + f_1(t),$$
$$m_2 x_2'' = -k_3 x_2 + k_2(x_1 - x_2) + f_2(t).$$

Let $k_1 = 6, k_2 = 2,\ k_3 = 3, m_1 = m_1 = 1,\ f_1(t) = 2$ and $f_2(t) = 0$. Simplify the resulting system and take apply the Laplace transform to obtain

$$(s^2+8)X_1 - 2X_2 = \frac{2}{s},$$
$$-2X_1 + (s^2+5)X_2 = 0.$$

The initial positions and velocities are zero. Then

$$X_1(s) = \frac{2(s^2+5)}{s(x^4+13s^2+36)} = \frac{5}{18s} - \frac{1}{10}\frac{s}{s^2+4} - \frac{8}{45}\frac{s}{s^2+9},$$
$$X_2(s) = \frac{s-4}{s(s^2+13s^2+36)} = -\frac{1}{9s} + \frac{1}{5}\frac{s}{s^2+4} - \frac{4}{45}\frac{s}{s^2+9}.$$

From the inverse Laplace transform we obtain the solution for the displacement functions

$$x_1(t) = \frac{5}{18} - \frac{1}{10}\cos(2t) - \frac{8}{45}\cos(3t),$$
$$x_2(t) = \frac{1}{9} - \frac{1}{5}\cos(2t) + \frac{4}{45}\cos(3t).$$

15. As in the solution to Problem 14, write the equations of motion

$$x_1'' + 8x_1 - 2x_2 = 1 - H(t-2),$$
$$x_2'' - 2x_1 + 5x_2 = 0.$$

Initial positions and velocities are zero. Transforming these yields

$$(s^2+8)X_1 - 2X_2 = \frac{1}{s}(1-e^{-2s}),$$
$$-2X_1 + (s^2+5)X_2 = 0.$$

Then

$$X_1(s) = \frac{s^2+5}{s(s^4+13s^2+36)}(1-e^{-2s}),$$
$$X_2(s) = \frac{2}{s(s^4+13s^2+36)}(1-e^{-2s}).$$

Invert these to obtain the solution

$$\begin{aligned}
x_1(t) &= \frac{5}{36} - \frac{1}{20}\cos(2t) - \frac{4}{45}\cos(3t) \\
&\quad - \left[\frac{5}{36} - \frac{1}{20}\cos(2(t-2)) - \frac{4}{45}\cos(3(t-2))\right] H(t-2), \\
x_2(t) &= \frac{1}{18} - \frac{1}{10}\cos(2t) + \frac{2}{45}\cos(3t) \\
&\quad - \left[\frac{1}{18} - \frac{1}{10}\cos(2(t-2)) + \frac{2}{45}\cos(3(t-2))\right] H(t-2).
\end{aligned}$$

16. (a) The equations of motion are

$$\begin{aligned}
&my_1'' + k_1y_1 - k_2(y_2 - y_1) + c_1y_1' = A\sin(\omega t), \\
&my_2'' - k_2(y_1 - y_2) = 0,
\end{aligned}$$

with initial conditions

$$y_1(0) = y_1'(0) = y_2(0) = y_2'(0) = 0.$$

Transform the system and solve for $Y_1(s)$ and $Y_2(s)$ to obtain

$$\begin{aligned}
Y_1(s) &= \frac{ms^2 + k_2}{k_2} Y_2(s), \\
Y_2(s) &= \frac{A\omega k_2}{(s^2+\omega^2)(Mms^4 + mc_1s^3 + (mk_1 + mk_2 + Mk_2)s^2 + k_2c_1s + k_1k_2)}.
\end{aligned}$$

(b) If $\omega = \sqrt{k_2/m}$ then

$$\begin{aligned}
Y_1(s) &= \frac{s^2+\omega^2}{\omega^2} Y_2(s) \\
&= \frac{A\omega}{Ms^4 + c_1s^3 + (k_1 + k_2 + M\omega^2)s^2 + \omega^2c_1s + k_1\omega^2}.
\end{aligned}$$

The absence of the factor $s^2 + \omega^2$ in the denominator indicates that the forced vibrations of frequency ω have been absorbed.

17. The equations of motion are

$$\begin{aligned}
my_1'' &= k(y_2 - y_1), \\
m_2y_2'' &= k(y_1 - y_2),
\end{aligned}$$

with initial conditions

$$y_1(0) = y_1'(0) = y_2'(0) = 0, y_2(0) = d.$$

Apply the transform to the system and solve for $Y_1(s)$ and $Y_2(s)$ to obtain

$$Y_1(s) = \frac{kd}{m_1 s\left(s^2 + \frac{m_1+m_2}{(m_1 m_2)}k\right)},$$
$$Y_2(s) = \frac{d(m_1 s^2 + k)}{m_1 s\left(s^2 + \frac{m_1+m_2}{(m_1 m_2)}k\right)}.$$

The quadratic factor in the denominator shows that the motion has frequency

$$\omega = \sqrt{\left(\frac{m_1+m_2}{m_1 m_2}\right)k},$$

and therefore period

$$2\pi\sqrt{\frac{m_1 m_2}{(m_1+m_2)k}}.$$

18. The equations for the loop currents can be written

$$20i_1' + 10(i_1 - i_2) = E(t) = 5H(t-5),$$
$$30i_2' + 10i_2 + 10(i_2 - i_1) = 0,$$

with initial conditions $i_1(0) = i_2(0) = 0$. Transform the system and solve for $I_1(s)$ and $I_2(s)$ to obtain

$$\begin{aligned} I_1(s) &= \frac{5(30s+20)e^{-5s}}{s(600s^2+700s+100)} \\ &= \left[\frac{1}{s} - \frac{1}{10(s+1)} - \frac{27}{5}\frac{1}{6s+1}\right]e^{-5s}, \\ I_2(s) &= \frac{50e^{-5s}}{s(600s^2+700s+100)} \\ &= \left[\frac{1}{2s} + \frac{1}{10(s+1)} - \frac{18}{5}\frac{1}{6s+1}\right]e^{-5s}. \end{aligned}$$

Invert these to obtain the solution for the currents:

$$i_1(t) = \left[1 - \frac{1}{10}e^{-(t-5)} - \frac{9}{10}e^{-(t-5)/6}\right]H(t-5),$$
$$i_2(t) = \left[\frac{1}{2} + \frac{1}{10}e^{-(t-5)} - \frac{3}{10}e^{-(t-5)/6}\right]H(t-5).$$

19. As in the solution of Problem 18, except with $E(t) = 5\delta(t-1)$, the trans-

formed equations yield

$$I_1(s) = \frac{5(30s+20)e^{-s}}{600s^2+700s+100} = \left[\frac{1}{10(s+1)} + \frac{9}{10(6s+1)}\right]e^{-s},$$

$$I_2(s) = \frac{50e^{-s}}{600s^2+700s+100} = \left[-\frac{1}{10(s+1)} + \frac{3}{5(6s+1)}\right]e^{-s}.$$

The currents are

$$i_1(t) = \left[\frac{1}{10}e^{-(t-1)} + \frac{3}{20}e^{-(t-1)/6}\right]H(t-1),$$

$$i_2(t) = \left[-\frac{1}{10}e^{-(t-1)} + \frac{1}{10}e^{-(t-1)/6}\right]H(t-1).$$

20. Let $x_1(t)$ and $x_2(t)$ be the amounts of salt (in pounds) in tanks 1 and 2 respectively, at time t. Now

$$x_1'(t) = \text{ rate of change of salt in tank 1}$$
$$= \text{ (rate salt is added) - (rate salt is removed)}$$

and similarly for $x_2'(t)$. Then x_1 and x_2 satisfy the system

$$x_1'(t) = \frac{1}{3} + \frac{3}{18}x_2 - \frac{5}{60}x_1 \text{ and}$$
$$x_2'(t) = \frac{5}{60}x_1 - \frac{5}{18}x_2 + 11(H(t-4) - H(t-6)),$$

with initial conditions

$$x_1(0) = 11, x_2(0) = 7.$$

Transform these differential equations to obtain

$$(12s+1)X_1 - 2X_2 = \frac{4}{s} + 132,$$
$$-3X_1 + (36s+10)X_2 = \frac{396}{s}(e^{-4s} - e^{-6s}) + 252.$$

Then

$$X_1(s) = \frac{4752s^2+1968s+40+792(e^{-4s}-e^{-6s})}{s(432s^2+156s+4)}$$
$$= \frac{10}{s} - \frac{6}{3s+1} + \frac{108}{36s+1} + 2\left[\frac{99}{s} + \frac{27}{3s+1} - \frac{3888}{36s+1}\right](e^{-4s} - e^{-6s}),$$

$$X_2(s) = \frac{3024s^2+648s+12+396(12s+1)(e^{-4s}-e^{-6s})}{s(432s^2+156s+4)}$$
$$= \frac{3}{s} + \frac{9}{3s+1} + \frac{36}{36s+1} + \left[\frac{99}{s} - \frac{81}{3s+1} - \frac{2592}{36s+1}\right](e^{-4s} - e^{-6s}).$$

Apply the inverse Laplace transform to write the solution:

$$\begin{aligned}x_1(t) &= 10 - 2e^{-t/3} + 3e^{-t/36} + 2(99 + 9e^{-(t-4)/3} - 108e^{-(t-4)/36})H(t-4)\\ &\quad - 2(99 + 9e^{-(t-6)/3} - 108e^{-(t-6)/36})H(t-6),\\ x_2(t) &= 3 + 3e^{-t/3} + e^{-t/36} + (99 - 27e^{-(t-4)/3} - 72e^{-(t-4)/36})H(t-4)\\ &\quad - (99 - 27e^{-(t-6)/3} - 72e^{-(t-6)/36})H(t-6).\end{aligned}$$

21. Using the notation of the solution of Problem 20, we can write the system

$$\begin{aligned}x_1' &= -\frac{6}{200}x_1 + \frac{3}{100}x_2,\\ x_2' &= \frac{4}{200}x_1 - \frac{4}{200}x_2 + 5\delta(t-3),\end{aligned}$$

with initial conditions

$$x_1(0) = 10, x_2(0) = 5.$$

Simplify these equations and apply the Laplace transform to obtain

$$\begin{aligned}(100s+3)X_1 - 3X_2 &= 1000,\\ -2X_1 + (100s+4)X_2 &= 500 + 500e^{-3s}.\end{aligned}$$

Then

$$\begin{aligned}X_1(s) &= \frac{100000s + 5500 + 1500e^{-3s}}{10000s^2 + 700s + 6}\\ &= \frac{50}{50s+3} + \frac{900}{100s+1} + \left[\frac{300}{100s+1} - \frac{150}{50s+3}\right]e^{-3s},\\ X_2(s) &= \frac{50000s + 3500 + (50000s + 1500)e^{-3s}}{10000s^2 + 700s + 6}\\ &= -\frac{50}{50s+3} + \frac{600}{100s+1} + \left[\frac{150}{50s+3} + \frac{200}{100s+1}\right]e^{-3s}.\end{aligned}$$

Invert these equations to obtain the solution

$$\begin{aligned}x_1(t) &= e^{-3t/50} + 9e^{-t/100} + 3(e^{-(t-3)/100} - e^{-3(t-3)/50})H(t-3),\\ x_2(t) &= -e^{-3t/50} + 6e^{-t/100} + (3e^{-3(t-3)/50} + 2e^{-(t-3)/100})H(t-3).\end{aligned}$$

3.7 Polynomial Coefficients

1. Before transforming the equation, make the change of variable $u = 1/t$. Let $z(u) = y(t(u)) = y(1/u)$. Then

$$\frac{dy}{dt} = \frac{dz}{du}\frac{du}{dt} = -\frac{1}{t^2}\frac{dz}{du},$$

and $t^2(dx/dt) - 2y = 2$ transforms to

$$-\frac{dz}{du} - 2z = 2.$$

Apply the transform to this differential equation to obtain

$$-sZ + z(0) - 2Z = \frac{2}{s}.$$

Then

$$Z(s) = -\frac{2}{s(s+2)} + \frac{z(0)}{s+2} = \frac{1+z(0)}{s+2} - \frac{1}{s}.$$

Invert this equation to obtain

$$z(u) = ce^{-2u} - 1$$

or

$$y(t) = -1 + ce^{-2/t}.$$

This problem can also be solved as a first order linear differential equation, after dividing it by t^2.

2. Transform the initial value problem to obtain

$$s^2Y - sy(0) - y'(0) - 4\frac{d}{ds}(sY - y(0)) - 4Y = 0.$$

Upon taking the derivative and inserting the initial values, we obtain

$$4sY' + (8 - s^2)Y = 7,$$

with the solution

$$Y(s) = -\frac{7}{s^2} + \frac{c}{s^2}e^{s^2/8}.$$

c is the constant of integration. In order to have $\lim_{s\to\infty} Y(s) = 0$ we must choose $c = 0$. Therefore

$$Y(s) = -\frac{7}{s^2}$$

and

$$y(t) = -7t.$$

In Problems 3 through 9, the details of the solution are like those of Problems 1 and 2, and only the solution is given.

3. $y(t) = 7t^2$
4. $y(t) = -4t$
5. $y(t) = ct^2e^{-t}$

6. $y(t) = 3t^2$
7. $y(t) = 4$
8. $y(t) = 10t$
9. $y(t) = 3t^2/2$
10. Transform the differential equation to obtain

$$s^2Y - sy(0) - y'(0) + \frac{d}{ds}(s^2Y(s) - sy(0) - y'(0)) - \frac{d}{ds}(sY(s) - y(0)) - Y = 0.$$

Since $y(0) = 3$ and $y'(0) = -1$, this is

$$(s^2 - s)Y' + (s^2 + 2s - 2)Y = 3s + 2,$$

a first order linear differential equation for $Y(s)$. An integrating factor is $\mu = se^s$. Multiplying by this factor gives us

$$\frac{d}{ds}(e^s(s^3 - s^2)Y) = 3s^2e^s + 2se^s.$$

Integrate this equation to obtain

$$\begin{aligned} Y(s) &= \frac{3s^2 - 4s + 4}{s^2(s-1)} + K\frac{e^{-s}}{s^2(s-1)} \\ &= \frac{3}{s-1} - \frac{4}{s^2} + K\left(\frac{1}{s-1} - \frac{1}{s} - \frac{1}{s^2}\right)e^{-s}. \end{aligned}$$

Invert this equation to obtain the solution

$$y(t) = 3e^t - 4t + K(e^{t-1} - t)H(t-1).$$

K is arbitrary and can be given any real value. This illustrates a *bifurcation* in the solution. At $t = 1$, the solution splits off and travels along different curves, depending on the choice of K. Notice that the existence/uniqueness theorem for solutions of this differential equation does not apply at $t = 1$, which is a singular point.

11. When we wrote factorials and inverted terms of the form $1/s^{2n+k}$ in the binomial expansion used in the derivation, we assumed that n is a nonnegative integer.

Chapter 4

Series Solutions

4.1 Power Series Solutions

1. Put $y(x) = \sum_{n=0}^{\infty} a_n x^n$ into the differential equation to obtain

$$\begin{aligned} y' - xy &= \sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^{n+1} \\ &= a_1 + (2a_2 - a_0)x + \sum_{n=3}^{\infty} (n a_n - a_{n-2}) x^{n-1} \\ &= 1 - x. \end{aligned}$$

Then a_0 is arbitrary, $a_1 = 1$, $2a_2 - a_0 = -1$ and

$$a_n = \frac{a_{n-2}}{n} \text{ for } n = 3, 4, \cdots .$$

This is the recurrence relation. If we set $a_0 = c_0 + 1$, we obtain the coefficients

$$a_2 = \frac{c_0}{2}, a_4 = \frac{c_0}{2 \cdot 4}, a_6 = \frac{c_0}{2 \cdot 4 \cdot 6},$$

and so on, and $a_1 = 1$, $a_3 = 1/3$, $a_5 = 1/(3 \cdot 5)$, $a_7 = 1/(3 \cdot 5 \cdot 7)$, and so on. In general, we obtain

$$y(x) = 1 + \sum_{n=0}^{\infty} \frac{1}{1 \cdot 3 \cdot 5 \cdots (2n+1)} x^{2n+1} + c_0 \left(1 + \sum_{n=1}^{\infty} \frac{1}{2 \cdot 4 \cdot 6 \cdots 2n} x^{2n} \right).$$

2. Write

$$\begin{aligned} y' - x^3 y &= \sum_{n=1}^{\infty} n x_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^{n+3} \\ &= a_1 + 2a_2 x + 3a_3 x^2 + \sum_{n=4}^{\infty} (n a_n - a_{n-4}) x^{n-1} = 4. \end{aligned}$$

The recurrence relation is

$$a_n = \frac{1}{n} a_{n-4} \text{ for } n \geq 4,$$

with a_0 arbitrary, $a_1 = 4$, and $a_2 = a_3 = 0$. This yields the solution

$$y = 4 \sum_{n=0}^{\infty} \frac{1}{1 \cdot 5 \cdot 9 \cdots (4n+1)} x^{4n+1} + a_0 \left(1 + \sum_{n=1}^{\infty} \frac{1}{4 \cdot 8 \cdot 12 \cdots 4n} x^{4n} \right).$$

3. Write

$$\begin{aligned} y' + (1 - x^2)y &= \sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n - \sum_{n=0}^{\infty} a_n x^{n+2} \\ &= (a_1 + a_0) + (2a_2 + a_1)x + \sum_{n=3}^{\infty} (n a_n + a_{n-1} - a_{n-3}) x^{n-1} \\ &= x. \end{aligned}$$

The recurrence relation is

$$n a_n + a_{n-1} - a_{n-3} = 0 \text{ for } n \geq 3$$

and we also have a_0 arbitrary, $a_1 + a_0 = 0$, and $2a_2 + a_1 = 1$. This yields the solution

$$\begin{aligned} y = a_0 &\left(1 - x + \frac{1}{2!} x^2 + \frac{1}{3!} x^3 - \frac{7}{4!} x^4 + \frac{19}{5!} x^5 + \cdots \right) \\ &+ \frac{1}{2!} x^2 - \frac{1}{3!} x^3 + \frac{1}{4!} x^4 + \frac{11}{5!} x^5 - \frac{31}{6!} x^6 + \cdots . \end{aligned}$$

4. Begin with

$$\begin{aligned} &y'' + 2y' + xy = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=1}^{\infty} 2n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^{n+1} \\ &= (2a_2 + 2a_1) + (3 \cdot 2a_3 + 2 \cdot 2a_2 + a_0)x \\ &+ \sum_{n=4}^{\infty} (n(n-1)a_n + 2(n-1)a_{n-1} + a_{n-3}) x^{n-2} = 0. \end{aligned}$$

The recurrence relation is

$$n(n-1)a_n + 2(n-1)a_{n-1} + a_{n-3} = 0 \text{ for } n \geq 4$$

along with a_0 and a_1 arbitrary, $a_2 = -a_1$, and $6a_3 + 4a_2 + a_0 = 0$. Taking $a_0 = 1$ and $a_1 = 0$ gives us one solution

$$y_1(x) = 1 - \frac{1}{6} x^3 + \frac{1}{12} x^4 - \frac{1}{30} x^5 + \frac{1}{60} x^6 + \cdots ,$$

and taking $a_0 = 0$ and $a_1 = 1$ yields a second, linearly independent solution

$$y_2(x) = x - x^2 + \frac{2}{3}x^3 - \frac{5}{12}x^4 + \frac{7}{60}x^5 + \cdots .$$

The general solution has the form $y(x) = a_0 y_1(x) + a_1 y_2(x)$, where $a_0 = y(0)$ and $a_1 = y'(0)$ are initial conditions (arbitrary constants).

5. Write

$$\begin{aligned} y'' - xy' + y &= \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} a_n x^n \\ &= (2a_2 + a_0) + \sum_{n=3}^{\infty} (n(n-1)a_n - (n-3)a_{n-2})x^{n-2} = 3. \end{aligned}$$

Then a_0 and a_1 are arbitrary, $a_2 = -(a_0 - 3)/2$, and, as the recurrence relation,

$$a_n = \frac{(n-3)}{n(n-1)} a_{n-2} \text{ for } n = 3, 4, \cdots .$$

This yields the general solution

$$y(x) = 3 + a_1 x + (a_0 - 3)\left(1 + \sum_{n=1}^{\infty} \frac{(-1)(1)(3)\cdot(2n-3)}{(2n)!} x^{2n}\right).$$

Here $a_1 = y'(0)$ and $a_0 = y(0)$.

6. Write

$$\begin{aligned} y'' + xy' + xy &= \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} a_n x^{n+1} \\ &= 2a_2 + \sum_{n=3}^{\infty} (n(n-1)a_n + (n-2)a_{n-2} + a_{n-3})x^{n-2} = 0. \end{aligned}$$

Then a_0 and a_1 are arbitrary, $a_2 = 0$ and, for the recurrence relation,

$$a_n = \frac{-(n-2)a_{n-2} - a_{n-3}}{n(n-1)} \text{ for } n = 3, 4, \cdots .$$

Taking $a_0 = 1$ and $a_1 = 0$ we obtain one solution

$$\begin{aligned} y_1(x) = 1 &- \frac{2}{3}x^3 + \frac{3}{2\cdot 3\cdot 4\cdot 5}x^5 \\ &+ \frac{1}{2\cdot 3\cdot 5\cdot 6}x^6 - \frac{3\cdot 5}{2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7}x^7 + \cdots , \end{aligned}$$

and, taking $a_0 = 0$ and $a_1 = 1$, we obtain a second, linearly independent solution

$$\begin{aligned} y_2(x) = x &- \frac{1}{2\cdot 3}x^3 - \frac{1}{3\cdot 4}x^4 \\ &+ \frac{3}{2\cdot 3\cdot 4\cdot 5}x^5 + \frac{3\cdot 5}{2\cdot 3\cdot 5\cdot 6}x^6 + \cdots . \end{aligned}$$

7. Write

$$y'' - x^2y' + 2y = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}$$
$$- \sum_{n=1}^{\infty} na_n x^{n+1} + \sum_{n=0}^{\infty} 2a_n x^n = (2a_2 + 2a_0) + (6a_3 + 2a_1)x$$
$$+ \sum_{n=4}^{\infty}(n(n-1)a_n - (n-3)a_{n-3} + 2a_{n-2})x^{n-2} = x.$$

Then a_0 and a_1 are arbitrary, $a_2 = -a_0$, $6a_3 + 2a_1 = 1$, and, for the recurrence relation,

$$a_n = \frac{(n-3)a_{n-3} - 2a_{n-2}}{n(n-1)} \text{ for } n = 4, 5, \cdots .$$

The general solution has the form

$$y(x) = a_0\left[1 - x^2 + \frac{1}{6}x^4 - \frac{1}{10}x^5 - \frac{1}{90}x^6 + \cdots\right]$$
$$+ a_1\left[x - \frac{1}{3}x^3 + \frac{1}{12}x^4 + \frac{1}{30}x^5 - \frac{7}{180}x^6 + \cdots\right]$$
$$+ \frac{1}{6}x^3 - \frac{1}{6}x^5 + \frac{1}{60}x^6 + \frac{1}{1260}x^7 - \frac{1}{480}x^8 + \cdots .$$

Here $a_0 = y(0)$ and $a_1 = y'(0)$. The third series in the solution represents a particular solution obtained from the recurrence by putting $a_0 = a_1 = 0$.

8. Write

$$y' + xy = \sum_{n=1}^{\infty} na_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^{n+1}$$
$$= a_1 + \sum_{n=0}^{\infty}(2na_{2n} + a_{2n-2})x^{2n-1} + \sum_{n=1}^{\infty}((2n+1)a_{2n+1} + a_{2n-1})x^{2n}$$
$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}.$$

Then a_0 is arbitrary, $a_1 = 1$, and

$$a_{2n} = -\frac{1}{2n}a_{2n-2} \text{ and } a_{2n+1} = \frac{-a_{2n-1} + (-1)^n/((2n)!)}{2n+1}$$

for $n = 1, 2, \cdots$. The solution is

$$y(x) = a_0\left[1 - \frac{1}{2}x^2 + \frac{1}{2\cdot 4}x^4 - \frac{1}{2\cdot 4\cdot 6}x^6 + \cdots\right]$$
$$+ \left[x - \frac{3}{3!}x^3 + \frac{13}{5!}x^5 - \frac{79}{7!}x^7 + \frac{633}{9!}x^9 - \cdots\right].$$

9. We have

$$y'' + (1-x)y' + 2y = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}$$

$$+ \sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{n=1}^{\infty} n a_n x^n + 2\sum_{n=0}^{\infty} 2a_n x^n$$

$$= (2a_2 + a_1 + 2a_0) + \sum_{n=3}^{\infty} (n(n-1)a_n + (n-1)a_{n-1} - (n-4)a_{n-2})x^{n-2}$$

$$= 1 - x^2,$$

Then a_0 and a_1 are arbitrary, $2a_2 + a_1 + 2a_0 = 1$, $6a_3 + 2a_2 + a_1 = 0$, $12a_4 + 3a_3 = -1$, and

$$a_n = \frac{-(n-1)a_{n-1} + (n-4)a_{n-2}}{n(n-1)}$$

for $n = 5, 6, \cdots$. The general solution is

$$y(x) = a_0 \left[1 - x^2 + \frac{1}{3}x^3 - \frac{1}{12}x^4 + \frac{1}{30}x^5 - \cdots\right]$$

$$+ a_1 \left(x - \frac{1}{2}x^2\right) + \frac{1}{2}x^2 - \frac{1}{6}x^3 - \frac{1}{24}x^4 - \frac{1}{360}x^6 + \frac{1}{2520}x^7 + \cdots,$$

where $a_0 = y(0)$ and $a_1 = y'(0)$. The last series is a particular solution of the nonhomogeneous equation.

10. We have

$$y'' + xy' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=1}^{\infty} n a_n x^n$$

$$= 2a_2 + \sum_{n=3}^{\infty} (n(n-1)a_n + (n-2)a_{n-2})x^{n-2}$$

$$= -\sum_{n=3}^{\infty} \frac{1}{(n-2)!} x^{n-2}.$$

Then a_0 and a_1 are arbitrary, $a_2 = 0$, and

$$a_n = \frac{-(n-2)a_{n-2} - 1/(n-2)!}{n(n-1)}$$

for $n = 3, 4, \cdots$. The solution is

$$y(x) = a_0 + a_1 \left[x - \frac{1}{3!}x^3 + \frac{3}{5!}x^5 - \frac{15}{7!}x^7 + \frac{105}{9!}x^9 + \cdots\right]$$

$$+ \left[-\frac{1}{3!}x^3 - \frac{1}{4!}x^4 + \frac{2}{5!}x^5 + \frac{3}{6!}x^6 - \frac{11}{7!}x^7 + \frac{19}{8!}x^8 + \cdots\right].$$

Here $a_0 = y(0)$ and $a_1 = y'(0)$.

4.2 Frobenius Solutions

1. Substitute $y(x) = \sum_{n=0}^{\infty} c_n x^{n+r}$ into the differential equation to obtain

$$\begin{aligned}
xy'' + (1-x)y' + y &= \sum_{n=0}^{\infty}(n+r)(n+r-1)c_n x^{n+r-1} \\
&+ \sum_{n=0}^{\infty}(n+r)c_n x^{n+r-1} - \sum_{n=0}^{\infty}(n+r)c_n x^{n+r} + \sum_{n=0}^{\infty} c_n x^{n+r} \\
&= r^2 c_0 x^{r-1} + \sum_{n=1}^{\infty}((n+r)^2 c_n - (n+r-2)c_{n-1})x^{n+r-1} \\
&= 0.
\end{aligned}$$

Since c_0 is assumed to be nonzero, then r must satisfy the indicial equation $r^2 = 0$, with equal roots $r_1 = r_2 = 0$. One solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} c_n x^n$$

while a second solution has the form

$$y_2(x) = y_1(x)\ln(x) + \sum_{n=1}^{\infty} c_n^* x^n.$$

For the first solution, choose the coefficients to satisfy $c_0 = 1$ and

$$c_n = \frac{n-2}{n^2} c_{n-1} \text{ for } n = 1, 2, \cdots.$$

This yields the solution $y_1(x) = 1 - x$. Therefore

$$y_2(x) = (1-x)\ln(x) + \sum_{n=1}^{\infty} c_n^* x^n.$$

Substitute this into the differential equation to obtain

$$\begin{aligned}
&x\left[-\frac{2}{x} - \frac{1-x}{x^2}\right] + (1-x)\left[-\ln(x) + \frac{1-x}{x}\right] \\
&+ (1-x)\ln(x) + \sum_{n=2}^{\infty} n(n-1)c_n^* x^{n-1} + (1-x)\sum_{n=1}^{\infty} c_n^* x^{n-1} + \sum_{n=1}^{\infty} c_n^* x^n \\
&= (-3 + c_1^*) + (1 + 4c_2^*)x + \sum_{n=3}^{\infty}(n^2 c_n^* - (n-2)c_{n-1}^*)x^{n-2} = 0.
\end{aligned}$$

The coefficients c_n^* are determined by $c_1^* = 3$, $c_2^* = -1/4$, and

$$c_n^* = \frac{n-2}{n^2} \text{ for } n \geq 3.$$

A second solution is

$$y_2(x) = (1-x)\ln(x) + 3x - \sum_{n=2}^{\infty} \frac{1}{n(n-1)n!} x^n.$$

For the remaining problems of this section we give the essential elements of the solution without all of the details of the calculations.

2. The indicial equation is $r(r-1) = 0$, with roots $r_1 = 1$ and $r_2 = 0$. There are solutions

$$y_1(x) = \sum_{n=0}^{\infty} c_n x^{n+1} \text{ and } y_2(x) = ky_1(x)\ln(x) + \sum_{n=0}^{\infty} c_n^* x^n.$$

For y_1 the recurrence relation is

$$c_n = \frac{2(n+r-2)}{(n+r)(n+r-1)} c_{n-1}$$

for $n = 1, 2, \cdots$. With $r = 1$ and $c_0 = 1$ this gives us $y_1(x) = x$, a solution that can be seen by inspection from the differential equation.

Now substitute $y_2(x)$ into the differential equation to obtain

$$(2c_0^* + k) + 2(c_2^* - k)x + \sum_{n=3}^{\infty} (n(n-1)c_n^* - 2(n-2)c_{n-1}^*)x^{n-1} = 0.$$

Take $c_0^* = 1$ to obtain $k = -2$. c_1^* is arbitrary (choose this to be zero), $c_2^* = -2$, and

$$c_n^* = \frac{2(n-2)}{n(n-1)} c_{n-1}^* \text{ for } n = 3, 4, \cdots.$$

This gives us a second solution

$$y_2(x) = -2x\ln(x) + 1 - \sum_{n=2}^{\infty} \frac{2^n}{n!(n-1)} x^n.$$

3. The indicial equation is $r^2 - 4r = 0$, with roots $r_1 = 4$ and $r_2 = 0$. There are solutions of the form

$$y_1(x) = \sum_{n=0}^{\infty} c_n x^{n+4} \text{ and } y_2(x) = ky_1(x)\ln(x) + \sum_{n=0}^{\infty} c_n^* x^n.$$

With $r = 4$ we obtain the recurrence relation

$$c_n = \frac{n+1}{n} c_{n-1}$$

and the first solution

$$\begin{aligned} y_1(x) &= x^4(1+2x+3x^2+4x^3+\cdots) \\ &= x^4 \frac{d}{dx}(1+x+x^2+x^3+x^4+\cdots) \\ &= x^4 \frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{x^4}{(1-x)^2}. \end{aligned}$$

A second solution is

$$y_2(x) = \frac{3-4x}{(1-x)^2}.$$

4. The indicial equation is $4r^2 - 9 = 0$, with roots $r_1 = 3/2$ and $r_2 = -3/2$. There are solutions

$$y_1(x) = \sum_{n=0}^{\infty} c_n x^{n+3/2} \text{ and } y_2(x) = k y_1(x)\ln(x) + \sum_{n=0}^{\infty} c_n^* x^{n-3/2}.$$

Upon substitution and solving for the coefficients, we obtain

$$y_1(x) = x^{3/2}\left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{2^n n!(5\cdot 7\cdot 9\cdots(2n+3))} x^{2n}\right]$$

and

$$y_2(x) = x^{-3/2}\left[1 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2^{n+1} n!(3)\cdots(2n-3)} x^{3n}\right].$$

5. The indicial equation is $4r^2 - 2r = 0$, with roots $r_1 = 1/2$ and $r_2 = 0$. There are solutions of the form

$$y_1(x) = \sum_{n=0}^{\infty} c_n x^{n+1/2}$$

and

$$y_2(x) = \sum_{n=0}^{\infty} c_n^* x^n.$$

Substitute these into the differential equation in turn to obtain

$$\begin{aligned} y_1(x) &= x^{1/2}\left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{2^n n!(3\cdot 5\cdot 7\cdots(2n+1))} x^n\right] \\ &= x^{1/2}\left[1 - \frac{1}{6}x + \frac{1}{120}x^2 - \frac{1}{5040}x^3 + \frac{1}{362880}x^4 + \cdots\right] \end{aligned}$$

and

$$\begin{aligned} y_2(x) &= 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{2^n n!(1\cdot 3\cdot 5\cdots(2n-1))} x^n \\ &= 1 - \frac{1}{2}x + \frac{1}{24}x^2 - \frac{1}{720}x^3 + \frac{1}{40320}x^4 - \cdots. \end{aligned}$$

6. The indicial equation is $4r^2 - 1 = 0$ with roots $r_1 = 1/2$ and $r_2 = -1/2$. There are solutions of the form

$$y_1(x) = \sum_{n=0}^{\infty} c_n x^{n+1/2} \text{ and } y_2(x) = ky_1(x)\ln(x) + \sum_{n=0}^{\infty} c_n^* x^{n-1/2}.$$

Upon substituting these into the differential equation, we obtain the simple solutions

$$y_1(x) = x^{1/2}, y_2(x) = x^{-1/2}.$$

We could also have observed that the differential equation in this problem is an Euler equation.

7. The indicial equation is $r^2 - 3r + 2 = 0$ with roots $r_1 = 2$ and $r_2 = 1$. There are solutions

$$y_1(x) = \sum_{n=0}^{\infty} c_n x^{n+2} \text{ and } y_2(x) = ky_1\ln(x) + \sum_{n=0}^{\infty} c_n^* x^{n+1}.$$

Substitute these in turn into the differential equation to obtain

$$y_1(x) = x^2 + \frac{1}{3!}x^4 + \frac{1}{5!}x^6 + \frac{1}{7!}x^8 + \cdots = x\sinh(x)$$

and

$$y_2(x) = x - x^2 + \frac{1}{2!}x^3 - \frac{1}{3!}x^4 + \frac{1}{4!}x^4 - \cdots = xe^{-x}.$$

8. The indicial equation is $r^2 - 2r = 0$, with roots $r_1 = 2$ and $r_2 = 0$. There are solutions

$$y_1(x) = \sum_{n=0}^{\infty} c_n x^{n+2} \text{ and } y_2(x) = ky_1(x)\ln(x) + \sum_{n=0}^{\infty} c_n^* x^n.$$

The recurrence relation for the $c_n\,'s$ is

$$c_n = \frac{-2c_{n-1}}{n(n-2)} \text{ for } n > 2.$$

With $c_0 = 1$, we obtain a first solution

$$y_1(x) = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{n+1}}{n!(n+2)!} x^{n+2} = x^2 - \frac{2}{3}x^3 + \frac{1}{6}x^4 - \frac{1}{45}x^5 + \frac{1}{540}x^6 - \cdots.$$

Substitute the second solution into the differential equation to obtain

$$2c_0^* - c_1^* + \sum_{n=2}^{\infty}\left[n(n-2)c_n^* + c_{n-1}^* + \frac{(-1)^n 2^n k}{n((n-2)!)^2}\right]x^{n-1} = 0.$$

with $c_0^* = 1$ for simplicity, we obtain $c_1^* = 2$, $k = -2$, c_2^* arbitrary (we take this coefficient to be zero), and

$$c_n^* = -\frac{1}{n(n-2)}\left[2c_{n-1}^* + \frac{(-1)^n 2^{n+1}}{n((n-2)!)^2}\right]$$

for $n = 3, 4, \cdots$. We obtain the second solution

$$y_2(x) = -2y_1\ln(x) + 1 + 2x + \frac{16}{9}x^3 - \frac{25}{36}x^4 + \frac{157}{1350}x^5 - \cdots.$$

9. The indicial equation is $2r^2 = 0$ with roots $r_1 = r_2 = 0$. There are solutions of the form

$$y_1(x) = \sum_{n=0}^{\infty} c_n x^n$$

and

$$y_2(x) = y_1(x)\ln(x) + \sum_{n=1}^{\infty} c_n^* x^n.$$

Upon substituting these in turn into the differential equation, we obtain the simple solutions

$$y_1(x) = 1 - x \text{ and } y_2(x) = (1-x)\ln\left(\frac{x}{x-2}\right) - 2.$$

10. The indicial equation is $r^2 - 1 = 0$, with roots $r_1 = 1$ and $r_2 = -1$. There are solutions of the form

$$y_1(x) = \sum_{n=0}^{\infty} c_n x^{n+1} \text{ and } y_2(x) = ky_1(x)\ln(x) + \sum_{n=0}^{\infty} c_n^* x^{n-1}.$$

Substitute each of these into the differential equation to obtain

$$y_1(x) = x\left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n(1\cdot 4\cdot 7\cdots(3n-2))}{3^n n!(5\cdot 8\cdot 11\cdots(3n+2))}x^{3n}\right]$$

and

$$y_2(x) = \frac{1}{x}\left[1 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(1\cdot 2\cdot 5\cdots(3n-4))}{3^n n!(4\cdot 7\cdots(3n-2))}x^{3n}\right].$$

Chapter 5

Approximation of Solutions

5.1 Direction Fields

1. The direction field is shown in Figure 5.1.

2. The direction field is given in Figure 5.2.

3. The direction field is in Figure 5.3.

4. Figure 5.4 is the direction field for this problem.

5. The direction field is in Figure 5.5.

6. The direction field is in Figure 5.6.

5.2 Euler's Method

In each of Problems 1 through 6, approximate solutions were computed by Euler's method with $h = 0.05$ and $n = 10$. In problems 1 through 5 the exact solution can be written, allowing comparisons between the approximate and exact solution values. In problem 6 the exact solution cannot be written using methods developed to this point.

1. The exact solution is
$$y = e^{1-\cos(x)}.$$
See Table 5.1 for the approximate values.

2. The exact solution is
$$y(x) = e^{x-1} - x - 1.$$
Values are listed in Table 5.2.

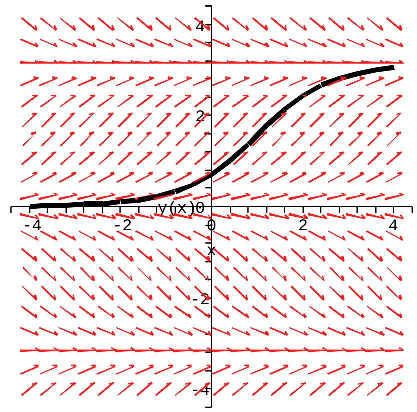

Figure 5.1: Problem 1, Section 5.1.

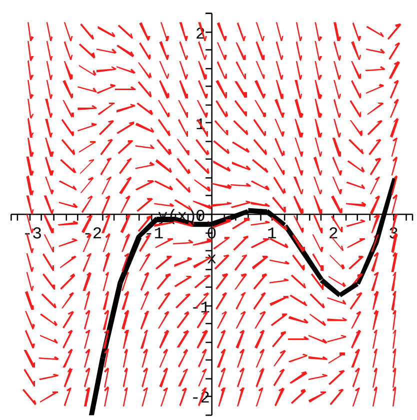

Figure 5.2: Problem 2, Section 5.1.

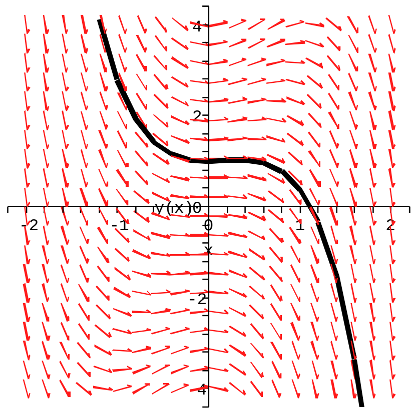

Figure 5.3: Problem 3, Section 5.1.

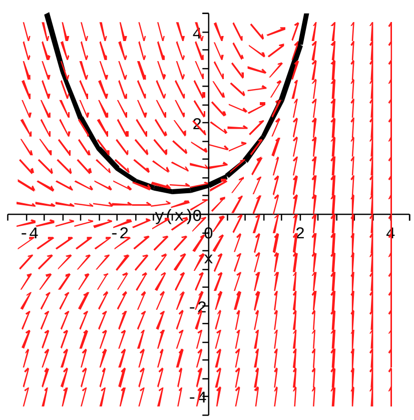

Figure 5.4: Problem 4, Section 5.1.

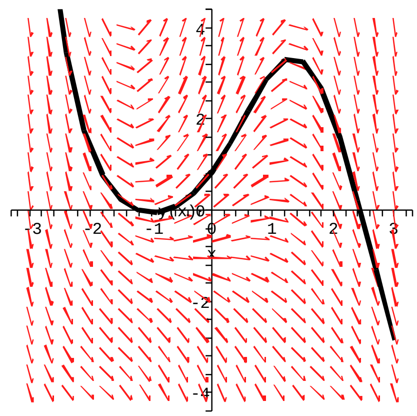

Figure 5.5: Problem 5, Section 5.1.

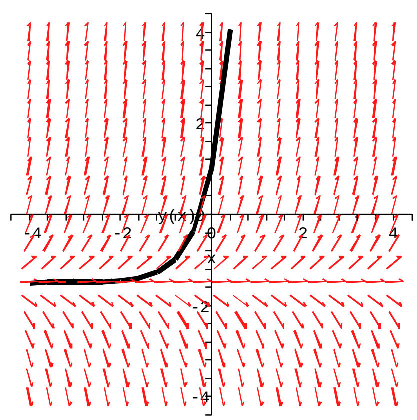

Figure 5.6: Problem 6, Section 5.1.

x_k	Euler approx. y_k	Exact $y(x_k)$
0.0	1	1
0.05	1	1.00125021
0.1	1.002498958	1.005008335
0.15	1.007503130	1.011292203
0.20	1.015031072	1.020133420
0.25	1.025113849	1.031575844
0.30	1.037794811	1.045675942
0.35	1.053129278	1.062502832
0.40	1.071185064	1.082138316
0.45	1.092042020	1.104676904
0.50	1.115792052	1.130225803

Table 5.1: Problem 1, Section 5.2.

x_k	Euler approx. y_k	Exact $y(x_k)$
1.0	-3	-3
1.05	-3.10	-3.101271096
1.1	-3.2025	-3.205170918
1.15	-3.307625	-3.311834243
1.20	-3.41550625	-3.421402758
1.25	-3.526281562	-3.534025417
1.30	-3.640095640	-3.649858808
1.35	-3.757100422	-3.769067549
1.40	-3.877445543	-3.891824698
1.45	-4.001328215	-3.891824698
1.50	-4.128894626	-4.148721271

Table 5.2: Problem 2, Section 5.2.

x_k	Euler approx. y_k	Exact $y(x_k)$
0.0	5	5
0.05	5	5.018785200
0.10	5.0375	5.075565325
0.15	5.1132605	5.171629965
0.20	5.228106406	5.309182735
0.25	5.384949598	5.491425700
0.30	5.586885208	5.722683920
0.35	5.838295042	6.008576785
0.40	6.141805532	6.356245750
0.45	6.513493864	6.774651405
0.50	6.953154700	7.274957075

Table 5.3: Problem 3, Section 5.3.

x_k	Euler approx. y_k	Exact $y(x_k)$
0.0	1	1
0.05	1.10	1.098750000
0.10	1.1975	1.195000000
0.15	1.2925	1.287750000
0.20	1.3850	1.380000000
0.25	1.4750	1.468750000
0.30	1.5625	1.555000000
0.35	1.6475	1.638750000
0.40	1.7300	1.720000000
0.45	1.8100	1.798750000
0.50	1.8875	1.875000000

Table 5.4: Problem 4, Section 5.2.

3. The exact solution is
$$y(x) = 5e^{3x^2/2}.$$
See Table 5.3 for computed values.

4. The exact solution is
$$y(x) = \frac{1}{2}(2 + 4x - x^2).$$
Values are listed in Table 5.4.

5. The exact solution is
$$y(x) = \left[\frac{\sin(1) - \cos(1)}{2} - 2\right] e^{x-1} + \frac{1}{2}(\cos(x) - \sin(x)).$$
See Table 5.5 for computed values.

x_k	Euler approx. y_k	Exact $y(x_k)$
1	-2	-2
1.05	-2.127015115	-2.129163317
1.10	-2.258244423	-2.262726022
1.15	-2.393836450	-2.400852694
1.20	-2.534057644	-2.543722054
1.25	-2.678878414	-2.691527844
1.30	-2.828588453	-2.844479698
1.35	-2.983392817	-3.002804084
1.40	-3.143512792	-3.166745253
1.45	-3.309186789	-3.336566226
1.50	-3.480671266	-3.512549830

Table 5.5: Problem 5, Section 5.2.

x_k	Euler approx. y_k
0.0	4
0.05	3.20
0.10	2.895
0.15	2.3324875
0.20	1.99078478
0.25	1.802625739
0.30	1.52652761
0.35	1.531089704
0.40	1.431377920
0.45	1.348435782
0.50	1.280454395

Table 5.6: Problem 6, Section 5.2.

6. We do not have the exact solution in closed form for this problem. Table 5.6 lists approximate solution values.

5.3 Taylor and Modified Euler Methods

In each of Problems 1 through 6, approximate solution values are computed using the Runge-Kutta method with $h = 0.2$ and $n = 10$.

1. Table 5.7 lists the approximate values for Problem 1.
2. Table 5.8 gives approximate values for Problem 2. Computed values taken from the exact solution, which we can obtain in this example, are also listed. This exact solution

$$y = -9e^{x-1} + x^2 + 2x + 2.$$

x_k	Runge-Kutta approximation
0.0	2
0.2	2.162573
0.4	2.27782433
0.6	2.34197299
0.8	2.35937518
1.0	2.33748836
1.2	2.28390814
1.4	2.20518645
1.6	2.10658823
1.8	1.99221666
2.0	1.86523474

Table 5.7: Problem 1, Section 5.3.

x_k	Runge-Kutta approximation	Exact
1.0	-4	-4
1.2	-5.15260667	-5.12562482
1.4	-6.66637645	-6.66642228
1.6	-8.63898286	-8.6386920
1.8	-11.1897243	-11.18986835
2.0	-14.464312	-14.46453645
2.2	-18.6407173	-18.64105231
2.4	-23.9363148	-23.93679970
2.6	-30.6166056	-30.61729182
2.8	-39.0058727	-39.00682718
2.0	-49.5001956	-49.50150489

Table 5.8: Problem 2, Section 5.3.

x_k	Runge-Kutta approximation
0.0	1
0.2	1.26465161
0.4	1.45389723
0.6	1.58483705
0.8	1.67216598
1.0	1.72743772
1.2	1.75944359
1.4	1.77479969
1.6	1.77846513
1.8	1.77414403
2.0	1.76458702

Table 5.9: Problem 3, Section 5.3.

x_k	Runge-Kutta approx.
3.0	2
3.2	0.927007472
3.4	0.281610758
3.6	0.0797232508
3.8	0.0218981447
4.0	5.92711033E-03
4.2	1.60552109E-03
4.4	4.42481887E-04
4.6	1.26188752E-04
4.8	3.78589406E-05
5.0	1.21355052E-05

Table 5.10: Problem 4, Section 5.3.

3. Table 5.9 gives approximate values for Problem 3.

4. Table 5.10 gives approximate values for Problem 4.

5. Table 5.11 lists approximate values for Problem 5, along with computed exact values, which can be obtained in this problem. The exact solution for this problem is
$$y(x) = (x+4)e^{-x}.$$

6. Table 5.12 lists approximate values for Problem 6, using RK4. An exact solution for comparison is not available for this problem.

x_k	Runge-Kutta approx.	Exact
0.0	4	4
0.2	3.34867474	3.43866916
0.4	2.94941776	2.9494082
0.6	2.52454578	2.52453353
0.8	2.15679297	2.15677903
1.0	1.83941205	1.83939721
1.2	1.56622506	1.5662099
1.4	1.33163683	1.33162361
1.6	1.13063507	1.1306205
1.8	0.958747437	0.958733552
2.0	0.812024757	0.812011699

Table 5.11: Problem 5, Section 5.3.

x_k	Runge-Kutta approx.
0.78	1
0.98	1.12897598
1.18	1.1538066
1.38	1.12922007
1.58	1.08698233
1.78	1.04366376
1.98	1.00569426
2.18	0.974171102
2.38	0.948175352
2.58	0.926448449
2.78	0.907969373
2.98	0.891968617
3.18	0.877955391
3.38	0.86554404
3.58	0.85445476
3.78	0.844473215
3.98	0.835431462
4.18	0.827195488
4.38	0.819656709
4.58	0.812725995
4.78	0.806329358

Table 5.12: Problem 6, Section 5.3.

Chapter 6

Vectors and Vector Spaces

6.1 Vectors in the Plane and 3-Space

1. $\mathbf{F}+\mathbf{G} = (2+\sqrt{2})\mathbf{i}+3\mathbf{j}$, $\mathbf{F}-\mathbf{G} = (2-\sqrt{2})\mathbf{i}-9\mathbf{j}+10\mathbf{k}$, $2\mathbf{F} = 4\mathbf{i}-6\mathbf{j}+10\mathbf{k}$, $3\mathbf{G} = 3\sqrt{2}\mathbf{i}+18\mathbf{j}-15\mathbf{k}$, $\| \mathbf{F} \| = \sqrt{38}$

2. $\mathbf{F}+\mathbf{G} = \mathbf{i}+4\mathbf{j}-3\mathbf{k}$, $\mathbf{F}-\mathbf{G} = \mathbf{i}-4\mathbf{j}-3\mathbf{k}$, $2\mathbf{F} = 2\mathbf{i}-6\mathbf{k}$, $3\mathbf{G} = 12\mathbf{j}$, $\| \mathbf{F} \| = \sqrt{10}$

3. $\mathbf{F}+\mathbf{G} = 3\mathbf{i}-\mathbf{k}$, $\mathbf{F}-\mathbf{G} = \mathbf{i}-10\mathbf{j}+\mathbf{k}$, $2\mathbf{F} = 4\mathbf{i}-10\mathbf{j}$, $3\mathbf{G} = 3\mathbf{i}+15\mathbf{j}-3\mathbf{k}$, $\| \mathbf{F} \| = \sqrt{29}$

4. $\mathbf{F}+\mathbf{G} = (\sqrt{2}+8)\mathbf{i}+\mathbf{j}-4\mathbf{k}$, $\mathbf{F}-\mathbf{G} = (\sqrt{2}-8)\mathbf{i}+\mathbf{j}-8\mathbf{k}$, $2\mathbf{F} = 2\sqrt{2}\mathbf{i}+2\mathbf{j}-12\mathbf{k}$, $3\mathbf{G} = 24\mathbf{i}+6\mathbf{k}$, $\| \mathbf{F} \| = \sqrt{41}$

5. $\mathbf{F}+\mathbf{G} = 3\mathbf{i}-\mathbf{j}+3\mathbf{k}$, $\mathbf{F}-\mathbf{G} = -\mathbf{i}-3\mathbf{j}-\mathbf{k}$, $2\mathbf{F} = 2\mathbf{i}+2\mathbf{j}+2\mathbf{k}$, $3\mathbf{G} = 6\mathbf{i}-6\mathbf{j}+6\mathbf{k}$, $\| \mathbf{F} \| = \sqrt{3}$

In each of Problems 6 through 9, we first use the given points to find a vector from the first point to the second. This vector may or may not be a unit vector, but at least it is in the right direction. Divide this vector by its length to obtain a unit vector in the direction from the first to the second point. If this vector is then multiplied by a positive scalar α, then we have a vector of length α in the direction from the first point to the second. We include these details for Problem 6 and give just the answer for Problems 7 - 9.

6. $-5\mathbf{i}+\mathbf{j}-2\mathbf{k}$ is a vector from the first point to the second. Divide this vector by its length to obtain a unit vector, then multiply by 5 to obtain a vector of length 5 in the direction from $(0,1,4)$ to $(-5,2,2)$:

$$\frac{5}{\sqrt{30}}(-5\mathbf{i}+\mathbf{j}-2\mathbf{k}).$$

7.
$$\frac{9}{\sqrt{45}}(-5\mathbf{i} - 4\mathbf{j} + 2\mathbf{k})$$

8.
$$\frac{12}{\sqrt{125}}(10\mathbf{i} - 3\mathbf{j} - 4\mathbf{k})$$

9.
$$\frac{4}{9}(-4\mathbf{i} + 7\mathbf{j} + 4\mathbf{k})$$

In each of Problems 10 through 15, we follow the procedure of the text to find parametric equations of a line through the given points. Details are provided for Problem 10 only. However, it must be understood that any line in three space can be described by infinitely many different parametric equations. For example, if in Example 6.1 we replace t by $2t$, we obtain slightly different looking parametric equations of the same line, since $2t$ takes on all real values as t does.

10. Let L be the line containing these points. A vector from $(1, 0, 4)$ to $(2, 1, 1)$ is $\mathbf{M} = \mathbf{i} + \mathbf{j} - 3\mathbf{k}$. A vector from $(1, 0, 4)$ to (x, y, z) on L is $(x - 1)\mathbf{i} + y\mathbf{j} + (z - 4)\mathbf{k}$. These two vectors are parallel, so for some scalar t,
$$(x - 1)\mathbf{i} + y\mathbf{j} + (z - 4)\mathbf{k} = t[\mathbf{i} + \mathbf{j} - 3\mathbf{k}].$$
Then
$$x - 1 = t, y = t, z = 4 - 3t.$$
Parametric equations of L are
$$x = 1 + t, y = t, z = 4 - 3t \text{ for } -\infty < t < \infty.$$

11. $x = 3 - 6t, y = t, z = 0$ for $-\infty < t < \infty$.

12. $x = 2, y = 1, z = 1 - 3t$ for $-\infty < t < \infty$. This line is parallel to the z - axis and passes through $(2, 1, 0)$.

13. $x = 0, y = 1 - t, z = 3 - 2t$ for $-\infty < t < \infty$.

14. $x = 1 - 3t, y = -2t, z = -4 + 9t$ for $-\infty < t < \infty$.

15. $x = 2 - 3t, y = -3 + 9t, z = 6 - 2t$ for $-\infty < t < \infty$.

6.2 The Dot Product

In 1 - 6, $\mathbf{F}$ is the first given vector, $\mathbf{G}$ the second, and θ is the angle between these vectors.

1. $\mathbf{F} \cdot \mathbf{G} = 2$ and
$$\cos(\theta) = \frac{\mathbf{F} \cdot \mathbf{G}}{\| \mathbf{F} \| \| \mathbf{G} \|} = \frac{2}{\sqrt{14}}.$$
The vectors are not orthogonal.

2. $\mathbf{F} \cdot \mathbf{G} = 8$, $\cos(\theta) = 8/\sqrt{82}$, not orthogonal

3. $\mathbf{F} \cdot \mathbf{G} = -23$, $\cos(\theta) = -23/\sqrt{29}\sqrt{41}$, not orthogonal

4. $\mathbf{F} \cdot \mathbf{G} = -63$, $\cos(\theta) = -63/\sqrt{75}\sqrt{74}$, not orthogonal

5. $\mathbf{F} \cdot \mathbf{G} = -18$, $\cos(\theta) = -9/10$, not orthogonal

6. $\mathbf{F} \cdot \mathbf{G} = 4$, $\cos(\theta) = 2/3$, not orthogonal

In Problems 7 - 12, if the given point is (x_0, y_0, z_0) and the normal vector is $\mathbf{N} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$, then the equation of the plane is

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0,$$

because (x, y, z) is on the plane if and only if the vector $(x - x_0)\mathbf{i} + (y - y_0)\mathbf{j} + (z - z_0)\mathbf{k}$ is orthogonal to $\mathbf{N}$. It is common practice to accumulate the constant term $ax_0 + by_0 + cz_0$ on the other side of the equation to write the plane in the form

$$ax + by + cz = k.$$

7. If (x, y, z) is in the plane, then $(x + 1)\mathbf{i} + (y - 1)\mathbf{j} + (z - 2)\mathbf{k}$ is orthogonal to $3\mathbf{i} - \mathbf{j} + 4\mathbf{k}$, so
$$3(x + 1) - (y - 1) + 4(z - 2) = 0.$$
This is one equation of the plane. We can write this equation as
$$3x - y + 4z = 4.$$

8. $x - 2y = -1$

9. $4x - 3y + 2z = 25$

10. $-3x + 2y = 1$

11. $7x + 6y - 5z = -26$

12. $4x + 3y + z = -6$

For each of Problems 13, 14 and 15, the projection of $\mathbf{v}$ onto $\mathbf{u}$ is calculated as

$$\frac{\mathbf{u} \cdot \mathbf{v}}{\| \mathbf{u} \|^2}\mathbf{u}.$$

13.
$$\text{proj}_{\mathbf{u}}\mathbf{v} = \frac{-9}{14}\mathbf{u}$$

14.
$$-\frac{11}{30}\mathbf{u}$$

15.
$$\frac{1}{62}\mathbf{u}$$

6.3 The Cross Product

1.
$$\mathbf{F} \times \mathbf{G} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & 6 & 1 \\ -1 & -2 & 1 \end{vmatrix} = 8\mathbf{i} + 2\mathbf{j} + 12\mathbf{k}.$$

$$\mathbf{G} \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & -2 & 1 \\ -3 & 6 & 1 \end{vmatrix} = -8\mathbf{i} - 2\mathbf{j} - 12\mathbf{k} = -\mathbf{F} \times \mathbf{G}.$$

2. $\mathbf{F} \times \mathbf{G} = \mathbf{i} + 12\mathbf{j} + 6\mathbf{k}$

3. $\mathbf{F} \times \mathbf{G} = -8\mathbf{i} - 12\mathbf{j} - 5\mathbf{k}$

4. $\mathbf{F} \times \mathbf{G} = 112\mathbf{k}$

In Problems 5 through 9, the three points are used to find two vectors in the plane that is wanted. Their cross product produces a normal vector to this plane, and then, knowing a point on the plane and a normal vector, we can find an equation of the plane, as in Section 6.1. This procedure produces $\mathbf{N} = \mathbf{O}$ exactly when the three points are collinear and do not define a unique plane. The details of this procedure are included only for Problem 5.

5. Form vectors $\mathbf{F} = 4\mathbf{i} - \mathbf{j} - 6\mathbf{k}$ and $\mathbf{G} = \mathbf{i} - \mathbf{k}$. Take the cross product to form a normal vector:

$$\mathbf{N} = \mathbf{F} \times \mathbf{G} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & -1 & -6 \\ 1 & 0 & -1 \end{vmatrix} = \mathbf{i} - 2\mathbf{j} + \mathbf{k}.$$

Of course, other normals could be used. The fact that $\mathbf{N} \neq \mathbf{O}$ means that the points are not collinear. The plane containing these points has equation
$$x + 1 - 2(y - 1) + z - 6 = 0$$
or
$$x - 2y + z = 3.$$

6. The points are not collinear and the plane containing them has equation $x + 2y + 6z = 12$.

7. The points are not collinear and the plane containing them has equation $2x - 11y + z = 0$.

8. The points are not collinear and the plane containing them has equation $5x + 16y - 2z = -4$.

9. The points are not collinear and the plane containing them has equation $29x + 37y - 12z = 30$.

For Problems 10, 11 and 12, recall that the vector $a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ is normal to the plane $ax + by + cz = d$. Any nonzero scalar multiple of this normal vector is also a normal vector.

10. $\mathbf{N} = 8\mathbf{i} - \mathbf{j} + \mathbf{k}$

11. $\mathbf{N} = \mathbf{i} - \mathbf{j} + 2\mathbf{k}$

12. $\mathbf{N} = \mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$

13. The area of a parallelogram in which two incident sides have an angle of θ between them is the product of the lengths of the sides times the cosine of θ. If the sides are along the vectors $\mathbf{F}$ and $\mathbf{G}$, drawn from a common point, then this area is
$$\| \mathbf{F} \| \| \mathbf{G} \| \cos(\theta),$$
and this is exactly $\| \mathbf{F} \times \mathbf{G} \|$.

14. The vector $\mathbf{N} = \mathbf{G} \times \mathbf{H}$ is normal to the base of the parallelopiped having sides along the vectors $\mathbf{F}$ and $\mathbf{G}$ (Figure 6.1). We know (Problem 13) that the area of this base is $\| \mathbf{N} \|$. Now
$$(\mathbf{G} \times \mathbf{H}) \cdot \mathbf{F} = \mathbf{N} \cdot \mathbf{F} = \| \mathbf{N} \| \| \mathbf{F} \| \cos(\theta)$$
is in magnitude the volume of the box having incident edges $\mathbf{F}$, $\mathbf{G}$, $\mathbf{H}$ as incident sides, because
$$\| \mathbf{F} \| \cos(\theta) = \pm \text{ altitude of the box.}$$
This altitude is denoted h in Figure 6.1.

6.4 The Vector Space R^n

1. If $\alpha(3\mathbf{i} + 2\mathbf{j}) + \beta(\mathbf{i} - \mathbf{j}) = \mathbf{0}$, then $3\alpha + \beta = 0$ and $2\alpha - \beta = 0$. Then $\alpha = \beta = 0$, so the given vectors are linearly independent.

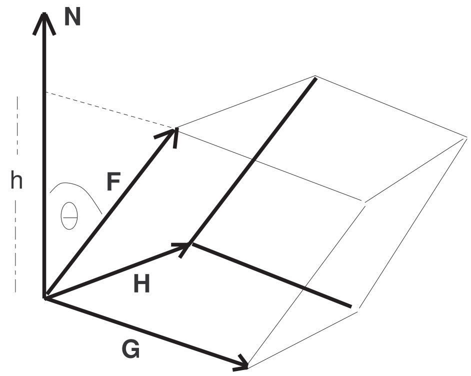

Figure 6.1: Parallelopiped in Problem 14, Section 6.3.

2. If
$$\alpha(2\mathbf{i}) + \beta(3\mathbf{j}) + \gamma(5\mathbf{i} - 12\mathbf{k}) + \delta(\mathbf{i} + \mathbf{j} + \mathbf{k}) = \mathbf{0}$$
then
$$\begin{aligned} 2\alpha + 5\gamma + \delta &= 0, \\ 3\beta + \delta &= 0, \\ -12\gamma + \delta &= 0. \end{aligned}$$
This system has nontrivial solution
$$\alpha = -17/2, \beta = -4, \gamma = 1, \delta = 12.$$
Therefore the given vectors are linearly dependent.

3. The vectors are linearly independent.

4. The vectors are linearly dependent, because
$$4 < 1, 0, 0, 0 > \, - 6 < 0, 1, 1, 0 > \, + < -4, 6, 6, 0 > = < 0, 0, 0, 0 > .$$

5. The vectors are linearly dependent because
$$2 < 1, 2, -3, 1 > \, + < 4, 0, 0, 2 > \, - < 6, 4, -6, 4 > = < 0, 0, 0, 0 > .$$

6. Suppose
$$\begin{aligned} &\alpha < 0, 1, 1, 1 > + \beta < -3, 2, 4, 4 > + \gamma < -2, 2, 34, 2 > \\ &+ \delta < 1, 1, -6, 2 > = < 0, 0, 0, 0 > . \end{aligned}$$

Then

$$\begin{aligned} -3\beta - 2\gamma + \delta &= 0, \\ \alpha + 2\beta + 2\gamma + \delta &= 0, \\ \alpha + 4\beta + 34\gamma - 6\delta &= 0, \\ \alpha + 4\beta + 2\gamma + 2\delta &= 0. \end{aligned}$$

It is routine to solve these equations to obtain

$$\alpha = \beta = \gamma = \delta = 0.$$

The only linear combination of the given vectors that equals the zero vector is the trivial linear combination (all coefficients zero). Therefore the vectors are linearly independent.

7. The vectors are linearly dependent, since

$$2 < 1, -2 > \; -2 \; < 4, 1 > \; + \; < 6, 6 > = < 0, 0 > .$$

8. The vectors are linearly independent.

9. The vectors are linearly independent.

10. The vectors are linearly independent.

In each of Problems 11 through 15 it is routine to check that S is not empty and that a linear combination of vectors in S is again in S. Thus S is a subspace of R^n for the appropriate n. We then produce a basis for the subspace.

11. By choosing $x = 1, y = 0$, then $x = 0, y = 1$ we obtain linearly independent vectors $< 1, 0, 0, -1 >$ and $< 0, 1, -1, 0 >$ that span S. These vectors form a basis for S.

12. The vectors $< 1, 0, 2, 0 >$ and $< 0, 1, 0, 3 >$ form a basis for S, which therefore has dimension 2.

13. The vectors $< 1, 0, 0, 0 >$, $< 0, 0, 1, 0 >$ and $< 0, 0, 0, 1 >$ form a basis for S, which has dimension 3.

14. The vectors $< 1, 1, 0, 0, 0, 0 >$, $< 0, 0, 1, 1, 0, 0 >$ and $< 0, 0, 0, 0, 0, 1 >$ form a basis for S, which has dimension 3.

15. Every vector in S is a scalar multiple of $< 0, 1, 0, 2, 0, 3, 0 >$, so S has dimension 1.

16. Write

$$\begin{aligned} < 4, 4, -1, 2, 0 > \; &= a < 2, 1, 0, 0, 0 > \; + b < 1, -2, 0, 0, 0 > \\ &+ c < 0, 0, 3, -2, 0 > \; + d < 0, 0, 2, -3, 0 > . \end{aligned}$$

By equating respective components, we have the system

$$\begin{aligned}2a + b &= 4,\\ a - 2b &= 4,\\ 3c + 2d &= -1,\\ -2c - 3d &= 2.\end{aligned}$$

Solve these for $a = 12/5$, $b = -4/5$, $c = 1/5$, $d = -4/5$. Then a, b, c, d are, in this order, the coordinates of $\mathbf{X}$ with respect to the given vectors in the subspace S.

17. Write

$$< -3, -2, 5, 1, -4 > = a < 1, 1, 1, 1, 0 > + b < -1, 1, 0, 0, 0 >$$
$$+ c < 1, 1, -1, -1, 0 > + d < 0, 0, 2, -2, 0 > + e < 0, 0, 0, 0, 2 > .$$

Then

$$\begin{aligned}a - b + c &= -3,\\ a + b + c &= -2,\\ a - c + 2d &= 5,\\ a - c - 2d &= 1,\\ 2e &= 4.\end{aligned}$$

Solve for the coordinates to obtain $a = 1/4$, $b = 1/2$, $c = -11/4$, $d = 1$, $e = 2$.

18. Proceeding as in Problems 16 and 17, we obtain the coordinates $-1/2, 1, 16/5, 2/5, 1$.

19. Since $\mathbf{V}_1, \cdots, \mathbf{V}_k$ span S, there are numbers $c_1, \cdots, c_k$ such that

$$\mathbf{U} = c_1\mathbf{V}_1 + \cdots + c_k\mathbf{V}_k.$$

Then $\mathbf{U}, \mathbf{V}_1, \cdots, \mathbf{V}_k$ are linearly dependent.

20. Because $\mathbf{V}_i \cdot \mathbf{V}_j = 0$ if $i \neq j$, then

$$\begin{aligned}&\| \mathbf{V}_1 + \cdots + \mathbf{V}_k \|^2 = (\mathbf{V}_1 + \cdots + \mathbf{V}_k) \cdot (\mathbf{V}_1 + \cdots + \mathbf{V}_k)\\ &= \mathbf{V}_1 \cdot (\mathbf{V}_1 + \cdots + \mathbf{V}_k) + \mathbf{V}_2 \cdot (\mathbf{V}_1 + \cdots + \mathbf{V}_k)\\ &+ \cdots + \mathbf{V}_k \cdot (\mathbf{V}_1 + \cdots + \mathbf{V}_k)\\ &= \mathbf{V}_1 \cdot \mathbf{V}_1 + \mathbf{V}_2 \cdot \mathbf{V}_2 + \cdots + \mathbf{V}_k \cdot \mathbf{V}_k\\ &= \| \mathbf{V}_1 \|^2 + \cdots + \| \mathbf{V}_k \|^2 .\end{aligned}$$

21. First,

$$\begin{aligned}(\mathbf{X} - \mathbf{Y}) \cdot (\mathbf{X} + \mathbf{Y}) &= \mathbf{X} \cdot \mathbf{X} + \mathbf{X} \cdot \mathbf{Y} - \mathbf{Y} \cdot \mathbf{X} - \mathbf{Y} \cdot \mathbf{Y}\\ &= \| \mathbf{X} \|^2 - \| \mathbf{Y} \|^2 = 0,\end{aligned}$$

so $\mathbf{X} - \mathbf{Y}$ is orthogonal to $\mathbf{X} + \mathbf{Y}$.

22. Let

$$\mathbf{Y} = \mathbf{X} - \sum_{j=1}^{k} (\mathbf{X} \cdot \mathbf{V}_j)\mathbf{V}_j.$$

Then

$$\begin{aligned}
0 &\leq \| \mathbf{Y} \|^2 = \mathbf{Y} \cdot \mathbf{Y} \\
&= \left(\mathbf{X} - \sum_{j=1}^{k} (\mathbf{X} \cdot \mathbf{V}_j)\mathbf{V}_j \right) \cdot \left(\mathbf{X} - \sum_{j=1}^{k} (\mathbf{X} \cdot \mathbf{V}_j)\mathbf{V}_j \right) \\
&= \mathbf{X} \cdot \mathbf{X} - 2\mathbf{X} \cdot \sum_{j=1}^{k} (\mathbf{X} \cdot \mathbf{V}_j)\mathbf{V}_j \\
&+ \left(\sum_{j=1}^{k} (\mathbf{X} \cdot \mathbf{V}_j)\mathbf{V}_j \right) \cdot \left(\sum_{j=1}^{k} (\mathbf{X} \cdot \mathbf{V}_j) \right) \mathbf{V}_j \\
&= \mathbf{X} \cdot \mathbf{X} - 2\mathbf{X} \cdot \sum_{j=1}^{k} (\mathbf{X} \cdot \mathbf{V}_j)\mathbf{V}_j \\
&+ \sum_{j=1}^{k} \sum_{r=1}^{k} (\mathbf{X} \cdot \mathbf{V}_j)(\mathbf{X} \cdot \mathbf{V}_r)\mathbf{V}_j \cdot \mathbf{V}_r.
\end{aligned}$$

We know that $\mathbf{V}_j \cdot \mathbf{V}_r = 0$ if $r \neq j$ and $\mathbf{V}_j \cdot \mathbf{V}_j = 1$. Therefore the double sum collapses to just those terms in which $r = j$ and we have

$$\begin{aligned}
0 &\leq \| \mathbf{X} \|^2 - 2 \sum_{j=1}^{k} (\mathbf{X} \cdot \mathbf{V}_j)^2 \\
&+ \sum_{j=1}^{k} (\mathbf{X} \cdot \mathbf{V}_j)^2 \\
&= \| \mathbf{X} \|^2 - \sum_{j=1}^{k} (\mathbf{X} \cdot \mathbf{V}_j)^2.
\end{aligned}$$

Therefore

$$\sum_{j=1}^{k} (\mathbf{X} \cdot \mathbf{V}_j)^2 \leq \| \mathbf{X} \|^2 .$$

23. If $\mathbf{V}_1, \cdots, \mathbf{V}_n$ is an orthonormal basis for R^n, and $\mathbf{X}$ is in R^n, then

$$\mathbf{X} = \sum_{j=1}^{n} (\mathbf{X} \cdot \mathbf{V}_j)\mathbf{V}_j.$$

Now reason as in the solution to Problem 22, using the fact

$$\mathbf{V}_j \cdot \mathbf{V}_k = \begin{cases} 0 & \text{if } j \neq k, \\ 1 & \text{if } j = k. \end{cases}$$

We have

$$\begin{aligned}
\| \mathbf{X} \|^2 &= \mathbf{X} \cdot \mathbf{X} \\
&= \left(\sum_{j=1}^{n} (\mathbf{X} \cdot \mathbf{V}_j)\mathbf{V}_j \right) \cdot \left(\sum_{j=1}^{n} (\mathbf{X} \cdot \mathbf{V}_j)\mathbf{V}_j \right) \\
&= \sum_{j=1}^{n} \sum_{r=1}^{n} (\mathbf{X} \cdot \mathbf{V}_j)(\mathbf{X} \cdot \mathbf{V}_r)\mathbf{V}_j \cdot \mathbf{V}_r \\
&= \sum_{j=1}^{n} (\mathbf{X} \cdot \mathbf{V}_j)^2.
\end{aligned}$$

24. $\mathbf{0}, \mathbf{V}_1, \cdots, \mathbf{V}_k$ are linearly dependent because

$$\mathbf{0} = 0\mathbf{V}_1 + \cdots + 0\mathbf{V}_k,$$

so $\mathbf{0}$ is a linear combination of the given vectors.

25. Let $\mathbf{V}_1, \cdots, \mathbf{V}_m$ be a spanning set for R^n. If these vectors are linearly independent, then they form a basis. Thus consider the case that the vectors are linearly dependent. In this case one of the vectors is a linear combination of the others, say (by renumbering if needed)

$$\mathbf{V}_m = c_1\mathbf{V}_1 + \cdots + c_{m-1}\mathbf{V}_{m-1}.$$

Then $\mathbf{V}_1, \cdots, \mathbf{V}_{m-1}$ span R^n. If these vectors are linearly independent, they form a basis. If not, one of the vectors is a linear combination of the others, say

$$\mathbf{V}_{m-1} = k_1\mathbf{V}_1 + \cdots + k_{m-1}\mathbf{V}_{m-2}.$$

But then $\mathbf{V}_1, \cdots, \mathbf{V}_{m-2}$ span R^n. Now keep repeating this argument. If $\mathbf{V}_1, \cdots, \mathbf{V}_{m-2}$ are linearly independent, they form a basis. If not, eliminate one of these vectors to form a spanning set with one less vector. Eventually we have removed $m - n$ vectors, and the remaining n form a basis for R^n.

26. Let S_1 be the subspace of R^n spanned by $\mathbf{u}_1, \cdots, \mathbf{u}_k$. Since $S_1 \neq R^n$, there is a vector $\mathbf{v}_1$ in R^n that is not in S_1. Since $\mathbf{v}_1$ is not a linear combination of $\mathbf{u}_1, \cdots, \mathbf{u}_k$, then $\mathbf{u}_1, \cdots, \mathbf{u}_k, \mathbf{v}_1$ are linearly independent. Suppose these vectors span S_2. If $S_2 = R^n$, we are done. If not, there is some $\mathbf{v}_2$ in R^n that is not in S_2. Since $\mathbf{v}_2$ is not a linear combination of $\mathbf{u}_1, \cdots, \mathbf{u}_k, \mathbf{v}_1$, then $\mathbf{u}_1, \cdots, \mathbf{u}_k, \mathbf{v}_1, \mathbf{v}_2$ are linearly independent. If $k+2 = n$ these vectors

form a basis for R^n. If not, there is some vector in R^n that is not a linear combination of $\mathbf{u}_1, \cdots, \mathbf{u}_k, \mathbf{v}_1, \mathbf{v}_2$, and we repeat the argument. After $n-k$ applications of this argument, we reach a linearly independent set of n vectors

$$\mathbf{u}_1, \cdots, \mathbf{u}_k, \mathbf{v}_1, \cdots, \mathbf{v}_{n-k}$$

spanning R^n, and these form a basis for R^n.

6.5 Orthogonalization

The arithmetic of carrying out the Gram-Schmidt process can be tedious and computations are most easily carried out using a software package such as MAPLE.

In each problem, the given vectors are denoted $\mathbf{X}_1, \cdots, \mathbf{X}_k$ in the given order.

1. Let $\mathbf{V}_1 = \mathbf{X}_1$ and then let

$$\begin{aligned} \mathbf{V}_2 &= \mathbf{X}_2 - \frac{\mathbf{X}_2 \cdot \mathbf{X}_1}{\mathbf{X}_1 \cdot \mathbf{X}_1}\mathbf{X}_1 \\ &= \mathbf{X}_2 + \frac{18}{17}\mathbf{X}_1 \\ &= < 52/17, -13/17, 0 > . \end{aligned}$$

2. Let $\mathbf{V}_1 = \mathbf{X}_1$ and

$$\mathbf{V}_2 = \mathbf{X}_1 + \frac{11}{5}\mathbf{X}_1 = < 0, 4/5, 2/5, 0 > .$$

3. Let $\mathbf{V}_1 = \mathbf{X}_1$, then

$$\mathbf{V}_2 = \mathbf{X}_1 - \frac{-7}{6}\mathbf{X}_1 = < 0, 4/3, 13/6, 29/6 > .$$

Finally,

$$\begin{aligned} \mathbf{V}_3 &= \mathbf{X}_3 - \frac{3}{6}\mathbf{V}_1 - \frac{43/2}{179/6}\mathbf{V}_2 \\ &= \mathbf{X}_3 - \frac{1}{2}\mathbf{V}_1 - \frac{129}{179}\mathbf{V}_2 \\ &= \frac{1}{179} < 0, 7, -11, 3 > . \end{aligned}$$

4. $\mathbf{V}_1 = \mathbf{X}_1$,

$$\begin{aligned} \mathbf{V}_2 &= \mathbf{X}_2 - \frac{5}{26}\mathbf{X}_1 \\ &= \frac{1}{26} < 109, 0, -41, 0, 58 >, \end{aligned}$$

$$\begin{aligned}\mathbf{V}_3 &= \mathbf{X}_3 - \frac{17}{26}\mathbf{X}_1 - \frac{331/26}{651/26}\mathbf{V}_2 \\ &= \mathbf{X}_3 - \frac{17}{26}\mathbf{x}_1 - \frac{331}{651}\mathbf{V}_2 \\ &= \frac{1}{651} < -962, 0, -1406, 0, 814 > .\end{aligned}$$

5. $\mathbf{V}_1 = \mathbf{X}_1$,

$$\begin{aligned}\mathbf{V}_2 &= \mathbf{X}_2 - \frac{5}{7}\mathbf{X}_1 \\ &= \frac{1}{9} < 0, 0, -1, -19, 40 >,\end{aligned}$$

$$\begin{aligned}\mathbf{V}_3 &= \mathbf{X}_3 + \frac{2}{9}\mathbf{V}_1 + \frac{17}{9}\mathbf{V}_2 \\ &= \frac{1}{218} < 0, 218, -341, 279, 62 >,\end{aligned}$$

$$\begin{aligned}\mathbf{V}_4 &= \frac{6}{9}\mathbf{X}_1 + \frac{13}{3}\mathbf{V}_2 - \frac{435}{1179}\mathbf{V}_3 \\ &= \frac{1}{393} < 0, 248, 88, -24, -32 > .\end{aligned}$$

6. $\mathbf{V}_1 = \mathbf{X}_1$,

$$\begin{aligned}\mathbf{V}_2 &= \mathbf{X}_2 - \frac{1}{10}\mathbf{X}_1 \\ &= \frac{1}{10} < 21, -8, -60, -31, -18, 0 >,\end{aligned}$$

$$\begin{aligned}\mathbf{V}_3 &= \mathbf{X}_3 - \frac{3}{10}\mathbf{X}_1 - \frac{163/10}{269/10}\mathbf{V}_2 \\ &= \frac{1}{269} < -423, -300, 489, -759, 132, 0 >,\end{aligned}$$

$$\begin{aligned}\mathbf{V}_4 &= \mathbf{X}_4 - \frac{-15}{10}\mathbf{X}_1 - \frac{13/2}{269/10}\mathbf{V}_2 - \frac{4455/269}{4095/269}\mathbf{V}_3 \\ &= \frac{1}{91} < 337, -145, 250, 29, -9, 0 > .\end{aligned}$$

7. $\mathbf{V}_1 = \mathbf{X}_1$,

$$\begin{aligned}\mathbf{V}_2 &= \mathbf{X}_2 + \frac{3}{2}\mathbf{X}_1 \\ &= \frac{1}{2} < 0, 0, -3, 3, 0, 0 > .\end{aligned}$$

8. $\mathbf{V}_1 = \mathbf{X}_1$, $\mathbf{V}_2 = \mathbf{X}_2$ because $\mathbf{X}_2$ and $\mathbf{X}_1$ are orthogonal. Finally,

$$\mathbf{V}_3 = \mathbf{X}_3 + \frac{4}{12}\mathbf{V}_1 + \frac{4}{2}\mathbf{V}_2 = < 0, -8/3, 0, -8/3, 0, 16/3 > .$$

6.6 Orthogonal Complements and Projections

1. Let $\mathbf{V}_1 =< 1, -1, 0, 0 >$ and $\mathbf{V}_2 =< 1, 1, 0, 0 >$. These form an orthogonal basis for S. Let

$$\begin{aligned}\mathbf{u}_S &= \frac{\mathbf{u}\cdot\mathbf{V}_1}{\mathbf{V}_1\cdot\mathbf{V}_1}\mathbf{V}_1 + \frac{\mathbf{u}\cdot\mathbf{V}_2}{\mathbf{V}_2\cdot\mathbf{V}_2}\mathbf{V}_2 \\ &= -4\mathbf{V}_1 + 2\mathbf{V}_2 =< -2, 6, 0, 0 >\end{aligned}$$

and

$$\mathbf{u}^\perp = \mathbf{u} - \mathbf{u}_S =< 0, 0, 1, 7 > .$$

Then $\mathbf{u}_S$ is in S and $\mathbf{u}^\perp$ is in $S^\perp$, and $\mathbf{u} = \mathbf{u}_S + \mathbf{u}^\perp$.

2. $$\mathbf{u}_S = \frac{2}{5}\mathbf{V}_1 + \frac{1}{5}\mathbf{V}_2 =< 0, 0, 0, 1, 0 >,$$

$$\mathbf{u}^\perp =< 0, -4, -4, 0, 3 > .$$

3. $$\mathbf{u}_S = \frac{7}{2}\mathbf{V}_1 + \mathbf{V}_2 - 3\mathbf{v}_3 =< 9/2, -1/2, 0, 5/2, -13/2 >,$$

$$\mathbf{u}^\perp =< -1/2, -1/2, 3, -1/2, -1/2 > .$$

4. $$\mathbf{u}_S = -3\mathbf{V}_1 + \frac{31}{39}\mathbf{V}_2 =< -86/39, 148/39, 62/13, 31/39 >,$$

$$\mathbf{u}^\perp =< 203/309, 203/309, -10/13, -226/39 > .$$

5. $$\mathbf{u}_S = 3\mathbf{V}_1 + \frac{1}{2}\mathbf{V}_2 =< 3, 1/2, 3, 1/2, 3, 0, 0 >,$$

$$\mathbf{u}^\perp =< 5, 1/2, -2, -1/2, -3, -3, 4 > .$$

6. $S^\perp$ consists of all vectors that are orthogonal to every vector in S. This is a symmetric relationship, because each vector in S is also orthogonal to each vector in $S^\perp$.

   Based on this observation, a vector is in S exactly when this vector is orthogonal to each vector in $S^\perp$, and a vector is in $(S^\perp)^\perp$ exactly when it is orthogonal to each vector in $S^\perp$. Thus the criterion for a vector to be in S and for a vector to be in $(S^\perp)^\perp$ is the same and we conclude that

$$S = (S^\perp)^\perp.$$

7. Let $\mathbf{v}_1, \cdots, \mathbf{v}_k$ be an orthogonal basis for S, and $\mathbf{u}_1, \cdots, \mathbf{u}_r$ an orthogonal basis for $S^{\perp}$. If $\mathbf{u}$ is any vector in R^n, then $\mathbf{u}$ has a unique representation as a sum of a vector in S and a vector in $S^{\perp}$, $\mathbf{u} = \mathbf{u}_S + \mathbf{u}^{\perp}$. Therefore every vector in R^n is a linear combination of the vectors

$$\mathbf{v}_1, \cdots, \mathbf{v}_k, \mathbf{u}_1, \cdots, \mathbf{u}_r.$$

Further, each $\mathbf{u}_i$ is orthogonal to each $\mathbf{v}_j$, because every vector in $S^{\perp}$ is orthogonal to each vector in S. Now $\mathbf{u}_S$ is a linear combination of $\mathbf{v}_1, \cdots, \mathbf{v}_k$ and $\mathbf{u}^{\perp}$ is a linear combination of $\mathbf{u}_1, \cdots, \mathbf{u}_r$, so $\mathbf{v}_1, \cdots, \mathbf{u}_r$ span R^n. Further,

$$\mathbf{v}_1, \cdots, \mathbf{v}_k, \mathbf{u}_1, \cdots, \mathbf{u}_r$$

are orthogonal, hence linearly independent. The vectors

$$\mathbf{v}_1, \cdots, \mathbf{v}_k, \mathbf{u}_1, \cdots, \mathbf{u}_r$$

form a basis for R^n. But then $k + r = n$. We conclude that

$$\text{dimension}(S) + \text{ dimension}(S^{\perp}) = \text{ dimension}R^n.$$

8. The idea of the solution is to use an orthogonal basis for S to produce $\mathbf{u}_S$, which is the vector we want. The given vectors $\mathbf{V}_1 =< 1, 0, 1, 0 >$ and $\mathbf{V}_2 =< -2, 0, 2, 1 >$ are orthogonal and form a basis for S. Compute

$$\begin{aligned}\mathbf{u}_S &= \frac{\mathbf{u} \cdot \mathbf{V}_1}{\mathbf{V}_1 \cdot \mathbf{V}_1}\mathbf{V}_1 + \frac{\mathbf{u} \cdot \mathbf{V}_2}{\mathbf{V}_2 \cdot \mathbf{V}_2}\mathbf{V}_2 \\ &= 2\mathbf{V}_1 + \frac{1}{9}\mathbf{V}_2 \\ &=< 16/9, 0, 20/9, 1/9 > .\end{aligned}$$

9. Let

$$\mathbf{V}_1 =< 2, 1, -1, 0, 0 >, \mathbf{V}_2 =< -1, 2, 0, 1, 0 > \text{ and } \mathbf{V}_3 =< 0, 1, 1, -2, 0 > .$$

These form an orthogonal basis for S. With $\mathbf{u} =< 4, 3, -3, 4, 7 >$, compute

$$\begin{aligned}\mathbf{u}_S &= \frac{7}{3}\mathbf{V}_1 + \mathbf{V}_2 - \frac{4}{3}\mathbf{V}_3 \\ &=< 11/3, 3, -11/3, 11/3, 0 > .\end{aligned}$$

10. Let

$$\mathbf{V}_1 =< 0, 1, 1, 0, 0, 1 >, \mathbf{V}_2 =< 0, 0, 3, 0, 0, -3 > \text{ and } \mathbf{V}_3 =< 6, 0, 0, -2, 0, 0 > .$$

These form an orthogonal basis for S. The vector in S closest to $\mathbf{u}$ is

$$\begin{aligned}\mathbf{u}_S &= \frac{8}{3}\mathbf{V}_1 - \frac{5}{8}\mathbf{V}_2 - \frac{1}{2}\mathbf{V}_3 \\ &=< -3, 8/3, 1/6, 1, 0, 31/6 > .\end{aligned}$$

6.7 The Function Space $C[a, b]$

Problems 1 through 4 involve the Gram-Schmidt orthogonalization process, except the setting is now a function space. The only difference this makes in applying the Gram-Schmidt expressions for the orthogonal vectors is that the vectors are now functions and the dot products are defined by integrals of the form

$$f \cdot g = \int_a^b p(x)f(x)g(x)\,dx,$$

in which the weight function $p(x)$ must be specified.

Problems 5, 6 and 7 involve finding a function "closest" to a given set of functions in the same sense that a vector $\mathbf{u}_S$ is closest to a subspace spanned by a given set of vectors. Again, the only difference is that now the vectors are functions and the dot products are integrals. Thus in these problems we must determine an orthogonal projection f_S, given $f(x)$ and a spanning set for the subspace S of $C[a, b]$.

1. Denote $X_1(x) = e^x$ and $X_2(x) = e^{-x}$. These span a subset of $C[0, 1]$ consisting of all functions of the form $ae^x + be^{-x}$. However, these functions are not orthogonal, since

$$X_1 \cdot X_2 = \int_0^1 X_1(x)X_2(x)\,dx = \int_0^1 dx = 1 \neq 0.$$

For an orthogonal basis, first choose

$$V_1(x) = X_1(x) = e^x.$$

Next choose

$$\begin{aligned} V_2(x) &= X_2(x) - \frac{X_2 \cdot X_1}{X_1 \cdot X_1} X_1 \\ &= e^{-x} - \frac{\int_0^1 1\,dx}{\int_0^1 e^{2x}\,dx} e^x \\ &= e^{-x} - \frac{2}{e^2 - 1} e^x. \end{aligned}$$

It is routine to check that indeed V_1 and V_2 are orthogonal, since

$$V_1 \cdot V_2 = \int_0^1 V_1(x)V_2(x)\,dx = 0.$$

2. Choose $V_1(x) = \sin(x)$, and

$$V_2(x) = \cos(x) - \frac{\cos(x) \cdot \sin(x)}{\sin(x) \cdot \sin(x)} \sin(x) = \cos(x),$$

since $\cos(x) \cdot \sin(x) = 0$ (these functions are orthogonal on $[-\pi, \pi]$ with the given dot product). Finally,

$$\begin{aligned} V_3(x) &= \sin(2x) - \frac{\sin(2x) \cdot \cos(x)}{\cos(x) \cdot \cos(x)} \cos(x) - \frac{\sin(2x) \cdot \sin(x)}{\sin(x) \cdot \sin(x)} \sin(x) \\ &= \sin(2x). \end{aligned}$$

3. Let $X_1(x) = 1$, $X_2(x) = x$ and $X_3(x) = x^2$. Choose $V_1(x) = 1$,

$$V_2(x) = x - \frac{x \cdot 1}{1 \cdot 1}(1) = x - \frac{2}{3}$$

and

$$\begin{aligned} V_3(x) &= x^2 - \frac{x^2 \cdot x}{x \cdot x} x - \frac{x^2 \cdot 1}{1 \cdot 1}(1) \\ &= x^2 - \frac{6}{5}\left(x - \frac{2}{3}\right) - \frac{1}{2}. \end{aligned}$$

4. Let $X_1(x) = 1$, $X_2(x) = \cos(\pi x/2)$ and $X_3 = \sin(\pi x/2)$. Here the weighted dot product is

$$f \cdot g = \int_0^2 x f(x) g(x)\, dx.$$

The formulas for the orthogonal basis functions is the same, but now the dot product that appears in the coefficients is different. Choose $V_1(x) = X_1(x) = 1$, and then

$$\begin{aligned} V_2(x) &= \cos(\pi x/2) - \frac{X_2 \cdot X_1}{X_1 \cdot X_1} X_1 \\ &= \cos(\pi x/2) + \frac{4}{\pi^2}. \end{aligned}$$

Finally,

$$\begin{aligned} V_3(x) &= X_3 - \frac{X_3 \cdot X_1}{X_1 \cdot X_1} X_1 - \frac{X_3 \cdot X_2}{X_2 \cdot X_2} X_2 \\ &= \sin(\pi x/2) - \frac{2}{\pi} - \frac{\pi(16 - \pi^2)}{\pi^4 - 32}\left(\cos(\pi x/2) + \frac{4}{\pi^2}\right). \end{aligned}$$

5. Here we are computing the orthogonal projection of $f(x) = x^2$ onto the subspace of $C[0, \pi]$ spanned by $1, \cos(x)$, $\cos(2x)$, $\cos(3x)$ and $\cos(4x)$. It is routine to verify that the given functions form an orthogonal basis for S with respect to the given dot product. This orthogonal projection is

$$f_S(x) = \sum_{k=0}^{n} c_k X_k(x),$$

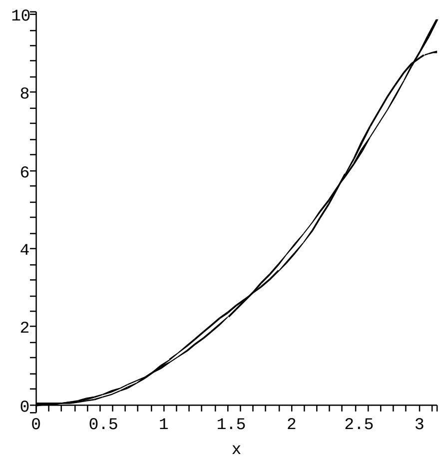

Figure 6.2: $f(x)$ and $f_S(x)$ in Problem 5, Section 6.7.

where $X_k(x) = \cos(kx)$ for $k = 0, 1, 2, 3, 4$, where

$$c_k = \frac{\int_0^\pi x^2 X_k(x)\,dx}{\int_0^\pi X_k^2(x)\,dx}.$$

Routine integrations yield

$$c_0 = \frac{\pi^2}{3} \text{ and } c_k = \frac{4(-1)^k}{k^2}$$

for $k = 1, 2, 3, 4$. Then

$$f_S(s) = \frac{\pi^2}{3} - 4\cos(x) + \cos(2x) - \frac{1}{2}\cos(3x) + \frac{1}{4}\cos(4x).$$

Figure 6.2 compares a graph of $f(x)$ and $f_S(x)$. It happens that these graphs are fairly close, but in applications $f(x)$ is probably not approximated closely enough by $f_S(x)$ for reliable calculations. The point, however, is that $f_S(x)$ is the function in $C[0, \pi]$ nearest to the subspace S spanned by the five given functions, in the sense of distance in this function space. If we wanted a better numerical approximation (graphs closer together), we could change S and include more functions $\cos(kx)$. This is the idea of a Fourier cosine expansion, treated in Chapter Fourteen.

6. As in Problem 5, we are finding the orthogonal projection of $f(x) = x^2$ onto the subspace S spanned by $X_k(x) = \sin(kx)$ for $k = 1, 2, 3, 4, 5$. We

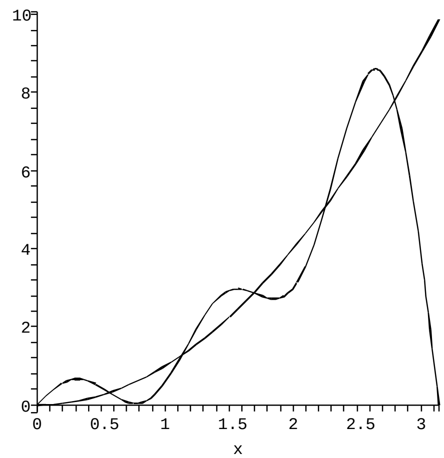

Figure 6.3: f and f_S in Problem 6, Section 6.7.

have

$$f_S(x) = \sum_{k=1}^{5} c_k \sin(kx)$$

where

$$\begin{aligned} c_k &= \frac{\int_0^\pi x^2 \sin(kx)\,dx}{\int_0^\pi \sin^2(kx)\,dx} \\ &= \frac{2}{\pi}\left(\frac{-2+2(-1)^k - k^2\pi^2(-1)^k}{k^3}\right). \end{aligned}$$

Figure 6.3 shows graphs of f and f_S. It is clear that f_S does not approximate f very well in the numerical sense that $f(x)$ and $f_S(x)$ are close, within some small error tolerance. However, it remains true that f_S is the function in $C[0,\pi]$ closest to S. If we want a better numerical approximation of $f(x)$ by a sum of multiples of functions $\sin(kx)$, we must choose k larger. Later we will see this as one idea behind Fourier sine series.

7. We want the function f_S in S that is closest (in the distance defined on this function space) to $f(x) = x(2-x)$, where S is the subspace spanned by the orthogonal functions 1, $\cos(k\pi x/2$ and $\sin(k\pi x/2$ for $k = 1, 2, 3$. This orthogonal projection has the form

$$f_S(x) = c_0 + \sum_{k=1}^{3}(c_k \cos(k\pi x/2) + d_k \sin(k\pi x/2)).$$

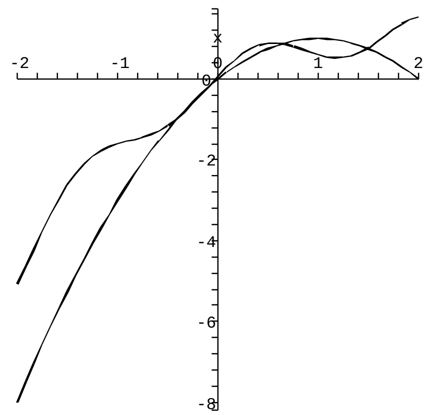

Figure 6.4: f and f_S in Problem 7, Section 6.7.

Routine integrations yield $c_0 = -4/3$ and, for $k = 1, 2, 3$,

$$c_k = \frac{16(-1)^{k+1}}{\pi^2 k^2} \text{ and } d_k = \frac{8(-1)^{k+1}}{\pi k}.$$

Figure 6.4 shows graphs of f and f_S.

Chapter 7

Matrices and Systems of Linear Equations

7.1 Matrices

1.

$$2\mathbf{A} - 3\mathbf{B} = \begin{pmatrix} 14 & -2 & 6 \\ 10 & -5 & -6 \\ -26 & -43 & -8 \end{pmatrix}$$

2.

$$-5\mathbf{A} + 3\mathbf{B} = \begin{pmatrix} 19 & 2 \\ 6 & -2 \\ -28 & 38 \\ -27 & 35 \end{pmatrix}$$

3.

$$\mathbf{A}^2 + 2\mathbf{AB} = \begin{pmatrix} 2 + 2x - x^2 & -12x + (1-x)(x + e^x + 2\cos(x)) \\ 4 + 2x + 2e^x + 2xe^x & -22 - 2x + e^{2x} + 2e^x\cos(x) \end{pmatrix}$$

4.

$$-3\mathbf{A} - 4\mathbf{B} = (18)$$

This is a 1×1 matrix, which we think of as just the number 18. Here the matrix structure serves no purpose, since there are no row and column locations to distinguish between.

5.

$$4\mathbf{A} + 8\mathbf{B} = \begin{pmatrix} -36 & 0 & 68 & 196 & 20 \\ 128 & -40 & -36 & -8 & 72 \end{pmatrix}$$

6.

$$\mathbf{A}^3 - \mathbf{B}^2 = \begin{pmatrix} -17 & 18 \\ 6 & 1 \end{pmatrix} - \begin{pmatrix} -40 & 8 \\ -5 & -39 \end{pmatrix} = \begin{pmatrix} 27 & 10 \\ 11 & 40 \end{pmatrix}$$

7.
$$\mathbf{AB} = \begin{pmatrix} -10 & -34 & -16 & -30 & -14 \\ 10 & -2 & -11 & -8 & -45 \\ -5 & 1 & 15 & 61 & -63 \end{pmatrix}; \mathbf{BA} \text{ is not defined.}$$

8.
$$\mathbf{AB} = \begin{pmatrix} -16 & 0 \\ 17 & 28 \end{pmatrix}; \mathbf{BA} = \begin{pmatrix} 12 & -32 \\ -14 & 0 \end{pmatrix}$$

9.
$$\mathbf{AB} = (115); \mathbf{BA} = \begin{pmatrix} 3 & -18 & -6 & -42 & 66 \\ -2 & 12 & 4 & 28 & -44 \\ -6 & 36 & 12 & 84 & -132 \\ 0 & 0 & 0 & 0 & 0 \\ 4 & -24 & 8 & -56 & 88 \end{pmatrix}$$

10.
$$\mathbf{AB} = \begin{pmatrix} 48 & 1 & 1 & -58 \\ -96 & 2 & 2 & 220 \\ -288 & -22 & -22 & -68 \\ -16 & 6 & 6 & 184 \end{pmatrix}; \mathbf{BA} = \begin{pmatrix} 76 & 152 \\ 50 & 136 \end{pmatrix}$$

11.
$$\mathbf{AB} \text{ is not defined}; \mathbf{BA} = \begin{pmatrix} 410 & 36 & -56 & 227 \\ 17 & 253 & 40 & -1 \end{pmatrix}$$

12.
$$\mathbf{AB} = \begin{pmatrix} -22 & 30 & -10 & -4 \\ -42 & 45 & 30 & 6 \end{pmatrix}; \mathbf{BA} \text{ is not defined.}$$

13. **AB** is not defined and
$$\mathbf{BA} = \begin{pmatrix} -16 & -13 & -5 \end{pmatrix}$$

14. Neither **AB** nor **BA** is defined.

15. **BA** is not defined,
$$\mathbf{AB} = \begin{pmatrix} 39 & -84 & 21 \\ -23 & 38 & 3 \end{pmatrix}$$

16. **AB** is not defined,
$$\mathbf{BA} = \begin{pmatrix} 28 & 30 \end{pmatrix}$$

17. **AB** is 14×14, **BA** is 21×21.

18. Neither **AB** nor **BA** is defined.

19. **AB** is not defined, **BA** is 4×2.

20. **AB** is 1×3, **BA** is not defined.

21. $\mathbf{AB}$ is not defined, $\mathbf{BA}$ is 7×6.

22. There are infinitely many examples, but here is one. Let

$$\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 8 & 4 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 6 & 0 \\ -1 & 1 \end{pmatrix}.$$

Then $\mathbf{B} \neq \mathbf{C}$, but

$$\mathbf{AB} = \mathbf{CA} = \begin{pmatrix} 12 & 6 \\ 6 & 3 \end{pmatrix}.$$

23. For the given graph G the adjacency matrix is

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{pmatrix}.$$

Compute

$$\mathbf{A}^3 = \begin{pmatrix} 2 & 7 & 7 & 4 & 4 \\ 7 & 8 & 9 & 9 & 9 \\ 7 & 9 & 8 & 9 & 9 \\ 4 & 9 & 9 & 6 & 7 \\ 4 & 9 & 9 & 7 & 6 \end{pmatrix} \text{ and } \mathbf{A}^4 = \begin{pmatrix} 14 & 17 & 17 & 18 & 18 \\ 17 & 34 & 33 & 26 & 26 \\ 17 & 33 & 34 & 26 & 26 \\ 18 & 26 & 26 & 25 & 24 \\ 18 & 26 & 26 & 24 & 25 \end{pmatrix}.$$

The number of $v_1 - v_4$ walks of length 3 is $(A^3)_{14} = 4$ and the number of $v_1 - v_4$ walks of length 4 is $(A^4)_{14} = 18$. The number of $v_2 - v_3$ walks of length 3 is 9, and the number of $v_2 - v_4$ walks of length 4 is 26.

24. The adjacency matrix is

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \end{pmatrix}.$$

Compute

$$\mathbf{A}^2 = \begin{pmatrix} 3 & 2 & 1 & 2 & 1 \\ 2 & 3 & 1 & 2 & 1 \\ 1 & 1 & 3 & 0 & 3 \\ 2 & 2 & 0 & 2 & 0 \\ 1 & 1 & 3 & 0 & 3 \end{pmatrix} \text{ and } \mathbf{A}^4 = \begin{pmatrix} 19 & 18 & 11 & 14 & 11 \\ 18 & 19 & 11 & 14 & 11 \\ 11 & 11 & 20 & 4 & 20 \\ 14 & 14 & 4 & 12 & 4 \\ 11 & 11 & 20 & 4 & 20 \end{pmatrix}.$$

The number of $v_1 - v_4$ walks of length 4 is $(\mathbf{A}^4)_{14} = 14$, the number of $v_2 - v_3$ walks of length 2 is $(\mathbf{A}^2)_{23} = 1$.

25. The adjacency matrix is

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \end{pmatrix}.$$

Then

$$\mathbf{A}^2 = \begin{pmatrix} 4 & 2 & 3 & 3 & 2 \\ 2 & 3 & 2 & 2 & 3 \\ 3 & 2 & 4 & 3 & 2 \\ 3 & 2 & 3 & 4 & 2 \\ 2 & 3 & 2 & 2 & 3 \end{pmatrix}, \mathbf{A}^3 = \begin{pmatrix} 10 & 10 & 11 & 11 & 10 \\ 10 & 6 & 10 & 10 & 6 \\ 11 & 10 & 10 & 11 & 10 \\ 11 & 10 & 11 & 10 & 10 \\ 10 & 6 & 10 & 10 & 6 \end{pmatrix},$$

and

$$\mathbf{A}^4 = \begin{pmatrix} 42 & 32 & 41 & 41 & 32 \\ 32 & 30 & 32 & 32 & 30 \\ 41 & 32 & 42 & 41 & 32 \\ 41 & 32 & 41 & 42 & 32 \\ 32 & 30 & 32 & 32 & 30 \end{pmatrix}.$$

The number of $v_4 - v_5$ walks of length 2 is 2, the number of $v_2 - v_3$ walks of length 3 is 10, the number of $v_1 - v_2$ walks of length 4 is 32, and the number of $v_4 - v_5$ walks of length 4 is 32.

26. (a) The i, i element of $\mathbf{A}^2$ is the number of $v_i - v_i$ walks of length 2 in the graph. Each such walk has the form $v_i - v_j - v_i$, for some $j \neq i$, hence corresponds to a vertex v_j adjacent to v_i in the graph. Therefore $\mathbf{A}_{ii}$ counts the number of vertices adjacent to v_i.

(b) The i, i element of $\mathbf{A}^3$ is the number of walks $v_i - v_i$ walks of length 3 in G. Any such walk has the form $v_i - v_j - v_k - v_i$, for some $j \neq k$, and neither j nor k equal to i. These three vertices therefore form the vertices of a triangle in the graph. However, each such triangle is counted twice in the i, i element of $\mathbf{A}^3$, because this triangle actually represents two $v_i - v_i$ walks, namely $v_i - v_j - v_k - v_i$ and (going the other way), $v_i - v_k - v_j - v_i$. Therefore

$$\mathbf{A}^3_{ii} = 2(\text{number of triangles in } G).$$

27. Let $\mathcal{M}$ be the set of all real $n \times m$ matrices.

First, each $n \times m$ matrix has nm elements in its n rows and m columns. If we string out the rows of an $n \times m$ real matrix $\mathbf{A}$ into one long row (row 2 following row 1, then row 3, and so on), we form an $nm-$ vector. In this way, we form a one-to-one correspondence matrices in $\mathcal{M}$ and vectors in R^{nm}.

Notice that we add two matrices by adding corresponding components, so the nm vector formed from $\mathbf{A} + \mathbf{B}$ is the sum of the nm vectors formed from **A** and **B**. Further, if we multiply **A** by a real number c, the rows of $c\mathbf{A}$, when strung out in this way, form the components of c times the nm vector formed from the rows of **A**. Thus we can identify the set of all real $n \times m$ matrices with R^{nm}, with this identification preserving the operations of addition of matrices (vectors) and multiplication by scalars. The dimension of this vector space of matrices is therefore the same as the dimension of R^{nm}, namely nm.

As an example of this correspondence, the 2 real matrix

$$\begin{pmatrix} 3 & 2 & -4 \\ 6 & 1 & 8 \end{pmatrix}$$

corresponds to the 6$-$ vector $< 3, 2, -4, 6, 1, 8 >$.

We can also see this dimension by explicitly constructing a basis for $\mathcal{M}$. Let $\mathbf{K}_{ij}$ be the matrix having a 1 in the i, j entry, and zeros everywhere else. These nm matrices correspond to the nm unit vectors in R^{nm} having one component 1 and all other components zero. The matrices $\mathbf{K}_{ij}$ form a basis for $\mathcal{M}$.

28. We can reason as in Problem 27, except, in stringing out the rows of an $n \times m$ matrix with complex entries, we can string out all the nm real parts of the entries, followed by the nm complex parts of the entries. This matches the set of all $n \times m$ complex matrices with R^{2nm}, with the operations of addition and scalar multiplication corresponding as in Problem 27. We conclude that this vector space of complex matrices has dimension $2nm$.

    As an example, the 2×3 complex matrix

$$\begin{pmatrix} 2-i & 4 & 6+7i \\ 1-i & 2i & 3-4i \end{pmatrix}$$

    corresponds to the 12$-$ vector

$$< 2, 4, 6, 1, 2, 3, -1, 0, 7, -1, 2, -4 > .$$

7.2 Elementary Row Operations

In each of Problems 1 - 8, if a single row operations is applied to **A**, then the resulting matrix is $\mathbf{\Omega A}$, where $\mathbf{\Omega}$ is the elementary matrix formed by performing the operation on $\mathbf{I}_n$. If a sequence of k elementary row operations is performed, then $\mathbf{\Omega} = \mathbf{E}_k \cdots \mathbf{E}_1$, where $\mathbf{E}_1$ is the elementary matrix performing the first operation, and so on.

1. **A** is 3×4. To multiply row two of **A** by $\sqrt{3}$, multiply **a** on the left by the 3×3 matrix $\mathbf{\Omega}$ formed from $\mathbf{I}_3$ by multiplying row two of this matrix by

$\sqrt{3}$. Thus form

$$\mathbf{\Omega} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sqrt{3} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

As a check, observe that

$$\mathbf{\Omega A} = \begin{pmatrix} -2 & 1 & 4 & 2 \\ 0 & \sqrt{3} & 16\sqrt{3} & 3\sqrt{3} \\ 1 & -2 & 4 & 8 \end{pmatrix}$$

2. Because $\mathbf{A}$ is 4×2, perform this row operation by adding 6 times row two to row three of $\mathbf{I}_4$ to obtaim

$$\mathbf{\Omega} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 6 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then

$$\mathbf{\Omega A} = \begin{pmatrix} 3 & -6 \\ 1 & 1 \\ 14 & 4 \\ 0 & 5 \end{pmatrix},$$

and this is the matrix obtained by performing the given row operation on $\mathbf{A}$.

3.

$$\mathbf{\Omega} = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & \sqrt{13} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 5 & 0 \\ 1 & 0 & \sqrt{13} \\ 0 & 0 & 1 \end{pmatrix}$$

and

$$\mathbf{\Omega A} = \begin{pmatrix} 40 & 5 & -15 \\ -2+2\sqrt{13} & 14+9\sqrt{13} & 6+5\sqrt{13} \\ 2 & 9 & 5 \end{pmatrix}$$

4.

$$\mathbf{\Omega} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

and

$$\mathbf{\Omega A} = \begin{pmatrix} -4 & 6 & -3 \\ 5 & -3 & 3 \\ 12 & 4 & -4 \end{pmatrix}$$

5.
$$\mathbf{\Omega} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 15 \end{pmatrix} \begin{pmatrix} 1 & \sqrt{3} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 15 \\ 1 & \sqrt{3} \end{pmatrix}$$

and
$$\mathbf{\Omega A} = \begin{pmatrix} 30 & 120 \\ -3+2\sqrt{3} & 15+8\sqrt{3} \end{pmatrix}$$

6.
$$\mathbf{\Omega} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ \sqrt{3} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 & 0 \\ \sqrt{3} & 1 & 0 \\ 4+\sqrt{3} & 1 & 4 \end{pmatrix}$$

and
$$\mathbf{\Omega A} = \begin{pmatrix} 3 & -4 & 5 & 9 \\ 2+3\sqrt{3} & 1-4\sqrt{3} & 3+5\sqrt{3} & -6+9\sqrt{3} \\ 18+3\sqrt{3} & 37-4\sqrt{3} & 31+5\sqrt{3} & 54+9\sqrt{3} \end{pmatrix}$$

7.
$$\mathbf{\Omega} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 14 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 4 \\ 14 & 1 & 0 \end{pmatrix}$$

and
$$\mathbf{\Omega A} = \begin{pmatrix} -1 & 0 & 3 & 0 \\ -36 & 28 & -20 & 28 \\ -13 & 3 & 44 & 9 \end{pmatrix}$$

8.
$$\mathbf{\Omega} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 1 & 3 \\ 0 & 0 & 1 \\ 5 & 0 & 0 \end{pmatrix}$$

and
$$\mathbf{\Omega A} = \begin{pmatrix} 28 & 50 & 2 \\ 9 & 15 & 0 \\ 0 & -45 & 70 \end{pmatrix}$$

In these and later problems, it is sometimes useful to use the delta notation, defined by
$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$
For example, $\mathbf{I}_n$ is the $n \times n$ matrix whose $i, j-$ element is δ_{ij}.

9. Let $\mathbf{A} = [a_{ij}]$ be $n \times m$. Since $\mathbf{B}$ and $\mathbf{E}$ are obtained, respectively, by interchanging rows s and t of $\mathbf{A}$ and $\mathbf{I}_n$ then, for $i \neq s$ and $i \neq t$, $b_{ij} = a_{ij}$ and $e_{ij} = \delta_{ij}$. For $i = s$, $b_{sj} = a_{tj}$ and $e_{sj} = \delta_{tj}$. And for $i = t$, $b_{ij} = a_{sj}$ and $e_{ij} = \delta_{sj}$.

   Now consider the $i, j-$ element of $\mathbf{EA}$. For $i \neq s$ and $i \neq t$,

$$(\mathbf{EA})_{ij} = \sum_{k=1}^{n} e_{ik}a_{kj} = a_{ij} = b_{ij}.$$

   For $i = s$,

$$(\mathbf{EA})_{sj} = \sum_{k=1}^{n} e_{sk}a_{kj} = \sum_{k=1}^{n} \delta_{tk}a_{kj} = a_{tj} = b_{sj}.$$

   And for $i = t$,

$$(\mathbf{EA})_{tj} = \sum_{k=1}^{n} e_{ik}a_{kj} = \sum_{k=1}^{n} \delta_{sk}a_{kj} = a_{sj} = b_{tj}$$

   for $j = 1, 2, \cdots, m$. Therefore $\mathbf{EA} = \mathbf{B}$.

10. Let $\mathbf{A}$ be $n \times m$. Since $\mathbf{B}$ and $\mathbf{E}$ are formed, respectively, by multiplying row s of $\mathbf{A}$ and $\mathbf{I}_n$ by α, then, for $i \neq s$, $b_{ij} = a_{ij}$ and, $e_{ij} = \delta_{ij}$, while for $i = s$, $b_{sj} = \alpha a_{sj}$ and $e_{sj} = \alpha\delta_{sj}$.

    Now consider the $i, j-$ element of $\mathbf{EA}$. For $i \neq s$,

$$(\mathbf{EA})_{ij} = \sum_{k=1}^{n} e_{ik}a_{kj} = \sum_{k=1}^{n} \alpha\delta_{ik}a_{kj} = a_{ij} = b_{ij},$$

    while

$$(\mathbf{EA})_{sj} = \sum_{k=1}^{n} e_{sk}a_{kj} = \sum_{k=1}^{n} \alpha\delta_{sk}a_{kj} = b_{sj}$$

    for $j = 1, 2, \cdots, m$. Therefore $\mathbf{EA} = \mathbf{B}$.

11. Let $\mathbf{A}$ be $n \times m$. Now $\mathbf{B}$ and $\mathbf{E}$ are obtained, respectively, from $\mathbf{A}$ and $\mathbf{I}_n$ by adding α times row s to row t. Then, for $i \neq t$, $b_{ij} = a_{ij}$ and $e_{ij} = \delta_{ij}$, while for $i = t$, $b_{tj} = a_{tj} + \alpha a_{sj}$ and $e_{tj} = \delta_{tj} + \alpha\delta_{sj}$.

    Now consider the $i, j-$ element of $\mathbf{EA}$. For $i \neq t$,

$$(\mathbf{EA})_{ij} = \sum_{k=1}^{n} e_{ik}a_{kj} = \sum_{k=1}^{n} \delta_{ik}a_{kj} = a_{ij}$$

    while, for $i = t$,

$$\begin{aligned}(\mathbf{EA})_{tj} &= \sum_{k=1}^{n} e_{tk}a_{kj} = \sum_{k=1}^{n} (\delta_{tk} + \alpha\delta_{sj})a_{kj} \\ &= a_{tj} + \alpha a_{sj} = b_{sj}.\end{aligned}$$

    Therefore $\mathbf{EA} = \mathbf{B}$.

7.3 Reduced Row Echelon Form

For the first three problems a sequence of row operations that reduces the matrix is given, along with $\mathbf{\Omega}$ that reduces $\mathbf{A}$ by multiplication on the left. $\mathbf{\Omega}$ is formed by applying the reducing sequence in order, beginning with $\mathbf{I}_n$. For Problems 4 - 12 only $\mathbf{\Omega}$ and the reduced matrix $\mathbf{A}_R$ are given.

It should be kept in mind that many different sequences of operations can be used to reduced a matrix. However, the final reduced matrix $\mathbf{A}_R$ will be the same regardless of the sequence used.

1. $\mathbf{A}$ is reduced simply by adding row two to row one. Thus

$$\mathbf{\Omega} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \mathbf{A}_R = \begin{pmatrix} 1 & 0 & 5 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

2. We can reduce $\mathbf{A}$ by first adding row two to row one of $\mathbf{I}_2$, then multiplying row one (of the new matrix) by 1/3. Thus proceed:

$$\mathbf{I}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1/3 & -1/3 \\ 0 & 1 \end{pmatrix} = \mathbf{\Omega}.$$

This yields

$$\mathbf{A}_R = \begin{pmatrix} 1 & 0 & 1/3 & 4/3 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

3. We can reduce $\mathbf{A}$ by the following sequence of operations, starting with $\mathbf{I}_4$: interchange rows one and two, then (on the resulting matrix), multiply row one by -1, then add row two to row one. Thus form

$$\mathbf{I}_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$\to \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = \mathbf{\Omega}.$$

Then

$$\mathbf{A}_R = \begin{pmatrix} -1 & -4 & -1 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

4. The matrix is in reduced form already, so $\mathbf{\Omega} = \mathbf{I}_2$ and $\mathbf{A}_R = \mathbf{A}$.

5.
$$\mathbf{\Omega} = \begin{pmatrix} 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & -6 & 17 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \mathbf{A}_R = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

6.
$$\mathbf{\Omega} = \begin{pmatrix} 0 & 1 \\ 1 & -2 \end{pmatrix}, \mathbf{A}_R = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

7.
$$\mathbf{\Omega} = \frac{1}{270}\begin{pmatrix} -8 & -2 & 38 \\ 37 & 43 & -7 \\ 19 & -29 & 11 \end{pmatrix}, \mathbf{A}_R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbf{I}_3$$

8.
$$\mathbf{\Omega} = \begin{pmatrix} -1/3 & 0 \\ 0 & 1 \end{pmatrix}, \mathbf{A}_R = \begin{pmatrix} 1 & -4/3 & -4/3 \\ 0 & 0 & 0 \end{pmatrix}$$

9.
$$\mathbf{\Omega} = \begin{pmatrix} 0 & 1 \\ 1/2 & 1/2 \end{pmatrix}, \mathbf{A}_R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 3/2 & 1/2 \end{pmatrix}$$

10.
$$\mathbf{\Omega} = \frac{1}{4}\begin{pmatrix} 0 & 0 & 1 \\ 4 & -4 & -8 \\ -4 & 8 & 8 \end{pmatrix}, \mathbf{A}_R = \begin{pmatrix} 1 & 0 & 0 & -3/4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

11.
$$\mathbf{\Omega} = \begin{pmatrix} 0 & 1/2 & -1 \\ 0 & 0 & 1 \\ -1/7 & 2/7 & -3/7 \end{pmatrix}, \mathbf{A}_R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbf{I}_3$$

12.
$$\mathbf{\Omega} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 3 & 0 \\ 1 & 0 & -6 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}, \mathbf{A}_R = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

7.4 Row and Column Spaces

1. We find that
$$\mathbf{A}_R = \begin{pmatrix} 1 & 0 & -3/5 \\ 0 & 1 & 3/5 \end{pmatrix}$$
so $\mathbf{A}$ has rank 2.

The rows of $\mathbf{A}$ are
$$\mathbf{R}_1 = (-4, 1, 3) \text{ and } \mathbf{R}_2 = (2, 2, 0).$$

These are linearly independent as vectors in R^3 and form a basis for the row space of **A**.

The columns of **A** are

$$\mathbf{C}_1 = \begin{pmatrix} -4 \\ 2 \end{pmatrix}, \mathbf{C}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \mathbf{C}_3 = \begin{pmatrix} 3 \\ 0 \end{pmatrix}.$$

$\mathbf{C}_1$ and $\mathbf{C}_2$ are linearly independent as vectors in R^2, while

$$\mathbf{C}_3 = -\frac{3}{5}\mathbf{C}_1 + \frac{3}{5}\mathbf{C}_2.$$

Therefore $\mathbf{C}_1$ and $\mathbf{C}_2$ form a basis for the column space, which also has dimension 2.

Note that we can actually read the row and column space dimensions from the reduced matrix, since the rank of **A** is the number of nonzero rows of $\mathbf{A}_R$, and this rank is equal to both the row and column ranks.

In addition, as an example, we looked at the row and column vectors explicitly in this solution, but this is not necessary if all we want is the rank of the matrix. For this, either the row rank or the column rank is sufficient, since these numbers must be equal.

2.

$$\mathbf{A}_R = \begin{pmatrix} 1 & 0 & 7 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix}.$$

Therefore the rank of **A** equals 2, and this is also the row rank and the column rank. The first two rows of **A** are independent in R^3, hence form a basis for the row space, and the first two columns are also independent in R^3 and form a basis for the column space.

3.

$$\mathbf{A}_R = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix},$$

so **A** has rank 2. The first two rows and the two columns of **A** are bases for the row and column spaces, respectively.

4.

$$\mathbf{A}_R = \begin{pmatrix} 1 & 0 & 0 & 1/6 & 1/6 \\ 0 & 1 & 0 & 1/6 & 1/6 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

so **A** has rank 2. The second row is 2 times the first row of **A**, but the first and third rows are independent and form a basis for the row space in R^5. The first two columns form a basis for the column space.

5.
$$\mathbf{A}_R = \begin{pmatrix} 1 & 0 & -1/4 & 1/2 \\ 0 & 1 & -5/4 & 1/2 \end{pmatrix},$$

so $\mathbf{A}$ has dimension 2. The two rows of $\mathbf{A}$ form a basis for the row space in R^4 and the first two columns form a basis for the column space in R^2.

6. $\mathbf{A}$ is in reduced form, so the rank of $\mathbf{A}$ is 2. The two rows form a basis for the row space in R^3 and the first and third columns form a basis for the column space in R^2.

7.
$$\mathbf{A}_R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

so $\mathbf{A}$ has rank 3. The first, second and fourth rows are linearly independent and form a basis for the row space in R^3. All three columns are linearly independent and form a basis for the column space in R^4.

8.
$$\mathbf{A}_R = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

so $\mathbf{A}$ has rank 2. The first two rows span the row space in R^3 and columns two and three span the column space in R^3.

9. We find that $\mathbf{A}_R = \mathbf{I}_3$, so $\mathbf{A}$ has rank 3. The row space has all the rows for a basis and the column space has all the columns.

10.
$$\mathbf{A}_R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

so $\mathbf{A}$ has rank 2. Rows one and three form a basis for the row space in R^3, and columns one and three form a basis for the column space in R^4.

11. $\mathbf{A}_R = \mathbf{I}_3$, so $\mathbf{A}$ has rank 3. All of the rows form a basis for the row space and all of the columns form a basis for the column space.

12.
$$\mathbf{A}_R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -13/2 \\ 0 & 0 & 1 & -7 \end{pmatrix},$$

so $\mathbf{A}$ has rank 3. The rows form a basis for the row space in R^4 and the first three columns for a basis for the column space in R^3.

13.
$$\mathbf{A}_R = \begin{pmatrix} 1 & 0 & -11 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{pmatrix},$$

so $\mathbf{A}$ has rank 2. The first two rows are linearly independent and form a basis for the row space in R^3, and the first two columns form a basis for the column space in R^3.

14.
$$\mathbf{A}_R = \begin{pmatrix} 1 & -2/3 & -1/3 & -1/3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

so $\mathbf{A}$ has rank 1. Either row forms a basis for the row space in R^5, and any of the first four columns forms a basis for the column space in R^2.

15. Use the fact that, for any matrix, the rank, row rank and column rank are the same. Since the rows of $\mathbf{A}$ are the columns of $\mathbf{A}^t$, then

$$\begin{aligned} \text{rank of } \mathbf{A} = & \text{ row rank of } \mathbf{A} \\ \text{column rank of } \mathbf{A}^t = & \text{ rank of } \mathbf{A}^t. \end{aligned}$$

7.5 Linear Homogeneous Systems

In Problems 1 - 12, we use the facts that (1) $\mathbf{AX} = \mathbf{O}$ has the same solutions as $\mathbf{A}_R\mathbf{X} = \mathbf{O}$, and (2) the solution of the reduced system can be read by inspection from the reduced coefficient matrix $\mathbf{A}_R$.

1. The coefficient matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & -1 & 1 \\ 0 & 1 & -1 & 1 \end{pmatrix}$$

has reduced form

$$\mathbf{A}_R = \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 1 \end{pmatrix}.$$

Since rank $(\mathbf{A}) = 2$, the general solution will have $m - \text{rank}(\mathbf{A}) = 4-2 = 2$ arbitrary constants. This is the dimension of the solution space. From the reduced system, we read that

$$x_1 = -x_3 + x_4,$$
$$x_2 = x_3 - x_4.$$

This system is solved by giving x_3 and x_4 any values (hence the solution space has dimension 2), and choosing x_1 and x_2 according to the last equations. Thus,

$$\mathbf{X} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -x_3 + x_4 \\ x_3 - x_4 \\ x_3 \\ x_4 \end{pmatrix} = x_3 \begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix}.$$

It looks nicer to write $x_3 = \alpha$ and $x_4 = \beta$ (both arbitrary numbers) and write the general solution as

$$\mathbf{X} == \alpha \begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix}.$$

2. The coefficient matrix

$$\mathbf{A} = \begin{pmatrix} -3 & 1 & -1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 4 \\ 0 & 0 & -3 & 2 & 1 \end{pmatrix}$$

has reduced form

$$\mathbf{A}_R = \begin{pmatrix} 1 & 0 & 0 & 1/9 & 11/9 \\ 0 & 1 & 0 & 2/3 & 13/3 \\ 0 & 0 & 1 & -2/3 & -1/3 \end{pmatrix}.$$

With $x_4 = \alpha$ and $x_5 = \beta$, the general solution is

$$\mathbf{X} = \alpha \begin{pmatrix} -1/9 \\ -2/3 \\ 2/3 \\ 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} -11/9 \\ -13/3 \\ 1/3 \\ 0 \\ 1 \end{pmatrix}.$$

The solution space has dimension $m - \text{rank}(\mathbf{A}) = 5 - 3 = 2$. We can see this from the fact that the general solution is in terms of two independent column vectors, which form a basis for the solution space.

3. The coefficient matrix

$$\mathbf{A} = \begin{pmatrix} -2 & 1 & 2 \\ 1 & -1 & 0 \\ 1 & 1 & 0 \end{pmatrix}$$

has reduced matrix

$$\mathbf{A}_R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The unique solution of the system is $\mathbf{X} = \mathbf{O}$, the trivial solution. Since $\text{rank}(\mathbf{A}) = 3$, the solution space has dimension $3 - 3 = 0$.

4. The coefficient matrix

$$\mathbf{A} = \begin{pmatrix} 4 & 1 & -3 & 1 \\ 2 & 0 & -1 & 0 \end{pmatrix}$$

has reduced matrix

$$\mathbf{A}_R = \begin{pmatrix} 1 & 0 & -1/2 & 0 \\ 0 & 1 & -1 & 1 \end{pmatrix}.$$

With $x_3 = \alpha$ and $x_4 = \beta$, the general solution is

$$\mathbf{X} = \alpha \begin{pmatrix} 1/2 \\ 1 \\ 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}.$$

These two column vectors form a basis for the solution space of $\mathbf{AX} = \mathbf{O}$, which has dimension $4 - 2 = 2$.

5. The coefficient matrix

$$\mathbf{A} = \begin{pmatrix} 1 & -1 & 3 & -1 & 4 \\ 2 & -2 & 1 & 1 & 0 \\ 1 & 0 & -2 & 0 & 1 \\ 0 & 0 & 1 & 1 & -1 \end{pmatrix}$$

has the reduced matrix

$$\mathbf{A}_R = \begin{pmatrix} 1 & 0 & 0 & 0 & 9/4 \\ 0 & 1 & 0 & 0 & 7/4 \\ 0 & 0 & 1 & 0 & 5/8 \\ 0 & 0 & 0 & 1 & -13/8 \end{pmatrix}.$$

With $x_5 = \alpha$ the general solution is

$$\mathbf{X} = \alpha \begin{pmatrix} -9/4 \\ -7/4 \\ -5/8 \\ 13/8 \\ 1 \end{pmatrix}.$$

The solution space has dimension 1, which is indeed equal to $m - \text{rank}(\mathbf{A}) = 5 - 4$.

6. The coefficient matrix is

$$\mathbf{A} = \begin{pmatrix} 6 & -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 2 \\ 1 & 0 & 0 & 0 & -2 \end{pmatrix}$$

with reduced matrix

$$\mathbf{A}_R = \begin{pmatrix} 1 & 0 & 0 & 0 & -2 \\ 0 & 1 & -1 & 0 & -12 \\ 0 & 0 & 0 & 1 & -4 \end{pmatrix}.$$

With $x_3 = \alpha$ and $x_5 = \beta$ the general solution is

$$\mathbf{X} = \alpha \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 2 \\ 12 \\ 0 \\ 4 \\ 1 \end{pmatrix}.$$

The dimension of the solution space is 2, which we can also determine (without knowing the general solution itself) as $m - \text{rank}(\mathbf{A}) = 5 - 3 = 2$.

7. The coefficient matrix

$$\mathbf{A} = \begin{pmatrix} -10 & -1 & 4 & -1 & 1 & -1 \\ 0 & 1 & -1 & 3 & 0 & 0 \\ 2 & -1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \end{pmatrix}$$

has reduced matrix

$$\mathbf{A}_R = \begin{pmatrix} 1 & 0 & 0 & 0 & 5/6 & 5/9 \\ 0 & 1 & 0 & 0 & 2/3 & 10/9 \\ 0 & 0 & 1 & 0 & 8/3 & 13/9 \\ 0 & 0 & 0 & 1 & 2/3 & 1/9 \end{pmatrix}.$$

With $x_5 = \alpha$ and $x_6 = \beta$ the general solution is

$$\mathbf{X} = \alpha \begin{pmatrix} -5/6 \\ -2/3 \\ -8/3 \\ -2/3 \\ 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} -5/9 \\ -10/9 \\ -13/9 \\ -1/9 \\ 0 \\ 1 \end{pmatrix}.$$

The solution space has dimension 2, which is also $m - \text{rank}(\mathbf{A}) = 6 - 4 = 2$.

8. The coefficient matrix

$$\mathbf{A} = \begin{pmatrix} 8 & 0 & -2 & 0 & 0 & 1 \\ 2 & -1 & 0 & 3 & 0 & -1 \\ 0 & 1 & 1 & 0 & -2 & -1 \\ 0 & 0 & 0 & 1 & -3 & 2 \end{pmatrix}$$

has reduced matrix

$$\mathbf{A}_R = \begin{pmatrix} 1 & 0 & 0 & 0 & 7/6 & -5/4 \\ 0 & 1 & 0 & 0 & -20/3 & 9/2 \\ 0 & 0 & 1 & 0 & 14/3 & -11/2 \\ 0 & 0 & 0 & 1 & -3 & 2 \end{pmatrix}.$$

Let $x_5 = \alpha$ and $x_6 = \beta$ to write the general solution

$$\mathbf{X} = \begin{pmatrix} -7/6 \\ 20/3 \\ -14/3 \\ 3 \\ 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 5/4 \\ -9/2 \\ 11/2 \\ -2 \\ 0 \\ 1 \end{pmatrix}.$$

The solution space has dimension 2. This dimension is also $m - \text{rank}(\mathbf{A}) = 6 - 4 = 2$.

9. Notice that the equations have unknowns x_1, x_2, x_4, x_5, but no x_3. Thus we have a system of three equations in four unknowns, but the unknowns are called x_1, x_2, x_4, x_5. The coefficient matrix is

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & -3 & 1 \\ 2 & -1 & 1 & 0 \\ 2 & -3 & 0 & 4 \end{pmatrix}$$

with reduced form

$$\mathbf{A}_R = \begin{pmatrix} 1 & 0 & 0 & -5/14 \\ 0 & 1 & 0 & -11/17 \\ 0 & 0 & 1 & -6/7 \end{pmatrix}.$$

The first three unknowns, x_1, x_2, x_4, depend on the fourth, x_5, which can be given any value α. The general solution is read from $\mathbf{A}_R$:

$$\mathbf{X} = \alpha \begin{pmatrix} 5/14 \\ 11/7 \\ 6/7 \\ 1 \end{pmatrix}.$$

The solution space is clearly one-dimensional. We can also see this dimension from $m -$ rank$(\mathbf{A}) = 4 - 3 = 1$.

10. The coefficient matrix

$$\mathbf{A} = \begin{pmatrix} 4 & -3 & 0 & 1 & 1 & -3 \\ 0 & 2 & 0 & 4 & -1 & -6 \\ 3 & -2 & 0 & 0 & 4 & -1 \\ 2 & 1 & -3 & 4 & 0 & 0 \end{pmatrix}$$

has reduced matrix

$$\mathbf{A}_R = \begin{pmatrix} 1 & 0 & 0 & 0 & -41/6 & -2/3 \\ 0 & 1 & 0 & 0 & -49/6 & -1/3 \\ 0 & 0 & 1 & 0 & -13/6 & -7/3 \\ 0 & 0 & 0 & 1 & 23/6 & -4/3 \end{pmatrix}.$$

Let $x_5 = \alpha$ and $x_6 = \beta$ to write the general solution

$$\mathbf{X} = \alpha \begin{pmatrix} 41/6 \\ 49/6 \\ 13/6 \\ -23/6 \\ 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 2/3 \\ 1/3 \\ 7/3 \\ 4/3 \\ 0 \\ 1 \end{pmatrix}.$$

Here $\text{rank}(\mathbf{A}) = 4$ and the solution space has dimension $6 - 4 = 2$.

11. The coefficient matrix is

$$\mathbf{A} = \begin{pmatrix} 1 & -2 & 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -1 & 1 & -2 & 3 \\ 1 & 0 & 0 & 0 & -1 & 2 & 0 \\ 2 & 0 & 0 & -3 & 1 & 0 & 0 \end{pmatrix}$$

and

$$\mathbf{A}_R = \begin{pmatrix} 1 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 1 & 0 & 0 & -1 & 3/2 & -1/2 \\ 0 & 0 & 1 & 0 & 0 & -2/3 & 3 \\ 0 & 0 & 0 & 1 & -1 & 4/3 & 0 \end{pmatrix}.$$

With $x_5 = \alpha$, $x_6 = \beta$ and $x_7 = \gamma$, the general solution is

$$\mathbf{X} = \alpha \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} -2 \\ -3/2 \\ 2/3 \\ -4/3 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \gamma \begin{pmatrix} 0 \\ 1/2 \\ -3 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

The solution space has dimension 3, consistent with $m - \text{rank}(\mathbf{A}) = 7 - 4 = 3$.

12. The coefficient matrix is

$$\mathbf{A} = \begin{pmatrix} 2 & 0 & 0 & 0 & -4 & 0 & 1 & 1 \\ 0 & 2 & 0 & 0 & 0 & -1 & 1 & -1 \\ 0 & 0 & 1 & -4 & 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 & -1 & 0 \end{pmatrix}$$

with reduced form

$$\mathbf{A}_R = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -3 & 7/2 & -1/2 \\ 0 & 1 & 0 & 0 & 0 & -1/2 & 1/2 & -1/2 \\ 0 & 0 & 1 & 0 & 0 & -2/3 & 2/3 & -1 \\ 0 & 0 & 0 & 1 & 0 & -1/6 & 1/6 & -1/2 \\ 0 & 0 & 0 & 0 & 1 & -3/2 & 3/2 & -1/2 \end{pmatrix}.$$

The general solution is in terms of x_6, x_7 and x_8, which can be assigned values arbitrarily:

$$\mathbf{X} = \alpha \begin{pmatrix} 3 \\ 1/2 \\ 2/3 \\ 1/6 \\ 3/2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} -7/2 \\ -1/2 \\ -2/3 \\ -1/6 \\ -3/2 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \gamma \begin{pmatrix} 1/2 \\ 1/2 \\ 1 \\ 1/2 \\ 1/2 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

The solution space has dimension 3, which is $m-\ \mathrm{rank}(\mathbf{A}) = 8-5=3$.

13. Yes. All that is required is that $m - \mathrm{rank}(\mathbf{A}) > 0$, so that the solution space has something in it. As a specific example, consider the system $\mathbf{AX} = \mathbf{O}$, with

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & -1 \\ 3 & 0 & 9 \end{pmatrix}.$$

This is a homogeneous system with three equations in three unknowns. We find that

$$\mathbf{A}_R = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix},$$

so $\mathbf{A}$ has rank 2. The solution space has dimension $3-2=1$, hence has nonzero vectors in it. The general solution is

$$\mathbf{X} = \alpha \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}.$$

14. Suppose $\mathbf{A}$ is $n \times m$. Let the columns of $\mathbf{A}$ be $\mathbf{C}_1, \cdots, \mathbf{C}_m$, written as column matrices. If

$$\mathbf{X} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{pmatrix}$$

then $\mathbf{AX} = \mathbf{O}$ is equivalent to

$$a_1\mathbf{C}_1 + a_2\mathbf{C}_2 + \cdots + a_m\mathbf{C}_m = \mathbf{O},$$

the $n \times 1$ zero matrix.

Now we can prove the proposition. If the columns of $\mathbf{A}$ are linearly dependent, then there are numbers $a_1, \cdots, a_m$, not all zero, such that $\mathbf{AX} = \mathbf{O}$.

This yields a linear combination of the columns equal to the zero vector, but with not all coefficients zero, so the columns are linearly dependent.

Conversely, if the columns are linearly dependent, then there are numbers $a_1, \cdots, a_m$, not all zero, such that

$$a_1\mathbf{C}_1 + a_2\mathbf{C}_2 + \cdots + a_m\mathbf{c}_m = \mathbf{O},$$

and then $x_1 = a_1, \cdots, x_m = a_m$ is a nontrivial solution of the system.

15. (a) Let $\mathbf{R}_1, \cdots, \mathbf{R}_n$ be the rows of $\mathbf{A}$. These vectors span R, the row space of the matrix. Now, $\mathbf{X}$ is in the solution space if and only if $\mathbf{X} = \mathbf{O}$, and this is true exactly when $\mathbf{R}_j \cdot \mathbf{X} = 0$ for $j = 1, \cdots, n$, which in turn is true if and only if $\mathbf{X}$ is orthogonal to each row of $\mathbf{A}$. But this is equivalent to $\mathbf{X}$ being orthogonal to every linear combination of the rows of $\mathbf{A}$, hence to every vector in the row space of $\mathbf{A}$. Therefore the solution space of $\mathbf{A}$ is the orthogonal complement of the row space, or

$$R^\perp = S(\mathbf{A}).$$

Since the columns of $\mathbf{A}^t$ are the rows of $\mathbf{A}$, the conclusion that $C^\perp = S(\mathbf{A}^t)$ follow immediately from the reasoning of part (a).

7.6 Nonhomogeneous Systems

1. The augmented matrix is

$$\begin{pmatrix} 3 & -2 & 1 & \vdots & 6 \\ 1 & 10 & -1 & \vdots & 2 \\ -3 & -2 & 1 & \vdots & 0 \end{pmatrix}$$

with reduced matrix

$$\begin{pmatrix} 1 & 0 & 0 & \vdots & 1 \\ 0 & 1 & 0 & \vdots & 1/2 \\ 0 & 0 & 1 & \vdots & 4 \end{pmatrix}.$$

Since $\text{rank}(\mathbf{A}) = \text{rank}([\mathbf{A} \vdots \mathbf{B}]) = 3$, and this is the number of unknowns, the system has the unique solution

$$\mathbf{X} = \begin{pmatrix} 1 \\ 1/2 \\ 4 \end{pmatrix}.$$

2. The augmented matrix is

$$\begin{pmatrix} 4 & -2 & 3 & 10 & \vdots & 1 \\ 1 & 0 & 0 & -3 & \vdots & 8 \\ 2 & -3 & 0 & 1 & \vdots & 16 \end{pmatrix}.$$

The reduced form of this matrix is

$$\begin{pmatrix} 1 & 0 & 0 & -3 & \vdots & 8 \\ 0 & 1 & 0 & -7/3 & \vdots & 0 \\ 0 & 0 & 1 & 52/9 & \vdots & -31/3 \end{pmatrix}.$$

Since

$$\text{rank}(\mathbf{A}) = \text{rank}([\mathbf{A}\vdots\mathbf{B}]) = 3,$$

the system has solutions. From the reduced augmented matrix we read the general solution

$$\mathbf{X} = \begin{pmatrix} 8 \\ 0 \\ -31/3 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} 3 \\ 7/3 \\ -52/9 \\ 1 \end{pmatrix}.$$

3. The augmented matrix is

$$\begin{pmatrix} 2 & -3 & 0 & 1 & 0 & -1 & \vdots & 0 \\ 3 & 0 & -2 & 0 & 1 & 0 & \vdots & 1 \\ 0 & 1 & 0 & -1 & 0 & 6 & \vdots & 3 \end{pmatrix}.$$

This reduces to

$$\begin{pmatrix} 1 & 0 & 0 & -1 & 0 & 17/2 & \vdots & 9/2 \\ 0 & 1 & 0 & -1 & 0 & 6 & \vdots & 3 \\ 0 & 0 & 1 & -3/2 & -1/2 & 51/4 & \vdots & 25/4 \end{pmatrix}.$$

Then $\text{rank}(\mathbf{A}) = \text{rank}([\mathbf{A}\vdots\mathbf{B}])$, the system has solutions. From the reduced augmented matrix we read the general solution

$$\mathbf{X} = \begin{pmatrix} 9/2 \\ 3 \\ 25/4 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} 1 \\ 1 \\ 3/2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 0 \\ 1/2 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \gamma \begin{pmatrix} -17/2 \\ -6 \\ -51/4 \\ 0 \\ 0 \\ 1 \end{pmatrix},$$

with α, β and γ arbitrary.

4. The augmented matrix is

$$\begin{pmatrix} 2 & -3 & \vdots & 1 \\ -1 & 3 & \vdots & 0 \\ 1 & -4 & \vdots & 3 \end{pmatrix}.$$

This has reduced form

$$\begin{pmatrix} 1 & 0 & \vdots & 0 \\ 0 & 1 & \vdots & 0 \\ 0 & 0 & \vdots & 1 \end{pmatrix}.$$

Since $\text{rank}(\mathbf{A}) = 2$ and $\text{rank}([\mathbf{A}\vdots\mathbf{B}]) = 3$, this system has no solution.

If you try to solve this relatively simple system by elimination of unknowns (high school algebra), you will reach an inconsistency that explains why this system has no solution.

5. The augmented matrix is

$$\begin{pmatrix} 0 & 3 & 0 & -4 & 0 & 0 & \vdots & 10 \\ 1 & -3 & 0 & 0 & 4 & -1 & \vdots & 8 \\ 0 & 1 & 1 & -6 & 0 & 1 & \vdots & -9 \\ 1 & -1 & 0 & 0 & 0 & 1 & \vdots & 0 \end{pmatrix}.$$

The reduced form of this is

$$\begin{pmatrix} 1 & 0 & 0 & 0 & -2 & 2 & \vdots & -4 \\ 0 & 1 & 0 & 0 & -2 & 1 & \vdots & -4 \\ 0 & 0 & 1 & 0 & -7 & 9/2 & \vdots & -38 \\ 0 & 0 & 0 & 1 & -3/2 & 3/4 & \vdots & -11/2 \end{pmatrix}.$$

Since $\text{rank}(\mathbf{A}) = \text{rank}([\mathbf{A}\vdots\mathbf{B}])$, the system has solutions, which we read from the reduced augmented matrix. The general solution is

$$\mathbf{X} = \begin{pmatrix} -4 \\ -4 \\ -38 \\ -11/2 \\ 0 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} 2 \\ 2 \\ 7 \\ 3/2 \\ 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} -2 \\ -1 \\ -9/2 \\ -3/4 \\ 0 \\ 1 \end{pmatrix},$$

with α and β arbitrary.

6. The augmented matrix is

$$\begin{pmatrix} 2 & -3 & 0 & 1 & \vdots & 1 \\ 0 & 3 & 1 & -1 & \vdots & 0 \\ 2 & -3 & 10 & 0 & \vdots & 0 \end{pmatrix}.$$

This has reduced matrix

$$\begin{pmatrix} 1 & 0 & 0 & 1/20 & \vdots & 11/20 \\ 0 & 1 & 0 & -9/30 & \vdots & 1/30 \\ 0 & 0 & 1 & -1/10 & \vdots & -1/10 \end{pmatrix}.$$

Since $\text{rank}(\mathbf{A}) = \text{rank}([\mathbf{A} \vdots \mathbf{B}])$, the system has solutions. We read from the reduced augmented matrix that the general solution has the form

$$\mathbf{X} = \begin{pmatrix} 11/20 \\ 1/30 \\ -1/10 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} -1/20 \\ 9/30 \\ 1/10 \\ 1 \end{pmatrix},$$

with α arbitrary.

7. The augmented matrix is

$$\begin{pmatrix} 8 & -4 & 0 & 0 & 10 & \vdots & 1 \\ 0 & 1 & 0 & 1 & -1 & \vdots & 2 \\ 0 & 0 & 1 & -3 & 2 & \vdots & 0 \end{pmatrix}.$$

(The x_1 column has been omitted since x_1 does not appear in the equations). The reduced form of this matrix is

$$\begin{pmatrix} 1 & 0 & 0 & 1/2 & 3/4 & \vdots & 9/8 \\ 0 & 1 & 0 & 1 & -1 & \vdots & 2 \\ 0 & 0 & 1 & -3 & 2 & \vdots & 0 \end{pmatrix}.$$

Since $\text{rank}(\mathbf{A}) = \text{rank}([\mathbf{A} \vdots \mathbf{B}])$, this system has solutions, which we read as

$$\mathbf{X} = \begin{pmatrix} 9/8 \\ 2 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} -1/2 \\ -1 \\ 3 \\ 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} -3/4 \\ 1 \\ -2 \\ 0 \\ 1 \end{pmatrix},$$

in which α and β are arbitrary.

8. The augmented matrix is

$$\begin{pmatrix} 2 & 0 & -3 & \vdots & 1 \\ 1 & -1 & 1 & \vdots & 1 \\ 2 & -4 & 1 & \vdots & 2 \end{pmatrix}.$$

This has reduced form

$$\begin{pmatrix} 1 & 0 & 0 & \vdots & 3/4 \\ 0 & 1 & 0 & \vdots & -1/12 \\ 0 & 0 & 1 & \vdots & 1/6 \end{pmatrix}.$$

Now $\text{rank}(\mathbf{A}) = \text{rank}([\mathbf{A}\vdots\mathbf{B}])$ = number of unknowns = 3, so the system has the unique solution

$$\mathbf{X} = \begin{pmatrix} 3/4 \\ -1/12 \\ 1/6 \end{pmatrix}.$$

9. The augmented matrix

$$\begin{pmatrix} 0 & 0 & 14 & 0 & -3 & 0 & 1 & \vdots & 2 \\ 1 & 1 & 1 & -1 & 0 & 1 & 0 & \vdots & -4 \end{pmatrix}.$$

This has reduced form

$$\begin{pmatrix} 1 & 1 & 0 & -1 & 3/14 & 1 & -1/14 & \vdots & -29/7 \\ 0 & 0 & 1 & 0 & -3/14 & 0 & 1/14 & \vdots & 1/7 \end{pmatrix}.$$

Note that $\text{rank}(\mathbf{A}) = \text{rank}(\mathbf{A}\vdots\mathbf{B}])$, so there are solutions. We read from the augmented matrix that the general solution has the form

$$\mathbf{X} =$$

$$\begin{pmatrix} -29/7 \\ 0 \\ 1/7 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \gamma \begin{pmatrix} -3/14 \\ 0 \\ 3/14 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \delta \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \epsilon \begin{pmatrix} 1/14 \\ 0 \\ -1/14 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix},$$

with α, β, γ, δ and ϵ arbitrary.

10. The augmented matrix is

$$\begin{pmatrix} 3 & -2 & \vdots & -1 \\ 4 & 3 & \vdots & 4 \end{pmatrix}.$$

This has reduced form

$$\begin{pmatrix} 1 & 0 & \vdots & 5/17 \\ 0 & 1 & \vdots & 16/17 \end{pmatrix}.$$

Since $\text{rank}(\mathbf{A}) = \text{rank}([\mathbf{A} \vdots \mathbf{B}])$ = number of unknowns = 2, the system has a unique solution, which is

$$\mathbf{X} = \begin{pmatrix} 5/17 \\ 16/17 \end{pmatrix}.$$

11. The augmented matrix is

$$\begin{pmatrix} 7 & -3 & 4 & 0 & \vdots & -7 \\ 2 & 1 & -1 & 4 & \vdots & 6 \\ 0 & 1 & 0 & -3 & \vdots & -5 \end{pmatrix}$$

with reduced form

$$\begin{pmatrix} 1 & 0 & 0 & 19/15 & \vdots & 22/15 \\ 0 & 1 & 0 & -3 & \vdots & -5 \\ 0 & 0 & 1 & -67/13 & \vdots & -121/15 \end{pmatrix}.$$

Now

$$\text{rank }(\mathbf{A}) = 3 = \text{ rank}([\mathbf{A} \vdots \mathbf{B}]),$$

so the system has solutions. We read from the reduced system that

$$\mathbf{X} = \begin{pmatrix} 22/15 \\ -5 \\ -121/15 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} -19/15 \\ 3 \\ 67/15 \\ 1 \end{pmatrix},$$

in which α is arbitrary.

12. The augmented coefficient matrix is

$$\begin{pmatrix} -4 & 5 & -6 & \vdots & 2 \\ 2 & -6 & 1 & \vdots & -5 \\ -6 & 16 & -11 & \vdots & 1 \end{pmatrix},$$

with reduced form

$$\begin{pmatrix} 1 & 0 & 0 & \vdots & -137/48 \\ 0 & 1 & 0 & \vdots & 1/6 \\ 0 & 0 & 1 & \vdots & 41/24 \end{pmatrix}.$$

Now, the coefficient matrix and its augmented matrix have the same rank 3, so the system has solution(s). Further, this rank is the number of unknowns, 3, so the solution is unique

$$\mathbf{X} = \begin{pmatrix} -137/48 \\ 1/6 \\ 41/24 \end{pmatrix}.$$

13. The augmented matrix is

$$\begin{pmatrix} 4 & -1 & 4 & \vdots & 1 \\ 1 & 1 & -5 & \vdots & 0 \\ -2 & 1 & 7 & \vdots & 4 \end{pmatrix},$$

with reduced form

$$\begin{pmatrix} 1 & 0 & 0 & \vdots & 16/57 \\ 0 & 1 & 0 & \vdots & 99/57 \\ 0 & 0 & 1 & \vdots & 23/57 \end{pmatrix}.$$

Since

$$\text{rank}(\mathbf{A}) = \text{ rank}([\mathbf{A}\vdots\mathbf{B}]) = \text{ number of unknowns } = 3,$$

the system has the unique solution

$$\mathbf{X} = \begin{pmatrix} 16/57 \\ 99/57 \\ 23/57 \end{pmatrix}.$$

14. The augmented matrix is

$$\begin{pmatrix} -6 & 2 & -1 & 1 & \vdots & 0 \\ 1 & 4 & 0 & -1 & \vdots & -5 \\ 1 & 1 & 1 & -7 & \vdots & 0 \end{pmatrix},$$

with reduced form

$$\begin{pmatrix} 1 & 0 & 0 & 21/23 & \vdots & -15/23 \\ 0 & 1 & 0 & -11/23 & \vdots & -25/23 \\ 0 & 0 & 1 & -171/23 & \vdots & 40/23 \end{pmatrix}.$$

Now $\mathbf{A}$ and $[\mathbf{A} \vdots \mathbf{B}]$ have the same rank, so the system has solutions which we read from the reduced augmented matrix

$$\mathbf{X} = \begin{pmatrix} -15/23 \\ -25/23 \\ 40/23 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} -21/33 \\ 11/23 \\ 171/23 \\ 1 \end{pmatrix}.$$

15. Write

$$\mathbf{X} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{pmatrix}$$

and let $\mathbf{C}_1, \cdots, \mathbf{C}_n$ be the columns of $\mathbf{A}$. Now $\mathbf{AX} = \mathbf{B}$ if and only if

$$a_1\mathbf{C}_1 + a_2\mathbf{C}_2 + \cdots + a_m\mathbf{C}_m = \mathbf{B}.$$

This means that the system has a solution $\mathbf{X}$ if and only if $\mathbf{X}$ is a linear combination of the columns of $\mathbf{A}$, hence is in the column space of $\mathbf{A}$.

7.7 Matrix Inverses

The most efficient way of computing a matrix inverse is by a software routine, such as MAPLE. In these problems we go through the reduction method in Problem 1 as an illustration, and then give just the inverse matrix for the remaining problems.

1. Reduce

$$\begin{pmatrix} -1 & 2 & \vdots & 1 & 0 \\ 2 & 1 & \vdots & 0 & 1 \end{pmatrix} \rightarrow \text{ add two times row one to row two } \rightarrow \begin{pmatrix} -1 & 2 & \vdots & 1 & 0 \\ 0 & 5 & \vdots & 2 & 1 \end{pmatrix}$$

$$\rightarrow \text{ multiply row one by } -1 \rightarrow \begin{pmatrix} 1 & -2 & \vdots & -1 & 0 \\ 0 & 5 & \vdots & 2 & 1 \end{pmatrix}$$

$$\rightarrow \text{ multiply row two by } 1/5 \rightarrow \begin{pmatrix} 1 & -2 & \vdots & -1 & 0 \\ 0 & 1 & \vdots & 2/5 & 1/5 \end{pmatrix}$$

$$\rightarrow \text{ add 2 times row two to row one } \rightarrow \begin{pmatrix} 1 & 0 & \vdots & -1/5 & 2/5 \\ 0 & 1 & \vdots & 2/5 & 1/5 \end{pmatrix}.$$

Because $\mathbf{I}_2$ has appeared on the left, the right two columns form the inverse matrix:

$$\mathbf{A}^{-1} = \frac{1}{5}\begin{pmatrix} -1 & 2 \\ 2 & 1 \end{pmatrix}.$$

2. The matrix is singular (has no inverse) because

$$\mathbf{A}_R = \begin{pmatrix} 1 & 1/4 \\ 0 & 0 \end{pmatrix} \neq \mathbf{I}_2.$$

3.

$$\mathbf{A}^{-1} = \frac{1}{12}\begin{pmatrix} -2 & 2 \\ 1 & 5 \end{pmatrix}$$

4.

$$\mathbf{A}^{-1} = -\frac{1}{4}\begin{pmatrix} 4 & 0 \\ -4 & -1 \end{pmatrix}$$

5.

$$\mathbf{A}^{-1} = \frac{1}{12}\begin{pmatrix} 3 & -2 \\ -3 & 6 \end{pmatrix}$$

6.

$$\mathbf{A}^{-1} = \frac{1}{56}\begin{pmatrix} 64 & -4 & 49 \\ -8 & 4 & -7 \\ 0 & 0 & 14 \end{pmatrix}$$

7.

$$\mathbf{A}^{-1} = \frac{1}{31}\begin{pmatrix} -6 & 11 & 2 \\ 3 & 10 & -1 \\ 1 & -7 & 10 \end{pmatrix}$$

8. $\mathbf{A}^{-1}$ does not exist, because

$$\mathbf{A}_R = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \neq \mathbf{I}_3.$$

9.

$$\mathbf{A}^{-1} = -\frac{1}{12}\begin{pmatrix} 6 & -6 & 0 \\ -3 & -9 & 2 \\ 3 & -3 & -2 \end{pmatrix}$$

10. $\mathbf{A}^{-1}$ does not exist because

$$\mathbf{A}_R = \begin{pmatrix} 1 & 0 & 28/27 \\ 0 & 1 & 14/9 \\ 0 & 0 & 0 \end{pmatrix} \neq \mathbf{I}_3.$$

11.

$$\mathbf{X} = \mathbf{A}^{-1}\mathbf{B} = \frac{1}{11}\begin{pmatrix} -1 & -1 & 8 & 4 \\ -9 & 2 & -5 & 14 \\ 2 & 2 & -5 & 3 \\ 3 & 3 & -2 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 0 \\ -5 \end{pmatrix} = \frac{1}{11}\begin{pmatrix} -23 \\ -75 \\ -9 \\ 14 \end{pmatrix}$$

12.
$$\mathbf{X} = \mathbf{A}^{-1}\mathbf{B} = \frac{1}{55}\begin{pmatrix} 5 & 5 & 5 \\ -10 & 34 & 23 \\ 5 & 6 & 17 \end{pmatrix}\begin{pmatrix} 4 \\ 0 \\ 5 \end{pmatrix} = \frac{1}{11}\begin{pmatrix} 9 \\ 15 \\ 13 \end{pmatrix}$$

13.
$$\mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$$
$$= -\frac{1}{28}\begin{pmatrix} -11 & -12 & -9 \\ -3 & -16 & -5 \\ -8 & -24 & -4 \end{pmatrix}\begin{pmatrix} -4 \\ 5 \\ 8 \end{pmatrix} = \frac{1}{7}\begin{pmatrix} 22 \\ 27 \\ 30 \end{pmatrix}$$

14.
$$\mathbf{X} = \mathbf{A}^{-1}\mathbf{B} = \frac{1}{52}\begin{pmatrix} 4 & 4 & 0 \\ 7 & -6 & 39 \\ 1 & 14 & 13 \end{pmatrix}\begin{pmatrix} 4 \\ -5 \\ 0 \end{pmatrix} = \frac{1}{52}\begin{pmatrix} -4 \\ 58 \\ -66 \end{pmatrix}$$

15.
$$\mathbf{X} = \mathbf{A}^{-1}\mathbf{B} = -\frac{1}{25}\begin{pmatrix} 5 & -15 & -15 \\ -10 & 15 & 10 \\ -5 & 10 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ -7 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} -21 \\ 14 \\ 0 \end{pmatrix}$$

7.8 Least Squares Vectors and Data Fitting

1. We have
$$\mathbf{A} = \begin{pmatrix} 1 & 1 \\ -2 & 3 \end{pmatrix} \text{ and } \mathbf{B} = \begin{pmatrix} 4 \\ -1 \end{pmatrix}.$$
Compute
$$\mathbf{A}^t\mathbf{A} = \begin{pmatrix} 5 & -5 \\ -5 & 10 \end{pmatrix} \text{ and } (\mathbf{A}^t\mathbf{A})^{-1} = \begin{pmatrix} 2/5 & 1/5 \\ 1/5 & 1/5 \end{pmatrix}.$$
Finally,
$$\mathbf{A}^t\mathbf{B} = \begin{pmatrix} 6 \\ 1 \end{pmatrix}.$$
The solution is
$$\mathbf{X}^* = (\mathbf{A}^t\mathbf{A})^{-1}(\mathbf{A}^t\mathbf{B}) = \begin{pmatrix} 13/5 \\ 7/5 \end{pmatrix}.$$

2. Compute
$$\mathbf{A}^t\mathbf{A} = \begin{pmatrix} 26 & -6 \\ -6 & 20 \end{pmatrix} \text{ and } (\mathbf{A}^t\mathbf{A})^{-1} = \begin{pmatrix} 5/121 & 3/242 \\ 3/242 & 13/242 \end{pmatrix}.$$
Further,
$$\mathbf{A}^t\mathbf{B} = \begin{pmatrix} -6 \\ -2 \end{pmatrix}.$$
Then
$$\mathbf{X}^* = (\mathbf{A}^t\mathbf{A})^{-1}(\mathbf{A}^t\mathbf{B}) = \begin{pmatrix} -3/11 \\ -2/11 \end{pmatrix}.$$

3. Compute

$$\mathbf{A}^t\mathbf{A} = \begin{pmatrix} 37 & 12 \\ 12 & 4 \end{pmatrix} \text{ and } (\mathbf{A}^t\mathbf{A})^{-1} = \begin{pmatrix} 1 & -3 \\ -3 & 37/4 \end{pmatrix}.$$

Next,

$$\mathbf{A}^t\mathbf{B} = \begin{pmatrix} -26 \\ -8 \end{pmatrix}.$$

Then

$$\mathbf{X}^* = (\mathbf{A}^t\mathbf{A})^{-1}(\mathbf{A}^t\mathbf{B}) = \begin{pmatrix} -2 \\ 4 \end{pmatrix}.$$

4. We find that

$$\mathbf{A}^t\mathbf{A} = \begin{pmatrix} 5 & -5 & -12 \\ -5 & 10 & 13 \\ -12 & 13 & 29 \end{pmatrix}$$

and this is a singular matrix. Thus obtain values of $\mathbf{X}^*$ as solutions of $\mathbf{AX}^* = \mathbf{B}_S$, where $\mathbf{B}_S$ is the orthogonal projection of $\mathbf{B}$ onto the column space S of $\mathbf{A}$. The first two columns of $\mathbf{A}$ are linearly independent and form a basis for R^2, so $S = R^2$. Since $\mathbf{B}$ is in R^2, then $\mathbf{B}_S = \mathbf{B}$. Therefore solve the nonhomogeneous system

$$\mathbf{AX}^* = \mathbf{B}$$

to obtain

$$\mathbf{X}^* = \alpha \begin{pmatrix} 7/3 \\ 1 \\ 5/3 \end{pmatrix} + \begin{pmatrix} -2 \\ 0 \\ -1 \end{pmatrix},$$

in which α is an arbitrary constant.

5. As in Problem 4, we find that $\mathbf{A}^t\mathbf{A}$ is singular. Further, we also find that $\mathbf{B}_S = \mathbf{B}$, so solve

$$\mathbf{AX}^* = \mathbf{B}$$

to obtain

$$\mathbf{X}^* = \alpha \begin{pmatrix} 7 \\ 6 \\ 7 \\ 1 \end{pmatrix} + \begin{pmatrix} -15 \\ -31/3 \\ -44/3 \\ 0 \end{pmatrix},$$

with α an arbitrary constant.

6. Form

$$\mathbf{A} = \begin{pmatrix} 1 & 1 \\ -2 & 3 \\ 0 & -1 \\ 2 & 2 \\ -3 & 7 \end{pmatrix},$$

so

$$\mathbf{A}^t = \begin{pmatrix} 1 & -2 & 0 & 2 & -3 \\ 1 & 3 & -1 & 2 & 7 \end{pmatrix}.$$

Then

$$\mathbf{A}^t\mathbf{A} = \begin{pmatrix} 18 & -22 \\ -22 & 64 \end{pmatrix} \text{ and } (\mathbf{A}^t\mathbf{A})^{-1} = \begin{pmatrix} 16/167 & 11/334 \\ 11/334 & 9/334 \end{pmatrix}.$$

Next, compute

$$\mathbf{A}^t\mathbf{B} = \begin{pmatrix} 6 \\ 6 \end{pmatrix}.$$

Then

$$\mathbf{X}^* = (\mathbf{A}^t\mathbf{A})^{-1}(\mathbf{A}^t\mathbf{B}) = \begin{pmatrix} -63/167 \\ -6/167 \end{pmatrix}.$$

7. We have

$$\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 1 & 3 \\ 1 & 5 \\ 1 & 7 \\ 1 & 9 \end{pmatrix} \text{ and } \mathbf{B} = \begin{pmatrix} 3.8 \\ 11.7 \\ 20.6 \\ 26.5 \\ 35.2 \end{pmatrix}.$$

Compute

$$\mathbf{A}^t\mathbf{A} = \begin{pmatrix} 5 & 25 \\ 25 & 165 \end{pmatrix} \text{ and } (\mathbf{A}^t\mathbf{A})^{-1} = \begin{pmatrix} 33/40 & -1/8 \\ -1/8 & 1/40 \end{pmatrix}.$$

Further,

$$\mathbf{A}^t\mathbf{B} = \begin{pmatrix} 97.80000 \\ 644.20000 \end{pmatrix}.$$

Then

$$\mathbf{X}^* = (\mathbf{A}^t\mathbf{A})^{-1}(\mathbf{A}^t\mathbf{B}) = \begin{pmatrix} 0.1599 \\ 3.8799 \end{pmatrix}.$$

The line has the equation $y = a + bx$, with $a = 3.8799$ and $b = 0.1599$.

8. We have

$$\mathbf{A} = \begin{pmatrix} 1 & -5 \\ 1 & -3 \\ 1 & -2 \\ 1 & 0 \\ 1 & 1 \\ 1 & 3 \\ 1 & 6 \end{pmatrix} \text{ and } \mathbf{B} = \begin{pmatrix} 21.2 \\ 13.6 \\ 10.7 \\ 4.2 \\ 2.4 \\ -3.7 \\ -14.2 \end{pmatrix}.$$

Then

$$\mathbf{A}^t\mathbf{A} = \begin{pmatrix} 7 & 0 \\ 0 & 84 \end{pmatrix} \text{ and } (\mathbf{A}^t\mathbf{A})^{-1} = \begin{pmatrix} 1/7 & 0 \\ 0 & 1/84 \end{pmatrix}.$$

Next,

$$\mathbf{A}^t\mathbf{B} = \begin{pmatrix} 34.20000 \\ -262.10000 \end{pmatrix}.$$

Finally, compute

$$\mathbf{X}^* = (\mathbf{A}^t\mathbf{A})^{-1}(\mathbf{A}^t\mathbf{B}) = \begin{pmatrix} 60.97750 \\ -10.82750 \end{pmatrix}.$$

The line has equation $y = a + bx$, where $a = -10.82750$ and $b = 60.97750$.

9. We have

$$\mathbf{A} = \begin{pmatrix} 1 & -3 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 4 \\ 1 & 7 \end{pmatrix} \text{ and } \mathbf{B} = \begin{pmatrix} -23 \\ -8.2 \\ -4.6 \\ -0.5 \\ 7.3 \\ 19.2 \end{pmatrix}.$$

Then

$$\mathbf{A}^t\mathbf{A} = \begin{pmatrix} 6 & 11 \\ 11 & 79 \end{pmatrix} \text{ and } (\mathbf{A}^t\mathbf{A})^{-1} = \begin{pmatrix} 79/353 & -11/353 \\ -11/353 & 6/353 \end{pmatrix}.$$

Next, compute

$$\mathbf{A}^t\mathbf{B} = \begin{pmatrix} -9.79999 \\ 227 \end{pmatrix}.$$

Then

$$\mathbf{X}^* = \begin{pmatrix} -9.266855 \\ 4.167394 \end{pmatrix}.$$

The equation of the line is $y = a + bx$, with $a = 4.167394$ and $b = -9.266855$.

10. We have

$$\mathbf{A} = \begin{pmatrix} 1 & -3 \\ 1 & -1 \\ 1 & 0 \\ 1 & 2 \\ 1 & 4 \\ 1 & 7 \\ 1 & 11 \end{pmatrix} \text{ and } \mathbf{B} = \begin{pmatrix} -7.4 \\ -4.2 \\ -3.7 \\ -1.9 \\ {}'0.3 \\ 2.8 \\ 7.2 \end{pmatrix}.$$

Then

$$\mathbf{A}^t\mathbf{A} = \begin{pmatrix} 7 & 20 \\ 20 & 200 \end{pmatrix} \text{ and } (\mathbf{A}^t\mathbf{A})^{-1} = \begin{pmatrix} 1/5 & -1/50 \\ -1/50 & 7/1000 \end{pmatrix}.$$

Finally, compute

$$\mathbf{A}^t\mathbf{B} = \begin{pmatrix} -6.89999 \\ 122.59999 \end{pmatrix}.$$

Then

$$\mathbf{X}^* = \begin{pmatrix} -5.364589 \\ 2.298867 \end{pmatrix}.$$

The equation of the line is $y = a + bx$, with $a = 2.298867$ and $b = -5.364589$.

7.9 LU Factorization

1. Given $\mathbf{A}$, first produce $\mathbf{U}$. Proceed

$$\mathbf{A} = \begin{pmatrix} \mathbf{2} & 4 & -6 \\ \mathbf{8} & 2 & 1 \\ \mathbf{-4} & 4 & 10 \end{pmatrix} \to \text{ add } -4 \text{ row one to row two, } 2 \text{ row one to row three}$$

$$\to \begin{pmatrix} 2 & 4 & -6 \\ 0 & \mathbf{-14} & 25 \\ 0 & \mathbf{12} & -2 \end{pmatrix}$$

$$\to \text{ add } 6/7 \text{ row two to row three } \to \begin{pmatrix} 2 & 4 & -6 \\ 0 & -14 & 25 \\ 0 & 0 & \mathbf{136/7} \end{pmatrix}.$$

This is $\mathbf{U}$:

$$\mathbf{U} = \begin{pmatrix} 2 & 4 & -6 \\ 0 & -14 & 25 \\ 0 & 0 & 136/7 \end{pmatrix}.$$

Now use the boldface entries in the formation of $\mathbf{U}$ to obtain $\mathbf{L}$. Start with

$$\mathbf{D} = \begin{pmatrix} 2 & 0 & 0 \\ 8 & -14 & 0 \\ -4 & 12 & 136/7 \end{pmatrix}.$$

Here we have listed the boldface elements from the formation of $\mathbf{U}$, with zeros above, to form a lower triangular matrix. This is not yet $\mathbf{L}$. In $\mathbf{D}$, divide each column by the reciprocal of the diagonal element of that column to obtain

$$\mathbf{L} = \begin{pmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ -2 & -6/7 & 1 \end{pmatrix}.$$

It is routine to check that $\mathbf{LU} = \mathbf{A}$.

2. Proceed

$$\mathbf{A} = \begin{pmatrix} \mathbf{1} & 5 & 2 \\ \mathbf{3} & -4 & 2 \\ \mathbf{1} & 4 & 10 \end{pmatrix} \to -3 \text{ times row one to row two, subtract row one from row three}$$

$$\begin{pmatrix} 1 & 5 & 2 \\ 0 & \mathbf{-14} & 4 \\ 0 & -1 & 8 \end{pmatrix} \to \text{ add } 1/19 \text{ row two to row three}$$

$$\begin{pmatrix} 1 & 5 & 2 \\ 0 & -19 & -4 \\ 0 & 0 & \mathbf{156/19} \end{pmatrix}.$$

Then

$$\mathbf{U} = \begin{pmatrix} 1 & 5 & 2 \\ 0 & -19 & -4 \\ 0 & 0 & 156/19 \end{pmatrix}.$$

To form $\mathbf{L}$, begin with

$$\mathbf{D} = \begin{pmatrix} 1 & 0 & 0 \\ 3 & -19 & 0 \\ 1 & -1 & 156/19 \end{pmatrix}$$

by using the boldface elements from the formation of $\mathbf{U}$. Multiply column two by $-1/19$ and column three by $19/156$ to obtain

$$\mathbf{L} = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 1 & 1/19 & 1 \end{pmatrix}.$$

Then $\mathbf{LU} = \mathbf{A}$.

For Problems 3, 4 and 5 the same algorithm is used and we give only the matrices $\mathbf{L}$ and $\mathbf{U}$.

3.

$$\mathbf{U} = \begin{pmatrix} -2 & 1 & 12 \\ 0 & -5 & 13 \\ 0 & 0 & 119/5 \end{pmatrix}, \mathbf{L} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & -3/5 & 1 \end{pmatrix}$$

4.

$$\mathbf{U} = \begin{pmatrix} 1 & 7 & 2 & -1 \\ 0 & -16 & -4 & 9 \\ 0 & 0 & 25/2 & 7/8 \end{pmatrix}, \mathbf{L} = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -3 & -7/8 & 1 \end{pmatrix}$$

5.

$$\mathbf{U} = \begin{pmatrix} 1 & 4 & 2 & -1 & 4 \\ 0 & -5 & 2 & 0 & 0 \\ 0 & 0 & 88/5 & 4 & 6 \\ 0 & 0 & 0 & 195/22 & -691/44 \end{pmatrix}, \mathbf{L} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -2 & -14/5 & 1 & 0 \\ 4 & 14/5 & -63/88 & 1 \end{pmatrix}$$

6. This problem has a little twist to it. It is the only problem in which the number of rows exceeds the number of columns. If the algorithm is carried out starting with $\mathbf{A}$, a difficulty occurs.

   However, we can still write the $LU-$ decomposition of $\mathbf{A}$ by working with $\mathbf{A}^t$, which is 3×4. The strategy is to find upper and lower triangular matrices $\mathbf{U}$ and $\mathbf{L}$ so that
$$\mathbf{A}^t = \mathbf{LU}.$$
   Then
$$\mathbf{A} = \mathbf{U}^t\mathbf{L}^t.$$
   But the transpose of an upper triangular matrix is lower triangular, and the transpose of a lower triangular matrix is upper triangular, so this is the decomposition we want for $\mathbf{A}$.

   Thus start with
$$\mathbf{A}^t = \begin{pmatrix} 4 & 2 & -3 & 0 \\ -8 & 24 & 2 & 1 \\ 2 & -2 & 14 & -5 \end{pmatrix}.$$
   Applying the algorithm to this matrix, we find that
$$\mathbf{U} = \begin{pmatrix} 4 & 2 & -3 & 0 \\ 0 & 28 & -4 & 1 \\ 0 & 0 & 211/14 & -137/28 \end{pmatrix}, \mathbf{L} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1/2 & -3/28 & 0 \end{pmatrix}.$$
   It is routine to check that
$$\mathbf{A}^t = \mathbf{LU}.$$
   Now take the transpose of this equation, recalling that the transpose of a product is the product of the transposes with the order reversed, to obtain the $LU-$ decomposition of $\mathbf{A}$:
$$\mathbf{A} = \begin{pmatrix} 4 & 0 & 0 \\ 2 & 28 & 0 \\ -3 & -4 & 211/14 \\ 0 & 1 & -137/28 \end{pmatrix} \begin{pmatrix} 1 & -2 & 1/2 \\ 0 & 1 & -3/28 \\ 0 & 0 & 0 \end{pmatrix}.$$

Problems 7 - 12 are small in the sense that the matrices are low-dimensional and the entries are integers. In such cases it would be just as efficient to solve the system $\mathbf{AX} = \mathbf{B}$ directly. The $LU-$ factorization method only reveals computational efficiencies when the systems are large. However, these problems are intended to promote familiarity with the method.

7. We want to solve $\mathbf{AX} = \mathbf{B}$, where
$$\mathbf{A} = \begin{pmatrix} 4 & 4 & 2 \\ 1 & -1 & 3 \\ 1 & 4 & 2 \end{pmatrix} \text{ and } \mathbf{B} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

We find that $\mathbf{A} = \mathbf{LU}$, where

$$\mathbf{U} = \begin{pmatrix} 4 & 4 & 2 \\ 0 & -2 & 5/2 \\ 0 & 0 & 21/4 \end{pmatrix} \text{ and } \mathbf{L} = \begin{pmatrix} 1 & 0 & 0 \\ 1/4 & 1 & 0 \\ 1/4 & -3/2 & 1 \end{pmatrix}.$$

Next solve the system $\mathbf{LY} = \mathbf{B}$ to obtain

$$\mathbf{Y} = \begin{pmatrix} 1 \\ -1/4 \\ 3/8 \end{pmatrix}.$$

Finally, solve $\mathbf{UX} = \mathbf{Y}$ to obtain

$$\mathbf{X} = \begin{pmatrix} 0 \\ 3/14 \\ 1/14 \end{pmatrix}.$$

8. The $LU-$ decomposition of $\mathbf{A}$ is

$$\mathbf{U} = \begin{pmatrix} 2 & 1 & 1 & 3 \\ 0 & 7/2 & 11/2 & 1/2 \end{pmatrix}, \mathbf{L} = \begin{pmatrix} 1 & 0 \\ 1/2 & 1 \end{pmatrix}.$$

Solve $\mathbf{LY} = \mathbf{B}$ to obtain

$$\mathbf{Y} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$

Now solve $\mathbf{UX} = \mathbf{Y}$ to obtain

$$\mathbf{X} = \alpha \begin{pmatrix} 10 \\ 1 \\ 0 \\ -7 \end{pmatrix} + \beta \begin{pmatrix} 16 \\ 0 \\ 1 \\ -11 \end{pmatrix} + \begin{pmatrix} -8 \\ 0 \\ 0 \\ 6 \end{pmatrix}.$$

9. We find that

$$\mathbf{U} = \begin{pmatrix} -1 & 1 & 1 & 6 \\ 0 & 3 & 2 & 16 \\ 0 & 0 & 17/3 & 52/3 \end{pmatrix}, \mathbf{L} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & -1/3 & 1 \end{pmatrix}.$$

Solve $\mathbf{LY} = \mathbf{B}$ to obtain

$$\mathbf{Y} = \begin{pmatrix} 2 \\ 5 \\ 29/3 \end{pmatrix}$$

and then solve $\mathbf{UX} = \mathbf{Y}$ for

$$\mathbf{X} = \alpha \begin{pmatrix} 1 \\ 28/3 \\ 26/3 \\ -17/6 \end{pmatrix} + \begin{pmatrix} 0 \\ -5/3 \\ -1/3 \\ 2/3 \end{pmatrix}.$$

10. Obtain

$$\mathbf{U} = \begin{pmatrix} 7 & 2 & -4 \\ 0 & 20/7 & 44/7 \\ 0 & 0 & 16 \end{pmatrix}, \mathbf{L} = \begin{pmatrix} 1 & 0 & 0 \\ -3/7 & 1 & 0 \\ 4/7 & 1 & 1 \end{pmatrix}.$$

Solve $\mathbf{LY} = \mathbf{B}$ to obtain

$$\mathbf{Y} = \begin{pmatrix} 7 \\ -1 \\ 3 \end{pmatrix}.$$

Solve $\mathbf{UX} = \mathbf{Y}$ to obtain

$$\mathbf{X} = \begin{pmatrix} 93/154 \\ 89/88 \\ -3/16 \end{pmatrix}.$$

11. Obtain

$$\mathbf{U} = \begin{pmatrix} 6 & 1 & -1 & 3 \\ 0 & 4/3 & 5/3 & 3 \\ 0 & 0 & 13/4 & 13/4 \\ 0 & 0 & 0 & 5 \end{pmatrix}, \mathbf{L} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2/3 & 1 & 0 & 0 \\ -2/3 & 5/4 & 1 & 0 \\ 1/3 & -1 & 4/13 & 1 \end{pmatrix}.$$

Solve $\mathbf{LY} = \mathbf{B}$:

$$\mathbf{Y} = \begin{pmatrix} 4 \\ 28/3 \\ -7 \\ 93/13 \end{pmatrix}$$

and then solve $\mathbf{UX} = \mathbf{Y}$:

$$\mathbf{X} = \begin{pmatrix} -263/130 \\ 537/65 \\ -233/65 \\ 93/65 \end{pmatrix}.$$

12. First obtain

$$\mathbf{U} = \begin{pmatrix} 1 & 2 & 0 & 1 & 1 & 2 & -4 \\ 0 & -3 & -3 & 3 & -8 & -4 & 17 \\ 0 & 0 & 8 & -10 & 8/3 & -14/3 & 4/3 \end{pmatrix}, \mathbf{L} = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 6 & 4/3 & 1 \end{pmatrix}.$$

The solution of $\mathbf{LY} = \mathbf{B}$ is

$$\mathbf{Y} = \begin{pmatrix} 0 \\ -4 \\ -10/3 \end{pmatrix}.$$

Solve $\mathbf{UX} = \mathbf{Y}$:

$$\mathbf{X} = \alpha \begin{pmatrix} 46 \\ -35 \\ 1 \\ 0 \\ 0 \\ 0 \\ -6 \end{pmatrix} + \beta \begin{pmatrix} -58 \\ 87/2 \\ 0 \\ 1 \\ 0 \\ 0 \\ 15/2 \end{pmatrix} + \gamma \begin{pmatrix} 19 \\ -14 \\ 0 \\ 0 \\ 1 \\ 0 \\ -2 \end{pmatrix} + \gamma \begin{pmatrix} -25 \\ 37/2 \\ 0 \\ 0 \\ 0 \\ 1 \\ 7/2 \end{pmatrix} + \begin{pmatrix} 47/3 \\ -73/6 \\ 0 \\ 0 \\ 0 \\ 0 \\ -5/2 \end{pmatrix}.$$

7.10 Linear Transformations

1. T is linear and

$$T(1,0,0) = <3,1,0>, T(0,1,0) = <0,-1,0>, T(0,0,1) = <0,0,2>$$

so

$$\mathbf{A}_T = \begin{pmatrix} 3 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Because $\mathbf{A}_T$ has rank 3, T is one-to-one and onto and the dimension of the null space is $3 - 3 = 0$ (contains only the zero vector).

2. T is linear and

$$\mathbf{A}_T = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix}.$$

T is not one-to-one (all vectors $<\alpha,\alpha,\beta,\beta>$ map to $<0,0,0,0>$). T is onto. Since $\mathbf{A}_T$ has rank 2, the dimension of the null space is $4 - 2 = 2$.

3. T is nonlinear because of the $2xy$ term.

4. T is linear and

$$\mathbf{A}_T = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ -1 & -3 & 0 & 0 & 1 \end{pmatrix}.$$

T is one-to-one and onto and the null space has dimension $5-5 = 0$, since $\mathbf{A}_T$ has rank 5.

5. T is linear and

$$\mathbf{A}_T = \begin{pmatrix} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}.$$

T is not one-to-one, but is onto. The null space has dimension $5 - 3 = 2$.

6. T is linear and
$$\mathbf{A}_T = \begin{pmatrix} 1 & 1 & 4 & -8 \\ -1 & 1 & -1 & 0 \end{pmatrix}.$$
T is onto but not one-to-one. The null space has dimension $4 - 2 = 2$.

7. T is not linear because of the $\sin(xy)$ term.

8. T is linear and
$$\mathbf{A}_T = \begin{pmatrix} -2 & 4 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$
T is not one-to-one and not onto and the dimension of the null space is $3 - 2 = 1$.

9. T is not linear because of the constant fourth and fifth components of $T(x, y, u, v, w)$. Note also that the zero vector does not map to the zero vector by T.

10. T is linear and
$$\mathbf{A}_T = \begin{pmatrix} 0 & -1 & 3 & 8 \\ 0 & 1 & 0 & -4 \end{pmatrix}.$$
T is not one-to-one and not onto. The null space has dimension $4 - 2 = 2$.

Chapter 8

Determinants

8.1 Definition of the Determinant

1. Each factor $a_{jp(j)}$ in a typical term of the sum defining $|\mathbf{A}|$ is replaced by $\alpha a_{jp(j)}$ in the corresponding term of $|\mathbf{B}|$. Since there are n factors in each such term, then each term in the sum defining $|\mathbf{B}|$ is α^n times the corresponding term in $|\mathbf{B}|$. Therefore $|\mathbf{B}| = \alpha^n |\mathbf{A}|$.

2. In the 2×2 case,
$$\mathbf{B} = \begin{pmatrix} a_{11} & (1/\alpha)a_{12} \\ \alpha a_{21} & a_{22} \end{pmatrix}$$
so
$$\begin{aligned} |\mathbf{B}| &= a_{11}a_{22} - \left(\frac{1}{\alpha}\right)(\alpha)(a_{12} - a_{21}) \\ &= a_{11}a_{22} - a_{12}a_{21} = |\mathbf{A}|. \end{aligned}$$

   In the 3×3 case,
$$\mathbf{B} = \begin{pmatrix} a_{11} & (1/\alpha)a_{12} & (1/\alpha^2)a_{13} \\ \alpha a_{21} & a_{22} & (1/\alpha)a_{23} \\ \alpha^2 a_{31} & (1/\alpha)a_{32} & a_{33} \end{pmatrix}$$
and by expanding this determinant we find that $|\mathbf{B}| = |\mathbf{A}|$.

   What we observe in these small cases is that each factor of α is matched with a factor of $1/\alpha$ in the terms of the sum defining the determinant, so these cancel. This leads us to conjecture that $|\mathbf{B}| = |\mathbf{A}|$ in the $n \times n$ case.

3. Suppose $\mathbf{A} = -\mathbf{A}^t$. We will use property (1) of determinants, and the conclusion of Problem 1. Since $\mathbf{A}^t = -\mathbf{A}$, then
$$|\mathbf{A}| = |\mathbf{A}^t| = |-\mathbf{A}| = (-1)^n |\mathbf{A}|.$$

If n is odd, then

$$|\mathbf{A}| = -|\mathbf{A}|,$$

and this implies that $|\mathbf{A}| = 0$.

4. From the definition of determinant,

$$|\mathbf{I}_n| = \sum_p \sigma(p)(\mathbf{I}_n)_{1p(1)}(\mathbf{I}_n)_{2p(2)} \cdots (\mathbf{I}_n)_{np(n)}.$$

Now $(\mathbf{I}_n)_{ij} = 0$ if $i \neq j$, so the only way a term of this sum can be nonzero is if each factor

$$(\mathbf{I}_n)_{jp(j)} \neq 0$$

in this term. But this can occur only if $p(j) = j$ for $j = 1, \cdots, n$, so

$$p(1) = 1, p(2) = 2, \cdots, p(n) = n.$$

This means that p must be the identity permutation that leaves each j unchanged for $j = 1, 2, \cdots, n$. But, if p is the identity permutation, than $\sigma(p) = 1$. Further, each $\mathbf{I}_{jj} = 1$. Therefore $\mathbf{I}_n$ has determine $1 \cdot 1 \cdots 1 = 1$.

5. If p is a permutation of $1, 2, \cdots, n$, then it is impossible for each $p(j) \geq j$ or for each $p(j) \leq j$ for $j = 1, 2, \cdots, n$, unless each $p(j) = j$ and p is the identity permutation. Thus, the only (possibly) nonzero term in the sum defining the determinant is

$$\begin{aligned}|\mathbf{A}| &= \sigma(p)\mathbf{A}_{1p(1)}\mathbf{A}_{2p(2)} \cdots \mathbf{A}_{np(n)} \\ &= \mathbf{A}_{11}\mathbf{A}_{22} \cdots \mathbf{A}_{nn}.\end{aligned}$$

6. A square $\mathbf{A}$ is nonsingular if and only if $|\mathbf{A}| = 0$. For an upper of lower triangular matrix, this means that, by the result of Problem 5, some diagonal element A_{jj} must be zero.

8.2 Evaluation of Determinants I

The most efficient way to evaluate a determinant is by using a software package. In many kinds of general computations, however, it is useful to understand row and column operations and cofactor expansions and how these are used to manipulate determinants, and this is the purpose of these problems.

There are many sequences of row and/or column operations that can be used to evaluate a given determinant. Of course, regardless of the sequence used, the value of the determinant depends only on the original matrix.

1. Add 2 times row two to row one and -7 times row two to row three to write

$$\begin{vmatrix} -2 & 4 & 1 \\ 1 & 6 & 3 \\ 7 & 0 & 4 \end{vmatrix} = \begin{vmatrix} 0 & 16 & 7 \\ 1 & 6 & 3 \\ 0 & -42 & -17 \end{vmatrix} = (-1)^{2+1}(1)\begin{vmatrix} 16 & 7 \\ -42 & -17 \end{vmatrix} = -22$$

2. Add 3 times row two to row one, then add row two to row three to obtain

$$\begin{vmatrix} 2 & -3 & 7 \\ 14 & 1 & 1 \\ -13 & -1 & 5 \end{vmatrix} = \begin{vmatrix} 44 & 0 & 10 \\ 14 & 1 & 1 \\ 1 & 0 & 6 \end{vmatrix} = (-1)^{2+2}(1)\begin{vmatrix} 44 & 10 \\ 1 & 6 \end{vmatrix} = 254$$

3. Add column two to column one, then 3 times column two to column three:

$$\begin{vmatrix} -4 & 5 & 6 \\ -2 & 3 & 5 \\ 2 & -2 & 6 \end{vmatrix} = \begin{vmatrix} 1 & 5 & 21 \\ 1 & 3 & 14 \\ 0 & -2 & 0 \end{vmatrix} = (-1)^{3+2}(-2)\begin{vmatrix} 1 & 21 \\ 1 & 14 \end{vmatrix} = -14$$

4. Add 2 times row three to row one and 2 times row three to row two to obtain

$$\begin{vmatrix} 2 & -5 & 8 \\ 4 & 3 & 8 \\ 13 & 0 & -4 \end{vmatrix} = \begin{vmatrix} 28 & -5 & 0 \\ 30 & 3 & 0 \\ 13 & 0 & -4 \end{vmatrix} = (-1)^{3+3}(-4)\begin{vmatrix} 28 & -5 \\ 30 & 3 \end{vmatrix} = -936$$

5. Add 2 times column three to column one and then add column three to column two to obtain

$$\begin{vmatrix} 17 & -2 & 5 \\ 1 & 12 & 0 \\ 14 & 7 & -7 \end{vmatrix} = \begin{vmatrix} 27 & 3 & 5 \\ 1 & 12 & 0 \\ 0 & 0 & -7 \end{vmatrix} = (-1)^{3+3}(-7)\begin{vmatrix} 27 & 3 \\ 1 & 12 \end{vmatrix} = -2,247$$

6. Add column one to column two, then 3 times column one to column three, then 2 times column one to column four to obtain

$$\begin{vmatrix} -3 & 3 & 9 & 6 \\ 1 & -2 & 15 & 6 \\ 7 & 1 & 1 & 5 \\ 2 & 1 & -1 & 3 \end{vmatrix} = \begin{vmatrix} -3 & 0 & 0 & 0 \\ 1 & -1 & 18 & 8 \\ 7 & 8 & 22 & 19 \\ 2 & 3 & 5 & 7 \end{vmatrix} = (-1)^{1+1}(-3)\begin{vmatrix} -1 & 18 & 8 \\ 8 & 22 & 19 \\ 3 & 5 & 7 \end{vmatrix}$$

$$= -3\begin{vmatrix} -1 & 18 & 8 \\ 8 & 22 & 19 \\ 3 & 5 & 7 \end{vmatrix}.$$

Now we have reduced the problem of evaluating a 4×4 determinant to one of evaluating a 3×3 determinant. In this 3×3 determinant, add 18 times column one to column two, then 8 times column one to column three to obtain

$$\begin{vmatrix} -1 & 18 & 8 \\ 8 & 22 & 19 \\ 3 & 5 & 7 \end{vmatrix} = \begin{vmatrix} -1 & 0 & 0 \\ 8 & 166 & 83 \\ 3 & 59 & 31 \end{vmatrix} = (-1)\begin{vmatrix} 166 & 83 \\ 59 & 31 \end{vmatrix} = -249$$

Putting the two steps together,

$$\begin{vmatrix} -3 & 3 & 9 & 6 \\ 1 & -2 & 15 & 6 \\ 7 & 1 & 1 & 5 \\ 2 & 1 & -1 & 3 \end{vmatrix} = (-3)(-249) = 747.$$

The determinants in Problems 7 - 10 are treated similarly, and we list only the value of the determinant.

7. -122
8. 293
9. 72
10. $-2,667$

8.3 Evaluation of Determinants II

For these problems, use a combination of row and column operations to obtain a row or column with some zeros, then expand by that row or column. Depending on the size of the resulting determinants, it may be useful to apply the cofactor method to each of these in turn.

For Problems 1 and 2 the cofactor expansion is written out in detail. For Problems 3 - 10 only the value of the determinant is given.

1. Expand the determinant by the third column:

$$\begin{vmatrix} -4 & 2 & -8 \\ 1 & 1 & 0 \\ 1 & -3 & 0 \end{vmatrix} = (-1)^{1+3}(-8)\begin{vmatrix} 1 & 1 \\ 1 & -3 \end{vmatrix} = (-8)(-4) = 32$$

2. Use row operations to reduce column one, then expand by this column:

$$\begin{vmatrix} 1 & 1 & 6 \\ 2 & -2 & 1 \\ 3 & -1 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 16 \\ 0 & -4 & -11 \\ 0 & -4 & -14 \end{vmatrix} = \begin{vmatrix} -4 & -11 \\ -4 & -14 \end{vmatrix} = \begin{vmatrix} -4 & -11 \\ 0 & -3 \end{vmatrix} = 12$$

3. 3
4. 124
5. -773
6. $3,775$
7. -152
8. $4,882$
9. $1,693$
10. $3,372$

11.

$$\begin{vmatrix} 1 & \alpha & \alpha^2 \\ 1 & \beta & \beta^2 \\ 1 & \gamma & \gamma^2 \end{vmatrix} = \begin{vmatrix} 1 & \alpha & \alpha^2 \\ 0 & \beta-\alpha & \beta^2-\alpha^2 \\ 0 & \gamma-\alpha & \gamma^2-\alpha^2 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & \alpha & \alpha^2 \\ 0 & \beta-\alpha & (\beta^{-}\alpha)(\beta+\alpha) \\ 0 & \gamma-\alpha & (\gamma-\alpha)(\gamma+\alpha) \end{vmatrix} = (\beta-\alpha)(\gamma-\alpha)\begin{vmatrix} 1 & \alpha & \alpha^2 \\ 0 & 1 & \beta+\alpha \\ 0 & 1 & \gamma+\alpha \end{vmatrix}$$

$$= (\beta-\alpha)(\gamma-\alpha)\begin{vmatrix} 1 & \beta+\alpha \\ 1 & \gamma+\alpha \end{vmatrix} = (\beta-\alpha)(\gamma-\alpha)(\gamma-\beta).$$

12. Add columns two, three and four to column one and factor $(\alpha+\beta+\gamma+\delta)$ out of column one to obtain

$$\begin{vmatrix} a & b & c & d \\ b & c & d & a \\ c & d & a & b \\ d & a & b & c \end{vmatrix} = (a+b+c+d)\begin{vmatrix} 1 & b & c & d \\ 1 & c & d & a \\ 1 & d & a & b \\ 1 & a & b & c \end{vmatrix}.$$

Now add

$$(-1)\text{row two} + \text{row three} - \text{row four}$$

to row one and factor out $(b-a+d-c)$ from the new row one to obtain

$$(a+b+c+d)(b-a+d-c)\begin{vmatrix} 0 & 1 & -1 & 1 \\ 1 & c & d & a \\ 1 & d & a & b \\ 1 & a & b & c \end{vmatrix}.$$

13. Define a function

$$L(x,y) = \begin{vmatrix} 1 & x & y \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix} = (y_2-y_3)x + (x_3-x_2)y + x_2y_3 - x_3y_2.$$

Thus $L(x,y)$ has the form $L(x,y) = ax+by+c$, with a, b and c constants. The graph of the equation $L(x,y)=0$ is a straight line in the plane. Since $L(x_2,y_2) = L(x_3,y_3) = 0$, both points (x_2,y_2) and $x_3,y_3)$ are on this line.

Finally, $L(x_1,y_1) = 0$ if and only if (x_1,y_1) is also on this line, and this occurs if and only if

$$\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix} = 0.$$

8.4 A Determinant Formula for $\mathbf{A}^{-1}$

1.
$$\mathbf{A}^{-1} = \frac{1}{13}\begin{pmatrix} 6 & 1 \\ -1 & 2 \end{pmatrix}$$

2.
$$\mathbf{A}^{-1} = \frac{1}{12}\begin{pmatrix} 4 & 0 \\ -1 & 3 \end{pmatrix}$$

3.
$$\mathbf{A}^{-1} = \frac{1}{5}\begin{pmatrix} -4 & 1 \\ 1 & 1 \end{pmatrix}$$

4.
$$\mathbf{A}^{-1} = \frac{1}{29}\begin{pmatrix} -3 & -5 \\ 7 & 2 \end{pmatrix}$$

5.
$$\mathbf{A}^{-1} = \frac{1}{32}\begin{pmatrix} 5 & 3 & 1 \\ -8 & -24 & 24 \\ -2 & -14 & 6 \end{pmatrix}$$

6.
$$\mathbf{A}^{-1} = \frac{1}{120}\begin{pmatrix} -10 & -10 & 0 \\ -11 & -95 & 36 \\ 3 & 15 & 12 \end{pmatrix}$$

7.
$$\mathbf{A}^{-1} = \frac{1}{29}\begin{pmatrix} -1 & 25 & -21 \\ -8 & -3 & 6 \\ -1 & -4 & 8 \end{pmatrix}$$

8.
$$\mathbf{A}^{-1} = \frac{1}{119}\begin{pmatrix} 9 & 35 & 5 \\ 0 & 119 & 0 \\ -4 & 77 & 11 \end{pmatrix}$$

9.
$$\mathbf{A}^{-1} = \frac{1}{378}\begin{pmatrix} 210 & -42 & 42 & 0 \\ 899 & -124 & 223 & -135 \\ 275 & -64 & 109 & -27 \\ -601 & 122 & -131 & 81 \end{pmatrix}$$

10.
$$\mathbf{A}^{-1} = \frac{1}{784}\begin{pmatrix} -52 & 131 & -62 & 54 \\ 208 & -132 & 248 & -216 \\ -496 & 360 & -320 & 304 \\ -212 & 127 & -102 & 190 \end{pmatrix}$$

8.5 Cramer's Rule

1. Since $|\mathbf{A}| = 47 \neq 0$, Cramer's rule applies. The solution is

$$x_1 = \frac{1}{47}\begin{vmatrix} 5 & -4 \\ -4 & 1 \end{vmatrix} = -\frac{11}{47},\; x_2 = \frac{1}{47}\begin{vmatrix} 15 & 5 \\ 8 & -4 \end{vmatrix} = -\frac{100}{47}$$

2. $|\mathbf{A}| = -3$ and the solution is

$$x_1 = -\frac{1}{3}\begin{vmatrix} 3 & 4 \\ 0 & 1 \end{vmatrix} = -1, x_2 = -\frac{1}{3}\begin{vmatrix} 1 & 3 \\ 1 & 0 \end{vmatrix} = 1.$$

3. $|\mathbf{A}| = 132$ and the solution is

$$x_1 = \frac{1}{132}\begin{vmatrix} 0 & -4 & 3 \\ -5 & 5 & -1 \\ -4 & 6 & 1 \end{vmatrix} = -\frac{66}{132} = -\frac{1}{2},$$

$$x_2 = \frac{1}{132}\begin{vmatrix} 8 & 0 & 3 \\ 1 & -5 & -1 \\ -2 & -4 & 1 \end{vmatrix} = -\frac{114}{132} = -\frac{19}{22},$$

$$x_3 = \frac{1}{132}\begin{vmatrix} 8 & -4 & 0 \\ 1 & 5 & -5 \\ -2 & 6 & -4 \end{vmatrix} = \frac{24}{132} = \frac{2}{11}$$

4. $|\mathbf{A}| = 108$ and the solution is

$$x_1 = -\frac{63}{108} = -\frac{7}{12}, x_2 = -\frac{165}{108} = -\frac{55}{36}, x_3 = -\frac{243}{108} = -\frac{9}{4}$$

5. $|\mathbf{A}| = -6$ and the solution is

$$x_1 = \frac{5}{6}, x_2 = -\frac{10}{3}, x_3 = -\frac{5}{6}$$

6. $|\mathbf{A}| = -130$ and the solution is

$$x_1 = \frac{197}{130}, x_2 = \frac{255}{130}, x_3 = \frac{1260}{130}, x_4 = \frac{42}{130}, x_5 = \frac{173}{130}$$

7. $|\mathbf{A}| = 4$ and the solution is

$$x_1 = -\frac{172}{2} = -86, x_2 = -\frac{109}{2}, x_3 = -\frac{43}{2}, x_4 = \frac{37}{2}$$

8. $|\mathbf{A}| = 12$ and the solution is

$$x_1 = \frac{117}{12}, x_2 = \frac{63}{12}, x_3 = \frac{3}{2}, x_4 = -\frac{21}{12}$$

9. $|\mathbf{A}| = 93 \neq 0$ and the solution is
$$x_1 = \frac{33}{93}, x_2 = -\frac{409}{33}, x_3 = -\frac{1}{93}, x_4 = \frac{116}{93}.$$

10. $|\mathbf{A}| = 42 \neq 0$, so by Cramer's rule,
$$x_1 = \frac{69}{21}, x_2 = \frac{162}{21}, x_3 = \frac{24}{21}, x_4 = -\frac{54}{21}.$$

8.6 The Matrix Tree Theorem

1. The tree matrix for this graph is
$$\mathbf{T} = \begin{pmatrix} 2 & 0 & -1 & 0 & -1 \\ 0 & 2 & -1 & -1 & 0 \\ -1 & -1 & 4 & -1 & -1 \\ 0 & -1 & -1 & 3 & -1 \\ -1 & 0 & -1 & -1 & 3 \end{pmatrix}.$$
Evaluate any 4×4 cofactor of T to obtain 21 as the number of spanning trees in G.

2.
$$\mathbf{T} = \begin{pmatrix} 4 & -1 & -1 & -1 & 0 & -1 \\ -1 & 3 & -1 & -1 & 0 & 0 \\ -1 & -1 & 2 & 0 & 0 & 0 \\ -1 & -1 & 0 & 4 & -1 & -1 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ -1 & 0 & 0 & -1 & -1 & 3 \end{pmatrix}$$
and evaluation of any cofactor yields 55 for the number of spanning trees in G.

3.
$$\mathbf{T} = \begin{pmatrix} 4 & -1 & 0 & -1 & -1 & -1 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 3 & -1 & -1 & 0 \\ -1 & 0 & -1 & 4 & -1 & -1 \\ -1 & 0 & -1 & -1 & 3 & 0 \\ -1 & 0 & 0 & -1 & 0 & 2 \end{pmatrix}$$
and each cofactor equals 61.

4.
$$\mathbf{T} = \begin{pmatrix} 4 & -1 & -1 & 0 & -1 & -1 \\ -1 & 3 & -1 & -1 & 0 & 0 \\ -1 & -1 & 3 & -1 & 0 & 0 \\ 0 & -1 & -1 & 3 & -1 & 0 \\ -1 & 0 & 0 & -1 & 3 & -1 \\ -1 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$
and each cofactor equals 64.

5.

$$\mathbf{T} = \begin{pmatrix} 3 & -1 & 0 & 0 & -1 & -1 \\ -1 & 3 & -1 & 0 & -1 & 0 \\ 0 & -1 & 4 & -1 & -1 & -1 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ -1 & 0 & -1 & 0 & 0 & 2 \end{pmatrix}$$

and each cofactor is equal to 61.

6. The tree matrix for the complete graph K_n is

$$\mathbf{T} = \begin{pmatrix} n-1 & -1 & -1 & \cdots & -1 \\ -1 & n-1 & -1 & \cdots & -1 \\ -1 & -1 & n-1 & \cdots & -1 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ -1 & -1 & -1 & \cdots & n-1 \end{pmatrix}.$$

To compute the number of spanning trees in K_n, evaluate any cofactor of T. We will compute $(-1)^{1+1}M_{11}$, which is the $n-1 \times n-1$ determinant formed by deleting row one and column one of $\mathbf{T}$. In M_{11}, add the last $n-2$ rows to row one to obtain a new $n-1 \times n-1$ determinant equal to M_{11}:

$$M_{11} = \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ -1 & n-1 & -1 & \cdots & -1 \\ -1 & -1 & n-1 & \cdots & -1 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ -1 & -1 & -1 & \cdots & n-1 \end{vmatrix}.$$

Subtract column one of this determinant from each other column. Again, this does not change the value of the determinant, so

$$M_{11} = \begin{vmatrix} 1 & 0 & 0 & \cdots & 0 \\ -1 & n & 0 & \cdots & 0 \\ -1 & 0 & n & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ -1 & 0 & 0 & \cdots & n \end{vmatrix}.$$

This is a lower triangular $n-1 \times n-1$ determinant, and is equal to the product of its diagonal elements, which consist of one 1 and $n-2$ entries of n. Thus $M_{11} = n^{n-2}$, and this is the number of spanning trees in K_n.

Chapter 9

Eigenvalues and Diagonalization

9.1 Eigenvalues and Eigenvectors

1.

$$p_{\mathbf{A}}(\lambda) = |\lambda \mathbf{I} - \mathbf{A}| = \lambda^2 - 2\lambda - 5$$

with roots (eigenvalues of $\mathbf{A}$) $\lambda_1 = 1+\sqrt{6}$ and $\lambda_2 = 1-\sqrt{6}$. Corresponding eigenvectors are

$$\mathbf{V}_1 = \begin{pmatrix} \sqrt{6} \\ 2 \end{pmatrix}, \mathbf{V}_2 = \begin{pmatrix} -\sqrt{6} \\ 2 \end{pmatrix}.$$

The Gershgorin circles are of radius 3 about $(1,0)$ and radius 2 about $(1,0)$. These enclose the eigenvalues.

2.

$$p_{\mathbf{A}}(\lambda) = \lambda^2 - 2\lambda - 8,$$

$$\lambda_1 = 4, \mathbf{V}_1 = \begin{pmatrix} 0 \\ 4 \end{pmatrix}, \lambda_2 = -2, \mathbf{V}_2 = \begin{pmatrix} 6 \\ -1 \end{pmatrix}.$$

The Gershgorin circle is of radius 1 about $(4,0)$. The other Gershgorin "circle" has radius 0 and so is not really a circle. We may think of this as a degenerate circle containing the eigenvalue -2 in its interior.

3.

$$p_{\mathbf{A}}(\lambda) = \lambda^2 + 3\lambda - 10,$$

$$\lambda_1 = -5, \mathbf{V}_1 = \begin{pmatrix} 7 \\ -1 \end{pmatrix}, \lambda_2 = 2, \mathbf{V}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The Gershgorin circle has radius 1 and center $(2,0)$.

4.
$$p_{\mathbf{A}}(\lambda) = \lambda^2 - 10\lambda + 18,$$
$$\lambda_1 = 5 + \sqrt{7}, \mathbf{V}_1 = \begin{pmatrix} 2 \\ 1 - \sqrt{7} \end{pmatrix}, \lambda_2 = 5 - \sqrt{7}, \mathbf{V}_2 = \begin{pmatrix} 2 \\ 1 + \sqrt{7} \end{pmatrix}.$$
The Gershgorin circles have radius 2, center $(6,0)$ and radius 3, center $(4,0)$.

5. $p_{\mathbf{A}}(\lambda) = \lambda^2 - 3\lambda + 14$,
$$\lambda_1 = (3 + \sqrt{14}i)/2, \mathbf{V}_1 = \begin{pmatrix} -1 + \sqrt{47}i \\ 4 \end{pmatrix}$$
$$\lambda_2 = (3 - \sqrt{14}i)/2, \mathbf{V}_2 = \begin{pmatrix} -1 - \sqrt{14}i \\ 4 \end{pmatrix}.$$
The Gershgorin circles have radius 6, center $(1,0)$ and radius 2, center $(2,0)$.

6.
$$p_{\mathbf{A}}(\lambda) = \lambda^2,$$
with roots $\lambda_1 = \lambda_2 = 0$. The only eigenvectors are nonzero scalar multiples of
$$\mathbf{V}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$
The Gershgorin circle has radius 1, center the origin.

7.
$$p_{\mathbf{A}}(\lambda) = \lambda^3 - 5\lambda^2 + 6\lambda,$$
$$\lambda_1 = 0, \mathbf{V}_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \lambda_2 = 2, \mathbf{V}_2 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \lambda_3 = 3, \mathbf{V}_3 = \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix}.$$
The Gershgorin circle has radius 3, center the origin.

8.
$$p_{\mathbf{A}}(\lambda) = (\lambda + 1)(\lambda^2 - \lambda - 7),$$
$$\lambda_1 = 1, \mathbf{V}_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \lambda_2 = (1 + \sqrt{29})/2, \mathbf{V}_2 = \begin{pmatrix} 2 \\ 5 + \sqrt{29} \\ 0 \end{pmatrix},$$
$$\lambda_3 = (1 - \sqrt{29})/2, \mathbf{V}_3 = \begin{pmatrix} 2 \\ 5 - \sqrt{29} \\ 0 \end{pmatrix}.$$
The Gershgorin circles have radius 1, center $(-2,0)$, and radius 1, center $(3,0)$.

9.

$$p_{\mathbf{A}}(\lambda) = \lambda^2(\lambda + 3),$$

$$\lambda_1 = -3, \mathbf{V}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \lambda_2 = \lambda_3 = 0, \mathbf{V}_2 = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}.$$

There is only one independent eigenvector associated with eigenvalue 0. The Gershgorin circle has radius 2, center $(-3, 0)$.

10.

$$p_{\mathbf{A}}(\lambda) = \lambda^3 + 2\lambda,$$

$$\lambda_1 = 0, \mathbf{V}_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \lambda_2 = \sqrt{2}i, \mathbf{V}_2 = \begin{pmatrix} 1 \\ -1 \\ -2\sqrt{2}i \end{pmatrix}, \lambda_3 = -\sqrt{2}i, \mathbf{V}_3 = \begin{pmatrix} 1 \\ -1 \\ \sqrt{2}i \end{pmatrix}.$$

The Gershgorin circles have center $(0, 0)$ and radii 1 and 2.

11.

$$p_{\mathbf{A}}(\lambda) = (\lambda + 14)(\lambda - 2)^2,$$

$$\lambda_1 = -14, \mathbf{V}_1 = \begin{pmatrix} -16 \\ 0 \\ 1 \end{pmatrix}$$

$$\lambda_2 = \lambda_3 = 2, \mathbf{V}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

with only one independent eigenvector associated with the multiple eigenvalue λ_2. The Gershgorin circles have radius 1, center $(-14, 0)$ and radius 3, center $(2, 0)$.

12.

$$p_{\mathbf{A}}(\lambda) = (\lambda - 3)(\lambda^2 + \lambda - 42),$$

$$\lambda_1 = 6, \mathbf{V}_1 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \lambda_2 = 3, \mathbf{V}_3 = \begin{pmatrix} 30 \\ -2 \\ 5 \end{pmatrix}, \lambda_3 = -7, \mathbf{V}_3 = \begin{pmatrix} 0 \\ 8 \\ 5 \end{pmatrix}.$$

The Gershgorin circles have radius 9, center $(-2, 0)$, and radius 5, center $(1, 0)$.

13.

$$p_{\mathbf{A}}(\lambda) = \lambda(\lambda^2 - 8\lambda + 7),$$

$$\lambda_1 = 0, \mathbf{V}_1 \begin{pmatrix} 14 \\ 7 \\ 10 \end{pmatrix}, \lambda_2 = 1, \mathbf{V}_2 = \begin{pmatrix} 6 \\ 0 \\ 5 \end{pmatrix}, \lambda_3 = 7, \mathbf{V}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

The Gershgorin circles have radius 2, center $(1, 0)$ and radius 5, center $(7, 0)$.

14.

$$p_{\mathbf{A}} = \lambda^2(\lambda^2 + 2\lambda - 1),$$
$$\lambda_1 = \lambda_2 = 0,$$

with two associated independent eigenvectors

$$\mathbf{V}_1 = \begin{pmatrix} 1 \\ 2 \\ 0 \\ -1 \end{pmatrix}, \mathbf{V}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

For the other two eigenvalues,

$$\lambda_3 = -1 + \sqrt{2}, \mathbf{V}_3 = \begin{pmatrix} 1 \\ 1 + \sqrt{2} \\ 0 \\ 0 \end{pmatrix}, \lambda_4 = -1 - \sqrt{2}, \mathbf{V}_4 = \begin{pmatrix} 1 \\ 1 - \sqrt{2} \\ 0 \\ 0 \end{pmatrix}.$$

The Gershgorin circles have radius 1, center $(-2, 0)$ and radius 2, center $(0, 0)$.

15.

$$p_{\mathbf{A}}(\lambda) = (\lambda - 1)(\lambda - 2)(\lambda^2 + \lambda - 13),$$
$$\lambda_1 = 1, \mathbf{V}_1 = \begin{pmatrix} -2 \\ -11 \\ 0 \\ 1 \end{pmatrix}, \lambda_2 = 2, \mathbf{V}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix},$$
$$\lambda_3 = \frac{-1 + \sqrt{53}}{2}, \mathbf{V}_3 = \begin{pmatrix} \sqrt{53} - 7 \\ 0 \\ 0 \\ 2 \end{pmatrix}, \lambda_4 = \frac{-1 - \sqrt{53}}{2}, \mathbf{V}_4 = \begin{pmatrix} -\sqrt{53} - 7 \\ 0 \\ 0 \\ 2 \end{pmatrix}.$$

The Gershgorin circles have radius 2, center $(-4, 0)$ and radius 1 and center $(3, 0)$.

16.

$$p_{\mathbf{A}}(\lambda) = \lambda^2(\lambda - 1)(\lambda - 5),$$
$$\lambda_1 = 1, \mathbf{V}_1 = \begin{pmatrix} 1 \\ -4 \\ 0 \\ 0 \end{pmatrix}, \lambda_2 = 5, \mathbf{V}_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$
$$\lambda_3 = \lambda_4 = 0, \mathbf{V}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

There is only one independent eigenvector associated with the double eigenvalue 0. The Gershgorin circles have radius 2, center $(0,0)$, radius 1, center $(2,0)$, and radius 1, center $(2,0)$. One of the Gershgorin circles is enclosed by the other in this example.

17.
$$p_{\mathbf{A}}(\lambda) = \lambda^2 - 5\lambda,$$
$$\lambda_1 = 0, \mathbf{V}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \lambda_2 = 5, \mathbf{V}_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}.$$
By taking the dot product of the eigenvectors, we see that they are orthogonal.

18.
$$p_{\mathbf{A}}(\lambda) = \lambda^2 - \lambda - 37,$$
so eigenvalues are $\lambda_1 = (1+\sqrt{149})/2$ and $\lambda_2 = (1-\sqrt{49})/2$. Corresponding eigenvectors are
$$\mathbf{V}_1 = \begin{pmatrix} 10 \\ 7+\sqrt{149} \end{pmatrix} \text{ and } \mathbf{V}_2 = \begin{pmatrix} 10 \\ 7-\sqrt{149} \end{pmatrix}.$$
These eigenvectors are orthogonal.

19.
$$p_{\mathbf{A}}(\lambda) = \lambda^2 - 10\lambda - 23,$$
so eigenvalues are $\lambda_1 = 5+\sqrt{2}$ and $\lambda_2 = 5-\sqrt{2}$. Corresponding eigenvectors are
$$\mathbf{V}_1 = \begin{pmatrix} 1+\sqrt{2} \\ 1 \end{pmatrix} \text{ and } \mathbf{V}_2 = \begin{pmatrix} 1-\sqrt{2} \\ 1 \end{pmatrix}.$$
These eigenvectors are orthogonal.

20.
$$p_{\mathbf{A}}(\lambda) = \lambda^2 + 9\lambda - 53$$
so the eigenvalues of $\mathbf{A}$ are $\lambda_1 = (-9+\sqrt{293})/2$ and $\lambda_2 = (-9-\sqrt{293})/2$. Corresponding eigenvectors are
$$\mathbf{V}_1 = \begin{pmatrix} 2 \\ 17+\sqrt{293} \end{pmatrix} \text{ and } \mathbf{V}_2 = \begin{pmatrix} 2 \\ 17-\sqrt{293} \end{pmatrix}.$$
These eigenvectors are orthogonal.

21.
$$p_{\mathbf{A}}(\lambda) = (\lambda - 3)(\lambda^2 + 2\lambda - 1),$$
so eigenvalues are $\lambda_1 = 3$, $\lambda_2 = 1+\sqrt{2}$ and $\lambda_3 = -1-\sqrt{2}$. Corresponding eigenvectors are
$$\mathbf{V}_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \mathbf{V}_2 = \begin{pmatrix} 1+\sqrt{2} \\ 1 \\ 0 \end{pmatrix}, \text{ and } \mathbf{V}_3 = \begin{pmatrix} 1-\sqrt{2} \\ 1 \\ 0 \end{pmatrix}.$$
These eigenvectors are mutually orthogonal.

22. $p_{\mathbf{A}}(\lambda) = (\lambda - 2)(\lambda^2 - 2\lambda - 2)$, so eigenvalues are $\lambda_1 = 2$, $\lambda_2 = 1 + \sqrt{3}$, $\lambda_3 = 1 - \sqrt{3}$. Corresponding eigenvectors are

$$\mathbf{V}_1 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, \mathbf{V}_2 = \begin{pmatrix} -1+\sqrt{3} \\ 1 \\ 1 \end{pmatrix} \text{ and } \mathbf{V}_3 = \begin{pmatrix} -1-\sqrt{3} \\ 1 \\ 1 \end{pmatrix}.$$

These eigenvectors are mutually orthogonal.

23. We know that $\mathbf{AE} = \lambda\mathbf{E}$. Then

$$\mathbf{A}^2\mathbf{E} = \mathbf{A}(\mathbf{AE}) = \mathbf{A}(\lambda\mathbf{E}) = \lambda(\mathbf{AE}) = \lambda^2\mathbf{E}.$$

This means that λ^2 is an eigenvalue of $\mathbf{E}^2$, with eigenvector $\mathbf{E}$. The general result follows now from an induction on k to show that $\mathbf{A}^k = \lambda^k\mathbf{E}$.

24. The characteristic polynomial of $\mathbf{A}$ is $p_{\mathbf{A}}(\lambda) = |\lambda\mathbf{I} - \mathbf{A}| = 0$. The constant term of this polynomial is obtained by setting $\lambda = 0$. This constant term is equal to $|-\mathbf{A}|$. This determinant is $(-1)^n|\mathbf{A}|$. Now $\lambda = 0$ is an eigenvalue of $\mathbf{A}$ (root of the characteristic polynomial) exactly when the constant term of $p_{\mathbf{A}}(\lambda)$ is zero, and we now know that this occurs exactly when $|\mathbf{A}| = 0$.

9.2 Diagonalization

1.
$$p_{\mathbf{A}}(\lambda) = \lambda^2 - 3\lambda + 4$$

is the characteristic polynomial, with roots $\lambda_1 = (3 + \sqrt{7}i)/2$ and $\lambda_2 = (3 - \sqrt{7}i)/2$. Corresponding eigenvectors are

$$\mathbf{V}_1 = \begin{pmatrix} -3+\sqrt{7}i \\ 8 \end{pmatrix} \text{ and } \mathbf{V}_2 = \begin{pmatrix} -3-\sqrt{7}i \\ 8 \end{pmatrix}.$$

The matrix

$$\mathbf{P} = \begin{pmatrix} -3+\sqrt{7}i & -3-\sqrt{7}i \\ 8 & 8 \end{pmatrix}$$

diagonalizes $\mathbf{A}$ and

$$\mathbf{P}^{-1}\mathbf{AP} = \begin{pmatrix} (3+\sqrt{7}i)/2 & 0 \\ 0 & (3-\sqrt{7}i)/2 \end{pmatrix}.$$

If we wrote the eigenvectors in the other order in forming $\mathbf{P}$, then the columns of $\mathbf{P}^{-1}\mathbf{AP}$ would be reversed.

2.
$$p_{\mathbf{A}}(\lambda) = \lambda^2 - 8\lambda + 12,$$

so the eigenvalues are $\lambda_1 = 2$ and $\lambda_2 = 6$. Corresponding eigenvectors are

$$\mathbf{V}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \text{ and } \mathbf{V}_2 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}.$$

We can diagonalize $\mathbf{A}$ with

$$\mathbf{P} = \begin{pmatrix} -1 & 3 \\ 1 & 1 \end{pmatrix},$$

obtaining

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{pmatrix} 2 & 0 \\ 0 & 6 \end{pmatrix}.$$

3.

$$p_{\mathbf{A}}(\lambda) = \lambda^2 - 2\lambda + 1,$$

so the eigenvalues are $\lambda_1 = \lambda_2 = 1$. Every eigenvector is a scalar multiple of

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

so $\mathbf{A}$ is not diagonalizable.

4.

$$p_{\mathbf{A}}(\lambda) = \lambda^2 - 4\lambda - 45,$$

so the eigenvalues are $\lambda_1 = -5$ and $\lambda_2 = 9$. Corresponding eigenvectors are

$$\mathbf{V}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } \mathbf{V}_2 = \begin{pmatrix} 3 \\ 14 \end{pmatrix}.$$

Then

$$\mathbf{P} = \begin{pmatrix} 1 & 3 \\ 0 & 14 \end{pmatrix}$$

diagonalizes $\mathbf{A}$ and

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{pmatrix} -5 & 0 \\ 0 & 9 \end{pmatrix}.$$

5.

$$p_{\mathbf{A}}(\lambda) = \lambda(\lambda - 5)(\lambda + 2),$$

and the eigenvalues of $\mathbf{A}$ are $\lambda_1 = 0$, $\lambda_2 = 5$ and $\lambda_3 = -2$. Corresponding eigenvectors are

$$\mathbf{V}_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \mathbf{V}_2 = \begin{pmatrix} 5 \\ 1 \\ 0 \end{pmatrix} \text{ and } \mathbf{V}_3 = \begin{pmatrix} 0 \\ -3 \\ 2 \end{pmatrix}.$$

Form

$$\mathbf{P} = \begin{pmatrix} 0 & 5 & 0 \\ 1 & 1 & -3 \\ 0 & 0 & 2 \end{pmatrix}$$

Then

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

6.

$$p_{\mathbf{A}}(\lambda) = \lambda(\lambda - 3\lambda - 2),$$

so the eigenvalues are $\lambda_1 = 0$, $\lambda_2 = (3 + \sqrt{17})/2$ and $\lambda_3 = (3 - \sqrt{17})/2$. Corresponding eigenvectors are

$$\mathbf{V}_1 = \begin{pmatrix} -2 \\ -3 \\ 1 \end{pmatrix}, \mathbf{V}_2 = \begin{pmatrix} 0 \\ 4 \\ 3 + \sqrt{17} \end{pmatrix} \text{ and } \mathbf{V}_3 = \begin{pmatrix} 0 \\ 4 \\ 3 - \sqrt{17} \end{pmatrix}.$$

Let

$$\mathbf{P} = \begin{pmatrix} -2 & 0 & 0 \\ -3 & 4 & 4 \\ 1 & 3 + \sqrt{17} & 3 - \sqrt{17} \end{pmatrix}.$$

Then

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & (3 + \sqrt{17})/2 & 0 \\ 0 & 0 & (3 - \sqrt{17})/2 \end{pmatrix}.$$

7.

$$p_{\mathbf{A}}(\lambda) = (\lambda + 2)^2(\lambda - 1),$$

so eigenvalues and corresponding eigenvectors are

$$\lambda_1 = 1, \mathbf{V}_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \lambda_2 = \lambda_3 = -2, \mathbf{V}_2 = \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix}.$$

$\mathbf{A}$ does not have three linearly independent eigenvectors (the repeated eigenvalue has only one independent eigenvector), and so is not diagonalizable.

8.

$$p_{\mathbf{A}}(\lambda) = (\lambda - 2)(\lambda^2 - 4\lambda + 5),$$

so eigenvalues and eigenvectors are

$$\lambda_1 = 2, \mathbf{V}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \lambda_2 = 2 + i, \mathbf{V}_2 = \begin{pmatrix} 0 \\ 1 \\ i \end{pmatrix}, \lambda_3 = 2 - i, \mathbf{V}_3 = \begin{pmatrix} 0 \\ 1 \\ -i \end{pmatrix}.$$

Let

$$\mathbf{P} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & i & -i \end{pmatrix}$$

and then

$$\mathbf{P}^{-1}\mathbf{AP} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2+i & 0 \\ 0 & 0 & 2-i \end{pmatrix}.$$

9.

$$p_{\mathbf{A}}(\lambda) = (\lambda - 1)(\lambda - 4)(\lambda^2 + 5\lambda + 5),$$

so eigenvalues and eigenvectors are $\lambda_1 = 1$, $\lambda_2 = 4$, $\lambda_3 = (-5+\sqrt{5})/2$ and $\lambda_4 = (-5-\sqrt{5})/2$. Corresponding eigenvectors are

$$\mathbf{V}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \mathbf{V}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix},$$

$$\mathbf{V}_3 = \begin{pmatrix} 0 \\ (2-3\sqrt{5})/41 \\ (-1+\sqrt{5})/2 \\ 1 \end{pmatrix} \text{ and } \mathbf{V}_4 = \begin{pmatrix} 0 \\ (2+3\sqrt{5})/41 \\ (-1-\sqrt{5})/2 \\ 1 \end{pmatrix}.$$

Let

$$\mathbf{P} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & (2-3\sqrt{5})/41 & (2+3\sqrt{5})/41 \\ 0 & 0 & (-1+\sqrt{5})/2 & (-1-\sqrt{5})/2 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

Then

$$\mathbf{P}^{-1}\mathbf{AP} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & (-5+\sqrt{5})/2 & 0 \\ 0 & 0 & 0 & (-5-\sqrt{5})/2 \end{pmatrix}.$$

10.

$$p_{\mathbf{A}}(\lambda) = (\lambda + 2)^4,$$

so the eigenvalues of $\mathbf{A}$ are -2, with multiplicity 4. We find that there are only three independent eigenvectors, namely

$$\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

Since $\mathbf{A}$ does not have four linearly independent eigenvectors, $\mathbf{A}$ is not diagonalizable.

11. Since $\mathbf{P}$ diagonalizes $\mathbf{A}$, $\mathbf{P}^{-1}\mathbf{AP} = \mathbf{D}$, a diagonal matrix having the eigenvalues of $\mathbf{A}$ along its main diagonal. Then $\mathbf{A} = \mathbf{PDP}^{-1}$, and

$$\begin{aligned}\mathbf{A}^k &= (\mathbf{PDP}^{-1})^k \\ &= (\mathbf{PDP}^{-1})(\mathbf{PDP}^{-1})\cdots(\mathbf{PDP}^{-1}) \\ &= \mathbf{PD}^k\mathbf{P}^{-1},\end{aligned}$$

with the interior pairings of $\mathbf{P}^{-1}\mathbf{P}$ canceling.

12.
$$p_{\mathbf{A}}(\lambda = \lambda^2 - \lambda - 18,$$

so the eigenvalues of $\mathbf{A}$ are $(1+\sqrt{73})/2$ and $(1-\sqrt{73})/2$. Form a matrix $\mathbf{P}$ with corresponding eigenvectors are columns, yielding

$$\mathbf{P} = \begin{pmatrix} -6 & -6 \\ 7+\sqrt{73} & 7-\sqrt{73} \end{pmatrix}.$$

Compute

$$\mathbf{P}^{-1} = \frac{1}{12\sqrt{73}} \begin{pmatrix} 7-\sqrt{73} & 6 \\ -7-\sqrt{73} & -6 \end{pmatrix}.$$

Then

$$\begin{aligned}\mathbf{A}^{16} &= \mathbf{P} \begin{pmatrix} ((1+\sqrt{73})/2)^{16} & 0 \\ 0 & ((1-\sqrt{73})/2)^{16} \end{pmatrix} \mathbf{P}^{-1} \\ &= \begin{pmatrix} 6(2^{16}) - 3^{16} & 3(2^{16}) - 3^{17} \\ -2^{17} + 2(3^{16}) & -2^{16} + 6(3^{16}) \end{pmatrix}.\end{aligned}$$

13.
$$p_{\mathbf{A}}(\lambda) = \lambda^2 + 6\lambda + 5,$$

so the eigenvalues are -1 and -5. Form $\mathbf{P}$ using corresponding eigenvectors as columns:

$$\mathbf{P} = \begin{pmatrix} 4 & 0 \\ 1 & 1 \end{pmatrix}.$$

Then

$$\begin{aligned}\mathbf{A}^{18} &= \mathbf{PAP}^{-1} \\ &= \begin{pmatrix} 4 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 5^{18} \end{pmatrix} \begin{pmatrix} 1/4 & 0 \\ -1/4 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ (1-5^{18})/4 & 5^{18} \end{pmatrix}.\end{aligned}$$

14. $\mathbf{A}$ has eigenvalues $-3+\sqrt{10}$ and $-3-\sqrt{10}$. Form $\mathbf{P}$ using corresponding eigenvectors as columns:

$$\mathbf{P} = \begin{pmatrix} 3 & 3 \\ 1-\sqrt{10} & 1+\sqrt{10} \end{pmatrix}.$$

Then

$$\mathbf{P}^{-1} = \begin{pmatrix} (10+\sqrt{10})/60 & -\sqrt{10}/20 \\ (10-\sqrt{10})/60 & \sqrt{10}/20 \end{pmatrix}.$$

Then

$$\begin{aligned} \mathbf{A}^{31} &= \mathbf{P} \begin{pmatrix} (-3+\sqrt{10})^{31} & 0 \\ 0 & (-3-\sqrt{10})^{31} \end{pmatrix} \mathbf{P}^{-1} \\ &= \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \end{aligned}$$

where

$$\begin{aligned} a &= \frac{1}{2\sqrt{10}} \left((1+\sqrt{10})(-3+\sqrt{10})^{31} + (\sqrt{10}-1)(-3-\sqrt{10})^{31} \right), \\ b &= \frac{3}{2\sqrt{10}} \left((-3+\sqrt{10})^{31} + (3+\sqrt{10})^{31} \right), \\ c &= \frac{1}{2\sqrt{10}} \left((-1+\sqrt{10})(-3+\sqrt{10})^{31} + (\sqrt{10}+1)(3+\sqrt{10})^{31} \right) \\ d &= \frac{3}{2\sqrt{10}} \left((-3+\sqrt{10})^{31} + (3+\sqrt{10})^{31} \right). \end{aligned}$$

15. Eigenvalues of $\mathbf{A}$ are $\lambda_1 = \sqrt{2}$ and $\lambda_2 = -\sqrt{2}$, with corresponding eigenvectors

$$\mathbf{V}_1 = \begin{pmatrix} \sqrt{2} \\ 1 \end{pmatrix}, \mathbf{V}_2 = \begin{pmatrix} -\sqrt{2} \\ 1 \end{pmatrix}.$$

Let

$$\mathbf{P} = \begin{pmatrix} \sqrt{2} & -\sqrt{2} \\ 1 & 1 \end{pmatrix}.$$

We find that

$$\mathbf{P}^{-1} = \begin{pmatrix} \sqrt{2}/4 & 1/2 \\ -\sqrt{2}/4 & 1/2 \end{pmatrix}.$$

Then

$$\begin{aligned} \mathbf{A}^{43} &= \begin{pmatrix} \sqrt{2} & -\sqrt{2} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} (\sqrt{2})^{43} & 0 \\ 0 & (-\sqrt{2})^{43} \end{pmatrix} \begin{pmatrix} \sqrt{2}/4 & 1/2 \\ -\sqrt{2}/4 & 1/2 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 2^{22} \\ 2^{21} & 0 \end{pmatrix}. \end{aligned}$$

16. Since $\mathbf{A}^2$ is diagonalizable, $\mathbf{A}^2$ has n linearly independent eigenvectors $\mathbf{X}_1, \cdots, \mathbf{X}_n$, with associated eigenvalues $\lambda_1, \cdots, \lambda_n$, respectively. (These eigenvalues need not be distinct). Now

$$p_{\mathbf{A}^2}(\lambda) = (\mathbf{A}^2 - \lambda_j \mathbf{I}_n)\mathbf{X}_j = \mathbf{O}$$

for $j = 1, 2, \cdots, n$. Then

$$\begin{aligned} p_{\mathbf{A}^2}(\lambda) &= (\mathbf{A} - \sqrt{\lambda_j}\mathbf{I}_n)(\mathbf{A} + \sqrt{\lambda_j}\mathbf{I}_n) \\ &= p_{\mathbf{A}}(\sqrt{\lambda_j})p_{\mathbf{A}}(-\sqrt{\lambda_j}) = O \end{aligned}$$

for $j = 1, 2, \cdots, n$. Then

$$p_{\mathbf{A}}(\sqrt{\lambda_j}) = 0 \text{ or } p_{\mathbf{A}}(-\sqrt{\lambda_j}) = 0.$$

But this means that $\sqrt{\lambda_j}$ or $-\sqrt{\lambda_j}$ is an eigenvalue of $\mathbf{A}$ with associated eigenvector $\mathbf{X}_j$. This implies that $\mathbf{A}$ has n linearly independent eigenvectors and is therefore diagonalizable.

9.3 Some Special Matrices

In Problems 1 - 12, begin by finding an orthogonal set of eigenvectors. Since any nonzero constant times an eigenvector is also an eigenvector, multiply each eigenvector by the reciprocal of its magnitude to obtain an orthonormal set of eigenvectors. The matrix $\mathbf{Q}$ is an orthogonal matrix that diagonalizes the given matrix.

For Problems 1 - 6, orthogonal eigenvectors were requested in Problems 17 - 22 of Section 9.1, so for these problems all that is needed is to normalize these eigenvectors.

1. In Problem 17 of Section 9.1 we found the orthogonal eigenvectors

$$\mathbf{V}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \mathbf{V}_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}.$$

Divide each by its length $\sqrt{5}$ and use the resulting orthonormal vectors as columns of $\mathbf{Q}$:

$$\mathbf{Q} = \begin{pmatrix} 1\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{pmatrix}.$$

$\mathbf{Q}$ is an orthogonal matrix that diagonalizes $\mathbf{A}$.

2. Previously we obtained the eigenvectors

$$\mathbf{V}_1 = \begin{pmatrix} 10 \\ 7 + \sqrt{149} \end{pmatrix} \text{ and } \mathbf{V}_2 = \begin{pmatrix} 10 \\ 7 - \sqrt{149} \end{pmatrix}.$$

Divide each eigenvector by its length to form

$$\mathbf{Q} = \begin{pmatrix} \frac{10}{\sqrt{298+14\sqrt{149}}} & \frac{10}{\sqrt{298-14\sqrt{149}}} \\ \frac{7+\sqrt{149}}{\sqrt{298+14\sqrt{149}}} & \frac{7-\sqrt{149}}{\sqrt{298-14\sqrt{149}}} \end{pmatrix}.$$

3. We know the eigenvectors

$$\mathbf{V}_1 = \begin{pmatrix} 1+\sqrt{2} \\ 1 \end{pmatrix} \text{ and } \mathbf{V}_2 = \begin{pmatrix} 1-\sqrt{2} \\ 1 \end{pmatrix}.$$

Divide each by its length to form

$$\mathbf{Q} = \begin{pmatrix} \frac{1+\sqrt{2}}{\sqrt{4+2\sqrt{2}}} & \frac{1-\sqrt{2}}{\sqrt{4-2\sqrt{2}}} \\ \frac{1}{\sqrt{4+2\sqrt{2}}} & \frac{1}{\sqrt{4-2\sqrt{2}}} \end{pmatrix}$$

.

4. We have the eigenvectors

$$\mathbf{V}_1 = \begin{pmatrix} 2 \\ 17+\sqrt{293} \end{pmatrix} \text{ and } \mathbf{V}_2 = \begin{pmatrix} 2 \\ 17-\sqrt{293} \end{pmatrix}.$$

Normalize these eigenvectors to form

$$\mathbf{Q} = \begin{pmatrix} \frac{2}{\sqrt{586-34\sqrt{298}}} & \frac{2}{\sqrt{586+34\sqrt{298}}} \\ \frac{17-\sqrt{298}}{\sqrt{586-34\sqrt{298}}} & \frac{17+\sqrt{298}}{\sqrt{586+34\sqrt{298}}} \end{pmatrix}.$$

5. Eigenvectors are

$$\mathbf{V}_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \mathbf{V}_2 = \begin{pmatrix} 1+\sqrt{2} \\ 1 \\ 0 \end{pmatrix}, \text{ and } \mathbf{V}_3 = \begin{pmatrix} 1-\sqrt{2} \\ 1 \\ 0 \end{pmatrix}.$$

Normalize these to form columns of $\mathbf{Q}$:

$$\mathbf{Q} = \begin{pmatrix} 0 & \frac{1+\sqrt{2}}{\sqrt{4+2\sqrt{2}}} & \frac{1-\sqrt{2}}{\sqrt{4-2\sqrt{2}}} \\ 0 & \frac{1}{\sqrt{4+2\sqrt{2}}} & \frac{1}{\sqrt{4-2\sqrt{2}}} \\ 1 & 0 & 0 \end{pmatrix}.$$

6. Eigenvectors are

$$\mathbf{V}_1 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, \mathbf{V}_2 = \begin{pmatrix} -1+\sqrt{3} \\ 1 \\ 1 \end{pmatrix} \text{ and } \mathbf{V}_3 = \begin{pmatrix} -1-\sqrt{3} \\ 1 \\ 1 \end{pmatrix}.$$

Normalize these to form

$$\mathbf{Q} = \begin{pmatrix} 0 & \frac{-1+\sqrt{3}}{\sqrt{6}} & \frac{-1-\sqrt{3}}{\sqrt{6}} \\ -1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{6} \\ 1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{6} \end{pmatrix}.$$

7.
$$p_{\mathbf{A}}(\lambda) = \lambda(\lambda^2 - 5\lambda - 4)$$
and $\mathbf{A}$ has eigenvalues 0, $(5+\sqrt{41})/2$ and $5-\sqrt{41})/2$, with corresponding eigenvectors
$$\mathbf{V}_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \mathbf{V}_2 = \begin{pmatrix} 5+\sqrt{41} \\ 0 \\ 4 \end{pmatrix}, \mathbf{V}_3 = \begin{pmatrix} 5-\sqrt{41} \\ 0 \\ 4 \end{pmatrix}.$$
Normalize these to find an orthogonal matrix that diagonalizes $\mathbf{A}$:
$$\mathbf{Q} = \begin{pmatrix} 0 & (5+\sqrt{41})/\sqrt{82+10\sqrt{41}} & (5-\sqrt{41})/\sqrt{82-10\sqrt{41}} \\ 1 & 0 & 0 \\ 0 & 4/\sqrt{82+10\sqrt{41}} & 4/\sqrt{82-10\sqrt{41}} \end{pmatrix}.$$

8.
$$p_{\mathbf{A}}(\lambda) = \lambda(\lambda^2 - 2\lambda - 16)$$
so 0, $1+\sqrt{17}$ and $1-\sqrt{17}$ are eigenvalues. We find the corresponding eigenvectors
$$\mathbf{V}_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \mathbf{V}_2 = \begin{pmatrix} 1+\sqrt{17} \\ -4 \\ 0 \end{pmatrix}, \mathbf{V}_3 = \begin{pmatrix} 1-\sqrt{17} \\ -4 \\ 0 \end{pmatrix}.$$
Then
$$\mathbf{Q} = \begin{pmatrix} 0 & (1+\sqrt{17})/\sqrt{34+2\sqrt{17}} & (1-\sqrt{17})/\sqrt{34-2\sqrt{17}} \\ 0 & -4/\sqrt{34+2\sqrt{17}} & -4/\sqrt{34-2\sqrt{17}} \\ 1 & 0 & 0 \end{pmatrix}.$$

9.
$$p_{\mathbf{A}}(\lambda) = \lambda(\lambda^2 - \lambda - 4)$$
and the eigenvalues are 0, $(1+\sqrt{17})/2$, $(1-\sqrt{17})/2$. Corresponding eigenvectors are
$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \mathbf{V}_2 \begin{pmatrix} 0 \\ -1-\sqrt{17} \\ 4 \end{pmatrix}, \mathbf{V}_3 = \begin{pmatrix} 0 \\ -1-\sqrt{17} \\ 4 \end{pmatrix}.$$
Then
$$\mathbf{Q} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & (-1-\sqrt{17})/\sqrt{34+2\sqrt{17}} & (-1+\sqrt{17})/\sqrt{34+2\sqrt{17}} \\ 0 & 4/\sqrt{34+2\sqrt{17}} & 4/\sqrt{34+2\sqrt{17}} \end{pmatrix}.$$

10.

$$p_{\mathbf{A}}(\lambda) = (\lambda - 1)(\lambda^2 - \lambda - 10)$$

and the eigenvalues are 1, $(1+\sqrt{41})/2$ and $(1-\sqrt{41})/2$. Corresponding eigenvectors are

$$\mathbf{V}_1 = \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix}, \mathbf{V}_2 = \begin{pmatrix} 6 \\ -1+\sqrt{41} \\ 2 \end{pmatrix}, \mathbf{V}_3 = \begin{pmatrix} 6 \\ -1-\sqrt{41} \\ 2 \end{pmatrix}.$$

Then

$$\mathbf{Q} = \begin{pmatrix} 1/\sqrt{10} & 6/\sqrt{82-2\sqrt{41}} & 6/\sqrt{82+2\sqrt{41}} \\ 0 & (-1+\sqrt{41})/\sqrt{82-2\sqrt{41}} & (-1-\sqrt{41})/\sqrt{82-2\sqrt{41}} \\ -2/\sqrt{10} & 2/\sqrt{82-2\sqrt{41}} & 2/\sqrt{82-2\sqrt{41}} \end{pmatrix}.$$

11.

$$p_{\mathbf{A}}(\lambda) = \lambda^2(\lambda^2 - 2\lambda - 3)$$

so the eigenvalues are 0, 0, -1 and 3. Corresponding eigenvectors are

$$\mathbf{V}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \mathbf{V}_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \mathbf{V}_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \mathbf{V}_4 = \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix}.$$

Then

$$\mathbf{Q} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ 0 & 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

12.

$$p_{\mathbf{A}} = \lambda(\lambda - 5)(\lambda^2 - 1)$$

and the eigenvalues are 0, 5, 1 and -1. Corresponding eigenvectors are

$$\mathbf{V}_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \mathbf{V}_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \mathbf{V}_4 = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}, \mathbf{V}_4 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}.$$

Then

$$\mathbf{Q} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 0 & -1/\sqrt{2} & 1/\sqrt{2} \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

13. The matrix is not hermitian, skew-hermitian or unitary. Compute

$$p_{\mathbf{A}}(\lambda) = (\lambda - 2)^2.$$

The eigenvalue is 2 with multiplicity 2 and only one independent eigenvector,

$$\begin{pmatrix} i \\ 1 \end{pmatrix}.$$

Therefore $\mathbf{A}$ is not diagonalizable.

14. $\mathbf{A}$ is not hermitian, skew-hermitian or unitary.

$$p_{\mathbf{A}}(\lambda) = (\lambda + 1)^2$$

with root -1 of multiplicity 2. Every eigenvector is a scalar multiple of

$$\begin{pmatrix} i \\ -1 \end{pmatrix},$$

so the matrix is not diagonalizable.

15. This matrix S is skew-hermitian, since

$$\mathbf{S}^t = -\overline{\mathbf{S}}.$$

$$p_{\mathbf{S}}(\lambda) = \lambda(\lambda^2 + 3)$$

so the eigenvalues are 0, $\sqrt{3}i$ and $-\sqrt{3}i$, with corresponding eigenvectors

$$\begin{pmatrix} 2 \\ 0 \\ 1+i, \end{pmatrix} \begin{pmatrix} 1 \\ \sqrt{3}i \\ -1-i \end{pmatrix}, \text{ and } \begin{pmatrix} 1 \\ -\sqrt{3}i \\ -1-i \end{pmatrix}.$$

Let $\mathbf{P}$ have these eigenvectors as columns (in the given order). Then

$$\mathbf{P}^{-1}\mathbf{SP} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \sqrt{3}i & 0 \\ 0 & 0 & -\sqrt{3}i \end{pmatrix}.$$

16. It is routine to check that

$$\overline{\mathbf{U}}\mathbf{U}^t = \mathbf{I},$$

so $\mathbf{U}$ is unitary.

$$p_{\mathbf{U}}(\lambda) = (\lambda - 1)\left(\lambda^2 - \frac{1+i}{\sqrt{2}}\lambda + i\right)$$

so the eigenvalues are

$$1, \frac{1+\sqrt{3}}{2\sqrt{2}} + \frac{1-\sqrt{3}}{2\sqrt{2}}i, \frac{1-\sqrt{3}}{2\sqrt{2}} + \frac{1+\sqrt{3}}{2\sqrt{2}}i,$$

with corresponding eigenvectors

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1+i \\ (1-\sqrt{3})i \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} 1+i \\ (1+\sqrt{3})i \\ 0 \end{pmatrix}.$$

Use these as the columns of $\mathbf{P}$. Then $\mathbf{P}$ diagonalizes $\mathbf{U}$.

17. The matrix is hermitian, since $\mathbf{H}^t = \overline{\mathbf{H}}$. The eigenvalues are approximately $\lambda_1 = 4.051374$, $\lambda_2 = 0.482696$, $\lambda_3 = -1.53407$. These are distinct, so the matrix is diagonalizable.

18. $\mathbf{H}$ is hermitian with eigenvalues 1, $(-1+\sqrt{41})/2$ and $(-1-\sqrt{41})/2$. Corresponding eigenvectors are

$$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 6-2i \\ 0 \\ 1+\sqrt{41} \end{pmatrix} \text{ and } \begin{pmatrix} 6-2i \\ 0 \\ 1-\sqrt{41} \end{pmatrix}.$$

The matrix having these eigenvectors as columns diagonalizes $\mathbf{H}$.

19. The matrix $\mathbf{S}$ is skew-hermitian with approximate eigenvalues $-2.164248i$, $0.772866i$ and $2.39182i$. Since these are distinct, $\mathbf{S}$ is diagonalizable.

20. The matrix is not hermitian, skew-hermitian or unitary. Eigenvalues are 1, -1 and $3i$, with corresponding eigenvectors

$$\begin{pmatrix} 0 \\ i \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ i \\ -1 \end{pmatrix} \text{ and } \begin{pmatrix} -10i \\ 3 \\ -1 \end{pmatrix}.$$

The matrix having these eigenvectors as columns diagonalizes the matrix.

21. $\mathbf{H}$ is hermitian with eigenvalues 0, $4+3\sqrt{2}$ and $4-3\sqrt{2}$. Corresponding eigenvectors are

$$\begin{pmatrix} 0 \\ i \\ 1 \end{pmatrix}, \begin{pmatrix} 4+3\sqrt{2} \\ -1 \\ -i \end{pmatrix} \text{ and } \begin{pmatrix} 4-3\sqrt{2} \\ -1 \\ -i \end{pmatrix}.$$

The matrix having these eigenvectors as columns diagonalizes $\mathbf{H}$.

22. The matrix of the quadratic form is

$$\mathbf{A} = \begin{pmatrix} -5 & 2 \\ 2 & 3 \end{pmatrix}.$$

Eigenvalues of $\mathbf{A}$ are $-1+2\sqrt{5}$ and $-1-2\sqrt{5}$ and the standard form of this quadratic form is

$$(-1+2\sqrt{5})y_1^2 + (-1-2\sqrt{5})y_2^2.$$

23. The matrix of this form is

$$\mathbf{A} = \begin{pmatrix} 4 & -6 \\ -6 & 1 \end{pmatrix}.$$

The eigenvalues are $(5+\sqrt{153})/2$ and $(5-\sqrt{153})/2$. The standard form is

$$\left(\frac{5+\sqrt{153}}{2}\right)y_1^2 + \left(\frac{5-\sqrt{153}}{2}\right)y_2^2.$$

24. The matrix is

$$\begin{pmatrix} -3 & 2 \\ 2 & 7 \end{pmatrix}$$

with eigenvalues $2 \pm \sqrt{29}$. The standard form is

$$(2+\sqrt{29})y_1^2 + (2-\sqrt{29})y_2^2.$$

25. The matrix is

$$\begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix}$$

with eigenvalues $(3+\sqrt{17})/2$ and $(3-\sqrt{17})/2$. The standard form is

$$\left(\frac{3+\sqrt{17}}{2}\right)y_1^2 + \left(\frac{3-\sqrt{17}}{2}\right)y_2^2.$$

26. The matrix is

$$\begin{pmatrix} 0 & -3 \\ -3 & 4 \end{pmatrix}$$

with eigenvalues $2 \pm \sqrt{13}$. The standard form is

$$(2+\sqrt{13})y_1^2 + (2-\sqrt{13})^2 y_2^2.$$

27. The matrix is

$$\begin{pmatrix} 5 & 2 \\ 2 & 2 \end{pmatrix}$$

with eigenvalues $1, 6$. The standard form is

$$y_1^2 + 6y_2^2.$$

28. The matrix is

$$\begin{pmatrix} 0 & -1 \\ -1 & 2 \end{pmatrix}$$

with eigenvalues $1 \pm \sqrt{2}$. The standard form is

$$(1+\sqrt{2})y_1^2 + (1-\sqrt{2})y_2^2.$$

In computations involving complex matrices, it is assumed that the conjugate of a product is the product of the conjugates, and the conjugate of a transpose is the transpose of the conjugate.

29. If $\mathbf{A}$ is hermitian, then $\mathbf{A}^t = \overline{\mathbf{A}}$, so

$$\overline{(\mathbf{A}\mathbf{A}^t)} = \overline{\mathbf{A}}(\overline{\mathbf{A}})^t = \overline{\mathbf{A}}(\mathbf{A}^t)^t = \overline{\mathbf{A}}\mathbf{A}.$$

30. If $\mathbf{H}$ is hermitian, then $\mathbf{H}^t = \overline{\mathbf{H}}$, so the diagonal elements satisfy $h_{jj} = \overline{h_{jj}}$, and therefore must be real.

31. If S is skew-hermitian, then $\mathbf{S}^t = -\overline{\mathbf{S}}$, so $s_{jj} = -\overline{s_{jj}}$ for $j = 1, \cdots, n$. Now write $s_{jj} = a_{jj} + ib_{jj}$. Then $a_{jj} = -a_{jj}$, so each $a_{jj} = 0$. This makes the diagonal elements zero (if $b = 0$) or pure imaginary (if $b \neq 0$).

32. Let $\mathbf{U}$ and $\mathbf{V}$ be unitary matrices. Then $\mathbf{U}^{-1} = \overline{\mathbf{U}}^t$ and $\mathbf{V}^{-1} = \overline{\mathbf{V}}^t$. Then

$$(\mathbf{U}\mathbf{V})^{-1} = \mathbf{V}^{-1}\mathbf{U}^{-1} = \overline{\mathbf{V}}^t\overline{\mathbf{U}}^t = ((\overline{\mathbf{U}})(\overline{\mathbf{V}}))^t = (\overline{\mathbf{U}\mathbf{V}})^t$$

and this implies that $\mathbf{U}\mathbf{V}$ is unitary.

Chapter 10

Systems of Linear Differential Equations

10.1 Systems of Linear Differential Equations

1. The system is $\mathbf{X}' = \mathbf{AX}$, where

$$\mathbf{A} = \begin{pmatrix} 5 & 3 \\ 1 & 3 \end{pmatrix}.$$

Two linearly independent solutions are

$$\mathbf{\Phi}_1(t) = \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^2 \text{ and } \mathbf{\Phi}_2(t) = \begin{pmatrix} 3 \\ 1 \end{pmatrix} e^{6t}.$$

Using $t = 0$, form the determinant having columns $\mathbf{\Phi}_1(0)$ and $\mathbf{\Phi}_1(0)$:

$$\begin{vmatrix} -1 & 3 \\ 1 & 1 \end{vmatrix} = -4 \neq 0,$$

Therefore these solutions are linearly independent. We can form the fundamental matrix using these solutions as columns:

$$\mathbf{\Omega}(t) = \begin{pmatrix} -e^{2t} & 3e^{6t} \\ e^{2t} & e^{6t} \end{pmatrix}.$$

In terms of the fundamental matrix, the general solution of the system is $\mathbf{X}(t) = \mathbf{\Omega}(t)\mathbf{C}$, where

$$\mathbf{C} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

To satisfy the initial condition $x_1(0) = 0, x_2(0) = 4$, solve for $\mathbf{C}$ in $\mathbf{X}(0) = \mathbf{\Omega}(0)\mathbf{C}$, which is

$$\begin{pmatrix} 0 \\ 4 \end{pmatrix} = \begin{pmatrix} -1 & 3 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

Then

$$\begin{pmatrix} 0 \\ 4 \end{pmatrix} = \mathbf{\Omega}^{-1}(0) \begin{pmatrix} 0 \\ 4 \end{pmatrix}$$
$$= -\frac{1}{4} \begin{pmatrix} 1 & -3 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 4 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}.$$

The solution of the initial value problem is

$$\mathbf{X}(t) = \mathbf{\Omega}(t) \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} -3e^{2t} + 3e^{6t} \\ 3e^{3t} + e^{6t} \end{pmatrix}.$$

2. The coefficient matrix is

$$\mathbf{A} = \begin{pmatrix} 2 & 1 \\ -3 & 6 \end{pmatrix}.$$

A fundamental matrix is

$$\mathbf{\Omega}(t) = \begin{pmatrix} e^{4t}\cos(t) & e^{4t}\sin(t) \\ 2e^{4t}(\cos(t) - \sin(t)) & 2e^{4t}(\cos(t) - \sin(t)) \end{pmatrix}.$$

Note that $|\mathbf{\Omega}(0)| = 2 \neq 0$. The general solution is $\mathbf{X}(t) = \mathbf{\Omega}(t)\mathbf{C}$. For the solution of the initial value problem, choose

$$\mathbf{C} = \mathbf{\Omega}^{-1}(0)\mathbf{X}(0) = \frac{1}{2} \begin{pmatrix} 2 & 0 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} -2 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 5/2 \end{pmatrix}.$$

The unique solution of the initial value problem is

$$\mathbf{X}(T) = \mathbf{\Omega}(t) \begin{pmatrix} -2 \\ 5/2 \end{pmatrix} = \begin{pmatrix} e^{4t}(-2\cos(t) + (5/2)\sin(t)) \\ e^{4t}(\cos(t) + \sin(t)) \end{pmatrix}.$$

3. The coefficient matrix is

$$\mathbf{A} = \begin{pmatrix} 3 & 8 \\ 1 & -1 \end{pmatrix}.$$

A fundamental matrix is

$$\mathbf{\Omega}(t) = \begin{pmatrix} 4e^{(1+2\sqrt{3})t} & 4e^{(1-2\sqrt{3})t} \\ (-1+\sqrt{3})e^{(1+2\sqrt{3})t} & (-1-\sqrt{3})e^{(1-2\sqrt{3})t} \end{pmatrix}.$$

Notice that $|\mathbf{\Omega}(0)| = -8\sqrt{3} \neq 0$. The general solution is $\mathbf{X}(t) = \mathbf{\Omega}(t)\mathbf{C}$ For the initial value problem, choose

$$\mathbf{C} = \mathbf{\Omega}^{-1}(0)\mathbf{X}(0) = -\frac{1}{8\sqrt{3}} \begin{pmatrix} -1-\sqrt{3} & -4 \\ 1-\sqrt{3} & 4 \end{pmatrix} \begin{pmatrix} 2 \\ 2 \end{pmatrix}$$
$$= \frac{1}{12} \begin{pmatrix} 3+5\sqrt{3} \\ 3-5\sqrt{3} \end{pmatrix}.$$

The unique solution of the initial value problem is (after some manipulation),

$$\mathbf{X}(t) = \mathbf{\Omega}(t)\mathbf{C} = \begin{pmatrix} 2e^t\cosh(2\sqrt{3}t) + (10/\sqrt{3})e^t\sinh(2\sqrt{3}t) \\ 2e^t\cosh(2\sqrt{3}t) - (1/\sqrt{3})e^t\sinh(2\sqrt{3}t) \end{pmatrix}.$$

4. The coefficient matrix is

$$\mathbf{A} = \begin{pmatrix} 1 & -1 \\ 4 & 2 \end{pmatrix}.$$

A fundamental matrix is

$$\mathbf{\Omega}(t) =$$
$$e^{3t/2}\begin{pmatrix} 2\cos(\sqrt{15}t/2) & 2\sin(\sqrt{15}t/2) \\ -\cos(\sqrt{15}t/2) + \sqrt{15}\sin(\sqrt{15}t/2) & -\sin(\sqrt{15}t/2) + \sqrt{15}\cos(\sqrt{15}t/2) \end{pmatrix}.$$

Note that $|\mathbf{\Omega}(0)| = 2\sqrt{15} \neq 0$. The general solution is $\mathbf{X} = \mathbf{\Omega}(t)\mathbf{C}$. For the solution of the initial value problem, choose

$$\mathbf{C} = \mathbf{\Omega}^{-1}(0)\mathbf{X}(0) = \frac{1}{2\sqrt{15}}\begin{pmatrix} \sqrt{15} & 0 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} -2 \\ 7 \end{pmatrix} = \begin{pmatrix} -1 \\ 2\sqrt{15}/5 \end{pmatrix}.$$

This gives the unique solution

$$\mathbf{X}(t) = \mathbf{\Omega}(t)\mathbf{C} = \begin{pmatrix} e^{3t/2}\left[(4\sqrt{15}/5)\sin(\sqrt{15}t/2) - 2\cos(\sqrt{15}t/2)\right] \\ e^{3t/2}\left[7\cos(\sqrt{15}t/2) - (7\sqrt{15}/5)\sin(\sqrt{15}t/2)\right] \end{pmatrix}.$$

5. The coefficient matrix is

$$\mathbf{A} = \begin{pmatrix} 5 & -4 & 4 \\ 12 & -11 & 12 \\ 4 & -4 & 5 \end{pmatrix}$$

A fundamental matrix is

$$\mathbf{\Omega}(t) = \begin{pmatrix} e^t & 0 & e^{-3t} \\ 0 & e^t & 3e^{-3t} \\ -e^t & e^t & e^{-3t} \end{pmatrix}.$$

Then $|\mathbf{\Omega}(0)| = -1 \neq 0$. The general solution is $\mathbf{X}(t) = \mathbf{\Omega}(t)\mathbf{C}$. For the initial value problem, choose

$$\mathbf{C} = \mathbf{\Omega}^{-1}(0)\mathbf{X}(0) = \begin{pmatrix} 2 & -1 & 1 \\ 3 & -2 & 3 \\ -1 & 1 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ -3 \\ 5 \end{pmatrix} = \begin{pmatrix} 10 \\ 24 \\ -9 \end{pmatrix}.$$

This gives us the unique solution

$$\mathbf{X}(t) = \mathbf{\Omega}(t) = \begin{pmatrix} 10e^t - 9e^{-3t} \\ 24e^t - 27e^{-3t} \\ 14e^t - 9e^{-3t} \end{pmatrix}.$$

10.2 Solution of $\mathbf{X}' = \mathbf{AX}$ for Constant $\mathbf{A}$

1. The coefficient matrix is

$$\mathbf{A} = \begin{pmatrix} 3 & 0 \\ 5 & -4 \end{pmatrix}.$$

The characteristic polynomial of $\mathbf{A}$ is $p_{\mathbf{A}}(\lambda) = (\lambda - 3)(\lambda + 4)$. Eigenvalues and corresponding eigenvectors are

$$3, \begin{pmatrix} 7 \\ 5 \end{pmatrix}, -4, \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

A fundamental matrix is

$$\mathbf{\Omega}(t) = \begin{pmatrix} 7e^{3t} & 0 \\ 5e^{3t} & e^{-4t} \end{pmatrix}.$$

The general solution is

$$\mathbf{X}(t) = \mathbf{\Omega}(t)\mathbf{C} = \begin{pmatrix} 7c_1e^{3t} \\ 5c_1e^{3t} + c_2e^{-4t} \end{pmatrix}.$$

For Problems 2 through 5, we give the solution in the form $\mathbf{X}(t) = \mathbf{\Omega}(t)\mathbf{C}$, with $\mathbf{\Omega}(t)$ a fundamental matrix. Note that we can read the eigenvalues and corresponding eigenvectors of the coefficient matrix from the fundamental matrix.

2.
$$\mathbf{X}(t) = \mathbf{\Omega}(t)\mathbf{C} = \begin{pmatrix} 2e^{t} & e^{6t} \\ -3e^{t} & e^{6t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 2c_1e^{t} + c_2e^{6t} \\ -3c_1e^{t} + c_2e^{6t} \end{pmatrix}$$

3.
$$\mathbf{X}(t) = \mathbf{\Omega}(t)\mathbf{C} = \begin{pmatrix} 1 & e^{2t} \\ -1 & e^{2t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} c_1 + c_2e^{2t} \\ -c_1 + c_2e^{2t} \end{pmatrix}$$

4.
$$\mathbf{X}(t) = \mathbf{\Omega}(t)\mathbf{C} = \begin{pmatrix} e^{t} & e^{-t} & e^{2t} \\ e^{t} & e^{-t} & 2e^{2t} \\ e^{t} & 2e^{-t} & e^{2t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} c_1e^{t} + c_2e^{-t} + c_3e^{2t} \\ c_1e^{t} + c_2e^{-t} + 2c_3e^{2t} \\ c_1e^{t} + 2c_2e^{-t} + c_3e^{2t} \end{pmatrix}$$

5.
$$\mathbf{X}(t) = \mathbf{\Omega}(t)\mathbf{C} = \begin{pmatrix} 1 & 2e^{3t} & -e^{-4t} \\ 6 & 3e^{3t} & 2e^{-4t} \\ -13 & -2e^{3t} & e^{-4t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$
$$= \begin{pmatrix} c_1 + 2c_2e^{3t} - c_3e^{-4t} \\ 6c_1 + 3c_2e^{3t} + 2c_3e^{-4t} \\ -13c_1 - 2c_2e^{3t} + c_3e^{-4t} \end{pmatrix}$$

In each of Problems 6 through 10 the unique solution of the initial value problem is given.

6.
$$\mathbf{X}(t) = \begin{pmatrix} 2e^t & e^{-t} \\ e^t & e^{-t} \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 7 \\ 5 \end{pmatrix} = \begin{pmatrix} 4e^t + 3e^{-t} \\ 2e^t + 3e^{-t} \end{pmatrix}$$

7.
$$\mathbf{X}(t) = \begin{pmatrix} 2e^{4t} & e^{-t} \\ -3e^{4t} & 2e^{-3t} \end{pmatrix} \begin{pmatrix} 2 & 1 \\ -3 & 2 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ -19 \end{pmatrix} = \begin{pmatrix} 6e^{4t} - 5e^{-3t} \\ -9e^{4t} - 10e^{-3t} \end{pmatrix}$$

8.
$$\mathbf{X}(t) = \begin{pmatrix} 2e^{-3t} & -5e^{4t} \\ e^{-3t} & e^{4t} \end{pmatrix} \begin{pmatrix} 2 & -5 \\ 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} -3 \\ 6 \end{pmatrix} = \frac{1}{7} \begin{pmatrix} 54e^{-3t} - 75e^{4t} \\ 27e^{-3t} + 15e^{4t} \end{pmatrix}$$

9.
$$\mathbf{X}(t) = \begin{pmatrix} 0 & e^{2t} & 3e^{3t} \\ 1 & e^{2t} & e^{3t} \\ 1 & 0 & e^{3t} \end{pmatrix} \begin{pmatrix} 0 & 1 & 3 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 5 \\ 1 \end{pmatrix}$$
$$= \begin{pmatrix} 4e^{2t} - 3e^{3t} \\ 2 + 4e^{2t} - e^{3t} \\ 2 - e^{3t} \end{pmatrix}$$

10.
$$\mathbf{X}(t) = \begin{pmatrix} e^t & e^{-t} & -e^{-3t} \\ e^t & 3e^{-t} & 3e^{-3t} \\ e^t & 3e^{-t} & -e^{-3t} \end{pmatrix} \begin{pmatrix} 1 & 1 & -1 \\ 1 & 3 & 3 \\ 1 & 3 & -1 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 7 \\ 3 \end{pmatrix}$$
$$= \begin{pmatrix} e^t + e^{-t} - e^{-3t} \\ e^t + 3e^{-t} + 3e^{-3t} \\ e^t + 3e^{-t} - e^{-3t} \end{pmatrix}$$

11. Eigenvalues of $\mathbf{A}$ are roots of $\lambda^2 - 4\lambda + 8$, and are $\lambda_1 = 2 + 2i$ and $\lambda_2 = 2 - 2i$, with corresponding eigenvectors

$$\begin{pmatrix} 2 \\ -i \end{pmatrix} \text{ and } \begin{pmatrix} 2 \\ i \end{pmatrix}.$$

A real fundamental matrix is

$$\mathbf{\Omega}(t) = \begin{pmatrix} 2e^{2t}\cos(2t) & 2e^{2t}\sin(2t) \\ e^{2t}\sin(2t)) & -e^{2t}\cos(2t) \end{pmatrix}.$$

12. Eigenvalues of $\mathbf{A}$ are roots of $\lambda^2 + 2\lambda + 5$, and are $\lambda_1 = -1 + 2i$ and $\lambda_2 = -1 - 2i$, with corresponding eigenvectors

$$\begin{pmatrix} 5 \\ -1+2i \end{pmatrix} \text{ and } \begin{pmatrix} 5 \\ -1-2i \end{pmatrix}.$$

A real fundamental matrix is

$$\mathbf{\Omega}(t) = \begin{pmatrix} 5e^{-t}\cos(2t) & 5e^{-t}\sin(2t) \\ e^{-t}(-\cos(2t) - 2\sin(2t)) & e^{-t}(2\cos(2t) - \sin(2t)) \end{pmatrix}.$$

13. Eigenvalues of $\mathbf{A}$ are roots of $\lambda^2 - 2\lambda + 2$, and are $\lambda_1 = 1+i$ and $\lambda_2 = 1-i$, with corresponding eigenvectors

$$\begin{pmatrix} 5 \\ 2-i \end{pmatrix} \text{ and } \begin{pmatrix} 5 \\ 2+i \end{pmatrix}.$$

A real fundamental matrix is

$$\mathbf{\Omega}(t) = \begin{pmatrix} 5e^{t}\cos(t) & 5e^{t}\sin(t) \\ e^{t}(2\cos(t) + \sin(t)) & e^{t}(2\sin(t) - \cos(t)) \end{pmatrix}.$$

14. Eigenvalues are $\mathbf{A}$ are roots of $(\lambda+1)(\lambda^2+1)$ and are $\lambda_1 = -1$, $\lambda_2 = i$ and $\lambda_3 = -i$, with corresponding eigenvectors

$$\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1+i \\ 1 \\ 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1-i \\ 1 \\ 1 \end{pmatrix}.$$

A real fundamental matrix is

$$\mathbf{\Omega}(t) = \begin{pmatrix} 0 & \cos(t) - \sin(t) & \sin(t) + \cos(t) \\ e^{-t} & \cos(t) & \sin(t) \\ -e^{-t} & \cos(t) & \sin(t) \end{pmatrix}.$$

15. Eigenvalues are $\mathbf{A}$ are roots of $(\lambda+2)(\lambda^2+2\lambda+5)$ and are $\lambda_1 = -2$, $\lambda_2 = -1+2i$ and $\lambda_3 = -1-2i$, with corresponding eigenvectors

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1+2i \\ 3 \end{pmatrix} \text{ and } \begin{pmatrix} 1 \\ 1-2i \\ 3 \end{pmatrix}.$$

A real fundamental matrix is

$$\mathbf{\Omega}(t) = \begin{pmatrix} 0 & e^{-t}\cos(2t) & e^{-t}\sin(2t) \\ 0 & e^{-t}(\cos(2t) - 2\sin(2t)) & e^{-t}(\sin(2t) + 2\cos(2t)) \\ e^{-2t} & 3e^{-t}\cos(2t) & 3e^{-t}\sin(2t) \end{pmatrix}.$$

16. The coefficient matrix is

$$\mathbf{A} = \begin{pmatrix} 2 & 0 \\ 5 & 2 \end{pmatrix}.$$

The eigenvalue is 2 with multiplicity 2 and one independent eigenvector, which we take to be

$$\mathbf{E}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

One solution is

$$\mathbf{\Phi}_1(t) = e^{2t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Try a second solution

$$\mathbf{\Phi}_2(t) = \mathbf{E}_1 t e^{2t} + \mathbf{E}_2 e^{2t}.$$

Substitute this into the differential equation $\mathbf{X}' = \mathbf{A}\mathbf{X}$ to obtain

$$\mathbf{E}_1 e^{2t} + 2t\mathbf{E}_1 e^{2t} + 2\mathbf{E}_2 e^{2t} = \mathbf{A}\mathbf{E}_1 t e^{2t} + \mathbf{A}\mathbf{E}_2 e^{2t}.$$

Divide by e^{2t}. Further, $\mathbf{A}\mathbf{E}_1 = 2\mathbf{E}_1$, so two terms in the last equation cancel. This leaves

$$\mathbf{E}_1 + 2\mathbf{E}_2 = \mathbf{A}\mathbf{E}_2.$$

The unknown here is

$$\mathbf{E}_2 = \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}.$$

This system of equations reduces to

$$\begin{aligned} 2e_1 &= 2e_1 \\ 1 + 2e_2 &= 5e_1 + 2e_2. \end{aligned}$$

Then $e_1 = 1/5$ and e_2 can be any number. Choose $e_2 = 1$. Then

$$\mathbf{E}_2 = \begin{pmatrix} 1/5 \\ 1 \end{pmatrix}.$$

The second solution is

$$\mathbf{\Phi}_2(t) = \mathbf{E}_1 t e^{2t} + \mathbf{E}_2 e^{2t} = \begin{pmatrix} (1/5)e^{2t} \\ te^{2t} + e^{2t} \end{pmatrix}.$$

A fundamental matrix has these two solutions as columns:

$$\mathbf{\Omega}(t) = \begin{pmatrix} 0 & (1/5)e^{2t} \\ e^{2t} & te^{2t} + e^{2t} \end{pmatrix}.$$

Different fundamental matrices can be obtained by making different choices of arbitrary constants in the derivation of this solution.

In Problems 17 through 21, we omit some of the details given in the solution of Problem 16.

17. The coefficient matrix has eigenvalue 3 of multiplicity 2, with eigenvector

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

A fundamental matrix is

$$\mathbf{\Omega}(t) = \begin{pmatrix} e^{3t} & 2te^{3t} \\ 0 & e^{3t} \end{pmatrix}.$$

18. The coefficient matrix has eigenvalue 1 of multiplicity 3 and an eigenvector

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

A fundamental matrix is

$$\mathbf{\Omega}(t) = \begin{pmatrix} e^{t} & 5te^{t} & 0 \\ 0 & e^{t} & 0 \\ 4te^{t} & (10t^2 + 8t)e^{t} & e^{t} \end{pmatrix}.$$

19. The coefficient matrix has eigenvalue 2 with eigenvector

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

and eigenvalue 5 of multiplicity 2, with eigenvector

$$\begin{pmatrix} -3 \\ -3 \\ 1 \end{pmatrix}.$$

A fundamental matrix is

$$\mathbf{\Omega}(t) = \begin{pmatrix} e^{2t} & 3e^{5t} & 27te^{5t} \\ 0 & 3e^{5t} & (3 + 27t)e^{5t} \\ 0 & -e^{5t} & (2 - 9t)e^{5t} \end{pmatrix}.$$

20. The coefficient matrix has a multiplicity 2 eigenvalue i, with single eigenvector

$$\begin{pmatrix} i \\ -1 \\ -i \\ 1 \end{pmatrix}$$

and a multiplicity 2 eigenvalue $-i$ with the single eigenvector

$$\begin{pmatrix} -i \\ -1 \\ i \\ 1 \end{pmatrix}.$$

A fundamental matrix is

$$\mathbf{\Omega}(t) = \begin{pmatrix} \cos(t) & \cos(t) + t\sin(t) & \sin(t) & \sin(t) - t\cos(t) \\ -\sin(t) & t\cos(t) & \cos(t) & t\sin(t) \\ -\cos(t) & \cos(t) - t\sin(t) & -\sin(t) & \sin(t) + t\cos(t) \\ \sin(t) & -t\cos(t) - 2\sin(t) & -\cos(t) & -t\sin(t) + 2\cos(t) \end{pmatrix}.$$

21. The coefficient matrix has eigenvalue 0 with eigenvector

$$\begin{pmatrix} 2 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

and eigenvalue 3 with eigenvector

$$\begin{pmatrix} 3 \\ 2 \\ 2 \\ 0 \end{pmatrix}$$

and eigenvalue 1 of multiplicity 2 and two linearly independent eigenvectors

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 \\ -2 \\ -2 \\ 1 \end{pmatrix}.$$

A fundamental matrix is

$$\mathbf{\Omega}(t) = \begin{pmatrix} 2 & 3e^{3t} & e^t & 0 \\ 0 & 2e^{3t} & 0 & -2e^t \\ 1 & 2e^{3t} & 0 & -2e^t \\ 0 & 0 & 0 & e^t \end{pmatrix}.$$

10.3 Solution of $\mathbf{X}' = \mathbf{AX} + \mathbf{G}$

For a linear system of diffeential equations, a fundamental matrix is not unique, and different fundamental matrices may be derived using different methods. The general solution can be written using any fundamental matrix.

1. The coefficient matrix $\mathbf{A}$ has eigenvalue 3 of multiplicity 2, with one independent eigenvector
$$\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$
Using the methods of Section 10.2, we find a fundamental matrix for the homogeneous system $\mathbf{X}' = \mathbf{A}\mathbf{X}$:
$$\mathbf{\Omega}(t) = e^{3t}\begin{pmatrix} 1+2t & 2t \\ -2t & 1-2t \end{pmatrix}.$$
Compute
$$\mathbf{\Omega}^{-1}(t) = e^{-3t}\begin{pmatrix} 1-2t & -2t \\ 2t & 1+2t \end{pmatrix}.$$
Now compute a particular solution of the given nonhomogeneous system as
$$\mathbf{u}(t) = \int \mathbf{\Omega}^{-1}(t)\mathbf{G}(t)\,dt = \int e^{-3t}\begin{pmatrix} 1-2t & -2t \\ 2t & 1+2t \end{pmatrix}\begin{pmatrix} -3e^{t} \\ e^{3t} \end{pmatrix} dt$$
$$= \int \begin{pmatrix} 6te^{-2t} - 3e^{-2t} - 2t \\ -6te^{-2t} + 1 + 2t \end{pmatrix} dt = \begin{pmatrix} -3te^{-2t} - t^2 \\ (3/2)(1+2t)e^{-2t} + t + t^2 \end{pmatrix}.$$
The general solution is
$$\mathbf{X}(t) = \mathbf{\Omega}(t)\mathbf{C} + \mathbf{\Omega}(t)\mathbf{u}(t)$$
$$= e^{3t}\begin{pmatrix} 1+2t & 2t \\ -2t & 1-2t \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$
$$+e^{3t}\begin{pmatrix} 1+2t & 2t \\ -2t & 1-2t \end{pmatrix}\begin{pmatrix} -3te^{-2t} - t^2 \\ (3/2)(1+2t)e^{-2t} + t + t^2 \end{pmatrix}$$
$$= \begin{pmatrix} e^{3t}(c_1(1+2t) + 2c_2t) + t^2e^{3t} \\ e^{3t}(-2c_1t + c_2(1-2t)) + (t-t^2)e^{3t} + 3e^{t}/2 \end{pmatrix}.$$

2. $\mathbf{A}$ has repeated eigenvalue 0 with eigenvector
$$\begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$
A fundamental matrix is
$$\mathbf{\Omega}(t) = \begin{pmatrix} 2 & 1+2t \\ 1 & t \end{pmatrix}.$$
The general solution is
$$\mathbf{X}(t) = \begin{pmatrix} 2c_1 + c_2(1+2t) + t + t^2 - 2t^3 \\ c_1 + c_2t + 2t^2 - t^3 \end{pmatrix}.$$

3. $\mathbf{A}$ has repeated eigenvalue 6 and eigenvector

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

A fundamental matrix is

$$\mathbf{\Omega}(t) = e^{6t} \begin{pmatrix} 1 & 1+t \\ 1 & t \end{pmatrix}$$

and the general solution is

$$\mathbf{X}(t) = \begin{pmatrix} e^{6t}\left(c_1 + c_2(1+t) + 2t + t^2 - t^3\right) \\ e^{6t}\left(c_1 + c_2 t + 4t^2 - t^3\right) \end{pmatrix}.$$

4. $\mathbf{A}$ has eigenvalue 2 of multiplicity 3, and linearly independent eigenvectors

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$$

A fundamental matrix is

$$\mathbf{\Omega}(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -4t \\ 0 & 1 & 1-4t \end{pmatrix} e^{2t}.$$

A general solution is

$$\mathbf{X}(t) = \begin{pmatrix} c_1 e^{2t} \\ (c_2 - 4tc_3)e^{2t} + 1 \\ (c_2 + c_3(1-4t))e^{2t} + 1 \end{pmatrix}.$$

5. $\mathbf{A}$ has eigenvalue 1 with multiplicity 2 and single associated independent eigenvector

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix},$$

and eigenvalue 3 with multiplicity 2 and two associated linearly independent eigenvectors

$$\begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} \text{ and } \begin{pmatrix} 0 \\ -9 \\ 2 \\ 0 \end{pmatrix}.$$

A fundamental matrix is

$$\mathbf{\Omega}(t) = \begin{pmatrix} 0 & e^t & 0 & 0 \\ 0 & -2e^t & e^{3t} & -9e^{3t} \\ 0 & 0 & 0 & 2e^{3t} \\ e^t & -5te^t & e^{3t} & 0 \end{pmatrix}$$

The general solution is

$$\mathbf{X}(t) = \begin{pmatrix} c_2e^t \\ -2c_2e^t + (c_3 - 9c_4)e^{3t} + e^t \\ 2c_4e^{3t} \\ (c_1 - 5c_2t)e^t + c_3e^{3t} + (1 + 3t)e^t \end{pmatrix}.$$

In Problems 6 through 9, where initial values are given, the method is to find the general solution of the system and then solve for the constants to satisfy the initial values. For these four problems only the solution is given.

6.
$$\mathbf{X}(t) = \begin{pmatrix} -1 + e^{2t} \\ -5t + (3 + 5t)e^{2t} \end{pmatrix}$$

7.
$$\mathbf{X}(t) = \begin{pmatrix} (-1 - 14t)e^t \\ (3 - 14t)e^t \end{pmatrix}$$

8.
$$\mathbf{X}(t) = \begin{pmatrix} 13t - (8 + 12t + 3t^2)e^{2t} \\ 4e^t + (7 + 2t)e^{2t} \\ -e^t - e^{2t} \end{pmatrix}$$

9.
$$\mathbf{X}(t) = \begin{pmatrix} (6 + 12t + (1/2)t^2)e^{-2t} \\ (2 + 12t + (1/2)t^2)e^{-2t} \\ (3 + 38t + 66t^2 + (13/6)t^3)e^{-2t} \end{pmatrix}$$

For the remaining problems, the solution is expressed in the form $\mathbf{X}(t) = \mathbf{PZ}(t)$, where $\mathbf{P}$ is a matrix having eigenvectors of $\mathbf{A}$ as columns, $\mathbf{A}(t)$ is the solution of the uncoupled system $\mathbf{Z}' = \mathbf{DZ} + \mathbf{P}^{-1}\mathbf{G}$, and $\mathbf{D}$ is a diagonal matrix having the eigenvalues of $\mathbf{A}$ on its main diagonal.

10. The coefficient matrix
$$\mathbf{A} = \begin{pmatrix} -2 & 1 \\ -4 & 3 \end{pmatrix}$$
has eigenvalues $-1, 2$ and we form a matrix of eigenvectors,
$$\mathbf{P} = \begin{pmatrix} 1 & 1 \\ 1 & 4 \end{pmatrix}.$$
We find that
$$\mathbf{P}^{-1} = \frac{1}{3}\begin{pmatrix} 4 & -1 \\ -1 & 1 \end{pmatrix}.$$
Then
$$\mathbf{P}^{-1}\mathbf{AP} = \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}$$

and $\mathbf{Z}$ satisfies

$$\mathbf{Z}' = \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix} \mathbf{Z} + \frac{1}{3} \begin{pmatrix} 4 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 10\cos(t) \end{pmatrix}.$$

The system for $\mathbf{Z}$ is the uncoupled system

$$\mathbf{Z}(t) = \begin{pmatrix} z_1' \\ z_2' \end{pmatrix} = \begin{pmatrix} -z_1 \\ 2z_2 \end{pmatrix} + \frac{1}{3} \begin{pmatrix} -10\cos(t) \\ 10\cos(t) \end{pmatrix},$$

which we solve by solving each of the uncoupled differential equations separately as a first order linear equation. This yields

$$\mathbf{Z}(t) = \begin{pmatrix} c_1 e^{-t} - (5/3)\cos(t) - (5/3)\sin(t) \\ c_2 e^{2t} - (4/3)\cos(t) + (2/3)\sin(t) \end{pmatrix}.$$

Then

$$\mathbf{X}(t) = \mathbf{PZ} = \begin{pmatrix} c_1 e^{t} + c_2 e^{-2t} - 3\cos(t) - \sin(t) \\ c_1 e^{t} + 4c_2 e^{-2t} - 7\cos(t) + \sin(t) \end{pmatrix}.$$

11. The coefficient matrix

$$\mathbf{A} = \begin{pmatrix} 3 & 3 \\ 1 & 5 \end{pmatrix}$$

has eigenvalues 2 and 6. Form a matrix of eigenvectors

$$\mathbf{P} = \begin{pmatrix} 3 & 1 \\ -1 & 1 \end{pmatrix}.$$

Then

$$\mathbf{P}^{-1} = \frac{1}{4} \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}$$

and the uncoupled system for $\mathbf{Z}$ is

$$\mathbf{Z}' = \begin{pmatrix} 2 & 0 \\ 0 & 6 \end{pmatrix} \mathbf{Z} + \frac{1}{4} \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 8 \\ 4e^{3t} \end{pmatrix}.$$

Solve for $\mathbf{Z}$ to obtain

$$\mathbf{Z}(t) = \begin{pmatrix} c_1 e^{2t} - 1 - e^{3t} \\ c_2 e^{6t} - 1/3 - e^{3t} \end{pmatrix}.$$

Then

$$\mathbf{X}(t) = \mathbf{PZ}(t) = \begin{pmatrix} 3c_1 e^{2t} + c_2 e^{6t} - 4e^{3t} - 10/3 \\ -c_1 e^{2t} + c_2 e^{6t} + 2/3 \end{pmatrix}.$$

12. Here

$$\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},$$

with eigenvalues 0 and 2. Use corresponding eigenvectors to form the columns of a diagonalizing matrix

$$\mathbf{P} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$$

Then

$$\mathbf{P}^{-1} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

and the uncoupled system for $\mathbf{Z}$ is

$$\mathbf{Z}' = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}\mathbf{Z} + \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 6e^{3t} \\ 4 \end{pmatrix}.$$

Solving for $\mathbf{Z}$ gives us

$$\mathbf{Z}(t) = \begin{pmatrix} c_1 - 2t + e^{3t} \\ c_2e^{2t} - 1 + 3e^{3t} \end{pmatrix}.$$

Then

$$\mathbf{X}(t) = \mathbf{PZ}(t) = \begin{pmatrix} c_1 + 2c_2e^{2t} - 1 - 2t + 4e^{3t} \\ -c_1 + c_2e^{2t} - 1 + 2t + 2e^{3t} \end{pmatrix}.$$

13. The coefficient matrix is

$$\mathbf{A} = \begin{pmatrix} 6 & 5 \\ 1 & 2 \end{pmatrix}$$

with eigenvalues $1, 7$. Form $\mathbf{P}$ from corresponding eigenvectors:

$$\mathbf{P} = \begin{pmatrix} 1 & 5 \\ -1 & 1 \end{pmatrix}.$$

Then

$$\mathbf{P}^{-1} = \frac{1}{6}\begin{pmatrix} 1 & -5 \\ 1 & 1 \end{pmatrix}$$

and the system for $\mathbf{Z}$ is

$$\mathbf{Z}' = \begin{pmatrix} 1 & 0 \\ 0 & 7 \end{pmatrix}\mathbf{Z} + \frac{1}{6}\begin{pmatrix} 1 & -5 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} -4\cos(3t) \\ 8 \end{pmatrix}.$$

Solving for $\mathbf{Z}$, we obtain

$$\mathbf{Z}(t) = \begin{pmatrix} c_1e^t + (1/15)\cos(3t) - (3/15)\sin(3t) + (20/3 \\ c_2e^{7t} + (7/87)\cos(3t) - (2/58)\sin(3t) - 4/21 \end{pmatrix}.$$

Then

$$\mathbf{X}(t) = \mathbf{PZ}(t) =$$

$$\begin{pmatrix} c_1e^t + 5c_2e^{7t} + (68/145)\cos(3t) - (54/145)\sin(3t) + 40/7 \\ -c_1e^t + c_2e^{7t} + (2/145)\cos(3t) + (24/145)\sin(3t) - 48/7 \end{pmatrix}.$$

14. The coefficient matrix

$$\mathbf{A} = \begin{pmatrix} 3 & -2 \\ 9 & -3 \end{pmatrix}$$

with eigenvalues $3, -3$. From corresponding eigenvectors we form

$$\mathbf{P} = \begin{pmatrix} 1+i & 1-i \\ 3 & 3 \end{pmatrix}.$$

We find that

$$\mathbf{P}^{-1} = \frac{1}{6}\begin{pmatrix} -3i & 1+i \\ 3i & 1-i \end{pmatrix}.$$

$\mathbf{Z}$ satisfies

$$\mathbf{Z}' = \begin{pmatrix} 3i & 0 \\ 0 & -3i \end{pmatrix}\mathbf{Z} + \frac{1}{6}\begin{pmatrix} -3i & 1+i \\ 3i & 1-i \end{pmatrix}\begin{pmatrix} 3e^{2t} \\ e^{2t} \end{pmatrix}.$$

Solve for $\mathbf{Z}$ to obtain

$$\mathbf{Z}(t) = \begin{pmatrix} d_1 e^{3it} + ((2-i)/6)e^{2t} \\ d_2 e^{-3it} + ((2+i)/6)e^{2t} \end{pmatrix}.$$

Write

$$d_1 = \frac{1}{2}(c_1 + c_2 i) \text{ and } d_2 = \frac{1}{2}(c_1 - c_2 i)$$

with c_1 and c_2 real, to obtain

$$\mathbf{Z}(t) = \begin{pmatrix} (1/2)(c_1\cos(3t) - c_2\sin(3t)) + (1/2)(c_2\cos(3t) + c_1\sin(3t))i + ((2-i)/6)e^{2t} \\ (1/2)(c_1\cos(3t) - c_2\sin(t)) - (1/2)(c_2\cos(3t) + c_1\sin(3t))i + ((2+i)/6)e^{2t} \end{pmatrix}.$$

Then

$$\mathbf{X}(t) = \mathbf{PZ}(t) =$$

$$\begin{pmatrix} c_1(\cos(3t) - \sin(3t)) - c_2(\sin(3t) + \cos(3t)) + e^{2t} \\ 3c_1\cos(3t) - 3c_2\sin(3t) + 2e^{2t} \end{pmatrix}.$$

15. The coefficient matrix is

$$\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

with eigenvalues 0 and 2. Use independent corresponding eigenvectors to form

$$\mathbf{P} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$$

Then

$$\mathbf{P}^{-1} = \begin{pmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{pmatrix}.$$

The uncoupled system is

$$\mathbf{Z}' = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}\mathbf{Z} + \begin{pmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{pmatrix}\begin{pmatrix} 6e^{2t} \\ 2e^{2t} \end{pmatrix},$$

with initial condition

$$\mathbf{Z}(0) = \mathbf{P}^{-1}\mathbf{X}(0) = \begin{pmatrix} 3 \\ 3 \end{pmatrix}.$$

Solve for $\mathbf{Z}$ to obtain

$$\mathbf{Z}(t) = \begin{pmatrix} 2 + e^{2t} \\ 3e^{2t} + 4te^{2t} \end{pmatrix}.$$

Then

$$\mathbf{X}(t) = \mathbf{PZ}(t) = \begin{pmatrix} 2 + 4(1+t)e^{2t} \\ -2 + 2(1+2t)e^{2t} \end{pmatrix}.$$

16. With

$$\mathbf{A} = \begin{pmatrix} 1 & -2 \\ -1 & 2 \end{pmatrix}$$

we find the eigenvalues 0 and 3 and, from corresponding eigenvectors,

$$\mathbf{P} = \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}.$$

Then

$$\mathbf{P}^{-1} = \frac{1}{3}\begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}.$$

The uncoupled system is

$$\mathbf{Z}' = \begin{pmatrix} 0 & 0 \\ 0 & 3 \end{pmatrix}\mathbf{Z} + \begin{pmatrix} 1/3 & 1/3 \\ -1/3 & 2/3 \end{pmatrix}\begin{pmatrix} 2t \\ 5 \end{pmatrix},$$

with initial condition

$$\mathbf{Z}(0) = \mathbf{P}^{-1}\mathbf{X}(0) = \begin{pmatrix} 25/3 \\ 11/3 \end{pmatrix}.$$

The solution for $\mathbf{Z}$ is

$$\mathbf{z}(t) = \begin{pmatrix} (1/3)t^2 + (5/3)t + 25/3 \\ (127/27)e^{3t} + (2/9)t - 28/27 \end{pmatrix}.$$

The solution of the original problem is

$$\mathbf{X}(t) = \mathbf{PZ}(t) = \begin{pmatrix} -(127/27)e^{3t} + (2/3)t^2 + (28/9)t + 478/27 \\ (127/27)e^{3t} + (1/3)t^2 + (17/9)t + 197/27 \end{pmatrix}.$$

17. With coefficient matrix

$$\mathbf{A} = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix}$$

the eigenvalues are $\pm i$ and, from the eigenvectors, we let

$$\mathbf{P} = \begin{pmatrix} 5 & 5 \\ 2-i & 2+i \end{pmatrix}.$$

With $\mathbf{X} = \mathbf{PZ}$ the uncoupled system is

$$\mathbf{Z}' = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \mathbf{Z} + \begin{pmatrix} (1-2i)/10 & i/2 \\ (1+2i)/10 & -i/2 \end{pmatrix} \begin{pmatrix} 5\sin(t) \\ 0 \end{pmatrix},$$

with initial condition

$$\mathbf{Z}(0) = \mathbf{P}^{-1}\mathbf{X}(0) = \begin{pmatrix} 1+i/2 \\ 1-i/2 \end{pmatrix}.$$

Solve for z_1 and z_2 and then obtain

$$\mathbf{X} = \mathbf{PZ} = \begin{pmatrix} 10\cos(t) - (5/2)t\sin(t) - 5t\cos(t) \\ 5\cos(t) + (5/2)\sin(t) - (5/2)t\cos(t) \end{pmatrix}.$$

18. The eigenvalues of the coefficient matrix are $1, 1, -3$. Form $\mathbf{P}$ having independent eigenvectors as columns:

$$\mathbf{P} = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & 3 \\ 0 & 1 & 1 \end{pmatrix}.$$

We find that

$$\mathbf{P}^{-1} = \begin{pmatrix} 3 & -2 & 3 \\ 1 & -1 & 2 \\ -1 & 1 & -1 \end{pmatrix}.$$

With $\mathbf{X} = \mathbf{PZ}$ we obtain the uncoupled system

$$\mathbf{Z}' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -3 \end{pmatrix} \mathbf{Z} + \begin{pmatrix} 3 & -2 & 3 \\ 1 & -1 & 2 \\ -1 & 1 & -1 \end{pmatrix} \begin{pmatrix} -3e^{-3t} \\ t \\ 0 \end{pmatrix}.$$

The initial conditions are

$$\mathbf{Z}(0) = \mathbf{P}^{-1}\mathbf{X}(0) = \begin{pmatrix} 11 \\ 6 \\ -4 \end{pmatrix}.$$

Solve this uncoupled system and obtain

$$\mathbf{X} = \mathbf{PZ} = \begin{pmatrix} (5/2)e^t - (8/3)e^{-3t} + 3te^{-3t} + (8/9) + (4/3)t \\ (27/4)e^t - (113/12)e^{-3t} + 9te^{-3t} + (5/3) + 3t \\ (17/4)e^t - (113/36)e^{-3t} + 3te^{-3t} + (8/9) + (4/3)t \end{pmatrix}.$$

19. The coefficient matrix has eigenvalues 1, 2, 2. A diagonalizing matrix is

$$\mathbf{P} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

Obtain

$$\mathbf{P}^{-1} = \begin{pmatrix} -1 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix}.$$

With $\mathbf{X} = \mathbf{PZ}$, the uncoupled system is

$$\mathbf{Z}' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \mathbf{Z} + \begin{pmatrix} -1 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ t \\ 2e^t \end{pmatrix}.$$

The initial conditions are

$$\mathbf{Z}(0) = \mathbf{P}^{-1}\mathbf{x}(0) = \begin{pmatrix} -1 \\ 3 \\ -1 \end{pmatrix}.$$

Solve for $\mathbf{Z}$ to obtain

$$\mathbf{X} = \mathbf{PZ} = \begin{pmatrix} (-1/4)e^{2t} + (2+2t)e^t - (3/4) - (1/2)t \\ e^{2t} + (2+2t)e^t - 1 - t \\ -(5/4)e^{2t} + 2te^t - (3/4) - (1/2)t \end{pmatrix}.$$

10.4 Exponential Matrix Solutions

The first five problems were done using MAPLE.

1.
$$e^{\mathbf{A}t} = \begin{pmatrix} \cos(2t) - (1/2)\sin(2t) & (1/2)\sin(2t) \\ -(5/2)\sin(2t) & \cos(2t) + (1/2)\sin(2t) \end{pmatrix}$$

2.
$$e^{\mathbf{A}t} = \begin{pmatrix} (2/3)e^{-3t} + 1/3 & 1/3 - (1/3)e^{-3t} \\ (2/3) - (2/3)e^{-3t} & (1/3)e^{-3t} + 2/3 \end{pmatrix}$$

3. $e^{\mathbf{A}t}$ has elements $a_{ij}(t)$, where

$$a_{11}(t) = e^{13t/2}\left(\cos(\sqrt{23}t/2) - \frac{3}{\sqrt{23}}\sin(\sqrt{23}t/2)\right),$$
$$a_{12}(t) = -\frac{4}{\sqrt{23}}e^{13t/2}\sin(\sqrt{23}t/2),$$
$$a_{21}(t) = \frac{8}{\sqrt{23}}e^{13t/2}\sin(\sqrt{23}t/2),$$
$$a_{22}(t) = e^{\sqrt{23}t/2}\left(\cos(\sqrt{23}t/2) + (3/\sqrt{23})\sin(\sqrt{23}t/2)\right).$$

4. $e^{\mathbf{A}t}$ has elements $a_{ij}(t)$, where

$$\begin{aligned}
a_{11}(t) &= e^{(1+\sqrt{7})t}(1/2 + 3\sqrt{7}/14) + e^{(1-\sqrt{7})t}(1/2 - 3\sqrt{7}/14),\\
a_{12}(t) &= (\sqrt{7}/14)e^{(1-\sqrt{7})t} - (\sqrt{7}/14)e^{(1+\sqrt{7})},\\
a_{21}(t) &= -(1/\sqrt{7})e^{(1-\sqrt{7})t} + (1/\sqrt{7})e^{(1+\sqrt{7})t},\\
a_{22}(t) &= e^{(1+\sqrt{7})t}(1/2 - 3\sqrt{7}/14) + e^{(1-\sqrt{7})t}(3\sqrt{7}/14 + 1/2).
\end{aligned}$$

5. $e^{\mathbf{A}t}$ has elements $a_{ij}(t)$, where

$$\begin{aligned}
a_{11}(t) &= \frac{2}{5}e^{2t} + \frac{2}{5}\cos(t) - \frac{1}{5}\sin(t),\\
a_{12}(t) &= -\frac{1}{5}e^{2t} + \frac{2}{5}\sin(t) + \frac{1}{5}\cos(t),\\
a_{13}(t) &= \frac{1}{5}e^{2t} + \frac{3}{5}\sin(t) - \frac{1}{5}\cos(t),\\
a_{21}(t) &= -\frac{3}{5}e^{2t} + \frac{3}{5}\cos(t) - \frac{4}{5}\sin(t),\\
a_{22}(t) &= \frac{1}{5}e^{2t} + \frac{4}{5}\cos(t) + \frac{3}{5}\sin(t),\\
a_{23}(t) &= -\frac{1}{5}e^{2t} + \frac{7}{5}\sin(t) + \frac{1}{5}\cos(t),\\
a_{31}(t) &= \frac{3}{5}e^{2t} - \frac{3}{5}\cos(t) - \frac{1}{5}\sin(t),\\
a_{32}(t) &= -\frac{1}{5}e^{2t} + \frac{1}{5}\cos(t) - \frac{3}{5}\sin(t),\\
a_{33}(t) &= \frac{1}{5}e^{2t} + \frac{4}{5}\cos(t) - \frac{2}{5}\sin(t).
\end{aligned}$$

6. If $\mathbf{D}$ is a diagonal matrix with diagonal elements d_{jj}, then $\mathbf{D}^n$ is the diagonal matrix with diagonal elements d_{jj}^n. Then

$$e^{\mathbf{D}t} = \sum_{n=0}^{\infty} \frac{1}{n!} D^n t.$$

This is a diagonal matrix whose jth diagonal element is

$$\sum_{n=0}^{\infty} \frac{1}{n!}(d_{jj})^n t,$$

and this diagonal element is $e^{d_{jj}t}$.

7. Notice that

$$\begin{aligned}
\mathbf{B}^n &= (\mathbf{P}^{-1}\mathbf{A}\mathbf{P})^n\\
&= (\mathbf{P}^{-1}\mathbf{A}\mathbf{P})(\mathbf{P}^{-1}\mathbf{A}\mathbf{P})\cdots(\mathbf{P}^{-1}\mathbf{A}\mathbf{P})\\
&= \mathbf{P}^{-1}\mathbf{A}^n\mathbf{P}.
\end{aligned}$$

Then

$$
\begin{aligned}
e^{\mathbf{B}t} &= \sum_{n=0}^{\infty} \frac{1}{n!}(\mathbf{P}^{-1}\mathbf{A}\mathbf{P})^n t \\
&= \sum_{n=0}^{\infty} \frac{1}{n!}\mathbf{P}^{-1}\mathbf{A}^n\mathbf{P}t \\
&= \mathbf{P}^{-1}\left((\sum_{n=0}^{\infty} \frac{1}{n!}\mathbf{A}^n t \right)\mathbf{P} \\
&= \mathbf{P}^{-1}e^{\mathbf{A}t}\mathbf{P}.
\end{aligned}
$$

8. From the result of Problem 7,

$$e^{\mathbf{A}t} = \mathbf{P}e^{\mathbf{D}t}\mathbf{P}^{-1},$$

where $\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{D}$. In the case that $\mathbf{P}$ diagonalizes $\mathbf{A}$, $\mathbf{D}$ is the diagonal matrix having the eigenvalues $d_1, \cdots, d_n$ of $\mathbf{A}$ down its diagonal. From Problem 6, $e^{\mathbf{D}t}$ is the $n \times n$ diagonal matrix having diagonal elements $e^{d_j t}$.

9. First deal with the matrix $\mathbf{A}$ of Problem 1. The eigenvalues, with corresponding eigenvectors, are

$$2i, \begin{pmatrix} 1-2i \\ 5 \end{pmatrix}, -2i, \begin{pmatrix} 1+2i \\ 5 \end{pmatrix}.$$

The matrix

$$\mathbf{P} = \begin{pmatrix} 1-2i & 1+2i \\ 5 & 5 \end{pmatrix}$$

diagonalizes $\mathbf{A}$, so

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{D} = \begin{pmatrix} 2i & 0 \\ 0 & -2i \end{pmatrix}.$$

Now,

$$e^{\mathbf{D}t} = \begin{pmatrix} e^{2it} & 0 \\ 0 & e^{-2it} \end{pmatrix}.$$

Further, we find that

$$\mathbf{P}^{-1} = \begin{pmatrix} (1/4)i & 1/10 - i/20 \\ -(1/4)i & 1/10 + i/20 \end{pmatrix}.$$

Then

$$
\begin{aligned}
e^{\mathbf{A}t} &= \mathbf{P}e^{\mathbf{D}t}\mathbf{P}^{-1} \\
&= \begin{pmatrix} 1-2i & 1+2i \\ 5 & 5 \end{pmatrix}\begin{pmatrix} e^{2it} & 0 \\ 0 & e^{-2it} \end{pmatrix}\begin{pmatrix} (1/4)i & 1/10 - i/20 \\ -(1/4)i & 1/10 + i/20 \end{pmatrix}
\end{aligned}
$$

$$= \begin{pmatrix} (1/2+i/4)e^{2it} + (1/2-i/4)e^{-2it} & (i/4)(e^{-2it} - e^{2it}) \\ (5i/4)(e^{2it} - e^{-2it}) & (1/2-i/4)e^{2it} + (1/2+i/4)e^{-2it} \end{pmatrix}.$$

This appears to be different from the solution obtained using MAPLE. However, recall that

$$e^{2it} = \cos(2t) + i\sin(2t) \text{ and } \sin(2t) = \cos(2t) - i\sin(2t).$$

If these are substituted into the exponential matrix we have just found, we obtain the exponential matrix produced by MAPLE.

Now turn to the matrix of Problem 2. Eigenvalues and eigenvectors are

$$-3, \begin{pmatrix} -1 \\ 1 \end{pmatrix}, 0, \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

Let

$$\mathbf{P} = \begin{pmatrix} -1 & 1 \\ 1 & 2 \end{pmatrix}.$$

Then

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{D} = \begin{pmatrix} -3 & 0 \\ 0 & 0 \end{pmatrix}.$$

Now

$$e^{\mathbf{D}t} = \begin{pmatrix} e^{-3t} & 0 \\ 0 & 1 \end{pmatrix}.$$

Then

$$\mathbf{P}e^{\mathbf{D}t}\mathbf{P}^{-1} = \frac{1}{3}\begin{pmatrix} 1+2e^{-3t} & 1-e^{-3t} \\ 2-2e^{-3t} & 2+e^{-3t} \end{pmatrix}.$$

Because only real quantities were involved in the computation, we obtain the same result as that returned by MAPLE.

10.5 Applications and Illustrations of Techniques

1. The capacitor charge is maximum when the capacitor voltage is maximum. This voltage is

$$V_C = \frac{q_2 - q_3}{10^{-1}} = 10(q_2 - q_3) = 5i_3.$$

Therefore

$$V_C = 180(e^{-2t} - e^{-20t/9}).$$

Then

$$\frac{dV_C}{dt} = 10(i_2 - i_3) = 40(1 - e^{-20t/9} - 9e^{-2t}) = 0.$$

This occurs if

$$t = \frac{9}{2}\ln\left(\frac{10}{9}\right) \approx 0.474$$

seconds. The capacitor voltage at this time is

$$V_C\left(\frac{9}{2}\ln\left(\frac{10}{9}\right)\right) = 20\left(\frac{9}{10}\right)^{10} \approx 6.97$$

volts.

2. Denote the amount (in pounds) of salt in tank j at time t by $x_j(t)$. Then

$$\begin{aligned} x_1' &= -16\left(\frac{x_1}{200}\right) + 12\left(\frac{x_2}{300}\right) + \frac{1}{4}(4), \\ x_2' &= 12\left(\frac{x_1}{200}\right) - 18\left(\frac{x_2}{300}\right). \end{aligned}$$

Together with the given initial conditions, we now have the initial value problem

$$\begin{aligned} x_1' &= -\frac{4}{50}x_1 + \frac{2}{50}x_2 + 1, \\ x_2' &= \frac{3}{50}x_1 - \frac{3}{50}x_2, \\ x_1(0) &= 200, x_2(0) = 150. \end{aligned}$$

This system has the solution

$$\begin{aligned} x_1(t) &= 120e^{-t/50} + 55e^{-3t/25} + 25, \\ x_2(t) &= 180e^{-t/50} - 55e^{-3t/25} + 25. \end{aligned}$$

3. Let $x_j(t)$ be the number of pounds of salt in tank j at time t. Then

$$\begin{aligned} x_1' &= -\frac{4}{50}x_1 + \frac{1}{50}x_2 + 1, \\ x_2' &= \frac{1}{50}x_1 - \frac{4}{50}x_2 + 2, \\ x_1(0) &= 40, x_2(0) = 0. \end{aligned}$$

This initial value problem has the unique solution

$$\begin{aligned} x_1(t) &= 20 + 25e^{-t/10} - 5e^{-3t/50}, \\ x_2(t) &= 30 - 25e^{-t/10} - 5e^{-3t/50}. \end{aligned}$$

The brine in tank 1 has minimum concentration when $t = 25\ln(25/3)$ minutes. At this time there is $20 - 6\sqrt{3}/125$ pounds of salt in tank 1 (about 19.9 pounds). The initial amount of salt in the tank is 40 pounds and this quantity decreases to this value and then rises toward the terminal amount of 20 pounds of salt (the limit as $t \to \infty$ of $x_1(t)$).

4. Using Kirchhoff's laws we obtain the equations

$$\begin{aligned} 5i_1 + i_1' - i_2' &= 5, \\ i_1' - i_2' &= 1000q_2, \\ 5i_1 - 1000q_2 &= 5. \end{aligned}$$

From the first equation and the derivative of the third, we have

$$\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} i_1 \\ i_2 \end{pmatrix}' = \begin{pmatrix} -50 & 0 \\ 0 & -20 \end{pmatrix} \begin{pmatrix} i_1 \\ i_2 \end{pmatrix} + \begin{pmatrix} 5 \\ 0 \end{pmatrix},$$

with the initial conditions

$$i_1(0+) = i_2(0+) = \frac{1}{10}.$$

This system has the solution

$$\begin{aligned} i_1(t) &= \frac{1}{10} - \frac{1}{15}e^{-10t}\sin(30t), \\ i_2(t) &= \frac{1}{30}e^{-2t}(3\cos(30t) - \sin(30t)). \end{aligned}$$

5. Designate down as positive, $y_1(t)$ the position of the upper weight relative to the equilibrium position of this weight, and $y_2(t)$ the position of the lower weight relative to the equilibrium position of the lower weight. Then

$$\begin{aligned} y_1'' &= -22y_1 + 6y_2, \\ y_2'' &= 6y_1 - 6y_2, \end{aligned}$$

with initial conditions

$$y_1(0) = y_2(0) = 1, y_1'(0) = y_2'(0) = 0.$$

Let $x_1 = y_1$, $x_2 = y_2$, $x_3 = y_1'$, and $x_4 = y_2'$. This converts the system of two second-order differential equations to a system of four first-order equations:

$$\begin{aligned} x_1' &= x_3, \\ x_2' &= x_4, \\ x_3' &= -22x_1 + 6x_2, \\ x_4' &= 6x_1 - 6x_2, \end{aligned}$$

with initial conditions

$$x_1(0) = x_2(0) = 1, x_3(0) = x_4(0) = 0.$$

The matrix of this system is

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -22 & 6 & 0 & 0 \\ 6 & -6 & 0 & 0 \end{pmatrix}$$

with eigenvalues $\pm 2i$ and $\pm 2\sqrt{6}i$. One eigenvector associated with $2i$ is

$$\begin{pmatrix} 1 \\ 3 \\ 2i \\ 6i \end{pmatrix}$$

and an eigenvector associated with $2\sqrt{6}i$ is

$$\begin{pmatrix} 3 \\ -1 \\ 6\sqrt{6}i \\ -2\sqrt{6}i \end{pmatrix}.$$

Use these to write the general solution of the system of differential equations in terms of real functions:

$$\mathbf{X}(t) = \begin{pmatrix} c_1\cos(2t) + c_2\sin(2t) + 3c_3\cos(2\sqrt{6}t) + 3c_4\sin(2\sqrt{6}t) \\ 3c_1\cos(2t) + 3c_2\sin(2t) - c_3\cos(2\sqrt{6}t) - c_4\sin(2\sqrt{6}t) \\ 2c_2\cos(2t) - 2c_1\sin(2t) + 6\sqrt{6}c_4\cos(2\sqrt{6}t) - 6\sqrt{6}c_3\sin(2\sqrt{6}t) \\ 6c_2\cos(2t) - 6c_1\sin(2t) - 2\sqrt{6}c_4\cos(2\sqrt{6}t) + 2\sqrt{6}c_3\sin(2\sqrt{6}t) \end{pmatrix}.$$

Substitute the initial conditions and recall that $y_1 = x_1$ and $y_2 = x_2$ to obtain

$$y_1(t) = \frac{2}{5}\cos(2t) + \frac{3}{5}\cos(2\sqrt{6}t),$$
$$y_2(t) = \frac{6}{5}\cos(2t) - \frac{1}{5}\cos(2\sqrt{6}t).$$

6. Using the same assignment of variables as in the solution to Problem 5, we have

$$y_1'' = -22y_1 + 6y_2,$$
$$y_2'' = 6y_1 - 6y_2 + 4\sin(3t),$$
$$y_1(0) = y_2(0) = y_1'(0) = y_2'(0) = 0.$$

Proceeding as in the solution to Problem 5, we obtain

$$y_1(t) = \frac{9}{25}\sin(2t) + \frac{3\sqrt{6}}{150}\sin(2\sqrt{6}t) - \frac{8}{25}\sin(3t),$$
$$y_2(t) = \frac{27}{25}\sin(2t) - \frac{\sqrt{6}}{150}\sin(2\sqrt{6}t) - \frac{52}{75}\sin(3t).$$

7. Consider the direction to the right as positive and let y_1 be the displacement of the left weight from the equilibrium position and y_2 the displacement of the right weight from its equilibrium position. The spring/mass system is modeled by the initial value problem

$$\begin{aligned} 2y_1'' &= -8y_1 + 5(y_2 - y_1), \\ 2y_2'' &= -5(y_2 - y_1) - 8y_2, \\ y_1(0) &= 1, y_2(0) = -1, y_1'(0) = y_2'(0) = 0. \end{aligned}$$

As we have done before, let

$$x_1 = y_1, x_2 = y_2, x_3 = y_1', x_4 = y_2'.$$

This gives us the first-order system

$$\begin{aligned} x_1' &= x_3, \\ x_2' &= x_4, \\ x_3' &= -\frac{13}{2}x_1 + \frac{5}{2}x_2, \\ x_4' &= \frac{5}{2}x_1 - \frac{13}{2}x_2. \\ x_1(0) &= 1, x_2(0) = -1, x_3(0) = x_4(0) = 0. \end{aligned}$$

The matrix of this system has eigenvalues $\pm 2i$ and $\pm 3i$. Eigenvectors corresponding to $2i$ and one for $3i$ are, respectively,

$$\begin{pmatrix} 1 \\ 1 \\ 2i \\ 2i \end{pmatrix} \text{ and } \begin{pmatrix} 1 \\ -1 \\ 3i \\ -3i \end{pmatrix}.$$

Using these, write the general solution of the system:

$$\mathbf{X(t)} = \begin{pmatrix} c_1\cos(2t) + c_2\sin(2t) + c_3\cos(3t) + c_4\sin(4t) \\ c_1\cos(2t) + c_2\sin(2t) - c_3\cos(3t) - c_4\sin(3t) \\ 2c_2\cos(2t) - 2c_1\sin(2t) + 3c_4\cos(3t) - 3c_3\sin(3t) \\ 6c_2\cos(2t) - 6c_1\sin(2t) - 3c_4\cos(3t) + 3c_3\sin(3t) \end{pmatrix}.$$

Upon using the initial conditions and setting $y_1 = x_1$ and $y_2 = x_2$ we obtain the solution for the displacement functions:

$$\begin{aligned} y_1(t) &= \cos(3t), \\ y_2(t) &= -\cos(3t). \end{aligned}$$

8. Using Kirchhoff's laws, we have

$$\begin{aligned} 40i_1 + 1000(q_1 - q_2) &= 5, \\ 1000(q_1 - q_2) &= 10i_2', \\ 40i_1 + 10i_2' &= 5, \\ i_1(0+) &= \frac{1}{8}, i_2(0+) = 0. \end{aligned}$$

Use the derivative of the first equation together with the third equation to obtain the system

$$\begin{pmatrix}1 & 0\\ 0 & 2\end{pmatrix}\begin{pmatrix}i_1\\ i_2\end{pmatrix}' = \begin{pmatrix}-25 & -25\\ -8 & 0\end{pmatrix}\begin{pmatrix}i_1\\ i_2\end{pmatrix} + \begin{pmatrix}0\\ 1\end{pmatrix},$$

with the initial conditions given previously. Upon multiplying this system on the left by

$$\begin{pmatrix}1 & 0\\ 0 & 2\end{pmatrix}^{-1},$$

which is the matrix

$$\begin{pmatrix}1 & 0\\ 0 & 1/2\end{pmatrix},$$

we obtain the system

$$\begin{pmatrix}i_1\\ i_2\end{pmatrix}' = \begin{pmatrix}-25 & 25\\ -4 & 0\end{pmatrix}\begin{pmatrix}i_1\\ i_2\end{pmatrix} + \begin{pmatrix}0\\ 1/2\end{pmatrix},$$

with the given initial conditions. This system has the solution

$$\begin{aligned} i_1(t) &= \frac{5}{24}e^{-20t} - \frac{5}{24}e^{-5t} + \frac{1}{8},\\ i_2(t) &= \frac{1}{24}e^{-20t} - \frac{1}{6}e^{-5t} + \frac{1}{8}. \end{aligned}$$

9. From Kirchhoff's laws,

$$\begin{aligned} 50i_1 + 100(i_1' - i_2') &= 5,\\ 50i_1 + 1000q_2 &= 5,\\ 10(i_1' - i_2') &= 1000q_2,\\ i_1(0+) = i_2(0+) &= \frac{1}{10}. \end{aligned}$$

Using the first equation and the derivative of the second, we have the system

$$\begin{pmatrix}2 & -2\\ 1 & 0\end{pmatrix}\begin{pmatrix}i_1\\ i_2\end{pmatrix}' = \begin{pmatrix}-10 & 0\\ 0 & -20\end{pmatrix}\begin{pmatrix}i_1\\ i_2\end{pmatrix} + \begin{pmatrix}1\\ 0\end{pmatrix},$$

with

$$i_1(0+) = i_2(0+) = \frac{1}{10}.$$

Multiply the system on the left by

$$\begin{pmatrix}2 & -2\\ 1 & 0\end{pmatrix}^{-1}$$

which is the matrix

$$\begin{pmatrix}0 & 1\\ -1/2 & 1\end{pmatrix}.$$

We obtain

$$\begin{pmatrix} i_1 \\ i_2 \end{pmatrix}' = \begin{pmatrix} 0 & -20 \\ 5 & -20 \end{pmatrix} - \begin{pmatrix} 0 \\ 1/2 \end{pmatrix},$$

with the given initial conditions. The coefficient matrix of this system has repeated eigenvalue -10, and only one independent eigenvector

$$\begin{pmatrix} 2 \\ 1 \end{pmatrix},$$

or any nonzero constant multiple of this eigenvector. One solution of the associated homogeneous system is

$$\begin{pmatrix} i_1 \\ i_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{-10t}.$$

To find a second, linearly independent solution, we can apply the method of Section 10.2, obtaining

$$\begin{pmatrix} 1+10t \\ 5t \end{pmatrix} e^{-10t}.$$

A fundamental matrix for the homogeneous system is

$$\mathbf{\Omega}(t) = \begin{pmatrix} 2e^{-10t} & (1+10t)e^{-10t} \\ e^{-10t} & 5te^{-10t} \end{pmatrix}.$$

In order to use variation of parameters, we need

$$\mathbf{\Omega}^{-1}(t) = \begin{pmatrix} -5te^{10t} & (1+10t)e^{10t} \\ e^{10t} & -2e^{10t} \end{pmatrix}.$$

We need to integrate

$$\mathbf{\Omega}^{-1}\mathbf{G}(t) = \mathbf{\Omega}^{-1}\begin{pmatrix} 0 \\ -1/2 \end{pmatrix} = \begin{pmatrix} -(1/2)(1+10t)e^{10t} \\ e^{10t} \end{pmatrix}.$$

This integration gives us

$$\mathbf{u}(t) = \int \mathbf{\Omega}^{-1}\mathbf{G}(t)\,dt = \begin{pmatrix} -(1/2)te^{10t} \\ (1/10)e^{10t} \end{pmatrix}.$$

Then

$$\mathbf{\Omega}(t)\mathbf{u}(t) = \begin{pmatrix} 1/10 \\ 0 \end{pmatrix}$$

is a particular solution. The general solution of the nonhomogeneous system is

$$i_1(t) = 2c_1e^{-10t} + c_2(1+10t)e^{-10t} + \frac{1}{10},$$
$$i_2(t) = c_1e^{-10t} + c_2te^{-10t}.$$

Upon inserting the initial conditions, we obtain the solution for the current functions:

$$
\begin{aligned}
i_1(t) &= \frac{1}{10} - 2te^{-10t}, \\
i_2(t) &= \left(\frac{1}{10} - t\right) e^{-10t}
\end{aligned}
$$

amperes.

10. The circuit can be modeled using any three of the following six equations:

$$
\begin{aligned}
20i_1 + 50(q_1 - q_2) &= 45, \\
20i_1 + 25i_2 + 10(i_2' - i_3') &= 45, \\
20i_1 + 25i_2 + 25i_3 &= 45, \\
50(q_1 - q_2) &= 25i_2 + 10(i_1' - i_2'), \\
50(q_1 - q_2) &= 25i_2 + 25i_3, \\
10(i_2' - i_3') &= 25i_3.
\end{aligned}
$$

Using the derivative of the first equation, the second equation, and the derivative of the third equation, we obtain the system

$$
\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & -2 \\ 4 & 5 & 5 \end{pmatrix} \begin{pmatrix} i_1 \\ i_2 \\ i_3 \end{pmatrix}' = \begin{pmatrix} -5 & 5 & 0 \\ -4 & -5 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} i_1 \\ i_2 \\ i_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 9 \\ 0 \end{pmatrix},
$$

with initial conditions

$$
i_1(0+) = \frac{9}{4}, i_2(0+) = i_3(0+) = 0.
$$

This is equivalent to the system

$$
\begin{pmatrix} i_1 \\ i_2 \\ i_3 \end{pmatrix}' = \begin{pmatrix} -5/2 & 5/2 & 0 \\ 0 & -9/4 & 0 \\ 2 & 1/4 & 0 \end{pmatrix} \begin{pmatrix} i_1 \\ i_2 \\ i_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 9/4 \\ -9/4 \end{pmatrix},
$$

with the above initial conditions. The coefficient matrix of this system has eigenvalues 0, $-5/2$, $-9/4$, with corresponding eigenvectors

$$
\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 5 \\ 0 \\ -4 \end{pmatrix}, \begin{pmatrix} 10 \\ 1 \\ -9 \end{pmatrix}.
$$

We can use these as columns of a matrix $\mathbf{P}$ that diagonalizes the system. A particular solution of the nonhomogeneous system is

$$
\begin{pmatrix} 1 \\ 1 \\ -9/5 \end{pmatrix}.
$$

We obtain the general solution of the nonhomogeneous system as

$$\begin{aligned} i_1(t) &= 5c_2e^{-5t/2} + 10c_3e^{-9t/4} + 1, \\ i_2(t) &= c_3e^{-9t/4} + 1, \\ i_3(t) &= c_1 - 4c_2e^{-5t/2} - 9c_3e^{-9t/4} - \frac{9}{5}. \end{aligned}$$

Upon applying the initial conditions to find the constants, we obtain the solution

$$\begin{aligned} i_1(t) &= \frac{45}{4}e^{-5t/4} - 10e^{-9t/4} + 1, \\ i_2(t) &= -e^{-9t/4} + 1, \\ i_3(t) &= -9e^{-5t/4} + 9e^{-9t/4}. \end{aligned}$$

The output voltage is just $25i_3(t)$, and this is maximum when $i_3(t)$ is a maximum. This occurs when $t = 4\ln(10/9)$ seconds. The output voltage at this time is

$$\mathrm{E_{out}} = \frac{45}{2}\left(\frac{9}{10}\right)^9,$$

which is approximately 8.7 volts.

11. The spring/mass system is modeled by the initial value problem

$$\begin{aligned} 5y_1'' &= -(65-\alpha)y_1 + \alpha(y_2 - y_1) - 30y_1', \\ 13y_2'' &= -\alpha(y_2 - y_1) - (65-\alpha)y_2 + 39\sin(t), \\ y_1(0) &= y_2(0) = y_1'(0) = y_2'(0) = 0. \end{aligned}$$

Let

$$x_1 = y_1, x_2 = y_2, x_3 = y_1', x_4 = y_2'.$$

This produces the first order system

$$\begin{aligned} x_1' &= x_3, \\ x_2' &= x_4, \\ x_3' &= -13x_1 + 2\sqrt{26}x_2 - 6x_3, \\ x_4' &= \frac{10\sqrt{26}}{13}x_1 - 5x_2 + 3\sin(t), \\ x_1(0) &= x_2(0) = x_3(0) = x_4(0) = 0. \end{aligned}$$

The coefficient matrix of this system has characteristic polynomial

$$p_{\mathbf{A}}(\lambda) = \lambda^4 + 6\lambda^3 + 18\lambda^2 + 30\lambda + 25$$

with roots (eigenvalues of $\mathbf{A}$) $-1\pm 2i$ and $-2\pm i$. An eigenvector associated with $-1+2i$ is

$$\begin{pmatrix} \sqrt{26} - 2\sqrt{26}i \\ 10 \\ 3\sqrt{26} + 4\sqrt{26}i \\ -10 + 20i \end{pmatrix}$$

and an eigenvector associated with $-2+i$ is

$$\begin{pmatrix} 2\sqrt{26}-\sqrt{26}i \\ 5 \\ -3\sqrt{26}+4\sqrt{26} \\ -10+5i \end{pmatrix}.$$

Use these to write the general solution in terms of real-valued functions of the associated homogeneous system:

$$\begin{aligned}
&x_1(t) = \sqrt{26}\left[e^{-t}(c_1-2c_2)\cos(2t)+(2c_1+c_2)\sin(2t)\right] \\
&+e^{-2t}\left[(2c_3-c_4)\cos(t)+(c_3+2c_4)\sin(t)\right], \\
&x_2(t) = 5e^{-t}\left[2c_1\cos(2t)+2c_2\sin(2t)\right] \\
&+e^{-2t}\left[c_3\cos(t)+c_4\sin(t)\right], \\
&x_3(t) = \sqrt{26}e^{-t}\left[(3c_1+4c_2)\cos(2t)\right]+(4c_1+3c_2)\sin(2t), \\
&+e^{-2t}\left[(-3c_3+4c_4)\cos(t)+(-4c_3-3c_4)\sin(t)\right], \\
&x_4(t) = 5e^{-t}\left[(-2c_1+4c_2)\cos(2t)+(-4c_1-2c_2)\sin(2t)\right] \\
&+e^{-2t}\left[(-2c_3+c_4)\cos(t)+(-c_3-2c_4)\sin(t)\right].
\end{aligned}$$

We also find the following solution of the nonhomogeneous system:

$$\begin{aligned}
x_1(t) &= -\frac{9\sqrt{26}}{40}\cos(t)+\frac{3\sqrt{26}}{40}\sin(t), \\
x_2(t) &= -\frac{9}{8}\cos(t)+\frac{9}{8}\sin(t), \\
x_3(t) &= \frac{3\sqrt{26}}{40}\cos(t)+\frac{9\sqrt{26}}{40}\sin(t), \\
x_4(t) &= \frac{9}{8}\cos(t)+\frac{9}{8}\sin(t).
\end{aligned}$$

Add this particular solution to the general solution of the associated homogeneous equation and then insert the initial conditions to solve for the constants, obtaining

$$c_1 = \frac{6}{100}, c_2 = \frac{3}{100}, c_3 = \frac{21}{200}, c_4 = -\frac{3}{200}.$$

Upon inserting these constants and putting $y_1 = x_1$ and $y_2 = x_2$ we obtain the displacement functions for the weights:

$$\begin{aligned}
y_1(t) = {} & \frac{3\sqrt{26}}{40}\left[2e^{-t}\sin(2t)\right. \\
& \left.+e^{-2t}(3\cos(t)+\sin(t))-3\cos(t)+\sin(t)\right] \\
y_2(t) = {} & \frac{3}{40}\left[e^{-t}(8\cos(2t)+4\sin(2t))\right. \\
& \left.+e^{-2t}(7\cos(t)-\sin(t))-15\cos(t)+15\sin(t)\right].
\end{aligned}$$

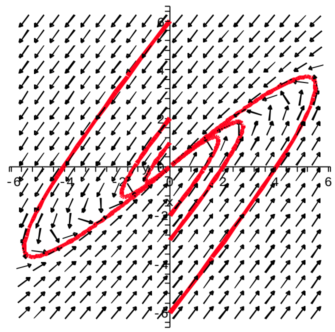

Figure 10.1: Phase portrait for Problem 1, Section 10.6.

10.6 Phase Portraits

1. Eigenvalues of $\mathbf{A}$ are $-2, -2$ and the origin is an improper node. The general solution is

$$\mathbf{X}(t) = \begin{pmatrix} c_1 e^{-2t} + 5(c_1 - c_2)te^{-2t} \\ c_2 e^{-2t} + 5(c_1 - c_2)te^{-2t} \end{pmatrix}$$

A phase portrait is given in Figure 10.1.

2. The eigenvalues of $\mathbf{A}$ are $-3, 4$ and the origin is a saddle point. The general solution is

$$\mathbf{X}(t) = \begin{pmatrix} -c_1 e^{-3t} + (4/3)c_2 e^{4t} \\ c_1 e^{-3t} + c_2 e^{4t} \end{pmatrix}.$$

Figure 10.2 shows a phase portrait.

3. Eigenvalues are $\pm 2i$; the origin is a center. The general solution is

$$\mathbf{X} = \begin{pmatrix} (c_1 - 2c_2)\sin(2t) + (2c_1 + c_2)\cos(2t) \\ c_1 \sin(2t) + c_2 \cos(2t) \end{pmatrix}$$

Figure 10.3 is a phase portrait.

4. The eigenvalues are $3, 2$ and the origin is a nodal source. The general solution is

$$\mathbf{X}(t) = \begin{pmatrix} 7c_1 e^{3t} + c_2 e^{2t} \\ 6c_1 e^{3t} + c_2 e^{2t} \end{pmatrix}.$$

A phase portrait consists of straight lines emanating from the origin.

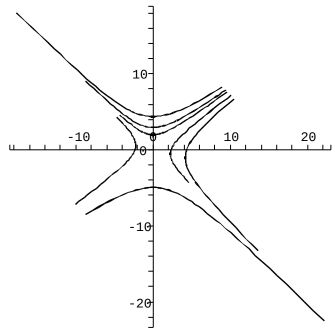

Figure 10.2: Phase portrait for Problem 2, Section 10.6.

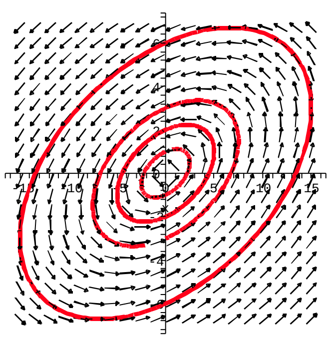

Figure 10.3: Phase portrait for Problem 3, Section 10.6.

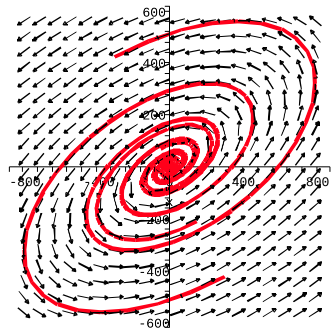

Figure 10.4: Phase portrait for Problem 5, Section 10.6.

5. The eigenvalues are $4 \pm 5i$ and the origin is a spiral point. The general solution is

$$\mathbf{X} = \begin{pmatrix} (3c_1 - 5c_2)e^{4t}\sin(5t) + (5c_1 + 3c_2)e^{4t}\cos(5t) \\ 2c_1 e^{4t}\sin(5t) + 2c_2 e^{4t}\cos(5t) \end{pmatrix}$$

Figure 10.4 is a phase portrait.

6. The eigenvalues are $-3, -5$ and the origin is a nodal sink. A phase portrait consists of straight lines moving into the origin. The general solution is

$$\mathbf{X}(t) = \begin{pmatrix} 7c_1 e^{-3t} + c_2 e^{-5t} \\ 5c_1 e^{-3t} + c_2 e^{-5t} \end{pmatrix}.$$

7. The eigenvalues are $3, 3$ and the origin is an improper node. The general solution is

$$\mathbf{X} = \begin{pmatrix} c_1 e^{3t} + c_2 t e^{3t} \\ (c_1 + c_2)e^{3t} + c_2 t e^{3t} \end{pmatrix}$$

8. The eigenvalues of the coefficient matrix are $\pm\sqrt{31}i$, so the origin is a center, with periodic closed orbits enclosing $(0, 0)$. A phase portrait for this system is shown in Figure 10.5.

   The general solution is

$$\mathbf{X}(t) = \begin{pmatrix} (3c_1 - \sqrt{31})\sin(\sqrt{31}t) + (\sqrt{31}c_1 + 3c_2)\cos(\sqrt{31}t) \\ 8c_1 \sin(\sqrt{31}t) + 8c_2 \cos(\sqrt{31}t) \end{pmatrix}$$

9. The eigenvalues are $-2 \pm \sqrt{3}i$, so the origin is a spiral point. The general solution is

$$\mathbf{X} = \begin{pmatrix} c_1 e^{-2t}\cos(\sqrt{3}t) - c_2 e^{-2t}\sin(\sqrt{3}t) \\ c_1 e^{-2t}\sin(\sqrt{3}t) + 3c_2 e^{-2t}\cos(\sqrt{3}t) \end{pmatrix}$$

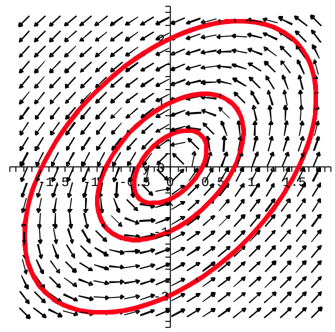

Figure 10.5: Phase portrait for Problem 8, Section 10.6.

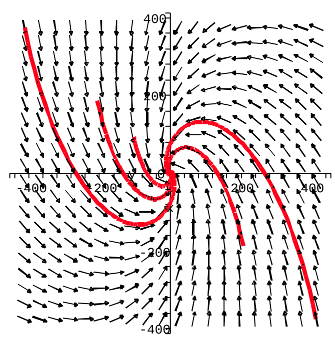

Figure 10.6: Phase portrait for Problem 9, Section 10.6.

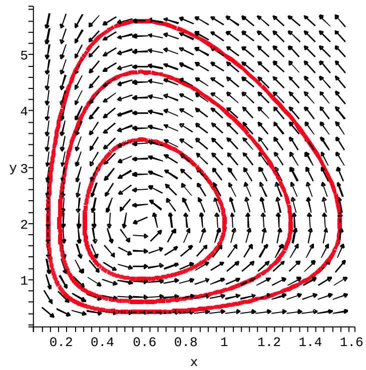

Figure 10.7: Phase portrait for Problem 12(a), Section 10.6.

Figure 10.6 is a phase portrait for this system.

10. The eigenvalues are $-13, -13$ and the origin is an improper node. The general solution is

$$\mathbf{X}(t) = \begin{pmatrix} (c_1 + c_2 t)e^{-13t} \\ (c_1 + c_2 t - (1/7)c_2)e^{-13t} \end{pmatrix}.$$

11. Let H be the constant of proportionality for the outside agent that at any time removes members of both species at a rate proportional to their population at that time. Coupling this term with a predator/prey model, we have the system.

$$x_1' = ax_1 - bx_1x_2 - Hx_1,$$
$$x_2' = -kx_2 + cx_1x_2 - Hx_2.$$

12. (a) Figure 10.7.
    (b) Figure 10.8.
    (c) Figure 10.9.
    (d) Figure 10.10.

13. (a) Figure 10.11.
    (b) Figure 10.12.
    (c) Figure 10.13.
    (d) Figure 10.14.

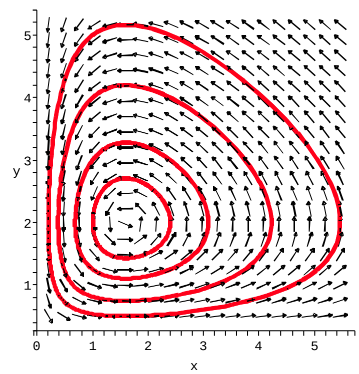

Figure 10.8: Phase portrait for Problem 12(b), Section 10.6.

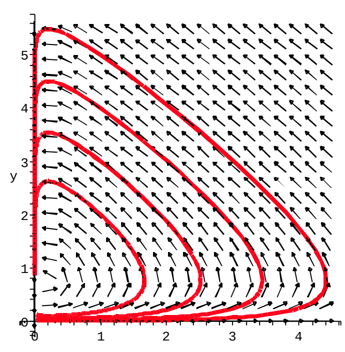

Figure 10.9: Phase portrait for Problem 12(c), Section 10.6.

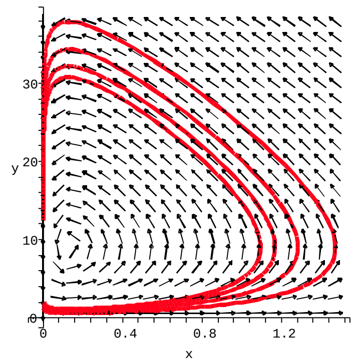

Figure 10.10: Phase portrait for Problem 12(d), Section 10.6.

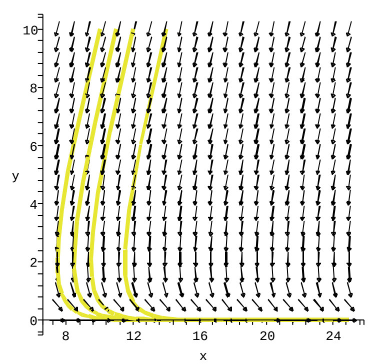

Figure 10.11: Phase portrait for Problem 13(a), Section 10.6.

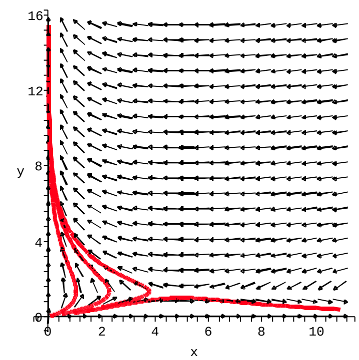

Figure 10.12: Phase portrait for Problem 13(b), Section 10.6.

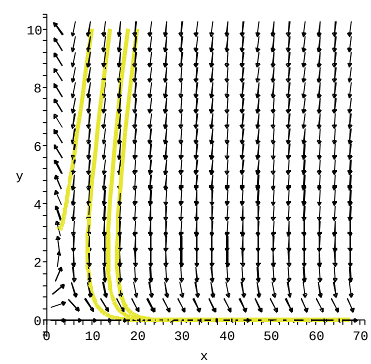

Figure 10.13: Phase portrait for Problem 13(c), Section 10.6.

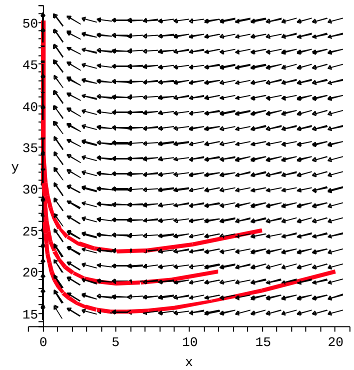

Figure 10.14: Phase portrait for Problem 13(d), Section 10.6.

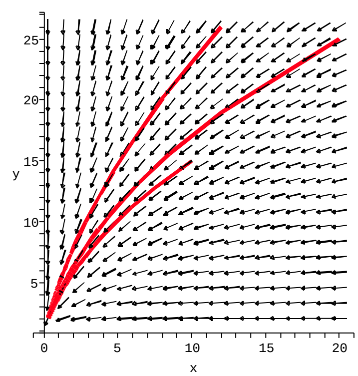

Figure 10.15: Phase portrait for Problem 14(a), Section 10.6.

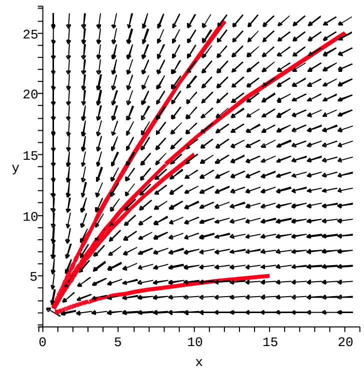

Figure 10.16: Phase portrait for Problem 14(b), Section 10.6.

14. (a) Figure 10.15.
    (b) Figure 10.16.
    (c) Figure 10.17.
    (d) Figure 10.18.

In generating phase portraits, it is sometimes necessary to experiment with various parameters, initial values, and values of the variable. Different choices can cause the program to terminate. For example, if we specify a value of t for which values of $x(t)$ and/or $y(t)$ are undefined, then no phase portrait will be generated. In such a case, try a different range of values for t. It may also be that the initial values do not correspond to points at which trajectories are interesting or informative. In such a case experiment with different initial values.

For some types of problems, phase portraits are quantitatively similar even for different choices of various constants occurring in the differential equations. We can see this with predator/prey models, whose phase portraits have certain similarities (closed, periodic orbits in the first quadrant). However, differences caused by different choices of coefficients become apparent if the scales on the axes are noted. Often, if phase portraits appearing to be similar for two systems were drawn on the same set of axes, we would see differences in scale in the orbits.

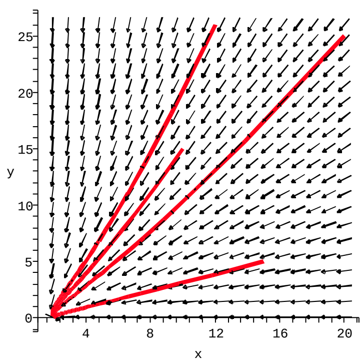

Figure 10.17: Phase portrait for Problem 14(c), Section 10.6.

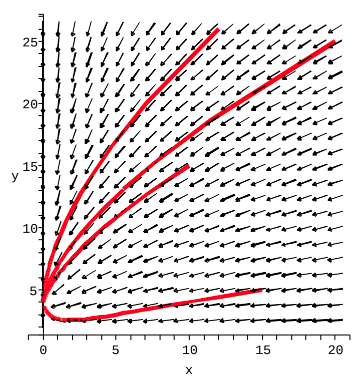

Figure 10.18: Phase portrait for Problem 14(d), Section 10.6.

Chapter 11

Vector Differential Calculus

11.1 Vector Functions of One Variable

For Problems 1, 2 and 3 we provide the details of the differentiation carried out both ways. For Problems 4 - 8 just the derivative is given.

1. First, applying the "product rule" for a scalar function times a vector function,

$$\begin{aligned}(f(t)\mathbf{F}'(t)) &= f'(t)\mathbf{F}(t) + f(t)\mathbf{F}'(t)\\ &= (-12\sin(3t))\mathbf{F}(t) + 4\cos(3t)[6t\mathbf{j} + 2\mathbf{k}]\\ &= -12\sin(3t)\mathbf{i} + [24t\cos(3t) - 36t^2\sin(3t)]\mathbf{j} + [8\cos(3t) - 24t\sin(3t)]\mathbf{k}.\end{aligned}$$

Now first carry out the product

$$f(t)\mathbf{F}(t) = 4\cos(3t)\mathbf{i} + 12t^2\cos(3t)\mathbf{j} + 8t\cos(3t)\mathbf{k},$$

so

$$\begin{aligned}(f(t)\mathbf{F}'(t)) = -12\sin(3t)\mathbf{i} &+ (24t\cos(3t) - 36t^2\sin(3t))\mathbf{j}\\ &+ (8\cos(3t) - 24t\sin(3t))\mathbf{k}.\end{aligned}$$

2. First,

$$\begin{aligned}(\mathbf{F}(t)\cdot\mathbf{G}(t))' &= \mathbf{F}'(t)\cdot\mathbf{G}(t) + \mathbf{F}(t)\cdot\mathbf{G}'(t)\\ &= (\mathbf{i} - 6t\mathbf{k})\cdot(\mathbf{i} + \cos(t)\mathbf{k})\\ &+ (t\mathbf{i} - 3t^2\mathbf{k})\cdot(-\sin(t)\mathbf{k})\\ &= 1 - 6t\cos(t) + 3t^2\sin(t).\end{aligned}$$

If we first take the dot product, then

$$\mathbf{F}(t)\cdot\mathbf{G}(t) = t - 3t^2\cos(t)$$

so

$$(\mathbf{F}(t) \cdot \mathbf{G}(t))' = 1 - 6t\cos(t) + 3t^2\sin(t).$$

3. Applying the product rule for cross products, we have

$$\begin{aligned}
(\mathbf{F}(t) \times \mathbf{G}(t))' &= \mathbf{F}'(t) \times \mathbf{G}(t) + \mathbf{F}(t) \times \mathbf{G}'(t) \\
&= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ 1 & -\cos(t) & t \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ t & 1 & 4 \\ 0 & \sin(t) & 1 \end{vmatrix} \\
&= -t\mathbf{j} - \cos(t)\mathbf{k} + ((1 - 4\sin(t))\mathbf{i} - t\mathbf{j} + t\sin(t)\mathbf{k}) \\
&= (1 - 4\sin(t))\mathbf{i} - 2t\mathbf{j} - (\cos(t) - t\sin(t))\mathbf{k}.
\end{aligned}$$

To carry out the cross product and then differentiate, first compute

$$\begin{aligned}
\mathbf{F}(t) \times \mathbf{G}(t) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ t & 1 & 4 \\ 1 & -\cos(t) & t \end{vmatrix} \\
&= (t + 4\cos(t))\mathbf{i} + (4 - t^2)\mathbf{j} - (t\cos(t) + 1)\mathbf{k}.
\end{aligned}$$

Then

$$\begin{aligned}
(\mathbf{F}(t) \times \mathbf{G}(t))' &= (1 - 4\sin(t))\mathbf{i} \\
&\quad - 2t\mathbf{j} - (\cos(t) - t\sin(t))\mathbf{k}.
\end{aligned}$$

4.

$$\begin{aligned}
(\mathbf{F}(t) \times \mathbf{G}(t))' = (3t^2 - 2t\sinh(t) - t^2\cosh(t))\mathbf{i} - 2t\mathbf{j} \\
- (\sinh(t) + t\cosh(t))\mathbf{k}
\end{aligned}$$

5.

$$\begin{aligned}
&(f(t)\mathbf{F}(t))' = (1 - 8t^3)\mathbf{i} + (6t^2\cosh(t) - (1 - 2t^3)\sinh(t))\mathbf{j} \\
&+ (e^t - 6t^2e^t - 2t^3e^t)\mathbf{k}
\end{aligned}$$

6.

$$(\mathbf{F}(t) \cdot \mathbf{G}(t))' = \sin(t) + t\cos(t) + 4 + 5t^4$$

7.

$$(\mathbf{F}(t) \times \mathbf{G}(t))' = te^t(2 + t)(\mathbf{j} - \mathbf{k})$$

8.

$$(\mathbf{F}(t) \cdot \mathbf{G}(t))' = -16\cos^2(t) + 16\sin^2(t)$$

9.
$$\mathbf{F}(t) = \sin(t)\mathbf{i} + \cos(t)\mathbf{j} + 45t\mathbf{k}, 0 \le t \le 2\pi$$

is a position vector for the curve C. Then

$$\mathbf{F}'(t) = \cos(t)\mathbf{i} - \sin(t)\mathbf{j} + 45\mathbf{k}$$

is a tangent vector. The distance function along the curve is

$$s(t) = \int_0^t \| \mathbf{F}'(\tau) \| \, d\tau = \int_0^t \sqrt{2026}\, d\tau = \sqrt{2026}t.$$

Then $t = s/\sqrt{2026}$, so in terms of the distance function, a position vector is

$$\mathbf{G}(s) = \mathbf{F}(t(s)) = \sin\left(\frac{s}{\sqrt{2026}}\right)\mathbf{i} + \cos\left(\frac{s}{\sqrt{2026}}\right)\mathbf{j} + \frac{45s}{\sqrt{2026}}\mathbf{k}.$$

This gives the tangent vector

$$\mathbf{G}'(s) = \frac{1}{\sqrt{2026}}\left[\cos\left(\frac{s}{\sqrt{2026}}\right)\mathbf{i} - \sin\left(\frac{s}{\sqrt{2026}}\right)\mathbf{j} + 45\mathbf{k}\right].$$

This is a unit tangent vector, since $\| \mathbf{G}'(s) \| = 1$.

10.
$$\mathbf{F}(t) = t^3(\mathbf{i} + \mathbf{j} + \mathbf{k})$$

for $-1 \le t \le 1$. A tangent vector is

$$\mathbf{F}'(t) = 3t^2(\mathbf{i} + \mathbf{j} + \mathbf{k}).$$

A distance function along the trajectory is given by

$$\begin{aligned} s(t) &= \int_{-1}^t \| \mathbf{F}'(\xi) \| \, d\xi \\ &= \int_{-1}^t \sqrt{3}3\xi^2 \, d\xi \\ &= \sqrt{3}\xi^3\Big]_{-1}^t = \sqrt{3}(t^3 + 1). \end{aligned}$$

Then
$$t^3 = \frac{s}{\sqrt{3}} - 1$$

so
$$t = \left(\frac{s}{\sqrt{3}} - 1\right)^{1/3}.$$

Set
$$\mathbf{G}(s) = \mathbf{f}(T(s)) = \left(\frac{s}{\sqrt{3}} - 1\right)^{1/3}(\mathbf{i} + \mathbf{j} + \mathbf{k}).$$

Then

$$\mathbf{G}'(s) = \frac{1}{\sqrt{3}}(\mathbf{i} + \mathbf{j} + \mathbf{k})$$

and this is a unit tangent vector.

11 .

$$\mathbf{F}(t) = t^2(2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k})$$

is a position vector, for $1 \le t \le 3$, and

$$\mathbf{F}'(t) = 2t(2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k})$$

is a tangent vector. A distance function along the curve is given by

$$s(t) = \int_1^t \parallel \mathbf{F}'(\tau) \parallel \, d\tau = 2\sqrt{29}\int_1^t \tau \, d\tau = \sqrt{29}(t^2 - 1).$$

Then $t = t(s) = \sqrt{1 + s/\sqrt{29}}$ for $0 \le s \le 8\sqrt{29}$. Let

$$\mathbf{F}(s) = \mathbf{G}(s) = \left(\frac{s}{\sqrt{29}} + 1\right)(2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}).$$

Then

$$\mathbf{G}'(s) = \frac{1}{\sqrt{29}}(2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k})$$

and this is a unit tangent vector because $\parallel \mathbf{G}'(s) \parallel = 1$.

12. Suppose $\mathbf{F}(t) \times \mathbf{F}'(t) = \mathbf{O}$ for all t. Then either $\mathbf{F}(t) = \mathbf{O}$, or $\mathbf{F}'(t) = \mathbf{O}$, or both vectors are nonzero and parallel, for all t. In the first case the particle sits at the origin for all time. In the second case there is no motion and the particle is at rest. In the last case, if the position and tangent vectors are always parallel, then the velocity vector is always directed along the path of motion and the motion is in a straight line.

We could also argue as follows in the last case. If $\mathbf{F}$ and $\mathbf{F}'$ are parallel for all t, then for some number c,

$$x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k} = c(x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k})$$

for all t. Then

$$x'(t) = cx(t), y'(t) = cy(t), z'(t) = cz(t).$$

This forces

$$x(t) = x_0e^{ct}, y(t) = y_0e^{ct}, z(t) = z_0e^{ct},$$

where $\mathbf{F}(0) = x_0\mathbf{i} + y_0\mathbf{j} + z_0\mathbf{k}$. These are parametric equations of a straight line.

11.2 Velocity and Curvature

In Problems 1 - 10, we can compute

$$\mathbf{v}(t) = \mathbf{F}'(t), \mathbf{a}(t) = \mathbf{F}''(t), v(t) = \parallel \mathbf{v}(t) \parallel$$

by straightforward differentiations and computation of a magnitude. In terms of t, we can compute the unit tangent vector as

$$\mathbf{T}(t) = \frac{1}{v(t)}\mathbf{v}(t) = \frac{1}{\parallel \mathbf{F}'(t) \parallel}\mathbf{F}'(t).$$

The tangential and normal components of the acceleration can be obtained as

$$a_T = \frac{dv}{dt} \text{ and } a_N = \sqrt{\parallel \mathbf{a} \parallel^2 - a_T^2}.$$

The unit normal is then

$$\mathbf{N}(t) = \frac{1}{a_N}(\mathbf{a}(t) - a_T\mathbf{T}(t)).$$

In this way we do not have to compute s and attempt to write vectors in terms of s, which is often quite awkward or even impossible in terms of elementary functions. We could also compute

$$\mathbf{N}(t) = \frac{d\mathbf{T}/dt}{\parallel d\mathbf{T}/dt \parallel},$$

which is a fairly straightforward calculation for finding a unit normal vector.

Finally, the curvature is conveniently computed in terms of t as

$$\kappa = \frac{a_N}{v^2}.$$

We can also compute curvature by

$$\kappa(t) = \frac{\parallel \mathbf{T}'(t) \parallel}{\parallel \mathbf{F}'(t) \parallel}$$

or, from the formula requested in Problem 13,

$$\kappa(t) = \frac{\parallel \mathbf{F}'(t) \times \mathbf{F}''(t) \parallel}{\parallel \mathbf{F}'(t) \parallel^3}.$$

In Problems 1 - 10, we will provide full details just for the first problem. The methodology is the same for the other problems.

1. The velocity is

$$\mathbf{v}(t) = \mathbf{F}'(t) = 3\mathbf{i} + 2t\mathbf{k},$$

the speed is $\| \mathbf{v}(t) \| = \sqrt{9+4t^2}$, acceleration is

$$\mathbf{a}(t) = \mathbf{F}''(t) = 2\mathbf{k},$$

and a unit tangent is

$$\mathbf{T}(t) = \frac{1}{\| \mathbf{F}'(t) \|}\mathbf{F}'(t) = \frac{1}{\sqrt{9+4t^2}}(3\mathbf{i} + 2t\mathbf{k}).$$

The curvature is

$$\kappa(t) = \frac{\| \mathbf{T}'(t) \|}{\| \mathbf{F}'(t) \|} = \frac{6}{(9+4t^2)^{3/2}},$$

in which we have omitted routine differentiations. The normal and tangential components of the acceleration are given by

$$a_T = \frac{dv}{dt} = \frac{4t}{\sqrt{9+4t^2}}$$

and

$$a_N = \sqrt{\| \mathbf{a} \|^2 - a_T^2} = \frac{6}{\sqrt{9+4t^2}}.$$

2.

$$\mathbf{v}(t) = (\sin(t) + t\cos(t))\mathbf{i} + (\cos(t) - t\sin(t))\mathbf{j},$$
$$\mathbf{a}(t) = (2\cos(t) - t\sin(t))\mathbf{i} - (2\sin(t) + t\cos(t))\mathbf{j},$$
$$\mathbf{T}(t) = \frac{1}{\sqrt{1+t^2}}\mathbf{v},$$
$$v(t) = \sqrt{1+t^2},$$
$$a_T = \frac{t}{\sqrt{1+t^2}}, a_N = \frac{2+t^2}{1+t^2}$$
$$\kappa = \frac{2+t^2}{(1+t^2)^{3/2}}$$

3.

$$\mathbf{v}(t) = 2\mathbf{i} - 2\mathbf{j} + \mathbf{k}, v = 3,$$
$$\mathbf{T} = \frac{1}{3}(2\mathbf{i} - 2\mathbf{j} + \mathbf{k})$$
$$a_T = a_N = \kappa = 0$$

4.

$$\mathbf{v}(t) = e^t(\sin(t) + \cos(t))\mathbf{i} + e^t(\cos(t) - \sin(t))\mathbf{k}, v = \sqrt{2}e^t,$$
$$\mathbf{a}(t) = 2e^t(\cos(t)\mathbf{i} - \sin(t)\mathbf{k}),$$
$$\mathbf{T}(t) = \frac{1}{\sqrt{2}}((\sin(t) + \cos(t))\mathbf{i} + (\cos(t) - \sin(t))\mathbf{k}),$$
$$a_T = \sqrt{2}e^t = a_N$$
$$\kappa = \frac{1}{\sqrt{2}}e^{-t}$$

5.

$$\mathbf{v}(t) = -3e^{-t}(\mathbf{i}+\mathbf{j}-2\mathbf{k}), \mathbf{a}(t) = 3e^{-t}(\mathbf{i}+\mathbf{j}-2\mathbf{k}),$$
$$v(t) = 3\sqrt{6}e^{-t}, \mathbf{T}(t) = \frac{1}{\sqrt{6}}(-\mathbf{i}-\mathbf{j}+2\mathbf{k}),$$
$$a_T = -3\sqrt{6}e^{-t}, a_N = 0, \kappa = 0$$

6.

$$\mathbf{v}(t) = -\alpha\sin(t)\mathbf{i} + \beta\mathbf{j} + \alpha\cos(t)\mathbf{k}, v(t) = \sqrt{\alpha^2+\beta^2},$$
$$\mathbf{a}(t) = -\alpha\cos(t)\mathbf{i} - \alpha\sin(t)\mathbf{k},$$
$$\mathbf{T}(t) = \frac{1}{\sqrt{\alpha^2+\beta^2}}(-\alpha\sin(t) + \beta\mathbf{j} + \alpha\cos(t)\mathbf{k}),$$
$$a_T = 0, a_N = \alpha, \kappa = \frac{\alpha}{\alpha^2+\beta^2}$$

7.

$$\mathbf{v}(t) = 2\cosh(t)\mathbf{j} - 2\sinh(t)\mathbf{k}, v(t) = 2\sqrt{\cosh(2t)},$$
$$\mathbf{a}(t) = 2\sinh(t)\mathbf{j} - 2\cosh(t)\mathbf{k},$$
$$\mathbf{T}(t) = \frac{1}{\sqrt{\cosh(t)}}(\cosh(t)\mathbf{j} - \sinh(t)\mathbf{k})$$
$$a_T = \frac{2\sinh(2t)}{\cosh(2t)}, a_N = \frac{2}{\sqrt{\cosh(2t)}}$$
$$\kappa = \frac{1}{2(\cosh(2t))^{3/2}}$$

Here we have used the hyperbolic identity

$$\cosh(2t) = \cosh^2(t) + \sinh^2(t).$$

8.

$$\mathbf{v}(t) = \frac{1}{t}(\mathbf{i}-\mathbf{j}+2\mathbf{k}), \mathbf{a}(t) = -\frac{1}{t^2}(\mathbf{i}-\mathbf{j}+2\mathbf{k})$$
$$v = \frac{\sqrt{6}}{t}, \mathbf{T}(t) = \frac{1}{\sqrt{6}}(\mathbf{i}-\mathbf{j}+2\mathbf{k}),$$
$$a_T = -\frac{\sqrt{6}}{t^2}, a_N = 0, \kappa = 0$$

9.

$$\mathbf{v}(t) = 2t(\alpha\mathbf{i} + \beta\mathbf{j} + \gamma\mathbf{k}),$$
$$\mathbf{a}(t) = 2(\alpha\mathbf{i} + \beta\mathbf{j} + \gamma\mathbf{k}),$$
$$v(t) = 2|t|\sqrt{\alpha^2+\beta^2+\gamma^2},$$

$$\mathbf{T}(t) = \frac{1}{\sqrt{\alpha^2+\beta^2+\gamma^2}}(\alpha\mathbf{i}+\beta\mathbf{j}+\gamma\mathbf{k}),$$

$$a_N = 0, \kappa = 0,$$

and

$$a_T = 2(\mathrm{sgn}(t))\sqrt{\alpha^2+\beta^2+\gamma^2},$$

where

$$\mathrm{sgn}(t) = \begin{cases} 1 & \text{if } t > 0, \\ -1 & \text{if } t < 0. \end{cases}$$

10.

$$\mathbf{v}(t) = (3\cos(t) - 3t\sin(t))\mathbf{j} - (3\sin(t) + 3t\cos(t))\mathbf{k},$$

$$v(t) = 3\sqrt{1+t^2},$$

$$\mathbf{a}(t) = (-6\sin(t) - 3t\cos(t))\mathbf{j} - (6\cos(t) - 3t\sin(t))\mathbf{k},$$

$$\mathbf{T}(t) = \frac{1}{\sqrt{1+t^2}}((\cos(t) - t\sin(t))\mathbf{j} - (\sin(t) + t\cos(t))\mathbf{k})$$

$$a_T = \frac{3t}{\sqrt{1+t^2}}, a_N = \frac{(3t^2+6)^2}{\sqrt{1+t^2}},$$

$$\kappa = \frac{(3t^2+6)^2}{9(1+t^2)^{3/2}}$$

11. A position vector for a straight line has the form

$$\mathbf{F}(t) = (a+bt)\mathbf{i} + (c+dt)\mathbf{j} + (p+ht)\mathbf{k}.$$

The tangent vector $\mathbf{F}'(t)$ is the constant vector $b\mathbf{i} + d\mathbf{j} + h\mathbf{k}$, so $\mathbf{T}(t)$ is a constant vector. Then $\mathbf{T}'(t) = \mathbf{O}$, so $\kappa = 0$.

Conversely, suppose a smooth curve has curvature zero. Then

$$\kappa = \| \mathbf{T}'(s) \| = \| \mathbf{F}''(s) \| = 0.$$

If

$$\mathbf{F}(s) = f(s)\mathbf{i} + g(s)\mathbf{j} + h(s)\mathbf{k},$$

then

$$f''(s) = g''(s) = h''(s) = 0$$

which means that $f(s) = a + bs$, $g(s) = c + ds$ and $h(s) = p + qs$ for some constants a, b, c, d, p, h. Then $\mathbf{F}$ is the position vector for a straight line.

12. As a convenience, let C be a circle of radius r about the origin in the $x, y-$ plane. Any circle in $3-$ space can be translated and rotated to such a position, and this will not change the curvature. A position vector for C is $\mathbf{F}(t) = r\cos(t)\mathbf{i} + r\sin(t)\mathbf{j}$. A tangent vector is $\mathbf{F}'(t) = -r\sin(t)\mathbf{i} + r\cos(t)\mathbf{j}$. This has length r, so a unit tangent is

$$\mathbf{T}(t) = -\sin(t)\mathbf{i} + \cos(t)\mathbf{j}.$$

Then

$$\mathbf{T}'(t) = -\cos(t)\mathbf{i} - \sin(t)\mathbf{j},$$

a unit vector. Finally,

$$\kappa = \frac{\| \mathbf{T}'(t) \|}{\| \mathbf{F}'(t) \|} = \frac{1}{r}.$$

13. First write

$$\mathbf{T}(t) = \frac{1}{\| \mathbf{F}'(t) \|}\mathbf{F}'(t) = \frac{1}{v(t)}\mathbf{F}'(t).$$

Thus

$$v\mathbf{T} = \mathbf{F}'.$$

Now $\mathbf{F}''(t)$ is the acceleration $\mathbf{a}(t)$, and $\mathbf{T} \times \mathbf{T} = \mathbf{O}$, so

$$\begin{aligned} v\mathbf{T} \times \mathbf{F}'' &= v\mathbf{T}(a_T\mathbf{T} + a_N\mathbf{N}) \\ &= va_T\mathbf{T} \times \mathbf{T} + va_N\mathbf{T} \times \mathbf{N} \\ &= va_N\mathbf{T} \times \mathbf{N} = v(v^2\kappa)\mathbf{T} \times \mathbf{N}. \end{aligned}$$

Now $\mathbf{T}$ and $\mathbf{N}$ are orthogonal unit vectors, so $\| \mathbf{T} \times \mathbf{N} \| = 1$ and we have

$$\| \mathbf{F}' \times \mathbf{F}'' \| = v^3\kappa.$$

Finally, $v = \| \mathbf{F}' \|$, so

$$\kappa = \frac{\| \mathbf{F}(\mathbf{t})' \times \mathbf{F}(\mathbf{t})'' \|}{\| \mathbf{F}(\mathbf{t})' \|^3}.$$

11.3 Vector Fields and Streamlines

1. The streamlines satisfy

$$dx = -\frac{dy}{y^2} = \frac{dz}{z}.$$

Integrate $dx = -(1/y^2)\,dy$ to obtain $x = 1/y + c_1$. Next integrate $dx = (1/z)\,dz$ to obtain $x = \ln|z| + c_2$. In terms of x, we can write parametric equations of the streamlines:

$$x = x, y = \frac{1}{x - c_1}, z = e^{x - c_2}.$$

For the streamline through $(2,1,1)$, we need $x=2$ and

$$1=\frac{1}{2-c_1}, 1=e^{2-c_2}.$$

Solve these to obtain $c_1=1$ and $c_2=2$. The streamline through $(2,1,1)$ has parametric equations

$$x=x, y=\frac{1}{x-1}, z=e^{x-2}.$$

2. Streamlines satisfy $dx=(-1/2)\,dy=dz$. Integrations yield

$$y=-2x+c_1, z=x+c_2.$$

For the streamline passing through $(0,1,1)$, we must have $c_1=c_2=1$.

3. Streamlines satisfy

$$x\,dx=\frac{dy}{e^x}=\frac{dz}{-1}.$$

Integration $xe^x\,dx=dy$ to obtain $y=xe^x-e^x+c_1$. Integrate $x\,dx=-dz$ to obtain $x^2=-2z+c_2$. Using x as parameter, streamlines are given by

$$y=xe^x-e^x+c_1, z=\frac{1}{2}(c_2-x^2).$$

For the streamline passing through $(2,0,4)$, we need

$$e^2+c_1=0 \text{ and } 4=\frac{1}{2}(c_2-4).$$

Then $c_1=-e^2$ and $c_2=12$. This yields the streamline

$$x=x, y=xe^x-e^x-e^2, z=\frac{1}{2}(12-x^2).$$

4. Streamlines satisfy

$$\frac{dx}{\cos(y)}=\frac{dy}{\sin(x)}, dz=0.$$

Integrate $\sin(x)\,dx=\cos(y)\,dy$ and $dz=0$ to obtain

$$-\cos(x)+c_1=\sin(y) \text{ and } z=c_2.$$

To pass through $(\pi/2,0,-4)$, we need $c_1=0$ and $c_2=-4$. The streamline through the given point is

$$x=x, y=\arcsin(-\cos(x)), z=-4.$$

5. Streamlines satisfy $dx = 0$ and

$$\frac{dy}{2e^z} = -\frac{dz}{\cos(y)}.$$

Integration of the separable differential equation

$$\cos(y)\,dy = -2e^z\,dz$$

gives $x = c_1$ and $\sin(y) = c_2 - 2e^z$. To pass through $(3, \pi/4, 0)$, we need $c_1 = 3$ and $c_2 = 2+\sqrt{2}/2$. With y as parameter, this curve has parametric equations

$$x = 3, y = y, z = \ln\left[\frac{\sqrt{2}}{4} + 1 - \frac{1}{2}\sin(y)\right].$$

6. Streamlines satisfy

$$\frac{dx}{3x^2} = -\frac{dy}{y} = \frac{dz}{z^3}.$$

Integrate the equations

$$\frac{1}{x^2}\,dx = -\frac{3}{y}\,dy \text{ and } \frac{1}{y}\,dy = -\frac{1}{z^3}\,dz$$

to obtain

$$\frac{1}{x} = -3\ln|y| + c_1 \text{ and } 2\ln|y| + c_2 = z^{-2}.$$

For the streamline passing through $(2, 1, 6)$, we need $c_1 = 1/2$ and $c_2 = 1/36$. Using y as the parameter, this streamline can be written

$$x = \frac{2}{1 + 6\ln(y)}, y = y, z = \frac{6}{\sqrt{1 + 72\ln(y)}}.$$

7. Circular streamlines about the origin in the $x, y-$ plane can be written as $x^2 + y^2 = r^2$, so $x\,dx + y\,dy = 0$, or

$$\frac{dx}{y} = -\frac{dy}{x}, dz = 0.$$

A vector field having these streamlines is

$$\mathbf{F(x, y)} = \frac{1}{x}\mathbf{i} - \frac{1}{y}\mathbf{j}.$$

11.4 The Gradient Field

1.

$$\begin{aligned}\nabla\varphi(x, y, z) &= \frac{\partial}{\partial x}(xyz)\mathbf{i} + \frac{\partial}{\partial y}(xyz)\mathbf{j} + \frac{\partial}{\partial z}(xyz)\mathbf{k} \\ &= yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}\end{aligned}$$

and

$$\nabla\varphi(1,1,1) = \mathbf{i} + \mathbf{j} + \mathbf{k}.$$

The maximum value of $D_{\mathbf{u}}\varphi(1,1,1)$ is $\parallel \nabla\varphi(1,1,1) \parallel = \sqrt{3}$. The minimum value is $-\sqrt{3}$.

2.

$$\nabla\varphi(x,y,z) = (2x - z\cos(zx))\mathbf{i} + x^2\mathbf{j} - x\cos(xz)\mathbf{k},$$
$$\nabla\varphi(1,-1,\pi/4) = \left(2 - \frac{\sqrt{2}\pi}{8}\right)\mathbf{i} + \mathbf{j} - \frac{\sqrt{2}}{2}\mathbf{k}$$

The maximum value of $D_{\mathbf{u}}\varphi(1,-1,\pi/4)$ is

$$\parallel \nabla\varphi(1,-1,\pi/4) \parallel = (176 + \pi - 16\sqrt{2}\pi)/32.$$

The minimum value is the negative of this maximum value.

3.

$$\nabla\varphi(x,y,z) = (2y + e^z)\mathbf{i} + 2x\mathbf{j} + xe^z\mathbf{k}$$
$$\nabla\varphi(-2,1,6) = (2 + e^6)\mathbf{i} - 4\mathbf{j} - 2e^6\mathbf{k}$$

The maximum value of $D_{\mathbf{u}}\varphi(-2,1,6)$ is

$$\parallel \nabla\varphi(-2,1,6) \parallel = \sqrt{20 + 4e^6 + 5e^{12}},$$

and the minimum value is the negative of this maximum value.

4.

$$\nabla\varphi(x,y,z) = -yz\sin(xyz)\mathbf{i} - xz\sin(xyz)\mathbf{j} - xy\sin(xyz)\mathbf{k},$$
$$\nabla\varphi(-1,1,\pi/2) = \frac{\pi}{2}\mathbf{i} - \frac{\pi}{2}\mathbf{j} - \mathbf{k}$$

The maximum value of $D_{\mathbf{u}}(-1,1,\pi/2)$ is

$$\parallel \nabla\varphi(-1,1,\pi/2) \parallel = \sqrt{1 + \frac{\pi^2}{2}}.$$

The minimum value is the negative of this maximum value.

5.

$$\nabla\varphi(x,y,z) = 2y\sinh(2xy)\mathbf{i} + 2x\sinh(2xy)\mathbf{j} - \cosh(z)\mathbf{k},$$
$$\nabla\varphi(0,1,1) = -\cosh(1)\mathbf{k},$$

$$D_{\mathbf{u}}(1,1,1)_{\text{max}} = \parallel \nabla\varphi(0,1,1) \parallel = \cosh(1),$$
$$D_{\mathbf{u}}(1,1,1)_{\text{min}} = - \parallel \nabla\varphi(0,1,1) \parallel = -\cosh(1)$$

6.

$$\nabla\varphi(x,y,z) = \frac{1}{\sqrt{x^2+y^2+z^2}}\left[x\mathbf{i}+y\mathbf{j}+z\mathbf{k}\right],$$

$$\nabla\varphi(2,2,2) = \frac{1}{\sqrt{3}}(\mathbf{i}+\mathbf{j}+\mathbf{k}),$$

$$\max D_{\mathbf{u}} = \parallel \nabla\varphi(2,2,2) \parallel = 1,$$
$$\min D_{\mathbf{u}} = - \parallel \nabla\varphi(2,2,2) \parallel = -1$$

7.

$$\begin{aligned} D_{\mathbf{u}}\varphi(x,y,z) &= \nabla\varphi(x,y,z)\cdot\mathbf{u} \\ &= ((8y^2-z)\mathbf{i}+16xy\mathbf{j}-x\mathbf{k})\cdot\frac{1}{\sqrt{3}}(\mathbf{i}+\mathbf{j}+\mathbf{k}) \\ &= \frac{1}{\sqrt{3}}(8y^2-z+16xy-x) \end{aligned}$$

8 .

$$\begin{aligned} D_{\mathbf{u}}\varphi(x,y,z) &= \nabla\varphi(x,y,z)\cdot\mathbf{u} \\ &= (-\sin(x-y)\mathbf{i}+\sin(x-y)\mathbf{j}+e^z\mathbf{k})\cdot\frac{1}{\sqrt{6}}(\mathbf{i}-\mathbf{j}+2\mathbf{k}) \\ &= \frac{1}{\sqrt{6}}(-2\sin(x-y)+2e^z) \end{aligned}$$

9.

$$\begin{aligned} D_{\mathbf{u}}\varphi(x,y,z) &= \nabla\varphi(x,y,z)\cdot\mathbf{u} \\ &= (2xyz^3\mathbf{i}+x^2y^3\mathbf{j}+3x^2yz^2\mathbf{k})\cdot\frac{1}{\sqrt{5}}(2\mathbf{j}+\mathbf{k}) \\ &= \frac{1}{\sqrt{5}}(2x^2z^3+3x^2yz^2) \end{aligned}$$

10.

$$\begin{aligned} D_{\mathbf{u}}\varphi(x,y,z) &= \nabla\varphi(x,y,z)\cdot\mathbf{u} \\ &= ((z+y)\mathbf{i}+(z+x)\mathbf{j}+(y+x)\mathbf{k})\cdot\frac{1}{\sqrt{17}}(\mathbf{i}-4\mathbf{k}) \\ &= \frac{1}{\sqrt{17}}(z-3y-4x) \end{aligned}$$

11. Let $\varphi(x,y,z) = x^2+y^2+z^2$, so the level surface is the locus of points satisfying $\varphi(x,y,z) = 4$. A normal vector at $(1,1,\sqrt{2})$ is

$$\mathbf{N} = \nabla\varphi(1,1,\sqrt{2}) = 2\mathbf{i}+2\mathbf{j}+2\sqrt{2}\mathbf{k}.$$

The tangent plane to the surface at $(1, 1, \sqrt{2})$ has equation

$$2(x-1) + 2(y-1) + 2\sqrt{2}(z - \sqrt{2}) = 0,$$

or

$$x + y + \sqrt{2}z = 4.$$

The normal line to the surface at this point has parametric equations

$$x = y = 1 + 2t, z = \sqrt{2}(1 + 2t) \text{ for } -\infty < t < \infty.$$

12. With the surface written as $x^2 + y - z = 0$, we have the level surface $\varphi(x, y, z) = 0$ with $\varphi(x, y, z) = x^2 + y - z$. A normal vector at $(-1, 1, 2)$is

$$\mathbf{N} = \nabla(x^2 + y - z)|_{(-1,1,2)} = -2\mathbf{i} + \mathbf{j} - \mathbf{k}.$$

The tangent plane at the given point has the equation

$$-2x + y - z = 1$$

and the normal line has parametric equations

$$x = -1 - 2t, y = 1 + t, z = 2 - t \text{ for } -\infty < t < \infty.$$

13. The normal vector is

$$\mathbf{N} = \nabla(x^2 - y^2 - z^2)|_{(1,1,0)} = 2\mathbf{i} - 2\mathbf{j}$$

and the tangent plane at $(1, 1, 0)$ has equation $2x - 2y = 0$, or $y = x$. The normal line at this point has parametric equations

$$x = 1 + 2t, y = 1 - 2t, z = 0 \text{ for } -\infty < t < \infty.$$

14. The normal vector is

$$\mathbf{N} = \nabla(x^2 - y^2 + z^2)|_{(1,1,0)} = 2\mathbf{i} - 2\mathbf{j}.$$

The tangent plane at $(1, 1, 0)$ has equation $y = x$ and the normal line has parametric equations

$$x = 1 + 2t, y = 1 - 2t, z = 0 \text{ for } -\infty < t < \infty.$$

15. The normal vector is

$$\mathbf{N} = \nabla(2x - \cos(x, y, z))|_{(1,\pi,1)} = 2\mathbf{i}.$$

The tangent plane has the equation $x = 1$ and the normal line has parametric equations

$$x = 1 + 2t, y = \pi, z = 1 \text{ for } -\infty < t < \infty.$$

16. The normal vector is

$$\mathbf{N} = \nabla(3x^4 + 3y^4 + 6z^4)|_{(1,1,1)} = 12\mathbf{i} + 12\mathbf{j} + 24\mathbf{k}.$$

The tangent plane has equation $x + y + 2z = 4$ and the normal line has parametric equations

$$x = 1 + 12t, y = 1 + 12t, z1 + 24t \text{ for } -\infty < t < \infty.$$

17. Since $\nabla\varphi(x, y, z) = \mathbf{i} + \mathbf{k}$ for all (x, y, z), the normal to the level surface $\varphi(x, y, z) = K$ is the constant vector $\mathbf{N} = \mathbf{i}+\mathbf{k}$, so the surface must be the plane $x + z = K$. The streamlines of the vector field $\nabla\varphi(x, y, z) = \mathbf{i} + \mathbf{k}$ are solutions of

$$dx = dz, dy = 0.$$

Integrate to obtain

$$x = z + c_1, y = c_2.$$

Using t as parameter,

$$x = t + c_1, y = c_2, z = t \text{ for } -\infty < t < \infty.$$

These streamlines are lines in $3-$ space which are orthogonal to the surface $x + z = K$.

11.5 Divergence and Curl

1.

$$\nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(2z) = 4$$

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x & y & 2z \end{vmatrix} = 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} = \mathbf{O}$$

$$\nabla \cdot (\nabla \times \mathbf{F}) = 0$$

2.

$$\nabla \cdot \mathbf{F} = xz\cosh(xyz),$$
$$\nabla \times \mathbf{F} = -xy\cosh(xyz)\mathbf{i} + yz\cosh(xyz)\mathbf{k}$$

$$\begin{aligned}\nabla \cdot (\nabla \times \mathbf{F}) &= \frac{\partial}{\partial x}(-xy\cosh(xyz)) + \frac{\partial}{\partial z}(yz\cosh(xyz)) \\ &= \cosh(xyz)(-y + y) + \sinh(xyz)(-xy^2z + zy^2z) = 0\end{aligned}$$

3.

$$\nabla \cdot \mathbf{F} = 2y + xe^y + 2$$
$$\nabla \times \mathbf{F} = (e^y - 2x)\mathbf{k}$$
$$\nabla \cdot (\nabla \times \mathbf{F}) = \frac{\partial}{\partial z}(e^y - 2x) = 0$$

4.

$$\nabla \cdot \mathbf{F} = 1 + 1 + 2 = 4$$
$$\nabla \times \mathbf{F} = \mathbf{O}$$
$$\nabla \cdot (\nabla \times \mathbf{F}) = 0$$

5.

$$\nabla \cdot \mathbf{F} = \cosh(x) + xz\sinh(xyz) - 1$$
$$\nabla \times \mathbf{F} = (-1 - xy\sinh(xyz))\mathbf{i} - \mathbf{j} + yz\sinh(xyz)\mathbf{k}$$

$$\begin{aligned}\nabla \cdot \nabla \times \mathbf{F} &= \frac{\partial}{\partial x}(-1 - xy\sinh(xyz)) + \frac{\partial}{\partial y}(yz\sinh(xyz)) \\ &= (-y + y)\sinh(xyz) + \cosh(xyz)(-xy^2z + xy^2z) = 0\end{aligned}$$

6.

$$\nabla \cdot \mathbf{F} = \cosh(x - z) + 2 + 1 = \cosh(x - z) + 3$$
$$\nabla \times \mathbf{F} = -2y\mathbf{i} - \cosh(x - z)\mathbf{j}$$
$$\nabla \cdot (\nabla \times \mathbf{F}) = \frac{\partial}{\partial x}(-2y) + \frac{\partial}{\partial y}(-\cosh(x - z)) = 0$$

7.

$$\nabla\varphi = \mathbf{i} - \mathbf{j} + 4z\mathbf{k}$$
$$\nabla \times (\nabla\varphi) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 1 & -1 & 4z \end{vmatrix} = \mathbf{O}$$

8.

$$\nabla\varphi = (18yz + e^x)\mathbf{i} + 18xz\mathbf{j} + 18xy\mathbf{k}$$
$$\nabla \times (\nabla\varphi) = (18x - 18x)\mathbf{i} + (18y - 18y)\mathbf{j} + (18z - 18z)\mathbf{k}$$

9.

$$\nabla\varphi = -6x^2yz^2\mathbf{i} - 2x^3z^2\mathbf{j} - 4x^3yz\mathbf{k}$$

and

$$\begin{aligned}\nabla \times (\nabla\varphi) &= (-4x^3z + 4x^3z)\mathbf{i} \\ &+ (-12x^2yz + 12x^2yz)\mathbf{j} + (-6x^2z^2 + 6x^2z^2)\mathbf{k} = \mathbf{O}\end{aligned}$$

10.

$$\nabla\varphi = z\cos(xz)\mathbf{i} + x\cos(xz)\mathbf{k}$$

$$\nabla \times (\nabla\varphi) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z\cos(xz) & 0 & x\cos(xz) \end{vmatrix}$$

$$= 0\mathbf{i} + (\cos(xz) - xz\sin(xz) - \cos(xz) + xz\sin(xz))\mathbf{j} + 0\mathbf{k} = \mathbf{O}$$

11.

$$\begin{aligned}\nabla\varphi = (&\cos(x+y+z) - x\sin(z+y+z))\mathbf{i} \\ &- x\sin(x+y+z)\mathbf{j} - x\sin(x+y+z)\mathbf{k}\end{aligned}$$

$$\begin{aligned}&\nabla \times (\nabla\varphi) = (-x\cos(x+y+z) + x\cos(x+y+z))\mathbf{i} \\ &+ (-\sin(x+y+z) - x\cos(x+y+z) + \sin(x+y+z) + x\cos(x+y+z))\mathbf{j} \\ &+ (-\sin(x+y+z) - x\cos(x+y+z) + \sin(x+y+z) + x\cos(x+y+z))\mathbf{k} \\ &= \mathbf{O}\end{aligned}$$

12.

$$\nabla\varphi = e^{x+y+z}(\mathbf{i} + \mathbf{j} + \mathbf{k})$$

$$\nabla \times (\nabla\varphi) = (e^{x+y+z} - e^{x+y+z})(\mathbf{i} + \mathbf{j} + \mathbf{k}) = \mathbf{O}$$

13. Let $\mathbf{F} = f\mathbf{i} + g\mathbf{j} + h\mathbf{k}$. Then

$$\begin{aligned}\nabla \cdot (\varphi\mathbf{F}) &= \nabla \cdot (\varphi f\mathbf{i} + \varphi g\mathbf{j} + \varphi h\mathbf{k}) \\ &= \frac{\partial}{\partial x}(\varphi f) + \frac{\partial}{\partial y}(\varphi g) + \frac{\partial}{\partial z}(\varphi h) \\ &= \left(\frac{\partial\varphi}{\partial x}f + \frac{\partial\varphi}{\partial y}g + \frac{\partial\varphi}{\partial z}h\right) \\ &+ \varphi\left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}\right) \\ &= \nabla\varphi \cdot \mathbf{F} + \varphi\nabla \cdot \mathbf{F}.\end{aligned}$$

Next,

$$\begin{aligned}
\nabla \times (\varphi \mathbf{F}) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \varphi f & \varphi g & \varphi h \end{vmatrix} \\
&= \left[\frac{\partial}{\partial y}(\varphi h) - \frac{\partial}{\partial z}(\varphi g)\right]\mathbf{i} \\
&+ \left[\frac{\partial}{\partial z}(\varphi f) - \frac{\partial}{\partial x}(\varphi h)\right]\mathbf{j} \\
&+ \left[\frac{\partial}{\partial x}(\varphi g) - \frac{\partial}{\partial y}(\varphi f)\right]\mathbf{k} \\
&= \left[\frac{\partial \varphi}{\partial y}h - \frac{\partial \varphi}{\partial z}g\right]\mathbf{i} + \left[\frac{\partial \varphi}{\partial z}f - \frac{\partial \varphi}{\partial x}h\right]\mathbf{j} \\
&+ \left[\frac{\partial \varphi}{\partial x}g - \frac{\partial \varphi}{\partial y}f\right]\mathbf{k} \\
&+ \varphi\left[\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right]\mathbf{i} + \left[\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}\right]\mathbf{j} + \left[\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right]\mathbf{k} \\
&= \nabla\varphi \times \mathbf{F} + \varphi(\nabla \times \mathbf{F}).
\end{aligned}$$

Chapter 12

Vector Integral Calculus

12.1 Line Integrals

1. On C, $x = t, y = t, z = t^3$, so

$$\begin{aligned}\int_C x\,dx - dy + z\,dz &= \int_0^1 (t(1) - (1) + t^3(3t^2))\,dt \\ &= \int_0^1 (t - 1 + 3t^5)\,dt = 0.\end{aligned}$$

2.

$$\begin{aligned}\int_C -4x\,dx + y^2\,dy - yz\,dz &= \int_0^1 (-4(-t^2)(-2t) + 0^2 - 0)\,dt \\ &= \int_0^1 -8t^3\,dt = -2.\end{aligned}$$

3.

$$\begin{aligned}\int_C (x+y)\,ds &= \int_0^2 (t+t)\sqrt{1+1+4t^2}\,dt \\ &= \int_0^2 2t\sqrt{2+4t^2}\,dt = \frac{1}{6}\left.(2+4t^2)^{3/2}\right]_0^2 = \frac{26\sqrt{2}}{3}\end{aligned}$$

4. Parametric equations of C are $x = t, y = 1 + t, z = 1 - 2t$ for $0 \le t \le 1$. Then

$$\begin{aligned}\int_C x^2 z\,ds &= \int_0^1 t^2(1-2t)\sqrt{6}\,dt \\ &= \sqrt{6}\int_0^1 (t^2 - 2t^3)\,dt = -\sqrt{6}/6.\end{aligned}$$

5.
$$\begin{aligned}\int_C \mathbf{F}\cdot\, d\mathbf{R} &= \int_0^3 (\cos(t)\mathbf{i} + t^2\mathbf{j} + t\mathbf{k})\cdot(\mathbf{i} - 2t\mathbf{j} + 0\mathbf{k})\, dt \\ &= \int_0^3 (\cos(t) - 2t^3)\, dt = \sin(3) - \frac{81}{2}.\end{aligned}$$

6.
$$\int_C 4xy\, ds = \int_1^2 4t^2\sqrt{6}\, dt = \frac{28\sqrt{6}}{3}.$$

7. Parametrize C as $x = 2\cos(t), y = 2\sin(t), z = 0$ for $0 \le t \le 2\pi$. Then
$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{R} &= \int_0^{2\pi} (2\cos(t)\mathbf{i} + 2\sin(t)\mathbf{j})\cdot(-2\sin(t)\mathbf{i} + 2\cos(t)\mathbf{j})\, dt \\ &= \int_0^{2\pi} (-4\cos(t)\sin(t) + 4\sin(t)\cos(t))\, dt = 0.\end{aligned}$$

8. Parametrize C by $x = 1, y = t, z = t^2$ for $0 \le t \le 2$. Then
$$\begin{aligned}\int_C yz\, ds &= \int_0^2 t(t^2)\sqrt{1+4t^2}\, dt \\ &= \int_0^2 t^3\sqrt{1+4t^2}\, dt.\end{aligned}$$
An integration by parts yields the value
$$\int_C yz\, ds = \frac{1}{120}(391\sqrt{17} + 1).$$

9.
$$\begin{aligned}\int_C -xyz\, dz &= \int_4^9 -z\sqrt{z}\, dz \\ &= -\frac{2}{5}z^{5/2}\Big]_4^9 = -\frac{422}{5}\end{aligned}$$

10.
$$\int_C xz\, dy = \int_1^3 t(-4t^2)\, dt = -80$$

11. Parametrize the line segment as $x = y = z = 1 + 3t$ for $0 \le t \le 1$. The work done is
$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{R} &= \int_0^1 ((1+3t)^2 - 2(1+3t)^2 + (1+3t))(3)\, dt \\ &= \left[\frac{(1+3t)^2}{2} - \frac{(1+3t)^3}{3}\right]_0^1 = -\frac{27}{2}.\end{aligned}$$

12. Parametrize the wire by $x = y = z = t$ for $0 \le t \le 3$. The mass M is

$$M = \int_C \delta(x, y, z)\, ds = \int_0^3 3t\sqrt{3}\, dt = \frac{27\sqrt{3}}{2}.$$

Because the density function and the position of the wire are symmetric in the first octant, we will have $\overline{x} = \overline{y} = \overline{z}$. Compute

$$\begin{aligned}\overline{x} &= \frac{2}{27\sqrt{3}} \int_0^3 x\delta(x, y, z)\, ds \\ &= \frac{2}{27\sqrt{3}} \int_0^3 t(3t)\sqrt{3}\, dt = 2.\end{aligned}$$

The centroid is $(2, 2, 2)$.

13. Take $\mathbf{F}(x) = f(x)\mathbf{i}$ and $\mathbf{R}(t) = t\mathbf{i}$, for $a \le t \le b$. The graph of the curve defined by this position vector is $[a, b]$, and

$$\int_C \mathbf{F} \cdot d\mathbf{R} = \int_a^b f(x)\, dx.$$

12.2 Green's Theorem

1. The work done by $\mathbf{F}$ is

$$\begin{aligned}\text{work } &= \oint_C xy\, dx + x\, dy = \iint_D \left[\frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}(xy)\right] dA \\ &= \int_0^1 \int_0^{6x} (1 - x)\, dy\, dx + \int_1^4 \int_0^{8-2x} (1 - x)\, dy\, dx \\ &= \int_0^1 6x(1 - x)\, dx + \int_1^4 (8 - 2x)(1 - x)\, dx = -8.\end{aligned}$$

2.

$$\begin{aligned}\text{work } &= \oint_C \mathbf{F} \cdot d\mathbf{R} \\ &= \oint_C (e^x - y + x\cosh(x))\, dx + (y^{3/2} + x)\, dy \\ &= \iint_D \left[\frac{\partial}{\partial x}(y^{3/2} + x) - \frac{\partial}{\partial y}(e^x - y + x\cosh(x))\right] dA \\ &= \iint_D 2\, dA = 2(\text{area of } D) = 2(6^2\pi) = 72\pi.\end{aligned}$$

3.

$$
\begin{aligned}
\text{work } &= \oint_C (-\cosh(4x^4) + xy)\,dx + (e^{-y} + x)\,dy \\
&\iint_D \left[\frac{\partial}{\partial x}(e^{-y} + x) - \frac{\partial}{\partial y}(-\cosh(4x^4) + xy)\right] dA \\
&= \iint_D (1 - x)\,dA = \int_1^3 \int_1^7 (1 - x)\,dy\,dx \\
&= \int_1^3 6(1 - x)\,dx = -12.
\end{aligned}
$$

4.

$$
\begin{aligned}
&\oint_C \mathbf{F} \cdot d\mathbf{R} \\
&= \iint_D \left[\frac{\partial}{\partial x}(-x) - \frac{\partial}{\partial y}(2y)\right] dA \\
&= \iint_D (-3)\,dA = -3(\text{area of } D) = -3(16\pi) = -48\pi.
\end{aligned}
$$

5.

$$
\begin{aligned}
&\oint_C \mathbf{F} \cdot d\mathbf{R} \\
&= \iint_D \left[\frac{\partial}{\partial x}(-2xy) - \frac{\partial}{\partial y}(x^2)\right] dA \\
&= \iint_D (-2y)\,dA = \int_1^6 \int_{(y+4)/5}^{(22-2y)/5} -2y\,dx\,dy \\
&= \int_1^6 (3y - 18)\frac{2y}{5}\,dy = -40.
\end{aligned}
$$

6.

$$
\begin{aligned}
\oint_C \mathbf{F} \cdot d\mathbf{R} &= \iint_D \left[\frac{\partial}{\partial x}(x - y) - \frac{\partial}{\partial y}(x + y)\right] dA \\
&= \iint_D 0\,dA = 0.
\end{aligned}
$$

7.

$$
\oint_C \mathbf{F} \cdot d\mathbf{R} = \iint_D \frac{\partial}{\partial x}(8xy^2) = \iint_D 8y^2\,dA.
$$

Change to polar coordinates $x = r\cos(\theta), y = r\sin(\theta)$, with $0 \le \theta \le 2\pi, 0 \le r \le 4$ to obtain

$$\begin{aligned}\iint_D 8y^2\,dA &= \int_0^{2\pi}\int_0^4 8r^2\sin^2(\theta)r\,dr\,d\theta \\ &= \int_0^{2\pi}\sin^2(\theta)\,d\theta\int_0^4 8r^3\,dr = 512\pi.\end{aligned}$$

8.

$$\begin{aligned}\oint_C \mathbf{F}\cdot d\mathbf{R} &= \iint_D\left[\frac{\partial}{\partial x}(\cos(2y) - e^{3y} + 4x) - \frac{\partial}{\partial y}(x^2 - y)\right] \\ &= \iint_D 5\,dA = 125.\end{aligned}$$

9.

$$\begin{aligned}\oint_C \mathbf{F}\cdot d\mathbf{R} &= \iint_D\left[\frac{\partial}{\partial x}(e^x\sin(y)) - \frac{\partial}{\partial y}(e^x\cos(y))\right] \\ &= \iint_D(-e^x\sin(y) + e^x\sin(y))\,dA = 0.\end{aligned}$$

10.

$$\begin{aligned}\oint_C \mathbf{F}\cdot d\mathbf{R} &= \left[\frac{\partial}{\partial x}(-xy^2) - \frac{\partial}{\partial y}(x^2y)\right] \\ &= \iint_D(-y^2 - x^2)\,dA = \int_0^{\pi/2}\int_0^2(r^2)r\,dr\,d\theta \\ &= \frac{\pi}{2}\int_0^2 -r^3\,dr = -2\pi.\end{aligned}$$

11.

$$\begin{aligned}\oint_C \mathbf{F}\cdot d\mathbf{R} &= \iint_D\left[\frac{\partial}{\partial x}(xy^2 - e^{\cos(y)}) - \frac{\partial}{\partial y}(xy)\right] \\ &= \iint_D(y^2 - x)\,dA = \int_0^3\int_0^{5-5x/3}(y^2 - x)\,dy\,dx \\ &= \int_0^3\frac{1}{3}\left(5 - \frac{5x}{3}\right)\,dx - \int_0^3 x\left(5 - \frac{5x}{3}\right)\,dx \\ &= \frac{95}{4}.\end{aligned}$$

12. (a) By Green's theorem, with $\mathbf{F} = -y\mathbf{i}$,

$$\oint_C -y\,dx = \iint_D\left[\frac{\partial}{\partial x}(0) - \frac{\partial}{\partial y}(-y)\right]\,dA = \iint_D\,dA = \text{ area of } D.$$

(b) This time apply Green's theorem with $\mathbf{F} = x\mathbf{j}$ to obtain

$$\oint_C \mathbf{F} \cdot d\mathbf{R} = \iint_D \left[\frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}(0) \right] dA = \iint_D dA = \text{ area of } D.$$

(c) Add the results of (a) and (b).

13. By Green's theorem,

$$\oint_C -\frac{\partial u}{\partial y}\, dx + \frac{\partial u}{\partial x}\, dy = \iint_D \left[\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) - \frac{\partial}{\partial y}\left(-\frac{\partial u}{\partial y}\right) \right] dA$$
$$= \iint_D \left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right] dA.$$

14. Assume that C is a join of two curves in two ways. First, C has an upper piece $y = p(x)$ and a lower piece $y = q(x)$ for $a \le x \le b$, so D consists of all (x, y) with

$$a \le x \le b, q(x) \le y \le p(x).$$

Second, C also has a right piece $y = \beta(x)$ and a left piece $y = \alpha(x)$ for $c \le y \le d$, so, looking left to right instead of bottom to top, D can also be described as consisting of all (x, y) with

$$c \le y \le d, \alpha(y) \le x \le \beta(y).$$

Now use both of these descriptions in turn as follows. Using the second description of C (look at C from left to right),

$$\oint_C g(x, y)\, dy = \int_c^d g(\beta(y), y)\, dy + \int_d^c g(\alpha(y), y)\, dy.$$

Note that, on the right part of C, y varies from c to d for a counterclockwise orientation, while, to retain this orientation, y varies from d to c on the left part of the boundary curve. Further,

$$\iint_D \frac{\partial g}{\partial x}\, dA = \int_c^d \int_{\alpha(y)}^{\beta(y)} \frac{\partial g}{\partial y}\, dy$$
$$= \int_c^d (g(\beta(y), y) - g(\alpha(y), y))\, dy.$$

Therefore

$$\oint_C g(x, y)\, dy = \iint_D \frac{\partial g}{\partial x}\, dA.$$

This is "half" of the conclusion of Green's theorem. For the rest, use the first description of C. Now, looking from bottom to top, we have (keeping in mind the counterclockwise orientation on C),

$$\oint_C f(x,y)\,dx = \int_b^a f(x,p(x))\,dx + \int_a^b f(x,q(x))\,dx$$
$$= -\int_a^b (f(x,p(x)) - f(x,q(x)))\,dx$$

and

$$\iint_D \frac{\partial f}{\partial y}\,dA = \int_a^b \int_{q(x)}^{p(x)} f(x,p(x)) - f(x,q(x))\,dA.$$

Then

$$\oint_C f(x,y)\,dx = -\iint_D \frac{\partial f}{\partial y}\,dA.$$

Upon adding these two equations, we obtain

$$\oint_C f(x,y)\,dx + g(x,y)\,dx = \iint_D \left[\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right]\,dA.$$

12.3 An Extension of Green's Theorem

1. If C does not enclose the origin, then by Green's theorem we have

$$\oint_C \mathbf{F}\cdot d\mathbf{R} = \iint_D \left[\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) - \frac{\partial}{\partial y}\left(\frac{x}{x^2+y^2}\right)\right]\,dA = 0,$$

because the integrand is identically zero. If C encloses the origin, use the extended form of Green's theorem, where K is a circle of radius r lying entirely within C and enclosing the origin. Then

$$\oint_C \mathbf{F}\cdot d\mathbf{R} = \oint_K \mathbf{F}\cdot d\mathbf{R}$$
$$= \int_0^{2\pi} \left[\frac{r\cos(\theta)}{r^2}(-r\sin(\theta)) + \frac{r\sin(\theta)}{r^2}(r\cos(\theta))\right]\,d\theta = 0.$$

2. If C does not enclose the origin, then by Green's theorem we have

$$\oint_C \mathbf{F}\cdot d\mathbf{R} = \iint_D \left[\frac{\partial}{\partial x}\left(\frac{y}{(x^2+y^2)^{3/2}}\right) - \frac{\partial}{\partial y}\left(\frac{x}{(x^2+y^2)^{3/2}}\right)\right]\,dA = 0.$$

because the two partial derivatives in this integral are equal. If C does enclose the origin, then choose a smaller circle K enclosed by C, which also encloses the origin. Then

$$\oint_C \mathbf{F}\cdot d\mathbf{F} = \oint_K \mathbf{F}\cdot d\mathbf{R}$$
$$= \int_0^{2\pi} \left[\frac{r\cos(\theta)}{r^3}(-r\sin(\theta)) + \frac{r\sin(\theta)}{r^3}(r\cos(\theta))\right]\,d\theta = 0.$$

3. If C does not enclose the origin, then by Green's theorem,

$$\begin{aligned}
\oint_C \mathbf{F}\cdot d\mathbf{R} &= \oint_K \mathbf{F}\cdot d\mathbf{R} \\
&= \int_0^{2\pi}\left[\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}-2y\right)-\frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}+x^2\right)\right]dA = 0
\end{aligned}$$

because the partial derivatives in this double integral are equal. If C encloses the origin, choose a smaller circle K, of radius r, enclosed by C and enclosing the origin. Then

$$\begin{aligned}
\oint_C \mathbf{F}\cdot d\mathbf{R} &= \oint_K \mathbf{F}\cdot d\mathbf{R} \\
&= \int_0^{2\pi}\left[\left(\frac{-r\sin(\theta)}{r^2}+r^2\cos^2(\theta)\right)(-r\sin(\theta))\right]d\theta \\
&+ \int_0^{2\pi}\left[\left(\frac{r\cos(\theta)}{r^2}-2r\sin(\theta)\right)(r\cos(\theta))\right]d\theta \\
&= \int_0^{2\pi}(1-r^3\cos^2(\theta)\sin(\theta)-2r^2\sin(\theta)\cos(\theta))\,d\theta \\
&= \theta+\frac{r^3}{3}\cos^3(\theta)-r^2\sin^2(\theta)\Big]_0^{2\pi} = 2\pi.
\end{aligned}$$

4. If C does not enclose the origin, then by Green's theorem,

$$\oint_C \mathbf{F}\cdot d\mathbf{R} = \iint_D\left[\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}-y\right)-\frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}+3x\right)\right]dA = 0$$

because the partial derivatives in the integrand cancel each other. If C does enclose the origin, use the extension of Green's theorem, with K a circle of radius r about the origin, completely enclosed by C. Now

$$\begin{aligned}
\oint_C \mathbf{F}\cdot d\mathbf{R} &= \oint_K \mathbf{F}\cdot d\mathbf{R} \\
&= \int_0^{2\pi}\left(\frac{-r\sin(\theta)}{r^2}+3r\cos(\theta)\right)(-r\sin(\theta))\,d\theta \\
&+ \int_0^{2\pi}\left(\frac{r\cos(\theta)}{r^2}-r\sin(\theta)\right)(r\cos(\theta))\,d\theta \\
&= \int_0^{2\pi}(1-4r^2\sin(\theta)\cos(\theta))\,d\theta = 2\pi.
\end{aligned}$$

5. If C does not enclose the origin, then by Green's theorem,

$$\oint_C \mathbf{F}\cdot d\mathbf{R} = \iint_D\left[\frac{\partial}{\partial x}\left(\frac{y}{\sqrt{x^2+y^2}}-3y^2\right)-\frac{\partial}{\partial y}\left(\frac{x}{\sqrt{x^2+y^2}}+2x\right)\right]dA = 0,$$

since the partial derivatives in the integral are equal and therefore cancel. If C does enclose the origin, use the extension of Green's theorem. If K is a circle of radius r enclosing the origin and enclosed by C, we obtain

$$\begin{aligned}
&\oint_C \mathbf{F}\cdot d\mathbf{R} = \oint_K \mathbf{F}\cdot d\mathbf{R}\\
&= \int_0^{2\pi}\left[\left(\frac{r\cos(\theta)}{r} + 2r\cos(\theta)\right)(-r\sin(\theta))\right] d\theta\\
&+ \int_0^{2\pi}\left[\left(\frac{r\sin(\theta)}{r} - 3r^2\sin^2(\theta)\right) r\cos(\theta)\right] d\theta\\
&= -r^2\int_0^{2\pi}(2\cos(\theta)\sin(\theta) + 3\sin^2(\theta)\cos(\theta))\, d\theta\\
&= -r^2(\sin^2(\theta) + \sin^3(\theta)\big]_0^{2\pi} = 0.
\end{aligned}$$

12.4 Potential Theory

1. Since
$$\frac{\partial}{\partial y}(y^3) = 3y^2 = \frac{\partial}{\partial x}(3x^2y - 4),$$
then $\mathbf{F}$ is conservative (in the entire plane, where the components of $\mathbf{F}$ are defined). To find a potential function φ, begin with $\partial\varphi/\partial x = y^3$ and integrate with respect to x to obtain
$$\varphi(x, y) = xy^3 + k(y)$$
in which $k(y)$ is the "constant" of the integration with respect to x. Next
$$\frac{\partial\varphi}{\partial y} = 3x^2y + k'(y) = 3x^2y - 4$$
so $k'(y) = -4$ and we can choose $k(y) = -4y$. A potential function is
$$\varphi(x, y) = xy^3 - 4y.$$
Of course $xy^3 - 4y + c$ is also a potential function for any constant c.

2. First,
$$\frac{\partial}{\partial y}(6y + ye^{xy}) = 6 + e^{xy} + xye^{xy} = \frac{\partial}{\partial x}(6x + xe^{xy}),$$
so $\mathbf{F}$ is conservative. To find a potential function φ, write
$$\frac{\partial\varphi}{\partial x} = 6y + ye^{xy} \text{ and } \frac{\partial\varphi}{\partial y} = 6x + xe^{xy}.$$
Choose one and integrate. If we choose the first equation, integrate with respect to x to obtain
$$\varphi(x, y) = 6xy + e^{xy} + k(y).$$

Then

$$\frac{\partial \varphi}{\partial y} = 6x + xe^{xy} + k'(y) = 6x + xe^{xy}.$$

Then $k'(y) = 0$ and we may choose $k(y) = 0$. A potential function is $\varphi(x, y) = 6xy + e^{xy}$.

3. Since

$$\frac{\partial}{\partial y}(16x) = 0 = \frac{\partial}{\partial x}(2 - y^2),$$

then **F** is conservative. Integrate $\partial\varphi/\partial x = 16x$ with respect to x to obtain

$$\varphi(x, y) = 8x^2 + k(y).$$

Then

$$\frac{\partial \varphi}{\partial y} = 2 - y^2 = k'(y)$$

so we may choose $k(y) = 2y - y^3/3$ to obtain the potential function

$$\varphi(x, y) = 8x^2 + 2y - \frac{1}{3}y^3.$$

4. Since

$$\frac{\partial}{\partial y}(2xy\cos(x^2)) = 2x\cos(x^2) = \frac{\partial}{\partial x}(\sin(x^2)),$$

then **F** is conservative. Integrate

$$\frac{\partial \varphi}{\partial x} = 2xy\cos(x^2)$$

to obtain

$$\varphi(x, y) = y\sin(x^2) + k(y).$$

Then

$$\frac{\partial \varphi}{\partial y} = \sin(x^2) = \sin(x^2) + k'(y),$$

and we may choose $k(y) = 0$. A potential function is

$$\varphi(x, y) = y\sin(x^2).$$

5. Since

$$\frac{\partial}{\partial y}\left(\frac{2x}{x^2 + y^2}\right) = -\frac{4xy}{(x^2 + y^2)^2} = \frac{\partial}{\partial x}\left(\frac{2y}{x^2 + y^2}\right),$$

we know that **F** is conservative on any region not containing the origin. A potential function $\varphi(x, y)$ must satisfy

$$\frac{\partial \varphi}{\partial x} = \frac{2x}{x^2 + y^2} \text{ and } \frac{\partial \varphi}{\partial y} = \frac{2y}{x^2 + y^2}.$$

Integrate one of these. If we integrate the second, we obtain

$$\varphi(x, y) = \ln(x^2 + y^2) + c(x).$$

Then we need

$$\frac{\partial \varphi}{\partial x} = \frac{2x}{x^2 + y^2} + c'(x) = \frac{2x}{x^2 + y^2}.$$

Then $c'(x) = 0$ and we may choose $c(x) = 0$, yielding the potential function

$$\varphi(x, y) = \ln(x^2 + y^2).$$

6. It is routine to compute that $\nabla \times \mathbf{F} = \mathbf{O}$, therefore $\mathbf{F}$ is conservative. A potential function φ must satisfy

$$\frac{\partial \varphi}{\partial x} = 2x, \frac{\partial \varphi}{\partial y} = -2y, \frac{\partial \varphi}{\partial z} = 2z.$$

From the first of these equations, $\varphi(x, y, z) = x^2 + k(y, z)$. Then, from the second equation,

$$\frac{\partial \varphi}{\partial y} = -2y = \frac{\partial k}{\partial y}.$$

Integrate this with respect to y to obtain

$$k(y, z) = -y^2 + c(z).$$

Finally, we also need

$$\frac{\partial \varphi}{\partial z} = 2z = c'(z),$$

so $c(z) = z^2$. Then

$$\varphi(x, y, z) = x^2 - y^2 + z^2.$$

7. By inspection in this simple case, $\varphi(x, y, z) = x - 2y + z$ is a potential function for $\mathbf{F}$.

8. A routine computation yields $\nabla \times \mathbf{F} = \mathbf{O}$, so $\mathbf{F}$ is conservative. We need a potential function to satisfy

$$\frac{\partial \varphi}{\partial x} = yz\cos(x), \frac{\partial \varphi}{\partial y} = z\sin(x) + 1, \text{ and } \frac{\partial \varphi}{\partial z} = y\sin(x).$$

Integrate the first with respect to x to obtain

$$\varphi(x, y, z) = yz\sin(x) + k(y, z).$$

Next we need

$$\frac{\partial \varphi}{\partial y} = z\sin(x) + 1 = z\sin(x) + \frac{\partial k}{\partial y}$$

so

$$\frac{\partial k}{\partial y} = 1.$$

Integrate this with respect to y to obtain

$$k(x, y) = y + c(z).$$

Thus far

$$\varphi(x, y, z) = yz\sin(x) + y + c(z).$$

Finally, we need

$$\frac{\partial\varphi}{\partial z} = y\sin(x) + c' = y\sin(x).$$

We may choose $c(x) = 0$, yielding the potential function

$$\varphi(x, y, z) = yz\sin(x) + y.$$

9. We find that

$$\nabla \times \mathbf{F} = (-z^2 - xy)\mathbf{i} + yz\mathbf{k} \neq \mathbf{O}$$

so $\mathbf{F}$ is not conservative and there is no potential function.

10. Compute

$$\nabla \times \mathbf{F} = e^{xyz}(2xy + x^2y^2z)\mathbf{j} + (2xz - e^{xyz}(2xz + x^2yz^2))\mathbf{k} \neq \mathbf{O}.$$

Therefore $\mathbf{F}$ is not conservative.

In Problems 11 - 20, we provide the potential function used to evaluate the integral, but omit the details of deriving this potential function.

11. By integrating, we find the potential function

$$\varphi(x, y) = x^3(y^2 - 4y).$$

Then

$$\int_C \mathbf{F} \cdot d\mathbf{R} = \varphi(2, 3) - \varphi(-1, 1) = -24 - 3 = -27.$$

12. $\varphi(x, y) = e^x\cos(y)$ is a potential function for $\mathbf{F}$. Then

$$\int_C \mathbf{F} \cdot d\mathbf{R} = \varphi(2, \pi/4) - \varphi(0, 0) = \frac{e^2}{\sqrt{2}} - 1.$$

13. In any region not containing the points of the $y-$ axis,

$$\varphi(x, y) = x^2y - \ln|y|$$

is a potential function for $\mathbf{F}$. If C does not cross the $x-$ axis, then

$$\int_C \mathbf{F} \cdot d\mathbf{R} = \varphi(2, 2) - \varphi(1, 3) = 8 - \ln(2) - 3 + \ln(3) = 5 + \ln(3/2).$$

14. $\varphi(x,y) = x + 3y^2 - \cos(y)$ is a potential function for $\mathbf{F}$, so

$$\int_C \mathbf{F} \cdot d\mathbf{R} = \varphi(1,3) - \varphi(0,0) = 29 - \cos(3).$$

15. $\mathbf{F}$ has potential function $\varphi(x,y) = x^3y^2 - 6xy^3$, so

$$\int_C \mathbf{F} \cdot d\mathbf{R} = \varphi(1,1) - \varphi(0,0) = -5.$$

16. The easiest way to find a potential function for $\mathbf{F}$ is to integrate $\partial\varphi/\partial y = x\cos(xz)$ with respect to y and obtain $\varphi(x,y,z) = xy\cos(xz) + k(x,z)$. Now observe that $\varphi(x,y,z) = xy\cos(xz)$ is a potential function for $\mathbf{F}$ if we choose $k(x,z) = 0$. Then

$$\int_C \mathbf{F} \cdot d\mathbf{R} = \varphi(1,1,7) - \varphi(1,0,\pi) = \cos(7).$$

17. $\varphi(x,y,z) = x - 3y^3z$ is a potential function for $\mathbf{F}$, so

$$\int_C \mathbf{F} \cdot d\mathbf{R} = \varphi(0,3,5) - \varphi(1,1,1) = -403.$$

18. $\varphi(x,y,z) = -8xy^2 - 4zy$ is a potential function for $\mathbf{F}$, so

$$\int_C \mathbf{F} \cdot d\mathbf{R} = \varphi(1,3,2) - \varphi(-2,1,1) = -108.$$

19. $\varphi(x,y,z) = 2x^3e^{yz}$ is a potential function for $\mathbf{F}$, so

$$\int_C \mathbf{F} \cdot d\mathbf{R} = \varphi(1,2,-1) - \varphi(0,0,0) = 2e^{-2}.$$

20. $\varphi(x,y,z) = xy - 2x^2z + z^3$, so

$$\int_C \mathbf{F} \cdot d\mathbf{R} = \varphi(3,1,4) - \varphi(1,1,1) = -5.$$

21. Let C be a smooth path of motion given by $\mathbf{R}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ and let L be the kinetic energy plus the potential energy. Then

$$L(t) = \frac{m}{2} \parallel \mathbf{R}'(t) \parallel^2 - \varphi(x(t),y(t),z(t)) = \frac{m}{2}\mathbf{R}'(t)\cdot\mathbf{R}'(t) - \varphi(x(t),y(t),z(t)).$$

Then

$$\begin{aligned}\frac{dL}{dt} &= \frac{m}{2}(2\mathbf{R}''(t)\cdot\mathbf{R}'(t)) - \frac{\partial\varphi}{\partial x}x'(t) - \frac{\partial\varphi}{\partial y}y'(t) - \frac{\partial\varphi}{\partial z}z'(t)\\ &= (m\mathbf{R}''(t)\cdot\mathbf{R}'(t)) - \nabla\varphi\cdot\mathbf{R}'(t)\\ &= (m\mathbf{R}''(t) - \nabla\varphi)\cdot\mathbf{R}'(t).\end{aligned}$$

But $\nabla\varphi$ is the force acting on the particle, so by Newton's second law, $m\mathbf{R}'' = \nabla\varphi$, and therefore $dL/dt = 0$. Therefore $L(t)$ is a constant of the motion.

22. We want to show that, in Theorem 12.5, a potential function exists if

$$\frac{\partial g}{\partial x} = \frac{\partial f}{\partial y}.$$

We will use this condition to explicitly construct a potential function. First observe that, if K is any closed path in D, then

$$\oint_K \mathbf{F} \cdot d\mathbf{R} = \iint_D \left[\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right] dA = 0.$$

This means that $\int_C \mathbf{F} \cdot d\mathbf{R}$ is independent of path in D. If we fix a point $P : (a, b)$ in D, we can define a function

$$\varphi(x, y) = \int_P^{(x,y)} \mathbf{F} \cdot d\mathbf{R}.$$

This is a function because its value depends only on (x, y) and not on the path in D from P to (x, y). We claim that $\nabla\varphi = \mathbf{F}$.

To show this, we will first show that

$$\frac{\partial \varphi}{\partial x} = f(x, y).$$

Choose Δx small enough that $(x + \Delta x, y)$ is in D. Now

$$\begin{aligned}
&\varphi(x + \Delta x, y) - \varphi(x, y)\\
&= \int_P^{(x+\Delta x,y)} \mathbf{F} \cdot d\mathbf{R} - \int_P^{(x,y)} \mathbf{F} \cdot d\mathbf{R}\\
&= \int_P^{(x,y)} \mathbf{F} \cdot d\mathbf{R} + \int_{(x,y)}^{(x+\Delta x,y)} \mathbf{F} \cdot d\mathbf{R} - \int_P^{(x,y)} \mathbf{F} \cdot d\mathbf{R}\\
&= \int_{(x,y)}^{(x+\Delta x,y)} \mathbf{F} \cdot d\mathbf{R}\\
&= \int_{(x,y)}^{(x+\Delta x,y)} f(\xi, \eta)\, d\xi + g(\xi, \eta)\, d\eta.
\end{aligned}$$

This is a line integral over over a horizontal line segment from (x, y) to $(x + \Delta x, y)$, with y fixed on this segment. Parametrize this segment by

$$\xi = x + t\Delta x, \eta = y \text{ for } 0 \leq t \leq 1.$$

On this segment,

$$d\xi = (\Delta x)\, dt \text{ and } d\eta = 0.$$

Then

$$\varphi(x + \Delta x, y) - \varphi(x, y) = \Delta x \int_0^1 f(x + t\Delta x, y)\, dt.$$

Then

$$\frac{\varphi(x+\Delta x,y)-\varphi(x,y)}{\Delta x}=\int_0^1 f(x+t\Delta x,y)\,dt.$$

By the mean value theorem for integrals, there is some t_0 in $(0,1)$ such that

$$\int_0^1 f(x+t\Delta x,y)\,dt=f(x+t_0\Delta x,y).$$

Therefore

$$\frac{\varphi(x+\Delta x,y)-\varphi(x,y)}{\Delta x}=f(x+t_0\Delta x,y).$$

As $\Delta x\to 0$, $x+t_0\Delta x\to x$ and, by continuity, $f(x+t_0\Delta x,y)\to f(x,y)$. Therefore,

$$\begin{aligned}\frac{\partial\varphi}{\partial x}&=\lim_{\Delta x\to 0}\frac{\varphi(x+\Delta x,y)-\varphi(x,y)}{\Delta x}\\&=\lim_{\Delta x\to 0}f(x+t_0\Delta x,y)=f(x,y).\end{aligned}$$

A similar argument, using a vertical path from (x,y) to $(x,y+\Delta y)$ shows that

$$\frac{\partial\varphi}{\partial y}=g(x,y).$$

12.5 Surface Integrals

1. On the surface, $z=10-x-4y$, so

$$d\sigma=\sqrt{1+(\partial z/\partial x)^2+(\partial z/\partial y)^2}\,dA=3\sqrt{2}\,dA,$$

and

$$\begin{aligned}\iint_\Sigma x\,d\sigma&=\iint_D 3\sqrt{2}x\,dA\\&=3\sqrt{2}\int_0^{5/2}\int_0^{10-4y}x\,dx\,dy=\frac{3\sqrt{2}}{2}\int_0^{5/2}(10-4y)^2\,dy\\&=-\frac{\sqrt{2}}{8}(10-4y)^3\Big]_0^{5/2}=125\sqrt{2}.\end{aligned}$$

2. On the surface, $z=x$ so $d\sigma=\sqrt{1^2+1^2+0^2}=\sqrt{2}\,dA$ and

$$\begin{aligned}\iint_\Sigma y^2\,d\sigma&=\iint_D\sqrt{2}y^2\,dA\\&=\sqrt{2}\int_0^2\int_0^4 y^2\,dx\,dy=\frac{128\sqrt{2}}{3}.\end{aligned}$$

3. On Σ,

$$d\sigma = \sqrt{1^2 + (2x)^2 + (2y)^2}\, dA = \sqrt{1 + 4(x^2 + y^2)}\, dA.$$

D is the annular region $2 \leq x^2 + y^2 \leq 7$. Then

$$\iint_\Sigma d\sigma = \iint_D \sqrt{1 + 4(x^2 + y^2)}\, dA.$$

Use polar coordinates. Now D is given by $\sqrt{2} \leq r \leq \sqrt{7}$, $0 \leq \theta \leq 2\pi$ and

$$\begin{aligned}\iint_\Sigma d\sigma &= \int_0^{2\pi} \int_{\sqrt{2}}^{\sqrt{7}} r\sqrt{1 + 4r^2}\, dr\, d\theta \\ &= 2\pi \left[\frac{1}{12}(1 + 4r^2)^{3/2}\right]_{\sqrt{2}}^{\sqrt{7}} = \frac{\pi}{6}((29)^{3/2} - 27).\end{aligned}$$

4. On the surface, $z = (25 - 4x - 8y)/10$, so

$$d\sigma = \sqrt{1 + (2/5)^2 + (4/5)^2}\, dA = \frac{3}{\sqrt{5}}\, dA.$$

Then

$$\begin{aligned}\iint_\Sigma (x + y)\, d\sigma &= \iint_D (x + y)\frac{3}{\sqrt{5}}\, dA \\ &= \frac{3}{\sqrt{5}} \int_0^1 \int_0^x (x + y)\, dy\, dx \\ &= \frac{3}{\sqrt{5}} \int_0^1 \frac{3}{2}x^2\, dx = \frac{3}{2\sqrt{5}}.\end{aligned}$$

5. On the surface, $z^2 = x^2 + y^2$, so

$$2z\frac{\partial z}{\partial x} = 2x \text{ and } 2z\frac{\partial z}{\partial y}z = 2y.$$

Then

$$\frac{\partial z}{\partial x} = \frac{x}{z} \text{ and } \frac{\partial z}{\partial y} = \frac{y}{z}.$$

Then

$$d\sigma = \sqrt{1 + \frac{x^2}{z^2} + \frac{y^2}{z^2}}\, dA = \sqrt{2}\, dA.$$

Then

$$\begin{aligned}\iint_\Sigma z\, d\sigma &= \int_D \sqrt{2}\sqrt{x^2 + y^2}\, dA \\ &= \sqrt{2}\int_0^{\pi/2} \int_2^4 r^2\, dr\, d\theta = \frac{28\pi}{3}\sqrt{2}.\end{aligned}$$

6. On Σ, $d\sigma = \sqrt{1+1+1}\,d\sigma = \sqrt{3}\,d\sigma$, and $z = x + y$, so

$$\begin{aligned}\iint_\Sigma xyz\,d\sigma &= \sqrt{3}\iint_D xy(x+y)\,dA \\ &= \sqrt{3}\int_0^1\int_0^1 (x^2y + xy^2)\,dy\,dx = \frac{\sqrt{3}}{3}.\end{aligned}$$

7. On Σ, $d\sigma = \sqrt{1+4x^2}\,dA$, so

$$\begin{aligned}&\iint_\Sigma y\,d\sigma = \iint_D y\sqrt{1+4x^2}\,dA \\ &= \int_0^2\int_0^3 y\sqrt{1+4x^2}\,dy\,dx = \frac{9}{2}\int_0^2 \sqrt{1+4x^2}\,dx = \frac{9}{8}(\ln(4+\sqrt{17}) + 4\sqrt{17}).\end{aligned}$$

8. On the surface, $d\sigma = \sqrt{1+4(x^2+y^2)}\,dA$, so

$$\begin{aligned}\iint_\Sigma x^2\,d\sigma &= \iint_D x^2\sqrt{1+4(x^2+y^2)}\,dA \\ &= \int_0^{2\pi}\int_0^2 (r^2\cos^2(\theta)\sqrt{1+4r^2}r\,dr\,d\theta \\ &= \int_0^{2\pi}\cos^2(\theta)\,d\theta\int_0^2 r^3\sqrt{1+4r^2}\,dr.\end{aligned}$$

For the first integral, use the identity $\cos^2(\theta) = (1+\cos(2\theta))/2$. For the second integral, use the substitution $u = 1+4r^2$, so $r^2 = (u-1)/4$ and $r\,dr = (1/8)du$. These yield

$$\begin{aligned}\iint_\Sigma x^2\,d\sigma &= \frac{1}{2}\int_0^{2\pi}(1+\cos(2\theta))d\theta\left(\frac{1}{32}\right)\int_1^{17}(u^{3/2} - u^{1/2})\,du \\ &= \frac{\pi}{240}(782\sqrt{17}+2).\end{aligned}$$

9. On this surface, $d\sigma = \sqrt{3}\,dA$ and $z = x - y$ so

$$\begin{aligned}\iint_\Sigma z\,d\sigma &= \iint_D \sqrt{3}(x-y)\,dA \\ &= \sqrt{3}\int_0^1\int_0^5 (x-y)\,dy\,dx = -10\sqrt{3}.\end{aligned}$$

10. On the surface, $d\sigma = \sqrt{1+4y^2}\,dA$ and $z = 1+y^2$, so

$$\begin{aligned}&\iint_\Sigma xyz\,d\sigma = \iint_D xy(1+y^2)\sqrt{1+4y^2}\,dA \\ &\quad= \int_0^1\int_0^1 xy(1+y^2)\sqrt{1+4y^2}\,dy\,dx = \frac{1}{2}\int_0^1 y(1+y^2)\sqrt{1+4y^2}\,dy.\end{aligned}$$

For the last integral let $u = 1+4y^2$, $y\,dy = (1/8)du$, and $1+y^2 = (u+3)/4$ to obtain

$$\iint_\Sigma xyz\,d\sigma = \frac{1}{2}\frac{1}{32}\int_1^5 (u^{3/2} + 3u^{1/2})\,du = \frac{1}{16}\left(5\sqrt{5} - \frac{3}{5}\right).$$

12.6 Applications of Surface Integrals

1. The triangular shell is part of the plane having equation $6x+2y+3z = 6$ (this is the plane through the given points). The projection of Σ onto the $x, y-$ plane is the set D of points (x, y) such that $0 \le y \le 3 - 3x$, $0 \le x \le 1$. On the surface,

$$z = 2 - \frac{2}{3}y - 2x$$

so $d\sigma = \sqrt{1 + \frac{4}{9} + 4}, dA = \frac{7}{3}\,dA$. The mass is

$$\begin{aligned} m = \iint_\Sigma (xz+1)\,d\sigma &= \frac{7}{3}\iint_D x\left[\left(2 - \frac{2}{3}y - 2x\right) + 1\right] dA \\ &= \frac{7}{3}\int_0^1\int_0^{3-3x} x\left[\left(2 - \frac{2}{3}y - 2x\right) + 1\right] dy\,dx. \end{aligned}$$

This integral is routine and we obtain $m = 49/12$. In similar fashion, evaluate

$$\overline{x} = \iint_\Sigma x(xz+1)\,d\sigma = \frac{12}{35},$$

$$\overline{y} = \iint_\Sigma y(xz+1)\,d\sigma = \frac{33}{35},$$

and

$$\overline{z} = \iint_\Sigma z(xz+1)\,d\sigma = \frac{24}{35}.$$

Observe that, because Σ is part of a plane, the center of mass is a point of Σ. This is not true of surfaces in general. For example, the center of mass of a homogeneous sphere is its center.

2. By symmetry of the sphere and the fact that the density function is constant, we conclude immediately that $\overline{x} = \overline{y} = 0$. On Σ,

$$d\sigma = \sqrt{1 + (x/z)^2 + (y/z)^2}\,dA = \frac{3}{z}\,dA.$$

The portion of the hemisphere lying above the plane $z = 1$ projects onto the $x, y-$ plane onto the region D given by (x, y) with $x^2 + y^2 \le 8$. Now compute the mass of the shell as

$$m = \iint_\Sigma K\left(\frac{3}{z}\right) dA = 3K\iint_D \frac{1}{\sqrt{9 - (x^2+y^2)}}\,dA.$$

To evaluate this integral use polar coordinates to obtain

$$\begin{aligned} m &= 3K \int_0^{2\pi} \int_0^{\sqrt{8}} \frac{r}{\sqrt{9 - r^2}}\, dr\, d\theta \\ &= 6\pi K \left[-\sqrt{9 - r^2} \right]_0^{\sqrt{8}} = 12K\pi. \end{aligned}$$

Finally,

$$\begin{aligned} \overline{z} &= \frac{1}{m} \iint_\Sigma Kz\, d\sigma = \frac{1}{m} \iint_D Kz \left(\frac{3}{z} \right) dA \\ &= \frac{3K}{m} (\text{area of })D = \frac{1}{m}(24K\pi) = 2. \end{aligned}$$

The center of mass is $(0, 0, 2)$.

3. By symmetry of the shell, and the fact that the density is constant, we have $\overline{x} = \overline{y} = 0$. On this surface, $d\sigma = \sqrt{1 + (x/z)^2 + (y/z)^2}\, dA = \sqrt{2}\, dA$. Then

$$\text{mass } = m = \iint_\Sigma K d\sigma = K\sqrt{2} \int_0^{2\pi} \int_0^3 r\, dr\, d\theta = 9\pi K \sqrt{2}.$$

Then

$$\begin{aligned} \overline{z} &= \frac{1}{m} \iint_\Sigma z\, d\sigma \\ &= \frac{\sqrt{2}K}{m} \int_0^{2\pi} \int_0^3 r^2\, dr\, d\theta = \frac{18K\pi\sqrt{2}}{m} = 2. \end{aligned}$$

The center of mass is $(0, 0, 2)$.

4. On Σ, $d\sigma = \sqrt{1 + 4(x^2 + y^2)}\, dA$. Σ projects onto the $x, y-$ plane to give the quarter annulus D consisting of points (x, y) with $x \geq 0, y \geq 0$ and $1 \leq x^2 + y^2 \leq 9$. The mass is

$$\begin{aligned} m &= \iint_\Sigma \frac{xy}{\sqrt{1 + 4(x^2 + y^2)}}\, d\sigma \\ &= \iint_D xy\, dA = \int_0^{\pi/2} \int_1^3 r^3 \cos(\theta) \sin(\theta)\, dr\, d\theta \\ &= \left(\frac{1}{2} \sin^2(\theta) \Big]_0^{\pi/2} \right) \left(\frac{r^4}{4} \Big]_1^3 \right) = 10. \end{aligned}$$

Since the region is symmetric about the line $y = x$ and $\delta(x, y, z) = \delta(r, \theta) = 1/\sqrt{1 + 4r^2}$ is independent of θ, then $\overline{x} = \overline{y}$. Compute

$$\begin{aligned} \overline{x} &= \frac{1}{m} \iint_\Sigma x\delta(x, y, z)\, d\sigma \\ &= \frac{1}{10} \iint_D x^2 y\, dA = \frac{1}{10} \int_0^{\pi/2} \int_1^3 r^4 \cos^2(\theta) \sin(\theta)\, dr\, d\theta = \frac{121}{75}. \end{aligned}$$

Finally,

$$\begin{aligned}\overline{z} &= \frac{1}{m}\iint_{\Sigma} z\delta(x,y,z)\,d\sigma = \frac{1}{10}\iint_{D}(16-x^2-y^2)xy\,dA \\ &= \frac{1}{10}\int_0^{\pi/2}\int_1^3 (16-r^2)r^3\sin(\theta)\cos(\theta)\,dr\,d\theta = \frac{331}{48}.\end{aligned}$$

The center of mass is

$$\left(\frac{121}{75}, \frac{121}{75}, \frac{331}{48}\right).$$

5. By symmetry of Σ and the density function, $\overline{x} = \overline{y} = 0$. Further, $d\sigma = \sqrt{1+4x^2+4y^2}\,dA$. For the mass, compute

$$\begin{aligned}m &= \iint_{\Sigma}\sqrt{1+4x^2+4y^2}\,d\sigma = \iint_{D}(1+4x^2+4y^2)\,dA \\ &= \int_0^{2\pi}\int_0^{\sqrt{6}}(1+4r^2+4r^2)r\,dr\,d\theta = 78\pi.\end{aligned}$$

Finally,

$$\begin{aligned}\overline{z} &= \frac{1}{m}\iint_{\Sigma} z\delta(x,y,z)\,d\sigma = \frac{1}{m}\iint_{D}(6-x^2-y^2)(1+4x^2+4y^2)\,dA \\ &= \frac{1}{m}\int_0^{2\pi}\int_0^{\sqrt{6}}(6-r^2)(1+4r^2)r\,dr\,d\theta = \frac{162\pi}{m} = \frac{27}{13}.\end{aligned}$$

6. By symmetry, $\overline{x} = \overline{y} = \overline{z}$. On Σ,

$$d\sigma = \sqrt{1+(x/z)^2+(y/z)^2}\,dA = (1/z)\,dA.$$

The mass is

$$m = \iint_{\Sigma} K\,d\sigma = K(\text{area of })\Sigma = \frac{4\pi K}{4} = K\pi.$$

Finally,

$$\overline{z} = \frac{1}{m}\iint_{\Sigma} Kz\,d\sigma = \frac{K}{K\pi}\iint_{D} z(1/z)\,dA = \frac{1}{\pi}(\text{area of })D = \frac{1}{4}.$$

7. A unit normal to the plane $x+2y+z=8$ is

$$\mathbf{n} = \frac{1}{\sqrt{6}}(\mathbf{i}+2\mathbf{j}+\mathbf{k}).$$

Then

$$\mathbf{F}\cdot\mathbf{n} = \frac{1}{\sqrt{6}}(x+2y-z).$$

On Σ, $z = 8 - x - 2y$, so

$$\mathbf{F} \cdot \mathbf{n} = \frac{1}{\sqrt{6}}(2x + 4y - 8).$$

Further, $d\sigma = \sqrt{1+4+1}\, dA = \sqrt{6}\, dA$. Therefore, the flux of $\mathbf{F}$ across Σ is

$$\iint_\Sigma \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iint_D (2x+4y-8)\, dA = \int_0^4 \int_0^{8-2y} (2x+4y-8)\, dx\, dy = \frac{128}{3}.$$

8. A unit normal to the sphere $x^2 + y^2 + z^2 = 4$ is

$$\mathbf{n} = \frac{1}{2}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}).$$

Then

$$\mathbf{F} \cdot \mathbf{n} = \frac{1}{2}(x^2 z - yz).$$

Now, $d\sigma = \sqrt{1 + (-x/z)^2 + (-y/z)^2}\, dA$. Therefore, the flux of $\mathbf{F}$ across Σ is

$$\iint_\Sigma \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iint_D (x^2 - y)\, dA.$$

Change to polar coordinates, in which D is the set of points (r, θ) with $0 \le r \le \sqrt{3}$ and $0 \le \theta \le 2\pi$. Then the flux is

$$\int_0^{2\pi} \int_0^{\sqrt{3}} (r^2 \cos^2(\theta) - r \sin(\theta)) r\, dr\, d\theta = \frac{9}{4} \int_0^{2\pi} \cos^2(\theta)\, d\theta = \frac{9\pi}{4}.$$

12.7 Lifting Green's Theorem to R^3

1. By Green's theorem,

$$\begin{aligned}
\oint_C -\varphi \frac{\partial \psi}{\partial y}\, dx + \varphi \frac{\partial \psi}{\partial x}\, dy &= \iint_D \left[\frac{\partial}{\partial x}\left(\varphi \frac{\partial \psi}{\partial x} \right) - \frac{\partial}{\partial y}\left(-\varphi \frac{\partial \psi}{\partial y} \right) \right] dA \\
&= \iint_D \left[\frac{\partial \varphi}{\partial x}\frac{\partial \psi}{\partial x} + \frac{\partial \varphi}{\partial y}\frac{\partial \psi}{\partial y} + \frac{\partial \varphi}{\partial z}\frac{\partial \psi}{\partial z} \right] dA \\
&+ \iint_D \varphi \left[\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} \right] dA \\
&= \iint_D \nabla\varphi \cdot \nabla\psi\, dA + \iint_D \varphi \nabla^2 \psi\, dA.
\end{aligned}$$

Upon rearranging terms at both ends of this equation, we obtain the requested identity.

2. Apply the result of Problem 1 to both integrals to write

$$\iint_D \varphi\nabla^2\psi\,dA = \oint_C -\varphi\frac{\partial\psi}{\partial y}\,dx + \varphi\frac{\partial\psi}{\partial x}\,dy - \iint_D \nabla\varphi\cdot\nabla\psi\,dA$$

and, interchanging φ and ψ,

$$\iint_D \psi\nabla^2\varphi\,dA = \oint_C -\psi\frac{\partial\varphi}{\partial y}\,dx + \psi\frac{\partial\varphi}{\partial x}\,dy - \iint_D \nabla\psi\cdot\nabla\varphi\,dA.$$

Subtract these equations to obtain

$$\oint_C (\varphi\nabla^2\psi - \psi\nabla^2\varphi)\,dA = \oint_C \left(\psi\frac{\partial\varphi}{\partial y}\,dx - \varphi\frac{\partial\psi}{\partial y}\right)\,dx + \oint_C \left(\varphi\frac{\partial\psi}{\partial x} - \psi\frac{\partial\varphi}{\partial x}\right)\,dy.$$

3. Under the given conditions,

$$\mathbf{N} = \frac{dy}{ds}\mathbf{i} - \frac{dx}{ds}\mathbf{j} \text{ and } \varphi_{\mathbf{N}} = \nabla\varphi\cdot\mathbf{N}.$$

Then

$$\begin{aligned}\oint_C \varphi_{\mathbf{N}}(x,y)\,ds &= \oint_C \left[\frac{\partial\varphi}{\partial x}\frac{dy}{ds} - \frac{\partial\varphi}{\partial y}\frac{dx}{ds}\right]\,ds \\ &= \oint_C -\frac{\partial\varphi}{\partial y}\,dx + \frac{\partial\varphi}{\partial x}\,dy.\end{aligned}$$

Apply Green's theorem to this line integral to obtain

$$\begin{aligned}\oint_C \varphi_{\mathbf{N}}(x,y)\,ds &= \iint_D \left[\frac{\partial}{\partial x}\left(\frac{\partial\varphi}{\partial x}\right) - \frac{\partial}{\partial y}\left(-\frac{\partial\varphi}{\partial y}\right)\right]\,dA \\ &= \iint_D \nabla^2\varphi\,dA.\end{aligned}$$

12.8 The Divergence Theorem of Gauss

1. $\nabla\cdot\mathbf{F} = 1$, so compute

$$\iiint_M \nabla\cdot\mathbf{F}\,dV = \text{volume of } M = \frac{4}{3}\pi(4^3) = \frac{256\pi}{3}.$$

2. $\nabla\cdot\mathbf{F} = 4 - 6 = -2$, so compute

$$\iiint_M \nabla\cdot\mathbf{F}\,dV = -2(\text{volume of } V) = -2\pi(2^2)(2) = -16\pi.$$

3. Since $\nabla\cdot\mathbf{F} = 0$, $\iiint_M \nabla\cdot\mathbf{F}\,dV = 0.$

4. Since $\nabla \cdot \mathbf{F} = 3x^2 + 3y^2 + 3z^2$, then

$$\iiint\limits_M \nabla \cdot \mathbf{F}\, dV = \iiint\limits_M 3(x^2 + y^2 + z^2)\, dV.$$

Convert this integral to spherical coordinates, obtaining

$$\begin{aligned}\iiint\limits_M \nabla \cdot \mathbf{F}\, dV &= \int_0^{2\pi} \int_0^{\pi} \int_0^1 3\rho^4 \sin(\varphi)\, d\rho\, d\varphi\, d\theta \\ &= \int_0^{2\pi} d\theta \int_0^{\pi} \sin(\varphi)\, d\varphi \int_0^1 3\rho^4\, d\rho = (2\pi)(2)\left(\frac{3}{5}\right) = \frac{12\pi}{5}.\end{aligned}$$

5. since $\nabla \cdot \mathbf{F} = 4$, compute

$$\iiint\limits_M \nabla \cdot \mathbf{F}\, dV = \iiint\limits_M 4\, dV = 4(\text{volume of } M) = \frac{8\pi}{3}.$$

6. $\nabla \cdot \mathbf{F} = 2 + x$, so

$$\iiint\limits_M \nabla \cdot \mathbf{F}\, dV = \int_0^3 \int_0^2 \int_0^4 (2 + x)\, dx\, dy\, dz = \left. 3(2 + x)^2\right]_0^4 = 96.$$

7. Compute $\nabla \cdot \mathbf{F} = 2(x + y + z)$, so, using cylindrical coordinates, we have

$$\iiint\limits_M \nabla \cdot \mathbf{F}\, dV = 2\int_0^{2\pi} \int_0^{\sqrt{2}} \int_r^{\sqrt{2}} (r\cos(\theta) + r\sin(\theta) + z)r\, dz\, dr\, d\theta.$$

We will do these integrations one at a time. First,

$$\int_r^{\sqrt{2}} (r^2(\cos(\theta)+\sin(\theta))+rz)\, dz = r^2(\cos(\theta)+\sin(\theta))(\sqrt{2}-r)+\frac{1}{2}r(2-r^2).$$

Next,

$$\int_0^{\sqrt{2}} \left[r^2(\cos(\theta) + \sin(\theta))(\sqrt{2} - r) + \frac{1}{2}r(2 - r^2)\right] d\,r = \frac{1}{3}(\cos(\theta)+\sin(\theta))+\frac{1}{2}.$$

Finally,

$$\int_0^{2\pi} \left(\frac{1}{3}(\cos(\theta) + \sin(\theta)) + \frac{1}{2}\right) d\theta = \pi.$$

Therefore

$$\iiint\limits_M \nabla \cdot \mathbf{F}\, dV = 2\pi.$$

8. With $\nabla \cdot \mathbf{F} = 1 + 2x$, compute

$$\iiint_M (1+2x)\,dV = \int_0^2 dz \iint_D (1+2x)\,dA,$$

where D is the region in the $x, y-$ plane described in polar coordinates by $0 \leq r \leq \sqrt{2}$ and $0 \leq \theta \leq 2\pi$. Then

$$\begin{aligned}\iiint_M (1+2x)\,dV &= 2\iint_D (1+2r\cos(\theta))r\,dr\,d\theta \\ &= 2\left[2\pi + \frac{4\sqrt{2}}{3}\int_0^{2\pi} \cos(\theta)\,d\theta\right] = 4\pi.\end{aligned}$$

9. With the given conditions on $\mathbf{F}$, Σ and M, we have

$$\iint_\Sigma (\nabla \times \mathbf{F}) \cdot \mathbf{n}\,d\sigma = \iiint_M (\nabla \cdot \nabla \times \mathbf{F})\,dV.$$

But $\nabla \cdot \nabla \times \mathbf{F} = 0$, so

$$\iint_\Sigma (\nabla \times \mathbf{F}) \cdot \mathbf{n}\,d\sigma = 0.$$

10. Apply Gauss's divergence theorem to obtain

$$\frac{1}{3}\iint_\Sigma \mathbf{R} \cdot \mathbf{n}\,d\sigma = \frac{1}{3}\iiint_M (\nabla \cdot \mathbf{R})\,dV = \frac{1}{3}\iiint_M 3\,dV = \text{ volume of } M.$$

12.9 The Integral Theorem of Stokes

1. The boundary curve C can be parametrized by $x = 2\cos(t)$, $y = 2\sin(t)$, $z = 0$ for $0 \leq t \leq 2\pi$. Further, on C,

$$\mathbf{R} = 2\cos(t)\mathbf{i} + 2\sin(t)\mathbf{j} + 0\mathbf{k}$$

so

$$\mathbf{F} \cdot d\mathbf{R} = (-16\cos^2(t)\sin^2(t) - 16\cos^2(t)\sin^2(t))\,dt = -32\cos^2(t)\sin^2(t)\,dt.$$

Then

$$\oint_C \mathbf{F} \cdot d\mathbf{R} = \int_0^{2\pi} -32\cos^2(t)\sin^2(t)\,dt = -8\pi.$$

If we use the surface integral, then evaluate $\int\int_\Sigma (\nabla \times \mathbf{F}) \cdot \mathbf{n}\,d\sigma$. Compute

$$\nabla \times \mathbf{F} = -(x^2 + y^2)\mathbf{k}.$$

Further, a normal to Σ is

$$\nabla(x^2 + y^2 + z^2) = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}.$$

A unit normal vector is

$$\mathbf{n} = \frac{1}{2}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}).$$

Finally, $d\sigma = \sqrt{1 + (x/z)^2 + (y/z)^2}\,dA = (2/z)\,dA$. Then

$$\iint_\Sigma (\nabla \times \mathbf{F}) \cdot \mathbf{n}\,d\sigma = -\iint_D (x^2 + y^2)\,dA = -\int_0^{2\pi}\int_0^2 r^3\,dr\,d\theta = -8\pi.$$

2. In this problem, C is a circle of radius 3 in the plane $z = 9$, and so has parametric equations $x = 3\cos(t)$, $y = 3\sin(t)$, $z = 9$ for $0 \le t \le 2\pi$. On C,

$$\mathbf{F} = 9\cos(t)\sin(t)\mathbf{i} + 27\sin(t)\mathbf{j} + 27\cos(t)\mathbf{k}.$$

Further,

$$d\mathbf{R} = (-3\sin(t)\mathbf{i} + 3\cos(t)\mathbf{j})\,dt.$$

Then

$$\mathbf{F} \cdot d\mathbf{R} = (-27\cos(t)\sin^2(t) + 81\cos(t)\sin(t))\,dt.$$

Immediately, $\oint_C \mathbf{F} \cdot d\mathbf{R} = 0$. Evaluation of $\int\int_\Sigma (\nabla \times \mathbf{F}) \cdot \mathbf{n} d\sigma$ involves more computation.

3. Compute $\nabla \times \mathbf{F} = \mathbf{i} + \mathbf{j} + \mathbf{k}$. A unit normal to Σ is

$$\mathbf{n} = \frac{1}{\sqrt{x^2 + y^2 + z^2}}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}).$$

This is

$$\mathbf{n} = \frac{1}{\sqrt{2}\sqrt{x^2 + y^2}}(x\mathbf{i} + y\mathbf{j} - 2\mathbf{k}).$$

Further

$$d\sigma = \sqrt{1 + \frac{x^2}{/}x^2 + y^2 + \frac{y^2}{x^2 + y^2}}\,dA = \sqrt{2}\,dA.$$

Then

$$\iint_\Sigma (\nabla \times \mathbf{F}) \cdot \mathbf{n}\,d\sigma = \iint_D \frac{x + y - z}{\sqrt{x^2 + y^2}}\,dA = \iint_D \left(\frac{x + y}{\sqrt{x^2 + y^2}} - 1\right)dA,$$

in which we used the fact that, on Σ, $z = \sqrt{x^2 + y^2}$. This integral is easily evaluated using polar coordinates, obtaining -16π.

For the line integral, parametrize C by $x = 4\sin(t)$, $y = 4\cos(t)$, $z = 4$ for $0 \le t \le 2\pi$. This orientation is consistent with the choice of the unit normal $\mathbf{n}$ on Σ. This gives us

$$\oint_C \mathbf{F} \cdot d\mathbf{R} = \int_0^{2\pi} (-16\cos(t)\sin(t) - 16\sin^2(t))\,dt = -16\pi.$$

4. The boundary curve of Σ is the circle $x^2+y^2=6$ in the $x,y-$ plane. Parametrize C by $x=\sqrt{6}\cos(t)$, $y=\sqrt{6}\sin(t)$, $z=0$, for $0\le t\le 2\pi$. Then

$$\oint_C \mathbf{F}\cdot d\mathbf{R}=\int_0^{2\pi} 6\cos^2(t)\sqrt{6}\cos(t)\,dt=6\sqrt{6}\int_0^{2\pi}(1-\sin^2(t))\cos(t)\,dt=0.$$

5. Notice that the boundary curve C is piecewise smooth and must be parametrized in three smooth curves. This is not difficult but is tedious. We therefore try to compute $\int\int_\Sigma(\nabla\times\mathbf{F})\cdot\mathbf{n}\,d\sigma$. First,

$$\nabla\times\mathbf{F}=(x-y)\mathbf{i}-y\mathbf{j}-x\mathbf{k}.$$

And

$$\mathbf{n}=\frac{1}{\sqrt{21}}(2\mathbf{i}+4\mathbf{j}+\mathbf{k}).$$

Finally, $d\sigma=\sqrt{21}\,dA$. Then

$$\iint_\Sigma(\nabla\times\mathbf{F})\cdot\mathbf{n}d\sigma=\iint_D(x-6y)\,dA=\int_0^2\int_0^{4-2y}(x-6y)\,dx\,dy=-\frac{32}{3}.$$

6. The circulation is $\oint_C\mathbf{F}\cdot d\mathbf{R}$. Take Σ to be the disk $0\le x^2+y^2\le 1$, with boundary C parametrized by $x=\cos(t)$, $y=\sin(t)$, $z=0$ for $0\le t\le 2\pi$. The proper unit normal to Σ is $\mathbf{n}=\mathbf{k}$. Now

$$\nabla\times\mathbf{F}=-za\mathbf{j}+(2xy+1)\mathbf{k}$$

so

$$(\nabla\times\mathbf{F})\cdot\mathbf{n}=2xy+1.$$

Further, $d\sigma=dA$. Then

$$\begin{aligned}\oint_C\mathbf{F}\cdot d\mathbf{R}&=\iint_\Sigma(\nabla\times\mathbf{F})\cdot\mathbf{n}\,d\sigma\\&=\iint_D(2xy+1)\,dA=\text{ area of }A=\pi\end{aligned}$$

since $\int\int_D 2xy\,dA=0$.

7. Compute

$$\nabla\times\mathbf{F}=-\mathbf{i}-\mathbf{j}-\mathbf{k}.$$

A normal to the surface is

$$\mathbf{N}=\mathbf{i}+4\mathbf{j}+\mathbf{k}$$

and the unit normal is

$$\mathbf{n}=\frac{1}{\sqrt{18}}(\mathbf{i}+4\mathbf{j}+\mathbf{k}).$$

Here C is the boundary of the part of the plane $x+4y+z=12$ in the first octant, consisting of three straight line segments: the line from $(0,0,12$ to $(12,0,0)$, then from $(12,0,0)$ to $(0,3,0)$, then from $(0,3,0)$ to $(0,0,12)$. We may think of this portion of the plane in the first octant as having equation $z=12-x-4y$, with (x,y) varying over the triangle D bounded by the segment $[0,12]$ on the $x-$ axis, the segment $[0,3]$ on the $y-$ axis, and the line $x+4y=12$. D has area $(1/2)(12)(3)=18$.

By Stokes's theorem, the circulation is

$$\oint_C \mathbf{F}\cdot\mathbf{T}\,ds = \iint_D (\nabla\times\mathbf{F})\cdot\mathbf{n}d\sigma.$$

Now

$$(\nabla\times\mathbf{F})\cdot\mathbf{n} = -\frac{6}{\sqrt{18}}$$

and

$$d\sigma = \|\ \mathbf{N}\ \|\ dx\,dy = \sqrt{18}$$

so

$$\begin{aligned}\iint_D (\nabla\times\mathbf{F})\cdot\mathbf{n}d\sigma &= \iint_D \frac{-6}{\sqrt{18}}\sqrt{18}\,dx\,dy\\ &= -6(\text{ area of } D) = -6(18) = -108.\end{aligned}$$

12.10 Curvilinear Coordinates

In these problems, the scale factors may be denoted h_1, h_2, h_3 or, h_u, h_v, h_w if the orthogonal coordinates are denotes u,v,w. The unit vectors along the axes in the new coordinate system (the orthogonal curvilinear coordinates version of $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$), are

$$\mathbf{u}_\alpha = \frac{1}{h_\alpha}\left(\frac{\partial x}{\partial q_\alpha}\mathbf{i} + \frac{\partial y}{\partial q_\alpha}\mathbf{j} + \frac{\partial z}{\partial z_\alpha}\mathbf{k}\right).$$

if the orthogonal coordinates are denotes q_1, q_2, q_3. Sometimes mildly clumsy notation is tolerated in the context of curvilinear coordinates. For example, if $q_1 = u$, we might write

$$\mathbf{u}_{q_1} = \mathbf{u}_u,$$

in which we have to use the boldface notation to distinguish the vector $\mathbf{u}$ from the coordinate u.

1. In cylindrical coordinates, we often see the notations

$$u_1 = u_r = r, u_2 = u_\theta = \theta, u_3 = u_z = z.$$

From Example 12.31, we know that, for cylindrical coordinates, the scale factors are

$$h_1 = h_r = 1, h_2 = h_\theta = r, h_3 = h_z = 1.$$

Given $g(r, \theta, z)$, we can compute the gradient and Laplacian in cylindrical coordinates as

$$\nabla g = \frac{\partial g}{\partial r}\mathbf{u}_r + \frac{1}{r}\frac{\partial \mathbf{g}}{\partial \theta}\mathbf{u}_\theta + \frac{\partial g}{\partial z}\mathbf{u}_z$$

and

$$\begin{aligned}\nabla^2 g &= \frac{1}{r}\left(\frac{\partial}{\partial r}\left(r\frac{\partial g}{\partial r}\right) + \frac{\partial}{\partial \theta}\left(\frac{1}{r}\frac{\partial g}{\partial \theta}\right) + \frac{\partial}{\partial z}\left(r\frac{\partial g}{\partial z}\right)\right)\\ &= \frac{1}{r}\frac{\partial g}{\partial r} + \frac{\partial^2 g}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 g}{\partial \theta^2} + \frac{\partial^2 g}{\partial z^2}.\end{aligned}$$

Given a vector field $\mathbf{F}(r, \theta, z)$ in cylindrical coordinates, write

$$\mathbf{F} = f_1\mathbf{u}_1 + f_2\mathbf{u}_2 + f_3\mathbf{u}_3.$$

The divergence is given by

$$\begin{aligned}\nabla \cdot \mathbf{F} &= \frac{1}{r}\left(\frac{\partial}{\partial r}(f_1 r) + \frac{\partial}{\partial \theta}(f_2) + \frac{\partial}{\partial z}(r f_1)\right)\\ &= \frac{1}{r}\left(f_1 + r\frac{\partial f_1}{\partial r} + \frac{\partial f_2}{\partial \theta} + r\frac{\partial f_3}{\partial r}\right).\end{aligned}$$

Finally, the curl is given by

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{u}_r & \mathbf{u}_\theta & \mathbf{u}_z \\ \partial/\partial r & \partial/\partial \theta & \partial/\partial z \\ f_1 & r f_2 & f_3 \end{vmatrix}.$$

2. Elliptic cylindrical coordinates u, v, z are defined by

$$x = a\cosh(u)\cos(v), a\sinh(u)\sin(v), z = z,$$

for $u \geq 0$ and $0 \leq v < 2\pi$, and z any real number. Here z is the usual rectangular coordinate and a is a positive constant. To see the coordinate surfaces, first suppose $u = k$, constant. Then

$$x = a\cosh(k)\cos(v), y = a\sinh(k)\sin(v), z = z.$$

If $k > 0$, then

$$\frac{x^2}{a^2\cosh^2(k)} + \frac{y^2}{a^2\sinh^2(k)} = 1,$$

and in $3-$ space this is an elliptical cylinder cutting the $x, y-$ plane in the given ellipse. If $v = k$ we get

$$\frac{x^2}{a^2\cos^2(k)} - \frac{y^2}{a^2\sin^2(k)} = \cosh^2(u) - \sinh^2(u) = 1$$

provided that $\cos(k) \neq 0$ and $\sin(k) \neq 0$. A graph of this equation in $3-$ space is a hyperbolic cylinder. Finally, the surfaces $z = k$ are planes parallel to the $x, y-$ plane.

Now compute the scale factors. First,

$$h_u = \sqrt{a^2 \sinh^2(u)\cos^2(v) + a^2\cosh^2(u)\sin^2(v)} = h_v$$

and

$$h_z = 1.$$

If $g(u, v, w)$ is a scalar-valued function, then

$$\nabla g = \frac{1}{h_1}\frac{\partial g}{\partial u}\mathbf{u}_u + \frac{1}{h_2}\frac{\partial g}{\partial v}\mathbf{u}_v + \frac{\partial g}{\partial z}\mathbf{u}_z,$$

where

$$\begin{aligned}
\mathbf{u}_u &= \frac{a}{h_1}(\sinh(u)\cos(v)\mathbf{i} + \cosh(u)\sin(v)\mathbf{j}),\\
\mathbf{u}_u &= \frac{a}{h_2}(-\cosh(u)\sin(v)\mathbf{i} + \sinh(u)\cos(v)\mathbf{j}),\\
\mathbf{u}_z &= \mathbf{k}.
\end{aligned}$$

The Laplacian of g is given by

$$\nabla^2 g = \frac{1}{h_1^2}\left(\frac{\partial^2 g}{\partial u^2} + \frac{\partial^2 g}{\partial v^2}\right) + \frac{\partial^2 g}{\partial z^2}.$$

The divergence and curl of a vector field $\mathbf{F}(u, v, z)$, with component functions f_1, f_2, f_3, are

$$\nabla \cdot \mathbf{F} = \frac{1}{h_1^2}\left(\frac{\partial}{\partial u}(f_1 h_1) + \frac{\partial}{\partial v}(f_2 h_2)\right) + \frac{\partial g}{\partial z},$$

and

$$\begin{aligned}
\nabla \times \mathbf{F} = &\left(\frac{1}{h_1}\frac{\partial f_3}{\partial v} - \frac{\partial f_2}{\partial z}\right)\mathbf{u}_u\\
&+ \left(\frac{\partial f_1}{\partial z} - \frac{1}{h_1}\frac{\partial f_3}{\partial u}\right)\mathbf{u}_v\\
&+ \frac{1}{h_1^2}\left(\frac{\partial}{\partial u}(f_2 h_2) - \frac{\partial}{\partial v}(f_1 h_1)\right)\mathbf{u}_z.
\end{aligned}$$

3. Bipolar coordinates are defined by

$$x = \frac{a\sinh(v)}{\cosh(v) - \cos(u)}, y = \frac{a\sin(u)}{\cosh(v) - \cos(u)}, z = z.$$

It is routine to compute the scale factors

$$h_u = \frac{a}{\cosh(v) - \cos(u)} = h_v, h_z = 1.$$

If $g(u,v,z)$ is a scalar function, then

$$\nabla g = \frac{1}{a}(\cosh(v)-\cos(u))\frac{\partial g}{\partial u}\mathbf{u}_u + \frac{1}{a}(\cosh(v)-\cos(u))\frac{\partial g}{\partial v}\mathbf{u}_v + \frac{\partial g}{\partial z}\mathbf{u}_z$$

and

$$\nabla^2 g = \frac{1}{a^2}(\cos(v)-\cos(u))^2\left[\frac{\partial^2 g}{\partial u^2} + \frac{\partial^2 g}{\partial v^2} + \frac{\partial}{\partial z}\left(\frac{a^2}{(\cosh(v)-\cos(u))^2}\frac{\partial g}{\partial z}\right)\right].$$

And, if $\mathbf{F}(u,v,z)$ is a vector field, then

$$\nabla\cdot\mathbf{F} = \frac{1}{a^2}(\cosh(v)-\cos(u))^2\left[\frac{\partial}{\partial u}\left(\frac{a}{\cosh(v)-\cos(u)}f_1\right)\right.$$
$$\left. + \frac{\partial}{\partial v}\left(\frac{a}{\cosh(v)-\cos(u)}f_2\right) + \frac{\partial}{\partial z}\left(\left(\frac{a}{\cosh(v)-\cos(u)}\right)^2 f_3\right)\right]$$

and

$$\nabla\times\mathbf{F} = \frac{(\cosh(v)-\cos(u))^2}{a^2}\begin{vmatrix} h_1\mathbf{u}_u & h_2\mathbf{u}_v & \mathbf{u}_z \\ \partial/\partial u & \partial/\partial v & \partial/\partial z \\ h_u f_1 & h_v f_2 & f_3 \end{vmatrix}.$$

4. Parabolic cylindrical coordinates are given by

$$x = uv, y = \frac{1}{2}(u^2-v^2), z = z.$$

We find that the scaling factors are

$$h_u = h_v = \sqrt{u^2+v^2}, h_z = 1.$$

If $g(u,v,z)$ is a scalar field, then

$$\nabla g = \frac{1}{\sqrt{u^2+v^2}}\frac{\partial g}{\partial u}\mathbf{u}_u + \frac{1}{\sqrt{u^2+v^2}}\frac{\partial g}{\partial v}\mathbf{u}_v + \frac{\partial g}{\partial z}\mathbf{u}_z$$

and

$$\nabla^2 g = \frac{1}{u^2+v^2}\left(\frac{\partial^2 g}{\partial u^2} + \frac{\partial^2 g}{\partial v^2}\right) + \frac{\partial}{\partial z}\left((u^2+v^2)\frac{\partial g}{\partial z}\right).$$

And, if $\mathbf{F}(u,v,z)$ is a vector field, then

$$\nabla\cdot\mathbf{F} = \frac{1}{u^2+v^2}\left(\frac{\partial}{\partial u}(\sqrt{u^2+v^2}f_1)\right.$$
$$\left. + \frac{\partial}{\partial v}(\sqrt{u^2+v^2}f_2) + \frac{\partial}{\partial z}((u^2+v^2)f_3)\right).$$

and

$$\nabla\times\mathbf{F} = \frac{1}{u^2+v^2}\begin{vmatrix} \sqrt{u^2+v^2}\mathbf{u}_u & \sqrt{u^2+v^2}\mathbf{u}_v & \mathbf{u}_z \\ \partial/\partial u & \partial/\partial v & \partial/\partial z \\ \sqrt{u^2+v^2}f_1 & \sqrt{u^2+v^2}f_2 & f_3 \end{vmatrix}.$$

Chapter 13

Fourier Series

13.1 Why Fourier Series?

1. Figure 13.1 shows graphs of $f(x)$ and $S_2(x)$, while Figure 13.2 has $f(x)$ and $S_{10}(x)$. $S_2(x)$ is not very close to $f(x)$, while the function and $S_{10}(x)$ appear indistinguishable in the scale of the graphs. As $N \to \infty$, $S_N(x) \to f(x)$ for all x in $[0, \pi]$, as is shown in Section 13.2.

   In general, convergence of Fourier series can be slow, and it might take large numbers of terms before the partial sums of the series close in on the function.

2. Let $s(x) = k\sin(nx)$. If $p(x)$ has degree k, then the $k+1$ derivative of $p(x)$ is identically zero on $[0, \pi]$, while the $k+1$ derivative of $s(x)$ is a multiple of $\sin(nx)$ or $\cos(nx)$, which is not identically zero on this interval.

3. The argument of Problem 2 can be applied to this problem.

13.2 The Fourier Series of a Function

1. The Fourier series of $f(x) = 4$ on $[-3, 3]$ has the form

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos(n\pi x/3) + b_n \sin(n\pi x/3)).$$

   We must compute the coefficients. First, because f is an even function, each $b_n = 0$. Next,

$$a_0 = \frac{2}{3}\int_0^3 4\,dx = 8,$$

   and, for $n = 1, 2, \cdots$,

$$a_n = \frac{2}{3}\int_0^3 4\cos(n\pi x/3)\,dx = 0.$$

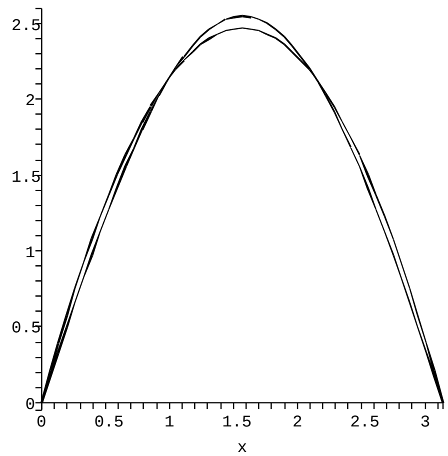

Figure 13.1: $f(x)$ and $S_2(x)$ in Problem 1, Section 13.1.

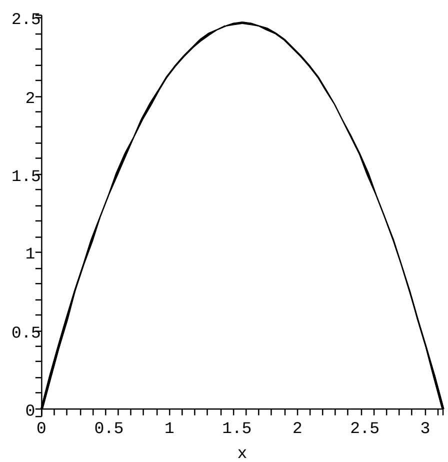

Figure 13.2: $f(x)$ and $S_{10}(x)$ in Problem 1, Section 13.1.

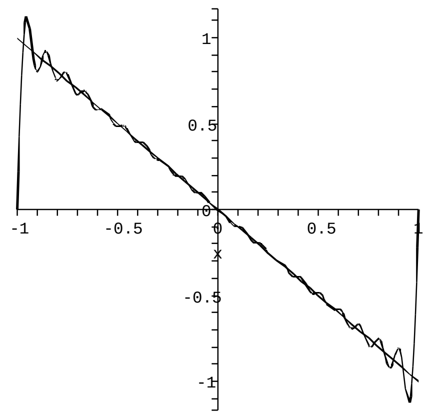

Figure 13.3: Twentieth partial sum of the Fourier series in Problem 2.

This Fourier series has just one term, 4 itself. Of course, this converges to 4 on $[-3, 3]$.

2. The Fourier series of $f(x) = -x$ on $[-1, 1]$ has the form

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos(n\pi x) + b_n \sin(n\pi x)).$$

Since f is an odd function, each $a_n = 0$. Compute

$$b_n = 2\int_0^1 -x \sin(n\pi x)\, dx = \frac{2}{n\pi}(-1)^n$$

for $n = 1, 2, \cdots$. The Fourier series is

$$\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin(n\pi x).$$

This series converges to $-x$ for $-1 < x < 1$, and to 0 at $x = \pm 1$. Figure 13.3 shows a graph of $f(x)$ compared to the twentieth partial sum of its Fourier series on $[-1, 1]$.

3. Since $f(x) = \cosh(\pi x)$ is an even function, each $b_n = 0$. Further,

$$a_0 = \int_0^1 \cosh(\pi x)\, dx = \frac{1}{\pi}\sinh(\pi)$$

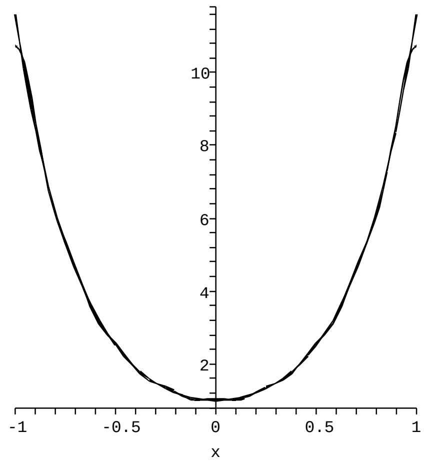

Figure 13.4: Eighth partial sum of the Fourier series in Problem 3.

and, for $n = 1, 2, \cdots$,,

$$a_n = 2\int_0^1 \cosh(\pi x)\cos(n\pi x)\,dx = \frac{2\sinh(\pi)}{\pi}\frac{(-1)^n}{1+n^2}.$$

The Fourier series is

$$\frac{1}{\pi}\sinh(\pi) + \frac{2}{\pi}\sinh(\pi)\sum_{n=1}^{\infty}\frac{(-1)^n}{1+n^2}\cos(n\pi x).$$

This series converges to $\cosh(\pi x)$ for $-1 \le x \le 1$. Figure 13.4 shows the function and eighth partial sum of this Fourier series.

For Problems 4 through 10, we give the Fourier series, analyze its convergence, and show a graph of one of its partial sums.

4. The series is

$$\frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos((2n-1)\pi x/2),$$

converging to $1 - |x|$ for $-2 \le x \le 2$. Figure 13.5 shows a graph of this function and the fifth partial sum of its Fourier series.

5. The series is

$$\frac{16}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\sin((2n-1)x),$$

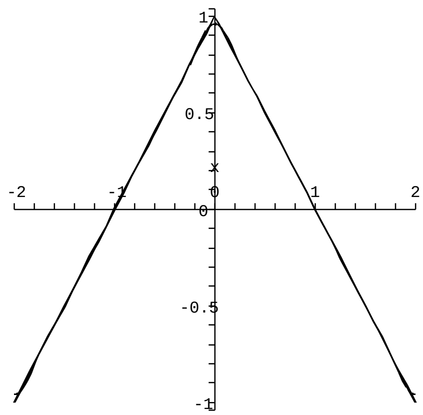

Figure 13.5: Fifth partial sum of the Fourier series in Problem 4.

converging to

$$\begin{cases} -4 & \text{for } -\pi < x < 0, \\ 4 & \text{for } 0 < x < 4, \\ 0 & \text{for } 0, \pi, -\pi. \end{cases}$$

Figure 13.6 shows a graph of this function and the twentieth partial sum of its Fourier series.

6. Since $f(x)$ is odd and periodic of period π, then $f(x) = \sin(2x)$ is its own Fourier series (the Fourier series has just one term, the function itself).

7. The Fourier series is

$$\frac{13}{3} + \sum_{n=1}^{\infty} (-1)^n \left[\frac{16}{(n\pi)^2} \cos(n\pi x/2) + \frac{4}{n\pi} \sin(n\pi x/2) \right],$$

converging to $f(x)$ for $-2 < x < 2$, and, at 2 and at -2, to

$$\frac{1}{2}(f(2+) + f(2-)) = \frac{1}{2}(9+5) = 7.$$

A graph of $f(x)$ and the twelfth partial sum of this Fourier series is shown in Figure 13.7.

8. The Fourier series is

$$\frac{71}{6} + \sum_{n=1}^{\infty} (a_n \cos(n\pi x/5) + b_n \sin(n\pi x/5)),$$

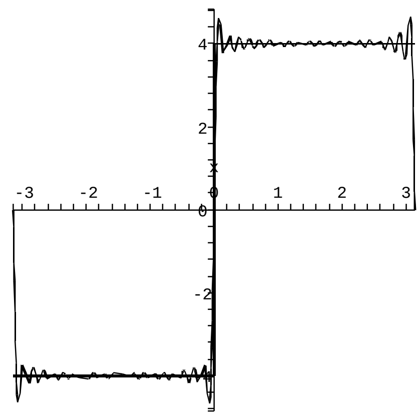

Figure 13.6: Twentieth partial sum of the Fourier series in Problem 5.

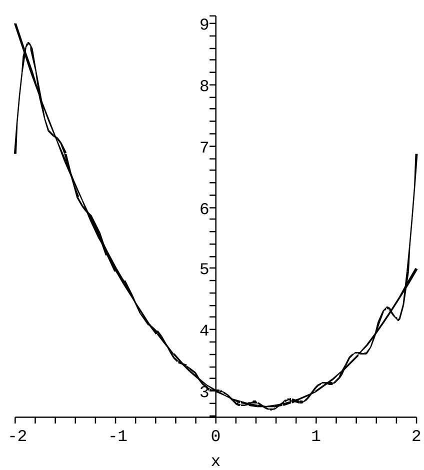

Figure 13.7: Twelfth partial sum of the Fourier series in Problem 7.

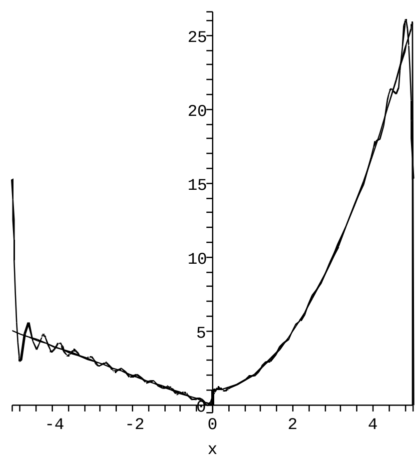

Figure 13.8: Twenty-fifth partial sum of the Fourier series in Problem 8.

where

$$a_n = \frac{25}{(n\pi)^2}(11(-1)^n - 1)$$

and

$$b_n = \frac{5}{n\pi}(1 - 21(-1)^n) + \frac{25}{(n\pi)^3}((-1)^n - 10).$$

The series converges to

$$\begin{cases} -x & \text{for } -5 < x < 0, \\ 1 + x^2 & \text{for } 0 < x < 5, \\ 1/2 & \text{for } x = 0, \\ 31/2 & \text{for } x = \pm 5. \end{cases}$$

A graph of $f(x)$ and the twenty-fifth partial sum of this Fourier series is shown in Figure 13.8.

9. The Fourier series of $f(x)$ on $[-\pi, \pi]$ is

$$\frac{3}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\sin((2n-1)x).$$

This converges to

$$\begin{cases} 1 & \text{for } -\pi < x < 0, \\ 2 & \text{for } 0 < x < \pi, \\ 3/2 & \text{for } x = 0, \pi, \text{ and } -\pi. \end{cases}$$

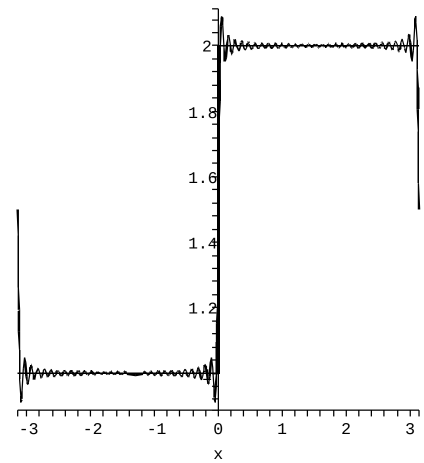

Figure 13.9: Thirtieth partial sum of the Fourier series in Problem 9.

Figure 13.9 shows the function and the thirtieth partial sum of this Fourier series.

10. The Fourier series is

$$\frac{2}{\pi} - \sin(x) - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n}{4n^2-1}\cos(nx),$$

converging to $\cos(x/2) - \sin(x)$ for $-\pi < x < \pi$ and to 0 for $x = \pm\pi$.

Figure 13.10 shows this function and the fourth partial sum of the Fourier series.

11. The Fourier series is

$$\frac{\sin(3)}{3} + 6\sin(3)\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2\pi^2 - 9}\cos\left(\frac{n\pi x}{3}\right),$$

converging to $\cos(x)$ on $[-3, 3]$.

Figure 13.11 compares the function to the fifth partial sum of this series.

12. The Fourier series is

$$\frac{3}{4} - \sum_{n=1}^{\infty}\left[\frac{1-(-1)^n}{n^2\pi^2}\cos(n\pi x) + \left(\frac{1-2(-1)^n}{n\pi}\right)\sin(n\pi x)\right],$$

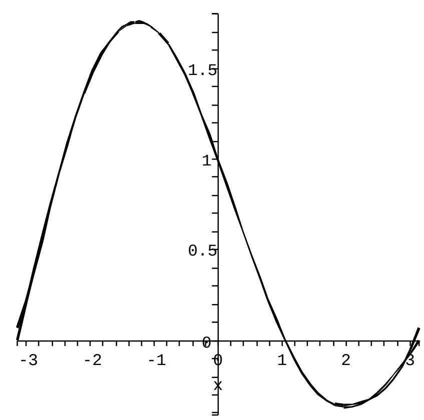

Figure 13.10: Fourth partial sum of the Fourier series in Problem 10.

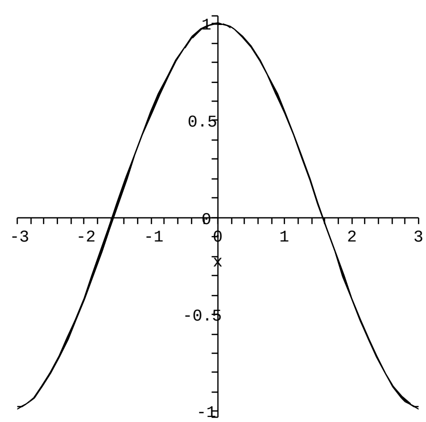

Figure 13.11: Fifth partial sum of the Fourier series in Problem 11.

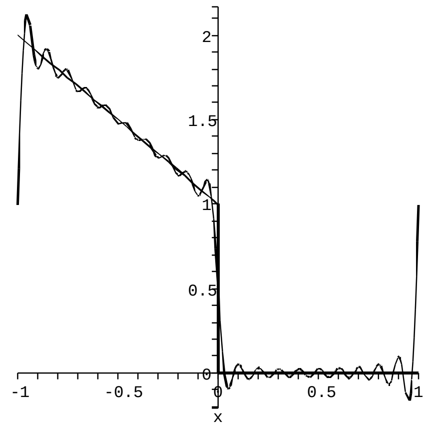

Figure 13.12: Twentieth partial sum of the Fourier series in Problem 12.

converging to

$$\begin{cases} 1-x & \text{for } -1 < x < 0, \\ 1/2 & \text{for } x = 0, \\ 1 & \text{for } x = -1, 1. \end{cases}$$

Figure 13.12 shows this function and the twentieth partial sum of its Fourier series on $[-1, 1]$.

For each of Problems 13 - 19, the convergence theorem is used to determine the sum of the Fourier series of the function on the interval. It is not necessary to write the series to obtain this information.

13. The Fourier series of $f(x)$ on this interval converges to

$$\begin{cases} 3/2 & \text{for } x = \pm 3, \\ 2x & \text{for } -3 < x < -2, \\ -2 & \text{for } x = -2, \\ 0 & \text{for } -2 < x < 1, \\ 1/2 & \text{for } x = 1, \\ x^2 & \text{for } 1 < x < 3. \end{cases}$$

14. The Fourier series converges to

$$\begin{cases} (1-2\pi)/2 & \text{for } x = \pm\pi, \\ 3/2 & \text{for } x = 1, \\ 2x-2 & \text{for } -\pi < x < 1, \\ 3 & \text{for } 1 < x < \pi. \end{cases}$$

15. The Fourier series converges to

$$\begin{cases} (2+\pi^2)/2 & \text{for } x = \pm\pi, \\ x^2 & \text{for } -\pi < x < 0, \\ 1 & \text{for } x = 0, \\ 2 & \text{for } 0 < x < \pi. \end{cases}$$

16. The Fourier series converges to

$$\begin{cases} (\cos(2)+\sin(2))/2 & \text{for } x = \pm 2, \\ \cos(x) & \text{for } -2 < x < 0, \\ 1/2 & \text{for } x = 0, \\ \sin(x) & \text{for } 0 < x < 2. \end{cases}$$

17. The Fourier series converges to

$$\begin{cases} -1 & \text{for } -4 < x < 0, \\ 0 & \text{for } x = \pm 4 \text{ and for } x = 0, \\ 1 & \text{for } 0 < x < 4. \end{cases}$$

18. The Fourier series converges to

$$\begin{cases} 1 & \text{for } x = -1, 1 \text{ and for } 1/2 < x < 3/4, \\ 0 & \text{for } -1 < x < 1/2, \\ 2 & \text{for } 3/4 < x < 1, \\ 1/2 & \text{for } x = 1/2, \\ 3/2 & \text{for } x = 3/4. \end{cases}$$

19. The Fourier series converges to

$$\begin{cases} -1 & \text{for } x = -4, 4, \\ 3/2 & \text{for } x = -2, \\ 5/2 & \text{for } x = 2, \\ f(x) & \text{for all other } x \text{ in } [-4, 4]. \end{cases}$$

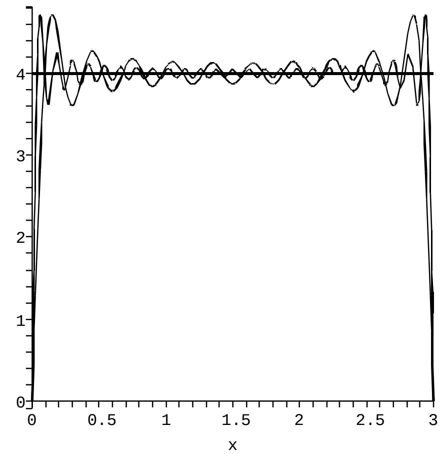

Figure 13.13: Partial sums of the sine series in Problem 1, Section 13.3.

13.3 Sine and Cosine Series

1. The cosine expansion is 4, just the constant term. The sine expansion is

$$\frac{16}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\sin((2n-1)\pi x/3),$$

converging to 4 if $x = 0, 3$ and to 4 if $0 < x < 3$. Figure 13.13 shows the tenth and twenty-fifth partial sums of this series compared to the function.

2. The cosine series is

$$-\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{2n-1}\cos((2n-1)\pi x/2),$$

converging to

$$\begin{cases} 1 & \text{for } 0 \leq x < 1, \\ 0 & \text{for } x = 1, \\ -1 & \text{for } 1 < x \leq 2. \end{cases}$$

Figure 13.14 shows a graph of the function and the tenth and twentieth partial sums of this expansion.

The sine series is

$$\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}(1+(-1)^n - 2\cos(n\pi/2))\sin(n\pi x/2),$$

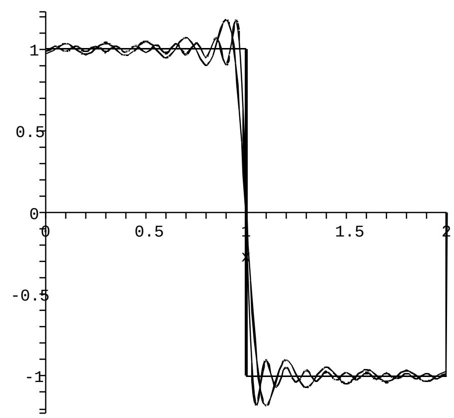

Figure 13.14: Partial sums of the cosine series in Problem 2, Section 13.3.

converging to

$$\begin{cases} 1 & \text{for } 0 < x < 1, \\ 0 & \text{for } x = 0, 1, 2, \\ -1 & \text{for } 1 < x < 2. \end{cases}$$

Figure 13.15 is a graph of $f(x)$ and the tenth and sixty-fifth partial sums of this sine expansion.

3. The cosine series is

$$\frac{1}{2}\cos(x) - \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n(2n-1)}{(2n-3)(2n+1)}\cos((2n-1)x/2),$$

converging to

$$\begin{cases} 0 & \text{for } 0 \le x < \pi, \\ -1/2 & \text{for } x = \pi, \\ \cos(x) & \text{for } \pi < x < 2\pi, \\ 0 & \text{for } x = 2\pi. \end{cases}$$

Figure 13.16 shows a graph of the function and the fifteenth partial sum of this cosine expansion.

The sine series is

$$-\frac{2}{3\pi}\sin(x/2) - \sum_{n=3}^{\infty} \frac{2n}{(n^2-4)\pi}((-1)^n + \cos(n\pi/2))\sin(nx/2),$$

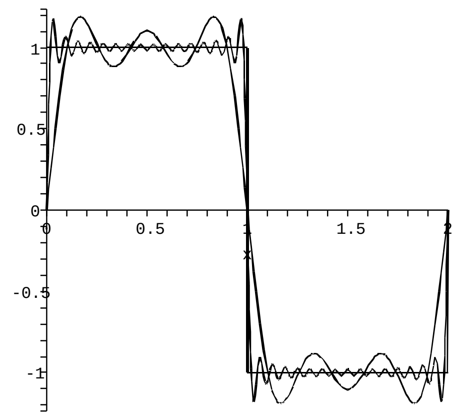

Figure 13.15: Partial sums of the sine series in Problem 2, Section 13.3.

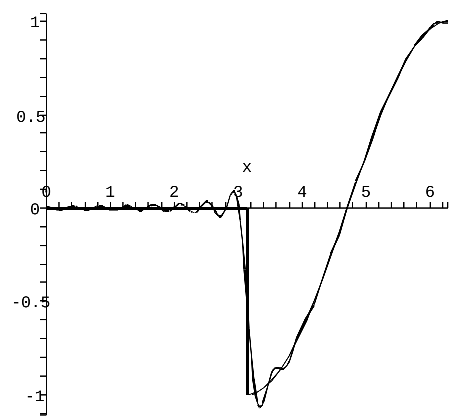

Figure 13.16: Partial sum of the cosine series in Problem 3, Section 13.3.

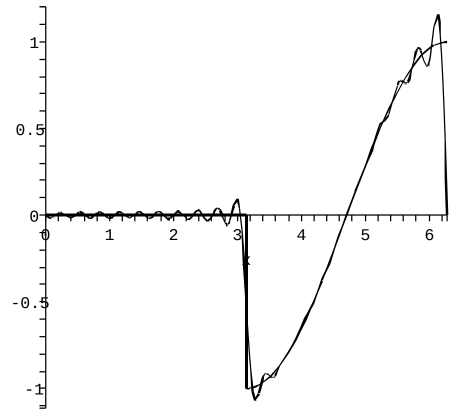

Figure 13.17: Partial sum of the sine series in Problem 3, Section 13.3.

converging to

$$\begin{cases} 0 & \text{for } 0 \le x < \pi, \\ -1/2 & \text{for } x = \pi, \\ \cos(x) & \text{for } \pi < x < 2\pi, \\ 0 & \text{for } x = 2\pi. \end{cases}$$

Figure 13.17 is a graph of the function and the fortieth partial sum of this sine series.

4. The cosine series is

$$1 - \frac{8}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \cos((2n-1)\pi x),$$

converging to $2x$ for $0 \le x \le 1$. Figure 13.18 is the fifth partial sum of this cosine series for $f(x)$.

The sine series is

$$-\frac{4}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin(n\pi x),$$

converging to $2x$ if $0 \le x < 1$ and to 0 for $x = 1$. Figure 13.19 is the fiftieth partial sum of this sine expansion.

5. The cosine series is

$$\frac{4}{3} + \frac{16}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos(n\pi x/2),$$

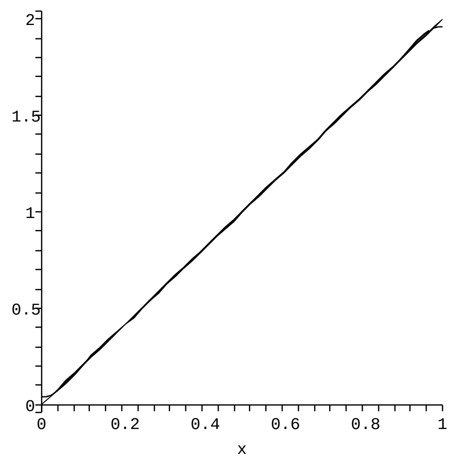

Figure 13.18: Fifth partial sum of the cosine expansion in Problem 4, Section 13.3.

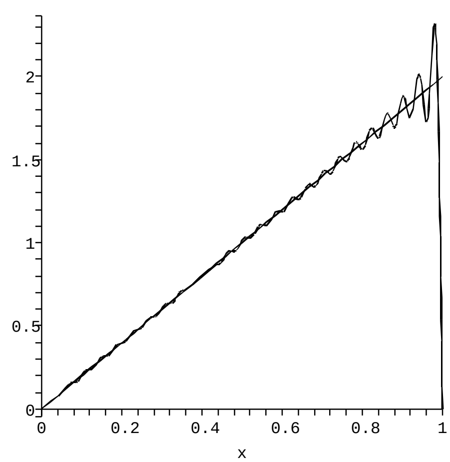

Figure 13.19: Fiftieth partial sum of the sine expansion in Problem 4, Section 13.3.

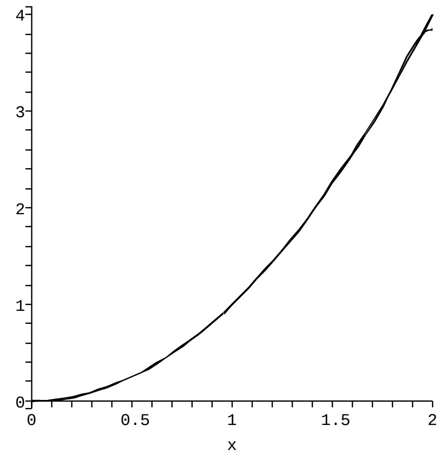

Figure 13.20: Tenth partial sum of the cosine expansion in Problem 5, Section 13.3.

converging to x^2 for $0 \leq x \leq 2$. Figure 13.20 compares $f(x)$ to the tenth partial sum of this cosine expansion.

The sine expansion is

$$-\frac{8}{\pi}\sum_{n=1}^{\infty}\left[\frac{(-1)^n}{n}+\frac{2(1-(-1)^n)}{n^3\pi^2}\right]\sin(n\pi x/2),$$

converging to x^2 for $0 \leq x < 2$ and to 0 for $x = 2$. Figure 13.21 shows the fiftieth partial sum of this sine expansion.

6. The cosine series is

$$-1-e^{-1}+2\sum_{n=1}^{\infty}\frac{1-(-1)^n e^{-1}}{1+n^2\pi^2}\cos(n\pi x),$$

converging to e^{-x} for $0 \leq x \leq 1$. Figure 13.22 shows the tenth partial sum of this cosine series.

The sine series is

$$2\pi\sum_{n=1}^{\infty}\left[\frac{n}{1+n^2\pi^2}(1-(-1)^n e^{-1})\right]\sin(n\pi x),$$

converging to e^{-x} for $0 < x < 1$ and to 0 for $x = 0, 1$. Figure 13.23 compares $f(x)$ with the sixtieth partial sum of this sine expansion.

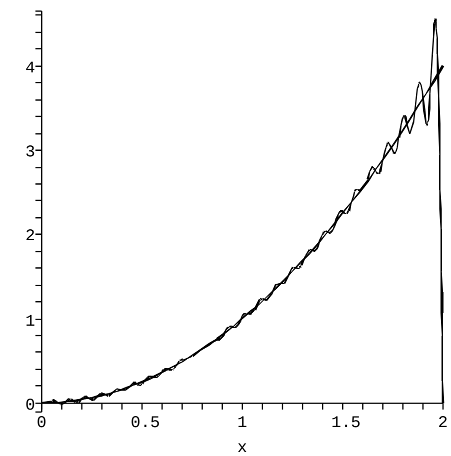

Figure 13.21: Fiftieth partial sum of the sine expansion in Problem 5, Section 13.3.

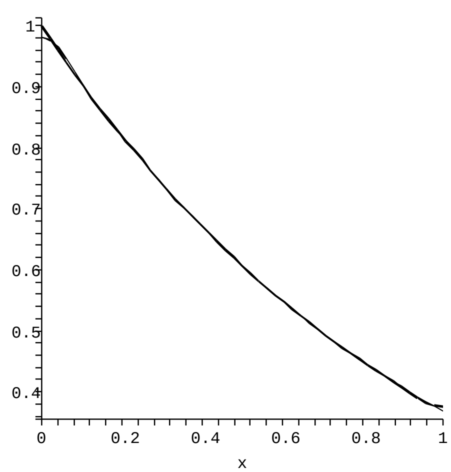

Figure 13.22: Tenth partial sum of the cosine expansion in Problem 6, Section 13.3.

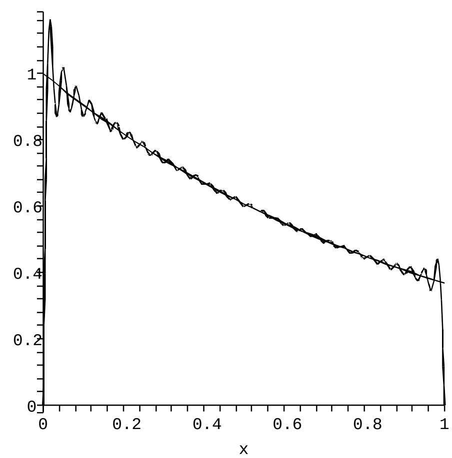

Figure 13.23: Sixtieth partial sum of the sine expansion in Problem 6, Section 13.3.

7. The cosine expansion is

$$\frac{1}{2}+\sum_{n=1}^{\infty}\left[\frac{4}{n\pi}\sin(n\pi/3)+\frac{12}{n^2\pi^2}\cos(2n\pi/3)-\frac{6}{n^2\pi^2}(1+(-1)^n)\right]\cos(n\pi x/3),$$

converging to

$$\begin{cases} x & \text{for } 0\le x<2, \\ 1 & \text{for } x=2 \\ 2-x & \text{for } 2<x\le 3. \end{cases}$$

Figure 13.25 compares $f(x)$ with the fortieth partial sum of this cosine series.

The sine expansion is

$$\sum_{n=1}^{\infty}\left[\frac{12}{n^2\pi^2}\sin(2n\pi/3)-\frac{4}{n\pi}\cos(2n\pi/3)+\frac{2}{n\pi}(-1)^n\right]\sin(n\pi x/3),$$

converging to

$$\begin{cases} x & \text{for } 0\le x<2, \\ 1 & \text{for } x=2, \\ 2-x & \text{for } 2<x<3, \\ 0 & \text{for } x=3. \end{cases}$$

Figure 13.24 shows the fifty-fifth partial sum of this sine series.

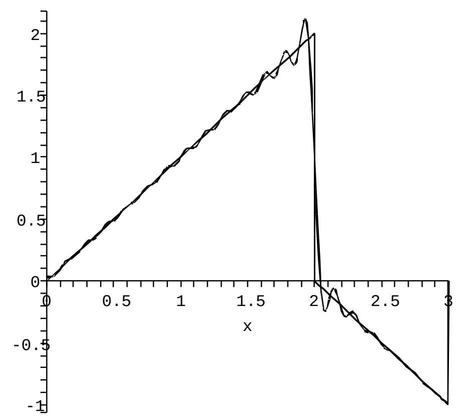

Figure 13.24: Fortieth partial sum of the cosine expansion in Problem 7, Section 13.3.

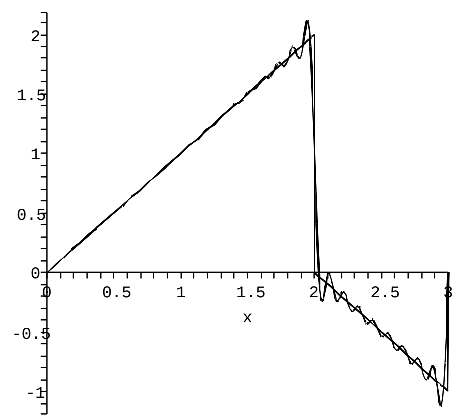

Figure 13.25: Fifty-fifth partial sum of the sine expansion in Problem 7, Section 13.3.

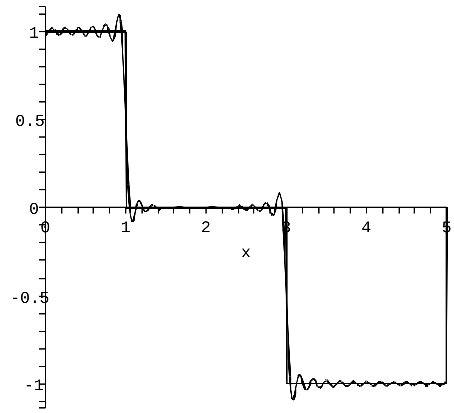

Figure 13.26: Sixtieth partial sum of the cosine expansion in Problem 8, Section 13.3.

8. The cosine expansion is

$$-\frac{1}{5}+\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\cos(n\pi/5)\sin(2n\pi/5)\cos(n\pi x/5),$$

converging to

$$\begin{cases} 1 & \text{for } 0 \le x < 1, \\ 1/2 & \text{for } x = 1, \\ 0 & \text{for } 1 < x < 3, \\ -1/2 & \text{for } x = 3, \\ -1 & \text{for } 3 < x < 5. \end{cases}$$

Figure 13.26 shows the sixtieth partial sum of this cosine expansion.

The sine expansion is

$$\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n}(1+(-1)^n-2\cos(n\pi/5)\cos(2n\pi/5))\sin(n\pi x/5),$$

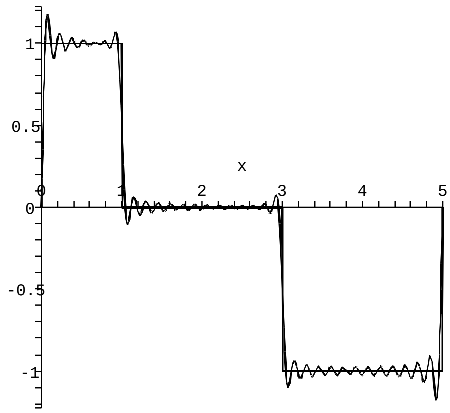

Figure 13.27: Sixty-fifth partial sum of the sine expansion in Problem 8, Section 13.3.

converging to

$$\begin{cases} 1 & \text{for } 0 < x < 1, \\ 1/2 & \text{for } x = 1, \\ 0 & \text{for } 1 < x < 3 \text{ or } x = 0 \text{ or } x = 5, \\ -1/2 & \text{for } x = 3, \\ -1 & \text{for } 3 < x < 5. \end{cases}$$

Figure 13.27 shows the sixty-fifth partial sum of this sine expansion.

9. The cosine expansion is

$$\frac{5}{6} + \frac{16}{\pi^2} \sum_{n=1}^{\infty} \left[\frac{1}{n^2} \cos\left(\frac{n\pi}{4}\right) - \frac{4}{n^3\pi} \sin\left(\frac{n\pi}{4}\right) \right] \cos\left(\frac{n\pi x}{4}\right)$$

and this converges to x^2 if $0 \le x \le 1$ and to 1 if $1 < x \le 4$. Figure 13.28 shows the tenth partial sum of this cosine expansion, compared to a graph of the function.

The sine expansion is

$$\sum_{n=1}^{\infty} \left[\frac{16}{n^2\pi^2} \sin\left(\frac{n\pi}{4}\right) + \frac{64}{n^3\pi^3} \left(\cos\left(\frac{n\pi}{4}\right) - 1 \right) - \frac{2(-1)^n}{n\pi} \right] \sin\left(\frac{n\pi x}{4}\right),$$

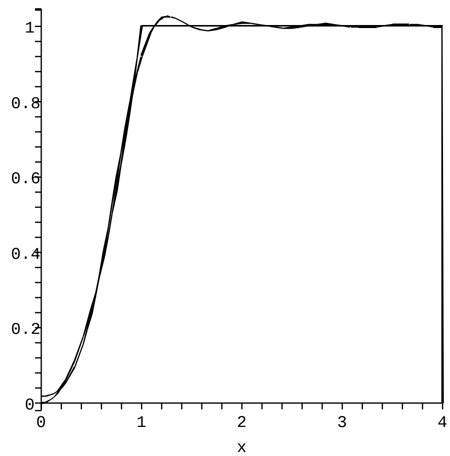

Figure 13.28: Tenth partial sum of the cosine expansion in Problem 9, Section 13.3.

converging to x^2 for $0 \leq x \leq 1$, to 1 if $1 < x < 4$, and to 0 if $x = 4$. Figure 13.29 shows a graph of the twentieth partial sum, compared to the function.

10. The cosine expansion of $f(x)$ is

$$-1 - \frac{24}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \left[2(-1)^n + \frac{4}{n^2\pi^2}(1 - (-1)^n) \right] \cos\left(\frac{n\pi x}{2}\right),$$

converging to $1 - x^3$ for $0 \leq x \leq 2$. Figure 13.30 is a graph of the function and the tenth partial sum of this cosine representation.

The sine series is

$$\frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \left[1 + 7(-1)^n - \frac{48}{n^2\pi^2}(-1)^n \right] \sin\left(\frac{n\pi x}{2}\right),$$

converging to $1 - x^2$ for $0 < x < 2$ and to 0 for $x = 0$ and for $x = 2$. Figure 13.31 compares the thirtieth partial sum of this series with the function.

11. The Fourier cosine expansion of $\sin(x)$ on $[0, \pi]$ is

$$\frac{2}{\pi} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{4n^2 - 1} \cos(2nx).$$

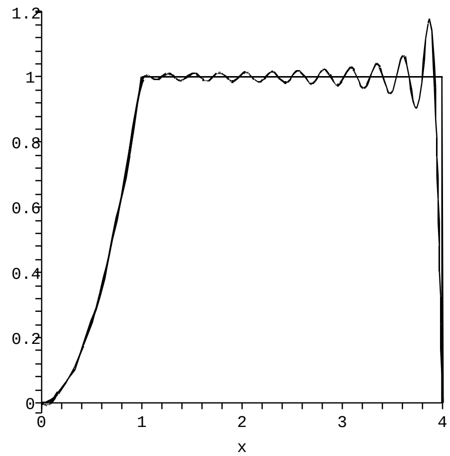

Figure 13.29: Twentieth partial sum of the sine expansion in Problem 9, Section 13.3.

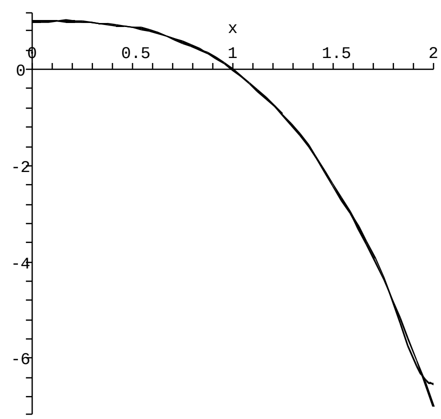

Figure 13.30: Tenth partial sum of the cosine expansion in Problem 10, Section 13.3.

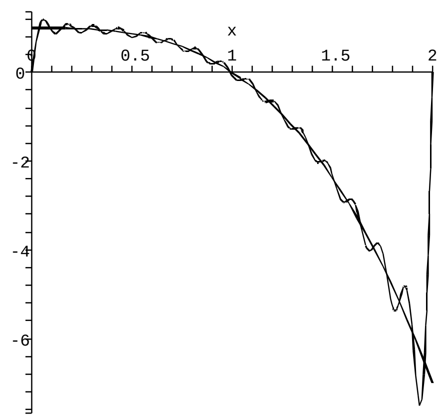

Figure 13.31: Thirtieth partial sum of the sine expansion in Problem 10, Section 13.3.

This converges to $\sin(x)$ for $0 \leq x \leq \pi$. Put $x = \pi/2$ in this series to obtain

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{4n^2-1} = \frac{\pi}{4}\left(\frac{2}{\pi} - 1\right) = \frac{1}{2} - \frac{\pi}{4}.$$

12. Write

$$f_e(x) = \frac{f(x) + f(-x)}{2}$$

and

$$f_o(x) = \frac{f(x) - f(-x)}{2}.$$

Then $f_e(x) = f_e(-x)$, so f_e is even. And $f_o(-x) = -f(x)$, so f_o is an odd function. Further,

$$f(x) = f_e(x) + f_o(x).$$

13. Suppose f is both even and odd on $[-L, L]$. Then, for any x in this interval,

$$f(x) = f(-x) = -f(x)$$

so $f(x) = 0$. To be both even and odd, the function must be identically zero on the interval.

13.4 Integration and Differentiation of Fourier Series

1. Since f is continuous on $[-\pi, \pi]$ and piecewise smooth on this interval. By the Fourier convergence theorem,

$$f(x) = \frac{\pi}{4} + \sum_{n=1}^{\infty} \left(\frac{1}{n^2\pi}((-1)^n - 1)\cos(nx) - \frac{(-1)^n}{n}\sin(nx) \right)$$

for $-\pi < x < \pi$. For $-\pi \leq x \leq \pi$, we can integrate the Fourier series term by term to obtain

$$\begin{aligned} \int_{\pi}^{x} f(t)\,dt &= \frac{\pi}{4}(x+\pi) \\ &\quad + \sum_{n=1}^{\infty} \left(\frac{1}{n^3\pi}((-1)^n - 1)\sin(nx) + \frac{(-1)^n}{n^2}\cos(nx) - \frac{1}{n^2} \right). \end{aligned}$$

2. f is continuous on $[-1, 1]$ and f' is piecewise continuous on $[-1, 1]$. Further $f(1) = f(-1)$. Further, $f''(x)$ exists on $(-1, 1)$ except at 0. The Fourier series of $f(x)$ is

$$\frac{1}{2} - \frac{4}{\pi^2}\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}\cos((2n-1)\pi x).$$

Termwise differentiation of this series yields the series

$$\frac{4}{\pi}\sum_{n=1}^{\infty} \frac{1}{2n-1}\sin((2n-1)\pi x).$$

It is routine to check that this is the Fourier expansion of

$$g(x) = \begin{cases} -1 & \text{for } -1 \leq x < 0, \\ 1 & \text{for } 0 < x \leq 1. \end{cases}$$

on $[-1, 1]$.

3. The Fourier expansion of $f(x)$ on $[-\pi, \pi]$ is

$$1 - \frac{1}{2}\cos(x) - 2\sum_{n=2}^{\infty} \frac{(-1)^n}{n^2 - 1}\cos(nx).$$

This converges to $x\sin(x)$ for $-\pi \leq x \leq \pi$. Note that f is continuous on $[-\pi, \pi]$, that $f(\pi) = f(-\pi)$, and that $f'(x)$ is continuous (hence piecewise continuous) on this interval. We can differentiate the Fourier series expansion to write, for $-\pi < x < \pi$,

$$\begin{aligned} f'(x) &= \sin(x) + x\cos(x) \\ &= \frac{1}{2}\sin(x) + 2\sum_{n=2}^{\infty} \frac{n(-1)^n}{n^2 - 1}\sin(nx). \end{aligned}$$

It is routine to check that the Fourier expansion of $g(x) = \sin(x) + x\cos(x)$ agrees with this result.

4. The Fourier expansion of $f(x) = x^2$ on $[-3, 3]$ is

$$3 + \frac{36}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos(n\pi x/3).$$

This series converges to x^2 for $-3 \le x \le 3$. Further, $f(-3) = f(3)$ and $f'(x) = 2x$ is continuous, hence piecewise continuous, on $[-3, 3]$. We may therefore differentiate this Fourier series representation term by term to obtain, for $-3 < x < 3$,

$$2x = -\frac{12}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin(n\pi x/3).$$

Expansion of $2x$ in a Fourier series on $[-3, 3]$ verifies this expansion.

5. Let the Fourier coefficients of f on $[-L, L]$ be a_n, b_n, as usual. From Bessel's inequality, the series

$$\sum_{n=0}^{\infty} a_n^2 \text{ and } \sum_{n=1}^{\infty} b_n^2$$

both converge. As with any convergent series, the general term has limit zero as $n \to \infty$, so

$$\lim_{n\to\infty} a_n^2 = \lim_{n\to\infty} b_n^2 = 0.$$

This means that a_n^2 and b_n^2 can be made as close to zero as we like, by choosing n sufficiently large. But then this will hold also for a_n and b_n, so

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = 0.$$

Inserting the integrals for the Fourier coefficients, we have

$$\lim_{n\to\infty}\frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right) = \lim_{n\to\infty}\frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right) = 0.$$

The positive factor of $1/L$ does not affect this limit, so

$$\lim_{n\to\infty}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right) = \lim_{n\to\infty}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right) = 0.$$

6. We will prove Theorem 13.8. Let the Fourier coefficients of f on $[-L, L]$ be a_n, b_n, and the Fourier coefficients of f', A_n, B_n. Notice that

$$A_0 = \frac{2}{L}\int_{-L}^{L} f'(x)\,dx = f(L) - f(-L) = 0$$

because $f(L) = f(-L)$. For $n = 1, 2, \cdots$, we claim that the Fourier coefficients of f and f' are related. First, integrate by parts to obtain

$$\begin{aligned} A_n &= \frac{1}{L}\int_{-L}^{L} f'(x)\cos\left(\frac{n\pi x}{L}\right)\,dx \\ &= \frac{1}{L}\left[f(x)\cos\left(\frac{n\pi x}{L}\right)\right]_{-L}^{L} + \frac{n\pi}{L}\frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\,dx. \end{aligned}$$

Now,

$$f(L)\cos(n\pi) - f(-L)\cos(-n\pi) = 0$$

because $f(L) = f(-L)$ by assumption. Therefore

$$A_n = \frac{n\pi}{L}\frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\,dx = \frac{n\pi}{L}a_n$$

for $n = 1, 2, \cdots$. A similar integration by parts yields

$$B_n = -\frac{n\pi}{L}a_n.$$

Now,

$$0 \le \left(|A_n| - \frac{1}{n}\right)^2 = A_n^2 - \frac{2}{n}|A_n| + \frac{1}{n^2}.$$

Similarly,

$$0 \le B_n^2 - \frac{2}{n}|B_n| + \frac{1}{n^2}.$$

Add these two inequalities to obtain

$$\frac{2}{n}\left(|A_n| + |B_n|\right) \le A_n^2 + B_n^2 + \frac{2}{n^2}.$$

Multiply this by $1/2$ to obtain

$$\frac{1}{n}\left(|A_n| + |B_n|\right) \le \frac{1}{2}\left(A_n^2 + B_n^2\right) + \frac{1}{n^2}.$$

On the left, insert $|A_n| = n\pi|a_n|/L$ and $|B_n| = n\pi|a_n|/L$ to obtain

$$|a_n| + |b_n| \le \frac{L}{2\pi}(A_n^2 + B_n^2) + \frac{L}{\pi}\frac{1}{n^2}.$$

From Bessel's inequality,

$$\sum_{n=1}^{\infty} A_n^2 \text{ and } \sum_{n=1}^{\infty} B_n^2$$

converge. Using the inequality of the preceding line in the comparison test for positive series, we conclude that

$$\sum_{n=1}^{\infty}(|a_n| + |b_n|)$$

converges. Finally, observe that, on $[-L, L]$,

$$|a_n \cos(n\pi x/L) + b_n \sin(n\pi x/L)| \leq |a_n| + |b_n|.$$

The nth term of the Fourier series of f is therefore bounded on the interval $[-L, L]$ by $M_n = |a_n| + |b_n|$, a nonnegative constant. Further, we know that

$$\sum_{n=1}^{\infty} M_n$$

converges. By a theorem of Weierstrass (sometimes called the $M-$ text), the Fourier series of f on $[-L, L]$ converges uniformly on this interval.

13.5 Phase Angle Form

1. For any t,

$$(\alpha f + \beta g)(t + p) = \alpha f(t + p) + \beta g(t + p)$$
$$= \alpha f(t) + \beta g(t) = (\alpha f + \beta g)(t).$$

2.

$$g(t + p/\alpha) = f(\alpha(t + p/\alpha)) = f(\alpha t + p) = f(\alpha t) = g(t),$$

and

$$h(t + \alpha p) = f\left(\frac{t + \alpha p}{\alpha}\right) = f(t/\alpha + p) = f(t/\alpha) = h(t).$$

3.

$$f'(t + p) = \lim_{h \to 0} \frac{f(t + p + h) - f(t + p)}{h}$$
$$= \lim_{h \to 0} \frac{f(t + h) - f(t)}{h} = f'(t).$$

4. Expanding f in a Fourier series on $[0, 2]$ yields the series

$$1 - \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin(n\pi x).$$

The trigonometric identity

$$\sin(n\pi x) = \cos\left(n\pi x - \frac{\pi}{2}\right)$$

enables us to write the phase angle form

$$1 - \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \cos\left(n\pi x - \frac{\pi}{2}\right).$$

The amplitude spectrum points are

$$(0, 1) \text{ and } (n\pi, -1/n\pi) \text{ for } n = 1, 2, \cdots .$$

5. The Fourier series of f is

$$\frac{1}{2}+\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\sin((2n-1)\pi x).$$

The phase angle form of this series is

$$1+\frac{2}{\pi}\frac{1}{2n-1}\cos\left((2n-1)\pi x-\frac{\pi}{2}\right).$$

Points of the amplitude spectrum are

$$(0,1),(n\pi,1/((2n-1)\pi)).$$

6. The Fourier series of f is

$$16+\frac{48}{\pi^2}\sum_{n=1}^{\infty}\left[\frac{1}{n^2}\cos\left(\frac{n\pi x}{2}\right)-\frac{\pi}{n}\sin\left(\frac{n\pi x}{2}\right)\right].$$

The phase angle form is

$$16+\frac{48}{\pi^2}\sum_{n=1}^{\infty}\frac{\sqrt{1+n^2\pi^2}}{n^2}\cos\left(\frac{n\pi x}{2}+\arctan(n\pi)\right).$$

Points of the amplitude spectrum are

$$(0,16),\left(\frac{n\pi}{2},\frac{24\sqrt{1+\pi^2n^2}}{\pi^2n^2}\right).$$

7. The Fourier series of f is

$$\frac{19}{8}+\sum_{n=1}^{\infty}\frac{2}{n^2\pi^2}\left[n\pi\sin\left(\frac{3n\pi}{2}\right)+\cos\left(\frac{3n\pi}{2}\right)-1\right]\cos\left(\frac{n\pi x}{2}\right)$$
$$+\left[\sin\left(\frac{3n\pi}{2}\right)-\frac{n\pi}{2}-n\pi\cos\left(\frac{3n\pi}{2}\right)\right]\sin\left(\frac{n\pi x}{2}\right).$$

The phase angle form is

$$\frac{19}{8}+\frac{1}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}d_n\cos\left(\frac{n\pi x}{2}+\delta_n\right),$$

where

$$d_n=\sqrt{8+5n^2\pi^2-12n\pi\sin(3n\pi/2)+4(n^2\pi^2-2)\cos(3n\pi/2)}$$

and

$$\delta_n=\arctan\left(\frac{n\pi/2+n\pi\cos(3n\pi/2)-\sin(3n\pi/2)}{n\pi\sin(3n\pi/2)+\cos(3n\pi/2)-1}\right).$$

8. The Fourier series is
$$\frac{8}{\pi}\sum_{n=1}^{\infty}\frac{n}{4n^2-1}\sin(2n\pi x).$$
The phase angle form is
$$\frac{8}{\pi}\sum_{n=1}^{\infty}\frac{n}{4n^2-1}\cos\left(2n\pi x-\frac{\pi}{2}\right).$$

9. We can write
$$f(x)=\begin{cases}x & \text{for } 0\le x<1,\\ x-2 & \text{for } 1<x\le 2,\end{cases}$$
and $f(x+2)=f(x)$, so f has period 2. The Fourier series is
$$\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin(n\pi x).$$
The phase angle form is
$$\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\cos\left(n\pi x+(-1)^{n+1}\frac{\pi}{2}\right).$$

10. We can write
$$f(x)=\begin{cases}k & \text{for } 0<x<1,\\ 0 & \text{for } 1<x<2,\end{cases}$$
with$f(x+2)=f(x)$. The Fourier series of this function has phase angle form
$$\frac{k}{2}+\frac{2k}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos((2n-1)\pi x-\pi).$$

11. Write
$$f(x)=\begin{cases}1 & \text{for } 0\le x<1,\\ 2 & \text{for } 1<x<3,\\ 1 & \text{for } 3<x<4,\end{cases}$$
with $f(x+4)=f(x)$. The Fourier series of this function is
$$\frac{3}{2}+\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{2n-1}\cos\left(\frac{(2n-1)\pi x}{2}\right).$$
This has phase angle form
$$\frac{3}{2}+\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\cos\left((2n-1)\frac{\pi x}{2}+\frac{\pi}{2}(1-(-1)^n)\right).$$

12. Write

$$f(x) = \begin{cases} k & \text{for } 0 < x < 1, \\ 0 & \text{for } 1 < x < 2, \end{cases}$$

and $f(x+2) = f(x)$. This function has Fourier series

$$\frac{k}{2} + \frac{2k}{\pi} \sum_{n=1}^{\infty} \frac{1}{2n-1} \sin((2n-1)\pi x)$$

with phase angle form

$$\frac{k}{2} + \frac{2k}{\pi} \sum_{n=1}^{\infty} \frac{1}{2n-1} \cos\left((2n-1)\pi x - \frac{\pi}{2}\right).$$

13.6 Complex Fourier Series

1. Compute

$$d_0 = \frac{1}{3} \int_0^3 2t\,dt = 3$$

and, for $n \neq 0$,

$$d_n = \frac{1}{3} \int_0^3 2xe^{-2n\pi it/3}\,dt = \frac{3}{n\pi} i.$$

The complex Fourier series expansion of $f(x)$ is

$$3 + \frac{3i}{\pi} \sum_{n=-\infty, n\neq 0}^{\infty} \frac{1}{n} e^{2n\pi ix/3}$$

$$= \begin{cases} 3 & \text{for } x = 0 \text{ or } x = 3, \\ 2x & \text{for } 0 < x < 3. \end{cases}$$

Points of the frequency spectrum are

$$(0,3), \left(\frac{2n\pi}{3}, \frac{3}{n\pi}\right).$$

2. The complex Fourier series of $f(x)$ is

$$\frac{4}{3} + \sum_{n=-\infty, n\neq 0}^{\infty} \left(\frac{2}{n^2\pi^2} - \frac{2i}{n\pi}\right) e^{n\pi ix}.$$

This converges to

$$\begin{cases} 2 & \text{for } x = 0 \text{ or } x = 2, \\ x^2 & \text{for } 0 < x < 2. \end{cases}$$

Points of the frequency spectrum are

$$(0,4/3),\ \left(n\pi, 2\sqrt{\frac{1}{n^4\pi^4}+\frac{1}{n^2\pi^2}}\right).$$

3. The complex Fourier expansion of $f(x)$ is

$$\frac{3}{4}-\frac{1}{2\pi}\sum_{n=-\infty,n\neq 0}^{\infty}\frac{1}{n}\left(\sin(n\pi/2)+(\cos(n\pi/2)-1)i\right)e^{n\pi ix/2}.$$

This converges to

$$\begin{cases} 1/2 & \text{for } x=0 \text{ or } x=1 \text{ or } x=4,\\ 0 & \text{for } 0<x<1,\\ 1 & \text{for } 1<x<4.\end{cases}$$

Points of the frequency spectrum are

$$(0,3/4),\ \left(\frac{n\pi}{2}, \frac{1}{2n\pi}\sqrt{\sin^2(n\pi/2)+(\cos(n\pi/2)-1)^2}\right).$$

4. The complex series is

$$-\frac{1}{2}-\frac{3i}{\pi}\sum_{n=-\infty,n\neq 0}^{\infty}\frac{1}{n}e^{n\pi ix/3}.$$

This converges to

$$\begin{cases} -2 & \text{for } x=0 \text{ or } x=6,\\ 1-x & \text{for } 0<x<6.\end{cases}$$

Points of the frequency spectrum are

$$(0,1/2),\ \left(\frac{n\pi}{2}, \frac{3}{n\pi}\right).$$

5. The complex Fourier series is

$$\frac{1}{2}+\frac{3i}{\pi}\sum_{n=-\infty,n\neq 0}^{\infty} e^{(2n-1)\pi ix/2},$$

converging to

$$\begin{cases} 1/2 & \text{for } x=0,2,4,\\ -1 & \text{for } 0<x<2,\\ 2 & \text{for } 2<x<4.\end{cases}$$

Points of the frequency spectrum are

$$(0,1/2),\ \left(\frac{n\pi}{2}, \frac{3}{(2n-1)\pi}\right).$$

6. The complex Fourier series of f is

$$\sum_{n=-\infty}^{\infty} \frac{1-e^{-5}}{5+2n\pi i} e^{2n\pi i x/5},$$

converging to

$$\begin{cases} (1-e^{-5})/2 & \text{for } x=0 \text{ or } x=5, \\ e^{-x} & \text{for } 0<x<5. \end{cases}$$

Points of the frequency spectrum are

$$\left(\frac{2n\pi}{3}, \frac{1-e^{-5}}{\sqrt{29}}\sqrt{25+4n^2\pi^2}\right).$$

7. The complex Fourier series of f is

$$\frac{1}{2} - \frac{2}{\pi^2} \sum_{n=-\infty, n\neq 0}^{\infty} \frac{1}{(2n-1)^2} e^{(2n-1)\pi i x},$$

converging to $f(x)$ for $0 \leq x \leq 2$.

Points of the frequency spectrum are

$$(0, 1/2), \left(n\pi, \frac{2}{\pi^2}\frac{1}{(2n-1)^2}\right).$$

13.7 Filtering of Signals

1. The complex Fourier coefficients of f are $d_0 = 0$ and, for nonzero n,

$$d_n = \frac{1}{4}\left[\int_{-2}^{0} -e^{-n\pi i t/2}\,dt + \int_0^2 e^{-n\pi i t/2}\,dt\right] = \frac{i}{\pi n}[(-1)^n - 1].$$

The complex Fourier series is

$$\sum_{n=-\infty, n\neq 0}^{\infty} \frac{i}{n\pi}[(-1)^n - 1] e^{n\pi i t/2}.$$

If we carry out a calculation like that of Example 13.17, we obtain the Fourier series

$$\frac{4}{\pi}\sum_{n=1}^{\infty} \frac{1}{2n-1}\sin\left(\frac{(2n-1)\pi t}{2}\right).$$

The Nth partial sum is therefore

$$S_N(t) = \frac{4}{\pi}\sum_{n=1}^{N} \frac{1}{2n-1}\sin\left(\frac{(2n-1)\pi t}{2}\right).$$

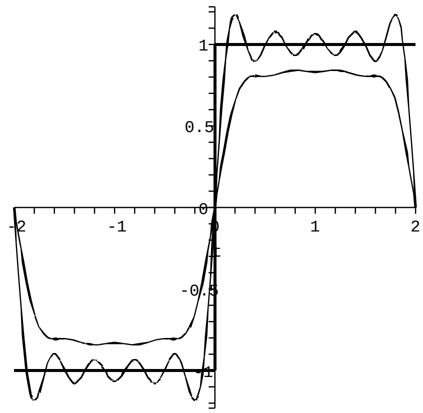

Figure 13.32: Fifth partial sum and Cesáro sum in Problem 1, Section 13.7.

The Nth Cesáro sum is formed by inserting factors $1 - |n|/N$:

$$\sigma_N(t) = \frac{4}{\pi}\sum_{n=1}^{N}\left(1 - \frac{2n-1}{N}\right)\frac{1}{2n-1}\sin\left(\frac{(2n-1)\pi t}{2}\right).$$

Figures 13.32, 13.33 and 13.34 compare $f(t)$, $S_N(t)$ and $\sigma_N(t)$ for $N = 5, 10, 25$, respectively. Notice that the Cesáro sums have the effect of smoothing the Gibbs effect seem at 0 and the ends of the interval.

2. The Nth partial sum of the Fourier series of f has the form

$$S_N(t) = \frac{13}{8} + \sum_{n=1}^{N}\left[a_n \sin\left(\frac{n\pi t}{2}\right) + b_n \cos\left(\frac{n\pi t}{2}\right)\right],$$

where

$$a_n = \frac{2}{n^3\pi^3}\left[\sin(n\pi/2)(-4 - n\pi^2) + n\pi\cos(n\pi/2) + 5n\pi(-1)^n\right]$$

and

$$b_n = -\frac{2}{n^3\pi^3}\left[-n\pi\sin(n\pi/2) + \cos(n\pi/2)(-4 - n^2\pi^2) + 4(-1)^n\right].$$

Form $\sigma_N(t)$ by inserting a factor of $1 - n/N$ into $S_N(t)$. Figures 13.35, 13.36 and 13.37 compare the Nth partial sums and Cesáro sums and the function for $N = 5, 10, 25$, respectively.

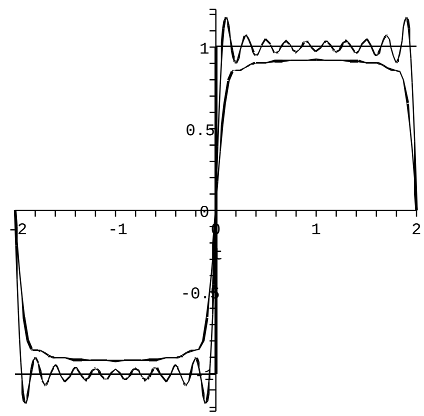

Figure 13.33: Tenth partial sum and Cesáro sum in Problem 1, Section 13.7.

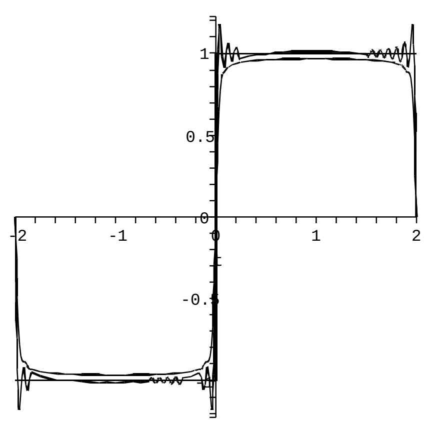

Figure 13.34: Twenty-fifth partial sum and Cesáro sum in Problem 1, Section 13.7.

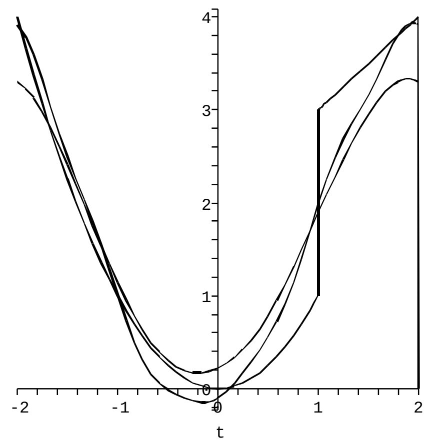

Figure 13.35: Fifth partial sum and Cesáro sum in Problem 2, Section 13.7.

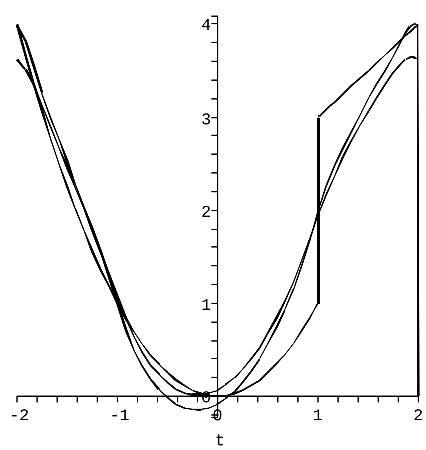

Figure 13.36: Tenth partial sum and Cesáro sum in Problem 2, Section 13.7.

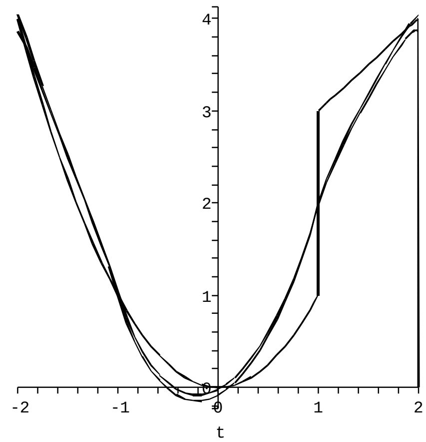

Figure 13.37: Twenty-fifth partial sum and Cesáro sum in Problem 2, Section 13.7.

3. We find that

$$S_N(t) = \sum_{n=1}^{N} \frac{2}{n\pi} \left[\cos(n\pi/2) - (-1)^n\right] \sin(n\pi t)$$

and

$$\sigma_N(t) = \sum_{n=1}^{N} \left(1 - \frac{n}{N} \frac{2}{n\pi}\right) \left[\cos(n\pi/2) - (-1)^n\right] \sin(n\pi t)$$

Figures 13.38, 13.39 and 13.40 compare the fifth, tenth and twenty-fifth partial sums of these sums with $f(t)$.

4. The Nth partial sums are

$$S_N(t) = \frac{1}{6}\sin(3)$$
$$+ \sum_{n=1}^{N} \frac{-1}{n^2\pi^2 - 9} \left[3\sin(3)(-1)^n \cos\left(\frac{n\pi t}{3}\right) + n\pi(-1 + (-1)^n \cos(3)) \sin\left(\frac{n\pi t}{3}\right)\right]$$

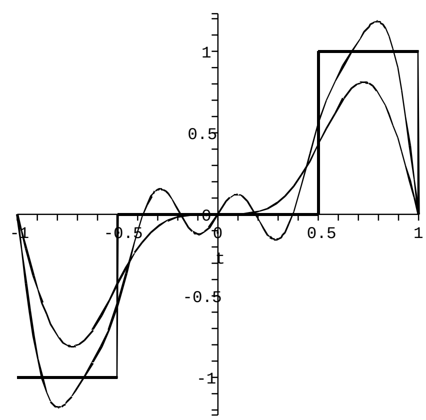

Figure 13.38: Fifth partial sum and Cesáro sum in Problem 3, Section 13.7.

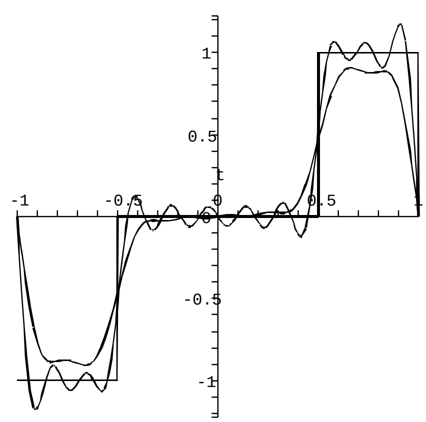

Figure 13.39: Tenth partial sum and Cesáro sum in Problem 3, Section 13.7.

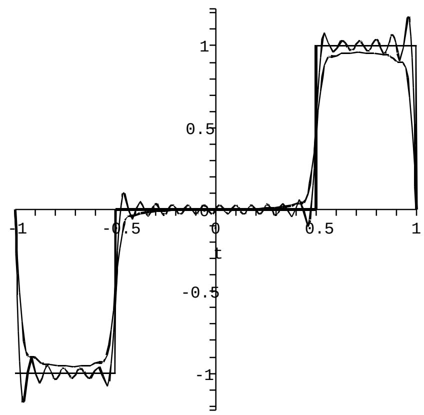

Figure 13.40: Twenty-fifth partial sum and Cesáro sum in Problem 3, Section 13.7.

and

$$\sigma_N(t) = \frac{1}{6}\sin(3)$$
$$+\sum_{n=1}^{N}\left(1-\frac{n}{N}\right)\frac{-1}{n^2\pi^2-9}\left[3\sin(3)(-1)^n\cos\left(\frac{n\pi t}{3}\right)\right.$$
$$\left.+n\pi(-1+(-1)^n\cos(3))\sin\left(\frac{n\pi t}{3}\right)\right]$$

Figures 13.41, 13.42 and 13.43 compare the fifth, tenth and twenty-fifth partial sums of these sums with $f(t)$.

5. We find the partial sums

$$S_N(t) = \frac{17}{4} + \sum_{n=1}^{N}\left[\frac{1-(-1)^n}{n^2\pi^2}\cos(n\pi t) + \frac{5-6(-1)^n}{n\pi}\sin(n\pi t)\right],$$

and

$$\sigma_N(t) = \frac{17}{4}$$
$$+\frac{17}{4} + \sum_{n=1}^{N}\left(1-\frac{n}{N}\right)\left[\frac{1-(-1)^n}{n^2\pi^2}\cos(n\pi t) + \frac{5-6(-1)^n}{n\pi}\sin(n\pi t)\right].$$

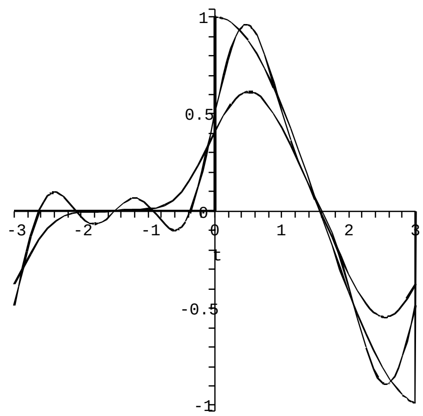

Figure 13.41: Fifth partial sum and Cesáro sum in Problem 4, Section 13.7.

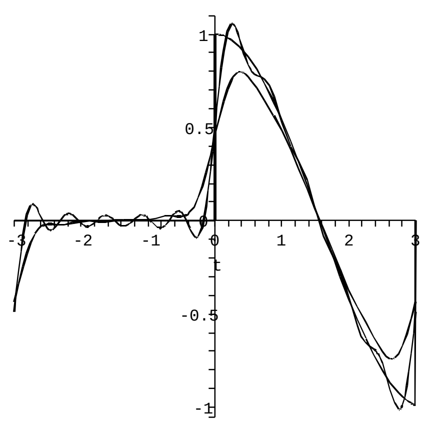

Figure 13.42: Tenth partial sum and Cesáro sum in Problem 4, Section 13.7.

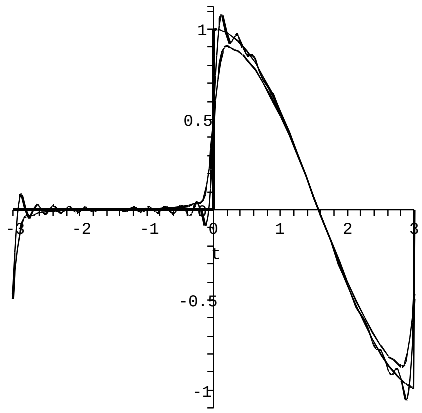

Figure 13.43: Twenty-fifth partial sum and Cesáro sum in Problem 4, Section 13.7.

Figures 13.44, 13.45 and 13.46 compare the fifth, tenth and twenty-fifth partial sums of these sums with $f(t)$.

6. The partial sums of the Fourier series are

$$S_N(t) = \sum_{n=1}^{N} \frac{2}{n\pi}(1-(-1)^n)\sin\left(\frac{n\pi t}{t}\right).$$

The Cesáro, Hamming and Gaussian filtered N partial sums are, respectively,

$$\sigma_N(t) = \sum_{n=1}^{N} \frac{2}{n\pi}\left(1-\frac{n}{N}\right)(1-(-1)^n)\sin\left(\frac{n\pi t}{t}\right),$$

$$H_N(t) = \sum_{n=1}^{N} \frac{2}{n\pi}(0.54+0.46\cos(\pi n/N))(1-(-1)^n)\sin\left(\frac{n\pi t}{t}\right),$$

$$G_N(t) = \sum_{n=1}^{N} \frac{2}{n\pi}e^{-n^2\pi^2/N^2}(1-(-1)^n)\sin\left(\frac{n\pi t}{t}\right),$$

with the understanding that we have used $\alpha = 1$ in the Gaussian filter function.

Figures 13.47, 13.48 and 13.49 compare these partial sums with the function, for $N = 5, 10, 25$ respectively.

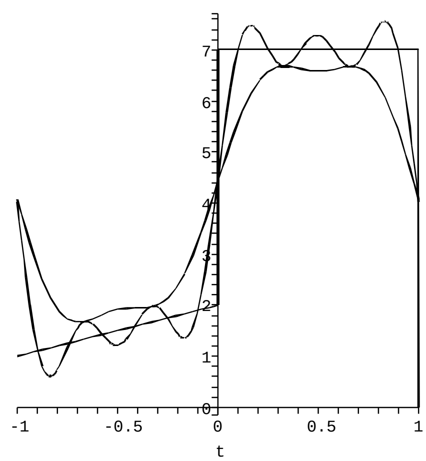

Figure 13.44: Fifth partial sum and Cesáro sum in Problem 5, Section 13.7.

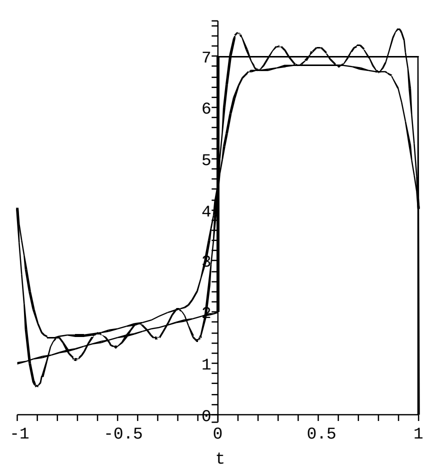

Figure 13.45: Tenth partial sum and Cesáro sum in Problem 5, Section 13.7.

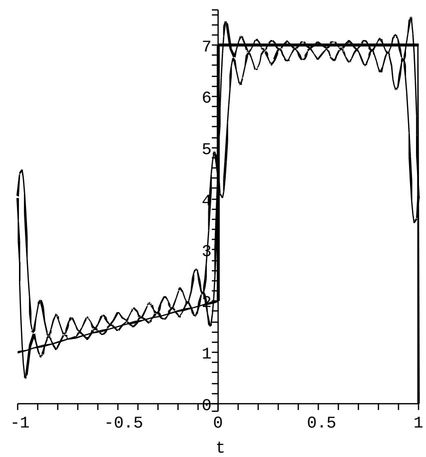

Figure 13.46: Twenty-fifth partial sum and Cesáro sum in Problem 5, Section 13.7.

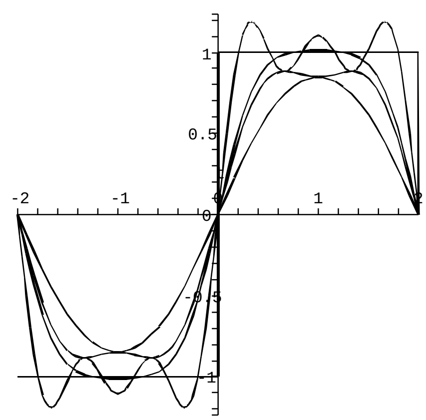

Figure 13.47: Fourier, Cesáro, Hamming, and Gauss partial sums, $N = 5$, in Problem 6, Section 13.7.

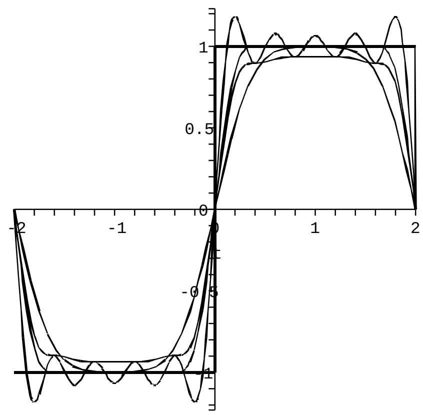

Figure 13.48: Fourier, Cesáro, Hamming, and Gauss partial sums, $N = 10$, in Problem 6, Section 13.7.

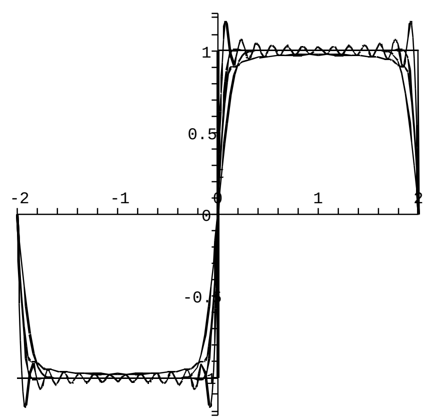

Figure 13.49: Fourier, Cesáro, Hamming, and Gauss partial sums, $N = 25$, in Problem 6, Section 13.7.

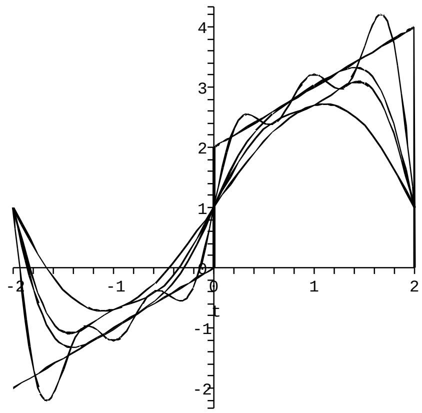

Figure 13.50: Fourier, Cesáro, Hamming, and Gauss partial sums, $N = 5$, in Problem 7, Section 13.7.

7. The partial sums are

$$S_N(t) = 1 + \sum_{n=1}^{N} \frac{2}{n\pi}(1 - 3(-1)^n)\sin\left(\frac{n\pi t}{2}\right),$$
$$\sigma_N(t) = 1 + \sum_{n=1}^{N} \frac{2}{n\pi}\left(1 - \frac{n}{N}\right)(1 - 3(-1)^n)\sin\left(\frac{n\pi t}{2}\right),$$
$$H_N(t) = 1 + \sum_{n=1}^{N} \frac{2}{n\pi}(0.54 + 0.46\cos(\pi n/N))(1 - 3(-1)^n)\sin\left(\frac{n\pi t}{2}\right),$$
$$G_N(t) = 1 + \sum_{n=1}^{N} \frac{2}{n\pi}e^{-n^2\pi^2/N^2}(1 - 3(-1)^n)\sin\left(\frac{n\pi t}{2}\right).$$

Graphs of these partial sums are given for $N = 5, 10, 25$, respectively, in Figures 13.50, 13.51 and 13.52.

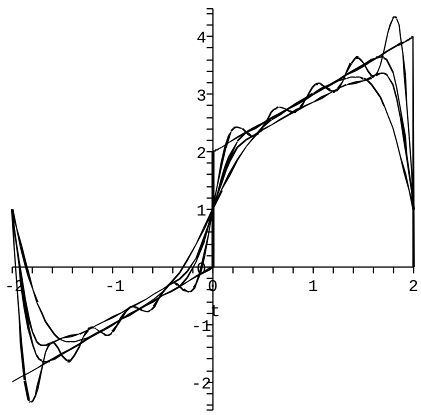

Figure 13.51: Fourier, Cesáro, Hamming, and Gauss partial sums, $N = 10$, in Problem 7, Section 13.7.

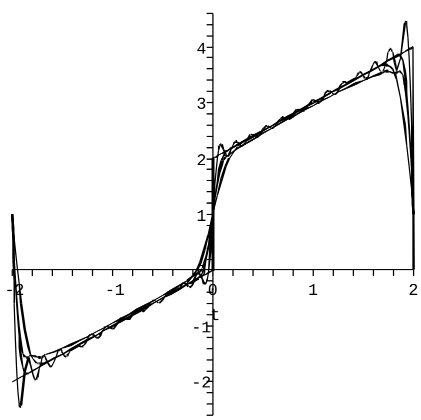

Figure 13.52: Fourier, Cesáro, Hamming, and Gauss partial sums, $N = 25$, in Problem 7, Section 13.7.

Chapter 14

The Fourier Integral and Transforms

14.1 The Fourier Integral

1. First,

$$\int_{-\infty}^{\infty} |f(x)|\, dx = \int_{-\pi}^{\pi} |x|\, dx = 2\int_{0}^{\pi} x\, dx = \pi^2.$$

Now $\xi\cos(\omega\xi)$ is an odd function of ξ, so each $A_\omega = 0$. Further,

$$\begin{aligned} B_\omega &= \frac{1}{\pi}\int_{-\infty}^{\infty} \xi\sin(\omega\xi)\, d\xi \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} \xi\sin(\omega\xi)\, d\xi = \frac{2}{\pi}\left[\frac{\sin(\pi\omega)}{\omega^2} - \frac{\pi}{\omega}\cos(\pi\omega)\right]. \end{aligned}$$

The Fourier integral representation of $f(x)$ is

$$\int_0^\infty \left[\frac{2\sin(\pi\omega)}{\pi\omega^2} - \frac{2\cos(\pi\omega)}{\omega}\right]\sin(\omega x)\, d\omega.$$

This representation converges to

$$\begin{cases} -\pi/2 & \text{for } x = -\pi, \\ x & \text{for } -\pi < x < \pi, \\ \pi/2 & \text{for } x = \pi, \\ 0 & \text{for } |x| > \pi. \end{cases}$$

2.

$$\int_{-\infty}^{\infty} |f(x)|\, dx = \int_{-10}^{10} k\, dx = 20k,$$

so this integral converges. Compute

$$A_\omega = \frac{1}{\pi}\int_{-10}^{10} k\cos(\omega t)\,dt = \frac{2k}{\pi\omega}\sin(10\omega)$$

if $\omega \neq 0$, and $A_0 = \lim_{\omega\to 0} A_\omega = 20k$. Further, $B_\omega = 0$ because $f(t)\sin(\omega t)$ is an odd function on the real line. The Fourier integral representation of $f(x)$ is

$$\int_0^\infty \frac{2k}{\pi\omega}\sin(10\omega)\cos(\omega x)\,d\omega.$$

This converges to

$$\begin{cases} k & \text{for } -10 < x < 10, \\ 0 & \text{for } |x| > 10, \\ k/2 & \text{for } x = -1 \text{ and for } x = 10. \end{cases}$$

3. Certainly $\int_{-\infty}^{\infty} |f(x)|\,dx$ converges, and each $A_\omega = 0$ because f is an odd function. Compute

$$B_\omega = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(\omega t)\,dt = \frac{2}{\pi\omega}(1-\cos(\pi\omega)).$$

The Fourier integral representation of $f(x)$ is

$$\int_0^\infty \frac{2}{\pi\omega}(1-\cos(\pi\omega))\sin(\omega x)\,d\omega.$$

This converges to

$$\begin{cases} -1/2 & \text{for } x = -\pi, \\ -1 & \text{for } -\pi < x < 0, \\ 0 & \text{for } x = 0 \text{ and for } |x| > \pi, \\ 1 & \text{for } 0 < x < \pi, \\ 1/2 & \text{for } x = \pi. \end{cases}$$

4. Certainly $\int_{-\infty}^{\infty} |f(x)|\,dx$ converges. Compute

$$\begin{aligned} A_\omega &= \frac{1}{\pi}\int_{-4}^{0} \sin(t)\cos(\omega t)\,dt + \frac{1}{\pi}\int_0^4 \cos(t)\cos(\omega t)\,dt \\ &= \frac{1}{2\pi(\omega-1)}(1+\sin(4(\omega-1))-\cos(4(\omega-1))) \\ &\quad - \frac{1}{2\pi(\omega+1)}(1-\sin(4(\omega+1))-\cos(4(\omega+1))) \end{aligned}$$

if $|\omega| \neq 1$, and

$$B_\omega = \frac{1}{2\pi(\omega-1)}(1-\cos(4(\omega-1))+\sin(4(\omega-1))) + \frac{1}{2\pi(\omega+1)}(1-\cos(4(\omega+1)))-\sin(4(\omega+1)))$$

if $|\omega| \neq 1$. We also take $A_1 = \lim_{\omega\to 1} A_\omega$, with a similar assignment if $\omega = -1$. B_1 and B_{-1} are treated the same way.

The Fourier integral representation of $f(x)$ is

$$\int_0^\infty (A_\omega \cos(\omega x) + B_\omega \sin(\omega x))\, d\omega.$$

This converges to

$$\begin{cases} 1/2 & \text{for } x = 0, \\ \cos(4)/2 & \text{for } x = 4, \\ -\sin(4)/2 & \text{for } x = -4, \\ f(x) & \text{for } -4 < x < 0 \text{ and for } 0 < x < 4. \end{cases}$$

5. Clearly $\int_{-\infty}^{\infty} |f(x)|\, dx$ converges. Since $f(x)$ is even, $B_\omega = 0$. Compute

$$\begin{aligned} A_\omega &= \frac{1}{\pi}\int_{-100}^{100} t^2\cos(\omega t)\, dt = \frac{2}{\pi}\int_0^{100} t^2 \cos(\omega t)\, dt \\ &= \frac{2}{\pi}\left[\frac{t^2\cos(\omega t)}{\omega} + \frac{2t\cos(\omega t)}{\omega^2} - \frac{2\sin(\omega t)}{\omega^3}\right]_0^{100} \\ &= \frac{20000\sin(100\omega)}{\pi\omega} - \frac{4\sin(100\omega)}{\pi\omega^3} + \frac{400\sin(100\omega)}{\pi\omega^2}. \end{aligned}$$

The Fourier integral representation of $f(x)$ is

$$\int_0^\infty \left[\frac{400\cos(100\omega)}{\pi\omega^2} + \frac{20000\omega^2 - 4}{\pi\omega^3}\sin(100\omega)\right]\cos(\omega x)\, d\omega.$$

This converges to

$$\begin{cases} x^2 & \text{for } -100 < x < 100, \\ 0 & \text{for } |x| > 100, \\ 5000 & \text{for } x = 100 \text{ and for } x = -100. \end{cases}$$

6. $\int_{-\infty}^{\infty} |f(x)|\, dx$ converges. Compute

$$\begin{aligned} A_\omega &= \frac{1}{\pi}\int_{-\pi}^{2\pi} |t|\cos(\omega t)\, dt \\ &= \frac{\cos(\pi\omega) + \pi\omega\sin(\pi\omega) + 2\cos^2(\pi\omega) - 3 + 4\pi\omega\sin(\pi\omega)\cos(\pi\omega)}{\pi\omega^2} \end{aligned}$$

and

$$B_\omega = \frac{1}{\pi}\int_{-\pi}^{2\pi} |t|\sin(\omega t)\,dt$$
$$= \frac{-\sin(\pi\omega) + \pi\omega\cos(\pi\omega) + 2\sin(\pi\omega)\cos(\pi\omega) - 4\pi\omega\cos^2(\pi\omega) + 2\pi\omega}{\pi\omega^2}.$$

The Fourier integral representation of $f(x)$ is

$$\int_0^\infty (A_\omega\cos(\omega x) + B_\omega\sin(\omega x))\,d\omega.$$

This converges to

$$\begin{cases} |x| & \text{for } -\pi < x < 2\pi, \\ 0 & \text{for } x < -\pi \text{ and for } x > 2\pi, \\ \pi/2 & \text{for } x = -\pi, \\ \pi & \text{for } x = 2\pi. \end{cases}$$

7. Certainly $\int_{-\infty}^{\infty} |f(t)|\,dt$ converges. Compute

$$A_\omega = \frac{1}{\pi}\int_{-3\pi}^{\pi} \sin(t)\cos(\omega t)\,dt = \frac{4\cos(\pi\omega)(\cos^2(\pi\omega) - 1)}{\pi(\omega^2 - 1)}$$

and

$$B_\omega = \frac{1}{\pi}\int_{-3\pi}^{\pi} \sin(t)\sin(\omega t)\,dt = -\frac{4\sin(\pi\omega)\cos^2(\pi\omega)}{\pi(\omega^2 - 1)}.$$

The Fourier integral representation is

$$\int_0^\infty (A_\omega\cos(\omega x) + B_\omega\sin(\omega x))\,d\omega.$$

This converges to

$$\begin{cases} \sin(x) & \text{for } -3\pi \le x \le \pi, \\ 0 & \text{for } x < -3\pi \text{ and for } x > \pi. \end{cases}$$

8. The integral representation is

$$\int_0^\infty (A_\omega\cos(\omega x) + B_\omega\sin(\omega x))\,d\omega,$$

where

$$A_\omega = \frac{1}{\pi}\int_{-5}^{1} \frac{1}{2}\cos(\omega t)\,dt + \frac{1}{\pi}\int_1^5 \cos(\omega t)\,dt$$
$$= \frac{\sin(\omega)}{\pi\omega}\left(24\cos^4(\omega) - 18\cos^2(\omega) + 1\right)$$

and

$$\begin{aligned} B_\omega &= \frac{1}{\pi} \int_{-5}^{1} \frac{1}{2} \sin(\omega t)\, dt + \int_{1}^{5} \sin(\omega t)\, dt \\ &= -\frac{2\cos(\omega)}{\pi\omega} \left(4\cos^4(\omega) - 5\cos^2(\omega) + 1\right). \end{aligned}$$

This integral representation converges to

$$\begin{cases} 1/4 & \text{for } x = -5, \\ 3/2 & \text{for } x = 1, \\ 1/2 & \text{for } x = 5, \\ f(x) & \text{for all other } x. \end{cases}$$

9. With $f(x) = e^{-|x|}$, integrations yield the Fourier integral representation

$$\int_0^\infty \frac{1}{\pi(\omega^2+1)} \cos(\omega x)\, d\omega,$$

converging to $e^{-|x|}$ for all x.

10. With $f(x) = xe^{-4|x|}$, we obtain the Fourier integral representation

$$\int_0^\infty \frac{2(\omega^2-1)}{\pi(\omega^2+1)} \sin(\omega x)\, d\omega,$$

converging to $xe^{-4|x|}$ for all x.

11. First, we can write the Fourier integral representation of $f(x)$ as

$$\frac{1}{\pi} \int_0^\infty \int_{-\infty}^{\infty} f(t) \cos(\omega(t-x))\, dt\, d\omega.$$

Interchange the order of integration and use the fact that $f(t)\cos(\omega(t-x))$ is an even function of ω to write this integral representation as

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t) \cos(\omega(t-x))\, d\omega\, dt.$$

Now $f(t)\sin(\omega(t-x))$ is a odd function of ω, so

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} f(t) \sin(\omega(t-x))\, d\omega = 0.$$

We can therefore write the integral representation of $f(x)$ as

$$\begin{aligned}
&\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)[\cos(\omega(t-x)) + i\sin(\omega(t-x))]\,d\omega\,dt\\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\lim_{r\to\infty}\int_{-r}^{r} e^{i\omega(t-x)}\,d\omega\right] dt\\
&= \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\left[\lim_{r\to\infty}\frac{e^{ir(t-x)} - e^{-ir(t-x)}}{2i(t-x)}\right] dt\\
&= \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\frac{\sin(\omega(t-x))}{t-x}\,dt.
\end{aligned}$$

14.2 Fourier Cosine and Sine Integrals

For Problems 1 - 10 we will give the cosine and sine integral representations without all of the details of the integrations for the coefficients.

1. The Fourier cosine integral representation of $f(x)$ is

$$\int_0^{\infty}\frac{4}{\pi\omega^3}(10\omega\cos(10\omega) - (50\omega^2 - 1)\sin(10\omega))\cos(\omega x)\,d\omega.$$

The sine integral representation is

$$\int_0^{\infty}\frac{4}{\pi\omega^3}(10\omega\sin(10\omega) - (50\omega^2 - 1)\cos(10\omega) - 1)\sin(\omega x)\,d\omega.$$

Both integrals converge to

$$\begin{cases} x^2 & \text{for } 0 \le x < 10,\\ 0 & \text{for } x > 10,\\ 50 & \text{for } x = 10.\end{cases}$$

2. The cosine integral is

$$\int_0^{\infty}\frac{2(\cos(2\pi\omega) - 1)}{\pi(\omega^2 - 1)}\cos(\omega x)\,d\omega$$

and the sine integral is

$$\int_0^{\infty}\frac{2\sin(2\pi\omega)}{\pi(\omega^2 - 1)}\sin(\omega x)\,d\omega.$$

Both integrals converge to $f(x)$ for all x.

3. The cosine integral is

$$\int_0^{\infty}\frac{2}{\pi\omega}(2\sin(4\omega) - \sin(\omega))\cos(\omega x)\,d\omega,$$

converging to

$$\begin{cases} 1 & \text{for } 0 < x < 1, \\ 3/2 & \text{for } x = 1, \\ 2 & \text{for } 1 < x < 4, \\ 1 & \text{for } x = 0 \text{ and for } x = 4, \\ 0 & \text{for } x > 4. \end{cases}$$

The sine integral is

$$\int_0^\infty \frac{2}{\pi\omega}(1 + \cos(\omega) - 2\cos(4\omega))\sin(\omega x)\,d\omega,$$

converging to

$$\begin{cases} 1 & \text{for } 0 < x < 1, \\ 3/2 & \text{for } x = 1, \\ 2 & \text{for } 1 < x < 4, \\ 1 & \text{for } x = 4, \\ 0 & \text{for } x = 0 \text{ and for } x > 4. \end{cases}$$

4. The cosine integral representation is

$$\int_0^\infty \frac{2}{\pi(\omega^2+1)}(\sinh(5)\cos(5\omega) + \omega\cosh(5)\sin(5\omega))\cos(\omega x)\,d\omega,$$

converging to

$$\begin{cases} \cosh(x) & \text{for } 0 < x < 5, \\ \cosh(5)/2 & \text{for } x = 5, \\ 0 & \text{for } x > 5 \\ 1 & \text{for } x = 0. \end{cases}$$

The sine integral is

$$\int_0^\infty \frac{2}{\pi(\omega^2+1)}(5\omega\sinh(5) - \omega\cosh(5)\cos(5\omega) + \omega)\sin(\omega x)\,d\omega,$$

converging to

$$\begin{cases} \cosh(x) & \text{for } 0 < x < 5, \\ \cosh(5)/2 & \text{for } x = 5, \\ 0 & \text{for } x > 5 \text{ and for } x = 0. \end{cases}$$

5. The cosine integral is

$$\int_0^\infty \left[\frac{2}{\pi\omega}((2\pi-1)\sin(\pi\omega) + 2\sin(3\pi\omega)) + \frac{4}{\pi\omega^2}(\cos(\pi\omega)-1)\right]\cos(\omega x)\,d\omega,$$

converging to

$$\begin{cases} 1+2x & \text{for } 0 < x < \pi, \\ (3+2\pi)/2 & \text{for } x = \pi, \\ 2 & \text{for } \pi < x < 3\pi, \\ 1 & \text{for } x = 3\pi \text{ and for } x = 0, \\ 0 & \text{for } x > 3\pi. \end{cases}$$

The sine integral is

$$\int_0^\infty \left[\frac{2}{\pi\omega}(1 + (1-2\pi)\cos(\pi\omega) - 2\cos(3\pi\omega)) + \frac{4}{\pi\omega^2}\sin(\pi\omega) \right] \sin(\omega x)\, d\omega,$$

converging to

$$\begin{cases} 1+2x & \text{for } 0 < x < \pi, \\ (3+2\pi)/2 & \text{for } x = \pi, \\ 2 & \text{for } \pi < x < 3\pi, \\ 1 & \text{for } x = 3\pi, \\ 0 & \text{for } x > 3\pi \text{ and for } x = 0. \end{cases}$$

6. The cosine integral is

$$\int_0^\infty \left[\frac{2}{\pi\omega^2}(\cos(2\omega) - 1 + 3\omega\sin(2\omega) - \omega\sin(\omega)) \right] \cos(\omega x)\, d\omega.$$

The sine integral converges to

$$\int_0^\infty \left[\frac{2}{\pi\omega^2}(\sin(2\omega) - 3\omega\cos(2\omega)) + \omega\cos(\omega) \right] \sin(\omega x)\, d\omega.$$

Both integrals converge to

$$\begin{cases} f(x) & \text{for } 0 \le x < 1 \text{ and for } 1 < x < 2 \text{ and } x > 2, \\ 3/2 & \text{for } x = 1 \text{ and for } x = 2. \end{cases}$$

7. The cosine integral is

$$\int_0^\infty \frac{2}{\pi}\left(\frac{2+\omega^2}{4+\omega^4}\right)\cos(\omega x)\, d\omega,$$

converging to

$$\begin{cases} e^{-x}\cos(x) & \text{for } x > 0, \\ 1 & \text{for } x = 0. \end{cases}$$

The sine integral is

$$\int_0^\infty \frac{2}{\pi}\left(\frac{\omega^3}{4+\omega^4}\right)\sin(\omega x)\, d\omega,$$

converging to

$$\begin{cases} e^{-x}\cos(x) & \text{for } x > 0, \\ 0 & \text{for } x = 0. \end{cases}$$

8. The cosine integral is

$$\int_0^\infty \frac{2}{\pi}\left(\frac{9-\omega^2}{(\omega^2+9)^2}\right)\cos(\omega x)\,d\omega.$$

The sine integral is

$$\int_0^\infty \frac{1}{\pi}\left(\frac{36\omega}{(\omega^2+9)^2}\right)\sin(\omega x)\,d\omega.$$

Both integrals converge to $f(x)$ for $x \geq 0$.

9. The cosine representation is

$$\int_0^\infty \frac{2k}{\pi\omega}\sin(c\omega)\cos(\omega x)\,d\omega.$$

The sine integral representation is

$$\int_0^\infty \frac{2k}{\pi\omega}(1-\cos(c\omega))\sin(\omega x)\,d\omega.$$

Both integrals converge to

$$\begin{cases} k & \text{for } 0 < x < c, \\ k/2 & \text{for } x = c, \\ 0 & \text{for } x > c, \end{cases}$$

while the cosine expansion converges to k at 0, and the sine expansion converges to 0 at 0.

10. The cosine expansion is

$$\int_0^\infty \frac{2}{\pi}\left(\frac{\omega^2+5}{(\omega^2+5)^2-4}\right)\cos(\omega x)\,d\omega,$$

and the sine integral representation is

$$\int_0^\infty \frac{2}{\pi}\left(\frac{\omega^3+1}{(\omega^2+5)^2-4}\right)\sin(\omega x)\,d\omega.$$

Both converge to $e^{-2x}\cos(x)$ for $x > 0$, while the sine integral converges to 0 at 0, and the cosine integral converges to 1 at 0.

11. From the Laplace integrals and the convergence theorem, we can write

$$e^{-kx} = \frac{2k}{\pi}\int_0^\infty \frac{1}{k^2+\omega^2}\cos(\omega x) \text{ for } x \geq 0$$

and

$$e^{-kx} = \frac{2}{\pi}\int_0^\infty \frac{\omega}{k^2+\omega^2}\sin(\omega x)\,d\omega \text{ for } x > 0.$$

Put $k = 1$ and interchange the symbols x and ω to obtain

$$A_\omega = \frac{\pi e^{-\omega}}{2} = \int_0^\infty \frac{1}{1+x^2}\cos(\omega x)\,dx$$

and

$$B_\omega = \frac{\pi e^{-\omega}}{2} = \int_0^\infty \frac{x}{1+x^2}\sin(\omega x)\,dx.$$

From these it follows that the Fourier cosine integral representation of $1/(1+x^2)$ is

$$C(x) = \int_0^\infty e^{-\omega}\cos(\omega x)\,d\omega = \frac{1}{1+x^2} \text{ for } x \geq 0$$

and the Fourier sine integral for $x/(1+x^2)$ is

$$S(x) = \int_0^\infty e^{-\omega}\sin(\omega x)\,d\omega = \frac{x}{1+x^2} \text{ for } x > 0.$$

By direct computation, we also have $S(0) = 0$.

14.3 The Fourier Transform

1.

$$\hat{f}(\omega) = \int_{-1}^0 -e^{-i\omega t}\,dt + \int_0^1 e^{-i\omega t}\,dt = \frac{2i}{\omega}(\cos(\omega) - 1).$$

The amplitude spectrum is the graph of

$$|\hat{f}(\omega)| = \left|\frac{2}{\omega}(\cos(\omega) - 1)\right|,$$

shown in Figure 14.1.

2. Write $f(t) = \sin(t)(H(t+k) - H(t-k))$ and use the modulation theorem to write

$$\begin{aligned}\hat{f}(\omega) &= \frac{i}{2}\left[\frac{2\sin(k(\omega+1))}{\omega+1} - \frac{2\sin(k(\omega-1))}{\omega-1}\right]\\ &= \left[\frac{\sin(k(\omega+1))}{\omega+1} - \frac{\sin(k(\omega-1))}{\omega-1}\right]i.\end{aligned}$$

Figure 14.2 is a graph of the amplitude spectrum with $k = \sqrt{7}$.

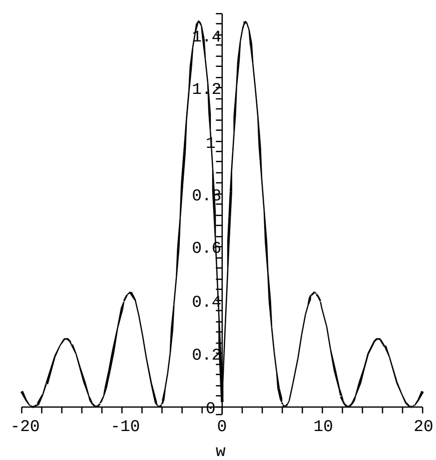

Figure 14.1: Amplitude spectrum in Problem 1, Section 14.3.

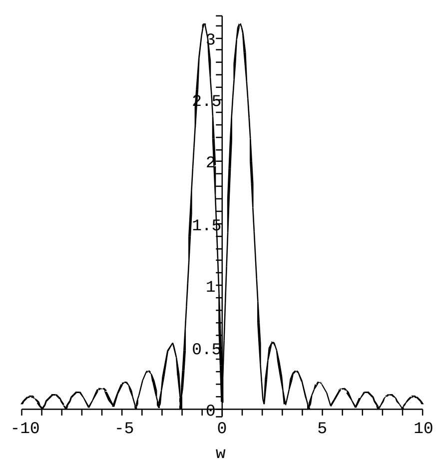

Figure 14.2: Amplitude spectrum in Problem 2, Section 14.3.

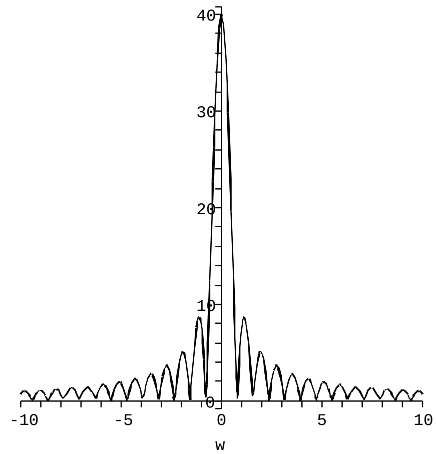

Figure 14.3: Amplitude spectrum in Problem 3, Section 14.3.

3. $f(t) = 5(H(t-3) - H(t-11)) = 5(H(t+4-7) - H(t+4+7))$, so

$$\hat{f}(\omega) = 5e^{-7i\omega}\left(\frac{2\sin(4\omega)}{\omega}\right) = \frac{10}{\omega}e^{-7i\omega}\sin(4\omega).$$

The amplitude spectrum is the graph of

$$|\hat{f}(\omega)|(\omega) = \left|\frac{10}{\omega}\sin(4\omega)\right|,$$

shown in Figure 14.3.

4. By time shifting,

$$\hat{f}(\omega) = 5\sqrt{\frac{\pi}{3}}e^{-\omega^2/12}e^{-5i\omega}.$$

Then

$$|\hat{f}(\omega)| = 5\sqrt{\frac{\pi}{3}}e^{-\omega^2/12}.$$

A graph of this function is given in Figure 14.4.

5.

$$\begin{aligned}\hat{f}(\omega) &= \int_k^\infty e^{-t/4}e^{-i\omega t}\,dt \\ &= \left.\frac{e^{-(i\omega+1/4)t}}{-(i\omega+1/4)}\right]_k^\infty = \frac{4e^{-(i\omega+1/4)k}}{1+4i\omega}.\end{aligned}$$

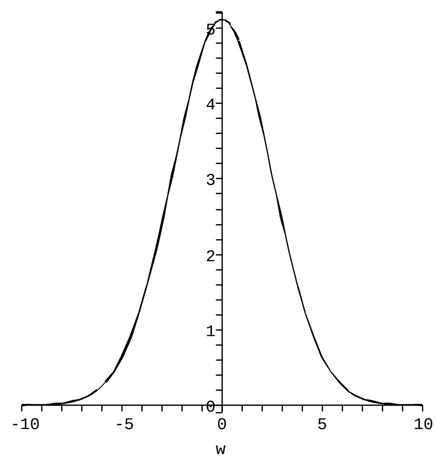

Figure 14.4: Amplitude spectrum in Problem 4, Section 14.3.

Then

$$|\hat{f}(\omega)|(\omega) = \frac{4e^{-k/4}}{\sqrt{1+16\omega^2}}.$$

The amplitude spectrum is shown in Figure 14.5 for $k = 4$.

6.

$$\hat{f}(\omega) = \frac{2}{\omega^3}(k^2\omega^2 \sin(k\omega) + 2k\omega\cos(k\omega) - 2\sin(k\omega)).$$

Figure14.6 shows the amplitude spectrum for $k = 2$ and $\omega > 0$.

7.

$$\hat{f}(\omega) = \pi e^{-|\omega|}.$$

The amplitude spectrum is shown in Figure 14.7.

8. Write

$$f(t) = 3e^{-6}H(t-2)e^{-3(t-2)}$$

so

$$\hat{f}(\omega) = 3e^{-6}\left(\frac{e^{-2i\omega}}{3+i\omega}\right).$$

We can also write

$$\hat{f}(\omega) = \frac{e^{-2(3+i\omega)}}{3+i\omega}.$$

Then

$$|\hat{f}(\omega)| = \frac{3e^{-6}}{9+\omega^2}.$$

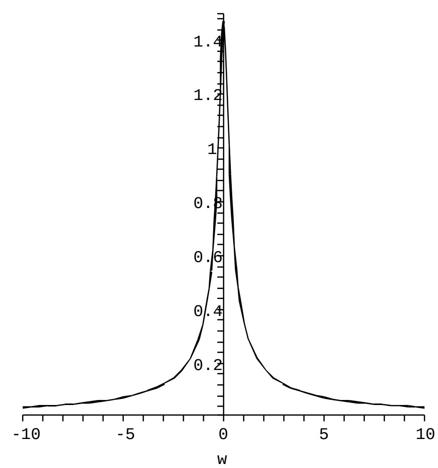

Figure 14.5: Amplitude spectrum in Problem 5, Section 14.3.

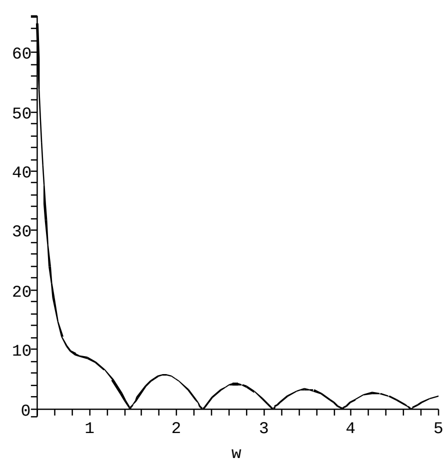

Figure 14.6: Amplitude spectrum in Problem 6, Section 14.3.

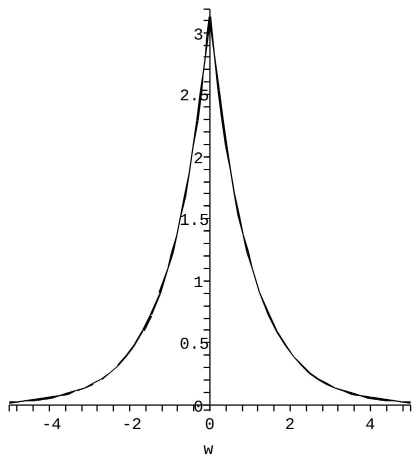

Figure 14.7: Amplitude spectrum in Problem 7, Section 14.3.

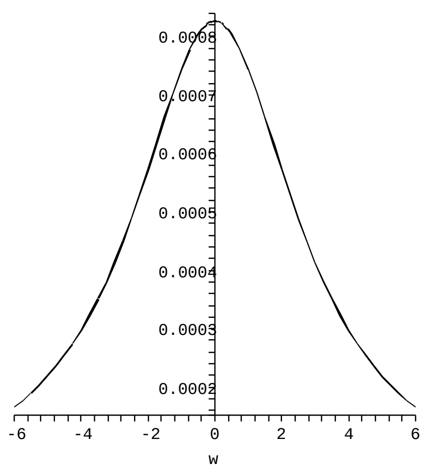

Figure 14.8: Amplitude spectrum in Problem 8, Section 14.3.

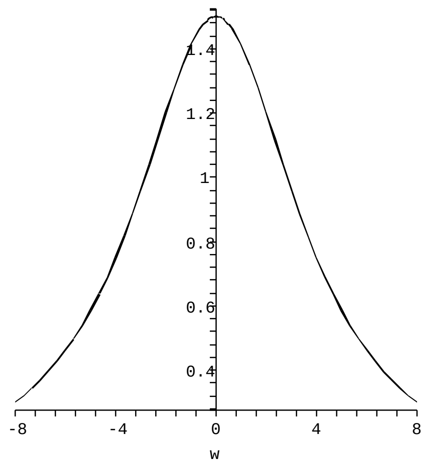

Figure 14.9: Amplitude spectrum in Problem 9, Section 14.3.

A graph of this amplitude spectrum is given in Figure 14.8.

9.
$$\hat{f}(\omega) = \frac{24}{16+\omega^2}e^{2i\omega}$$

The amplitude spectrum is the graph of

$$|\hat{f}(\omega)| = \frac{24}{16+\omega^2},$$

shown in figure 14.9.

10. Similar to Problem 8, write

$$f(t) = e^{-6}H(t-3)e^{-2(t-3)},$$

so

$$\hat{f}(\omega) = \frac{e^{-3(2+i\omega)}}{2+i\omega}.$$

Then

$$|\hat{f}(\omega)| = \frac{e^{-6}}{4+\omega^2}.$$

The amplitude spectrum has the same appearance as that of Problem 8.

11.
$$f(t) = 18\sqrt{\frac{2}{\pi}}e^{-4it}e^{-8t^2}$$

12. Write

$$\hat{f}(\omega) = \frac{e^{-4(\omega-5)i}}{3+(\omega-5)i},$$

so

$$f(t) = e^{5it}\hat{f}^{-1}\left[\frac{e^{-4i\omega}}{3+i\omega}\right] = e^{5it}H(t-4)e^{-3(t-4)}.$$

13. Write

$$\hat{f}(\omega) = \frac{e^{2(\omega-3)i}}{5+(\omega-3)i},$$

so

$$\begin{aligned} f(t) &= e^{3it}\hat{f}^{-1}\left[\frac{e^{2i\omega}}{5+i\omega}\right] \\ &= e^{3it}H(t+2)e^{-5(t+2)} = H(t+2)e^{-(10+(5-3i)t)}. \end{aligned}$$

14. Write

$$\hat{f}(\omega) = \frac{10\sin(3\omega)}{\omega+\pi} = -\frac{10\sin(3(\omega+\pi))}{\omega+\pi},$$

so

$$f(t) = -5e^{-\pi it}\hat{f}^{-1}\left[\frac{2\sin(3\omega)}{\omega}\right] = 5e^{-\pi it}(H(t+3)-H(t-3)).$$

15. Write

$$\hat{f}(\omega) = \frac{1+i\omega}{(3+i\omega)(2+i\omega)} = \frac{2}{3+i\omega} - \frac{1}{2+i\omega}.$$

Then

$$f(t) = H(t)(2e^{-3t} - e^{-2t}).$$

16.

$$\begin{aligned} \hat{f}^{-1}\left[\frac{1}{(1+i\omega)(2+i\omega)}\right] &= H(t)e^{-t} * H(t)e^{-2t} \\ &= \int_{-\infty}^{\infty} H(\tau)e^{-\tau}H(t-\tau)e^{-2(t-\tau)} \\ &= H(t)e^{-2t}\int_0^t e^{\tau}\,d\tau = H(t)e^{-2t}(e^t-1) \\ &= H(t)(e^{-t}-e^{-2t}). \end{aligned}$$

17.

$$\begin{aligned} \hat{f}^{-1}\left(\frac{1}{(1+i\omega)^2}\right) &= H(t)e^{-t} * H(t)e^{-t} \\ &= \int_{-\infty}^{\infty} H(\tau)e^{-\tau}H(t-\tau)e^{-(t-\tau)}\,d\tau \\ &= H(t)e^{-t}\int_0^t d\tau = H(t)te^{-t}. \end{aligned}$$

18.

$$\begin{aligned}\hat{f}^{-1}\left[\frac{\sin(3\omega)}{(2+i\omega)\omega}\right] &= \frac{1}{2}(H(t+3)-H(t-3)) * H(t)e^{-2t} \\ &= \frac{1}{2}\int_{-\infty}^{\infty}(H(t+3)-H(t-3))H(t-\tau)e^{-2(t-\tau)}\,d\tau \\ &= \frac{1}{2}e^{-2t}\left[H(t+3)\int_{-3}^{t}e^{2\tau}\,d\tau - H(t-3)\int_{3}^{t}e^{2\tau}\,d\tau\right] \\ &= \frac{1}{4}(1-e^{-2(t+3)})H(t+3) - \frac{1}{4}(1-e^{-2(t-3)})H(t-3)\end{aligned}$$

19. Compute

$$\int_{-\infty}^{\infty}|f(t)|^2\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{f}(\omega)\overline{\hat{f}(\omega)}\,d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty}|\hat{f}(\omega)|^2\,d\omega.$$

20. One way to compute this energy is to start with

$$\hat{f}(\omega)[H(t)e^{-2t}](\omega) = \frac{1}{2+i\omega}.$$

By Parseval's theorem (Problem 19),

$$\begin{aligned}\int_{-\infty}^{\infty}|f(t)|^2\,dt &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left|\frac{1}{2+i\omega}\right|^2 d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{4+\omega^2}\,d\omega \\ &= \frac{1}{4\pi}\arctan^{-1}\left(\frac{\omega}{2}\right)\Big]_{-\infty}^{\infty} = \frac{1}{4}.\end{aligned}$$

Another way to compute the same result is to proceed directly:

$$\int_{-\infty}^{\infty}(H(t)e^{-2t})^2\,dt = \int_{0}^{\infty}e^{-2t}\,dt = \frac{1}{4}.$$

In this example the direct computation is clearly simpler, but for some problems it is useful to be aware of this use of Parseval's theorem.

21. Begin with

$$\begin{aligned}\hat{f}\left[\frac{H(t+3)-H(t-3)}{2}\right](\omega) &= \frac{1}{2}\int_{-3}^{3}e^{-i\omega t}\,dt \\ &= \frac{e^{3i\omega}-e^{-3i\omega}}{2i\omega} = \frac{\sin(3\omega)}{\omega}.\end{aligned}$$

Using the symmetry property of the transform,

$$\begin{aligned}\hat{f}\left[\frac{\sin(3t)}{t}\right](\omega) &= \pi[H(-\omega+3)-H(-\omega-3)] \\ &= \pi[H(\omega+3)-H(\omega-3)].\end{aligned}$$

Now use Parseval's identity to write

$$\int_{-\infty}^{\infty} \left(\frac{\sin(3t)}{t}\right)^2 dt = \frac{1}{2\pi}\int_{-3}^{3} \pi^2 \, d\omega = 3\pi.$$

22. Let $\hat{y}(\omega) = \hat{f}[y(t)](\omega)$ and transform the differential equation to obtain

$$\hat{y}(-\omega^2 + 6i\omega + 5) = \hat{f}[\delta(t-3)](\omega) = e^{-3i\omega}.$$

Then

$$\hat{y}(\omega) = \frac{e^{-3i\omega}}{-\omega^2 + 6i\omega + 5}$$
$$= \frac{e^{-3i\omega}}{(1+i\omega)(5+i\omega)} = \frac{1}{4}\left[\frac{e^{-3i\omega}}{1+i\omega} - \frac{e^{-3i\omega}}{5+i\omega}\right].$$

Invert this to obtain the solution

$$y(t) = \frac{1}{4}(H(t-3)e^{-(t-3)} - H(t-3)e^{-5(t-3)}).$$

23. Compute

$$\hat{f}_{\text{win}}(\omega) = \int_{-5}^{5} t^2 e^{-i\omega t} \, dt$$
$$= \frac{2}{\omega^3}(25\omega^2 \sin(5\omega) + 10\omega\cos(5\omega) - 2\sin(5\omega)).$$

Since $w(t) = 1$ and the support of g is $[-5, 5]$, then $t_C = 0$. For the RMS bandwidth of the window function,

$$w_{\text{RMS}} = 2\left(\frac{\int_{-5}^{5} t^2 \, dt}{\int_{-5}^{5} \, dt}\right)^{1/2} = \frac{10}{\sqrt{3}}.$$

24. Compute

$$\hat{f}_{\text{win}}(\omega) = \int_{-4\pi}^{4\pi} \cos(at)e^{-i\omega t} \, dt$$
$$= \frac{2}{\omega^2 - a^2}(\omega\sin(4\pi\omega)\cos(4a\pi) - a\cos(4\pi\omega)\sin(4a\pi)).$$

Since $w(t)$ is constant on $[-4\pi, 4\pi]$, $t_C = 0$. We also have

$$w_{\text{RMS}} = 2\left(\frac{\int_{-4\pi}^{4\pi} t^2 \, dt}{\int_{-4\pi}^{4\pi} \, dt}\right)^{1/2} = \frac{8\pi}{\sqrt{3}}.$$

25. Compute

$$\begin{aligned}
\hat{f}_{\text{win}}(\omega) &= \int_0^4 e^{-t} e^{-i\omega t}\, dt = \frac{1}{1+i\omega}(1 - e^{-4(1+i\omega)}) \\
&= \frac{1}{1+\omega^2}(1 - e^{-4}(\cos(4\omega) - i\sin(4\omega))(1 - i\omega) \\
&= \frac{1 - e^{-4}\cos(4\omega) + e^{-4}\sin(4\omega)}{1+\omega^2} \\
&\quad + i\left[\frac{e^{-4}\sin(4\omega) + (e^{-4}\cos(4\omega) - 1)\omega}{1+\omega^2}\right].
\end{aligned}$$

We also have

$$t_C = \frac{\int_0^4 t\, dt}{\int_0^4 dt} = 2$$

and

$$w_{\text{RMS}} = 2\left(\frac{\int_0^4 (t-2)^2\, dt}{\int_0^4 dt}\right)^{1/2} = \frac{4}{\sqrt{3}}.$$

26. Compute

$$\begin{aligned}
\hat{f}_{\text{win}}(\omega) &= \int_{-1}^1 e^t \sin(\pi t) e^{-i\omega t}\, dt \\
&= \int_{-1}^1 \sin(\pi t) e^{(1-i\omega)t}\, dt \\
&= \frac{\pi\left[2\sinh(1)(1+\pi^2) - 2\omega^2\cos(\omega) + \cosh(1)\omega\sin(\omega)\right]}{(1+(\pi+\omega)^2)(1+(\pi-\omega)^2)} \\
&\quad + i\frac{\pi\left[\sinh(1)(2\omega^2\sin(\omega) - (2+2\pi^2)\sin(\omega)) + \cosh(1)\omega\cos(\omega)\right]}{(1+(\pi+\omega)^2)(1+(\pi-\omega)^2)}.
\end{aligned}$$

Finally, compute $t_C = 0$ and

$$w_{\text{RMS}} = 2\left(\frac{\int_{-1}^1 t^2\, dt}{\int_{-1}^1 dt}\right)^{1/2} = \frac{2}{\sqrt{3}}.$$

27. First,

$$\begin{aligned}
\hat{f}_{\text{win}}(\omega) &= \int_{-2}^2 (t+2)^2 e^{-i\omega t}\, dt \\
&= \frac{4}{\omega^3}\left((4\omega^2 - 1)\sin(2\omega) + 2\omega\cos(2\omega)\right) \\
&\quad + \frac{8i}{\omega^2}\left(2\omega\cos(2\omega) - \sin(2\omega)\right).
\end{aligned}$$

With $w(t) = 1$ and support $[-2, 2]$, $t_C = 0$. Finally,

$$w_{\text{RMS}} = 2\left(\frac{\int_{-2}^{2} t^2\,dt}{\int_{-\infty}^{\infty} dt}\right)^{1/2} = \frac{4}{\sqrt{3}}.$$

28. We have

$$\begin{aligned}
\hat{f}_{\text{win}}(\omega) &= \int_{3\pi}^{5\pi} e^{-i\omega t}\,dt \\
&= -\frac{1}{i\omega}(e^{5\pi i\omega} - e^{3\pi i\omega}) \\
&= -\frac{2e^{\pi i\omega}}{\omega}\left[\frac{e^{4\pi i\omega} - e^{-4\pi i\omega}}{2i}\right] = -2e^{\pi i\omega}\sin(4\pi\omega) \\
&= -2\cos(\pi\omega)\sin(4\pi\omega) - 2i\sin(\pi\omega)\sin(4\pi\omega).
\end{aligned}$$

Finally,

$$t_C = \frac{\int_{3\pi}^{5\pi} t\,dt}{\int_{3\pi}^{5\pi} dt} = 4\pi$$

and

$$w_{\text{RMS}} = 2\left(\frac{\int_{3\pi}^{5\pi}(t-4\pi)^2\,dt}{\int_{3\pi}^{5\pi} dt}\right)^{1/2} = \frac{2\pi}{\sqrt{3}}.$$

14.4 Fourier Cosine and Sine Transforms

In these problems the integrations are straightforward and are omitted.

1.

$$\hat{f}_C(\omega) = \int_0^{\infty} e^{-t}\cos(\omega t)\,dt = \frac{1}{1+\omega^2}$$

$$\hat{f}_S(\omega) = \int_0^{\infty} e^{-t}\sin(\omega t)\,dt = \frac{\omega}{1+\omega^2}$$

2.

$$\hat{f}_C(\omega) = \frac{a^2 - \omega^2}{(a^2+\omega^2)^2}$$

$$\hat{f}_S(\omega) = \frac{2a\omega}{(a^2+\omega^2)^2}$$

3.

$$\hat{f}_C(\omega) = \frac{1}{2}\left[\frac{\sin(K(\omega+1))}{\omega+1} + \frac{\sin(K(\omega-1))}{\omega-1}\right] \text{ for } \omega \neq \pm 1$$

$$\hat{f}_C(1) = \hat{f}_C(-1) = \frac{K}{2} + \frac{1}{2}\sin(2K)$$

$$\hat{f}_S(\omega) = \frac{\omega}{\omega^2 - 1} - \frac{1}{2}\left[\frac{\cos((\omega+1)K)}{\omega+1} + \frac{\cos((\omega-1)K)}{\omega-1}\right] \text{ for } \omega \neq \pm 1$$

$$\hat{f}_S(1) = \frac{1}{4}(1 - \cos(2K)), \hat{f}_S(-1) = -\frac{1}{4}(1 - \cos(2K))$$

4.

$$\hat{f}_C(\omega) = \frac{1}{\omega}(2\sin(K\omega) - \sin(2K\omega))$$

$$\hat{f}_S(\omega) = \frac{1}{\omega}(1 - 2\cos(K\omega) + \cos(2K\omega))$$

5.

$$\hat{f}_C(\omega) = \frac{1}{2}\left[\frac{1}{1+(\omega+1)^2} + \frac{1}{1+(\omega-1)^2}\right]$$

$$\hat{f}_S(\omega) = \frac{1}{2}\left[\frac{\omega+1}{1+(\omega+1)^2} + \frac{\omega-1}{1+(\omega-1)^2}\right]$$

6.

$$\begin{aligned}\hat{f}_C(\omega) = &\frac{1}{1+\omega^2}(\cosh(2K)\cos(2K\omega) - \cosh(K)\cos(K\omega)) \\ &+ \frac{1}{1+\omega^2}(\omega\sinh(2K)\sin(2K\omega) - \omega\sinh(K)\sin(K\omega))\end{aligned}$$

and

$$\begin{aligned}\hat{f}_S(\omega) = &\frac{1}{1+\omega^2}\left(\cosh(2K)\cosh(2K\omega) - \cosh(K)\sin(K\omega)\right) \\ &+ \frac{1}{1+\omega^2}\left(-\omega\sinh(2K)\cos(2K\omega) + \omega\sinh(K)\cos(K\omega)\right).\end{aligned}$$

7. Suppose for each $L > 0$, $f^{(4)}(t)$ is piecewise continuous on $[0, L]$, $f^{(3)}(t)$ is continuous, and, as $t \to \infty$, $f^{(3)}(t) \to 0$, $f''(t) \to 0$ and $f(t) \to 0$. Then we can integrate by parts four times to obtain

$$\begin{aligned}&\mathcal{F}_S[f^{(4)}(t)](\omega) = \int_0^\infty f^{(4)}(t)\sin(\omega t)\,dt \\ &= \left[f^{(3)}(t)\sin(\omega t) - \omega f''(t)\cos(\omega t) - \omega^2 f'(t)\sin(\omega t) + \omega^3\cos(\omega t)f(t)\right]_0^\infty \\ &+ \omega^4\int_0^\infty \sin(\omega t)f(t)\,dt \\ &= \omega^4\hat{f}_S(\omega) - \omega^3 f(0) + \omega f''(0).\end{aligned}$$

8. Under the same conditions as in the solution to Problem 7, four integrations by parts give us

$$\mathcal{F}_C[f^{(4)}(t)](\omega) = \int_0^\infty f^{(4)}(t)\cos(\omega t)\,dt$$
$$= \left[f^{(3)}(t)\cos(\omega t) + \omega f''(t)\sin(\omega t) - \omega^2 f'(t)\cos(\omega t) - \omega^3 f(t)\sin(\omega t)\right]_0^\infty$$
$$+ \omega^4 \int_0^\infty f(t)\cos(\omega t)\,dt$$
$$= \omega^4 \hat{f}_C(\omega) + \omega^2 f'(0) - f^{(3)}(0).$$

14.5 The Discrete Fourier Transform

The six point discrete Fourier transform of $u(j)$ is calculated by

$$\mathcal{D}[u](k) = \sum_{j=0}^{5} u(j)e^{-\pi kji/3}$$

for $k = -4, -3, -2, -1, 0, 1, 2, 3, 4$. For Problems 1 through 6, these values were computed using MAPLE and rounded to the five decimal places.

1.

$$\begin{aligned}
&\mathcal{D}[u](-4) \approx 1.3292 - 0.01658i,\\
&\mathcal{D}[u](-3) \approx 0.09624 + 0.72830(10^{-9})i,\\
&\mathcal{D}[u](-2) \approx 0.13292 + 0.01658i,\\
&\mathcal{D}[u](-1) \approx 2.93687 + 0.42794i,\\
&\mathcal{D}[u](0) \approx 1.82396 + 0i,\\
&\mathcal{D}[u](1) \approx 2.93687 - 0.42794i,\\
&\mathcal{D}[u](2) \approx 0.13292 - 0.01658i,\\
&\mathcal{D}[u](3) \approx 0.09624 - 0.72830(10^{-9})i,\\
&\mathcal{D}[u](4) \approx 0.13292 + 0.01658i
\end{aligned}$$

2.

$$\begin{aligned}
&\mathcal{D}[u](-4) \approx 0.24922 + 0.10702i,\\
&\mathcal{D}[u](-3) \approx 0.09624 + 0.12883i,\\
&\mathcal{D}[u](-2) \approx 0.01662 + 0.14018i,\\
&\mathcal{D}[u](-1) \approx -0.06520 + 0.15184i,\\
&\mathcal{D}[u](0) \approx 1.82396 + 8.17616i,\\
&\mathcal{D}[u](1) \approx 5.93894 - 0.70403i,\\
&\mathcal{D}[u](2) \approx 0.2492 + 0.10702i,\\
&\mathcal{D}[u](3) \approx 0.09624 + 0.12883i,\\
&\mathcal{D}[u](4) \approx 0.01662 + 0.14018i
\end{aligned}$$

3.

$$\begin{aligned}
&\mathcal{D}[u](-4) \approx 0.65000 - 0.17321i,\\
&\mathcal{D}[u](-3) \approx 0.61667 - 0.25346(10^{-9})i,\\
&\mathcal{D}[u](-2) \approx 0.65000 + 0.17321i,\\
&\mathcal{D}[u](-1) \approx 0.81667 + 0.40415i,\\
&\mathcal{D}[u](0) \approx 2.45000 + 0i,\\
&\mathcal{D}[u](1) \approx 0.81667 - 0.40415i,\\
&\mathcal{D}[u](2) \approx 0.65000 - 0.17321i,\\
&\mathcal{D}[u](3) \approx 0.61667 + 0.25346(10^{-9})i,\\
&\mathcal{D}[u](4) \approx 0.65000 + 0.17321i
\end{aligned}$$

4.

$$\begin{aligned}
&\mathcal{D}[u](-4) \approx 0.84806 - 0.13087i,\\
&\mathcal{D}[u](-3) \approx 0.81083 - 0.14161(10^{-9})i,\\
&\mathcal{D}[u](-2) \approx 0.84806 + 0.13087i,\\
&\mathcal{D}[u](-1) \approx 1.0008 + 0.25403i,\\
&\mathcal{D}[u](0) \approx 1.49139 + 0i,\\
&\mathcal{D}[u](1) \approx 1.0008 - 0.25303i,\\
&\mathcal{D}[u](2) \approx 0.84806 - 0.13087i,\\
&\mathcal{D}[u](3) \approx 0.81083 + 0.14161(10^{-9})i,\\
&\mathcal{D}[u](4) \approx 0.84806 + 0.13087i
\end{aligned}$$

5.

$$\begin{aligned}
&\mathcal{D}[u](-4) \approx -14.00000 + 10.39230i,\\
&\mathcal{D}[u](-3) \approx -15.00000 + 0.22023(10^{-7})i,\\
&\mathcal{D}[u](-2) \approx -14.00000 - 10.39230i,\\
&\mathcal{D}[u](-1) \approx -6.00000 - 31.17691i,\\
&\mathcal{D}[u](0) \approx 55.00000 + 0i,\\
&\mathcal{D}[u](1) \approx -6.00000 + 31.17691i,\\
&\mathcal{D}[u](2) \approx -14.00000 + 10.39230i,\\
&\mathcal{D}[u](3) \approx -15.00000 - 0.22023(10^{-7})i,\\
&\mathcal{D}[u](4) \approx -14.00000 - 10.39230i
\end{aligned}$$

6.

$$\begin{aligned}
&\mathcal{D}[u](-4) \approx 0.00932 + 0.09972i,\\
&\mathcal{D}[u](-3) \approx -0.03259 + 0.21350(10^{-8})i,\\
&\mathcal{D}[u](-2) \approx 0.00932 - 0.09972i,\\
&\mathcal{D}[u](-1) \approx 3.21296 - 2.57414i,\\
&\mathcal{D}[u](0) \approx -0.41198 + 0i,\\
&\mathcal{D}[u](1) \approx 3.21296 + 2.57414i,\\
&\mathcal{D}[u](2) \approx 0.00932 + 0.09972i,\\
&\mathcal{D}[u](3) \approx -0.03259 - 0.21350(10^{-8})i,\\
&\mathcal{D}[u](4) \approx 0.00932 - 0.09972i
\end{aligned}$$

For Problems 7 through 12, the $N-$ point inverse discrete Fourier transform of the sequence $[U_j]_{j=0}^{N-1}$ is the sequence computed by

$$u_j = \frac{1}{N}\sum_{k=0}^{N-1} U_k e^{2\pi ijk/N}.$$

Values were computed using MAPLE to nine decimal places, with results recorded below to six places.

7. For the given sequence, $N = 6$ and

$$u_j = \frac{1}{6}\sum_{k=0}^{5}(1+i)^k e^{2\pi ijk/6}.$$

We obtain

$$\begin{aligned}
u_0 &\approx -1.333333 + 0.166667i,\\
u_1 &\approx -0.427030 + 0.549038i,\\
u_2 &\approx -0.016346 + 0.561004i,\\
u_3 &\approx 0.333333 + 0.5000000i,\\
u_4 &\approx 0.849679 + 0.272329i,\\
u_5 &\approx 1.593696 - 2.049038i.
\end{aligned}$$

8. Here, $N = 6$ and

$$u_j = \frac{1}{5}\sum_{k=0}^{4}(i^{-k})e^{2n\pi ijk/5}.$$

We obtain

$$\begin{aligned}
u_(0) &\approx 0.200000,\\
u_1 &\approx 0.731375 - 0.531375i,\\
u_2 &\approx -0.096262 + 0.296261i,\\
u_3 &\approx 0.049047 + 0.150953i,\\
u_4 &\approx 0.115838 + 0.084162.
\end{aligned}$$

9. $N = 7$ and

$$u_j = \frac{1}{7}\sum_{k=0}^{6}(e^{-ik})e^{2n\pi ijk/7}.$$

Approximate values are

$$\begin{aligned}
u_0 &\approx 0.103479 + 0.014751i,\\
u_1 &\approx 0.933313 - 0.296094,\\
u_2 &\approx -0.094163 + 0.088785i,\\
u_3 &\approx -0.023947 + 0.062482i,\\
u_4 &\approx 0.004307 + 0.051899i,\\
u_5 &\approx 0.025788 + 0.043852i,\\
u_6 &\approx 0.051222 + 0.034325i.
\end{aligned}$$

10. $N = 5$ and

$$u_j = \frac{1}{5}\sum_{k=0}^{4}(k^2)e^{2\pi ijk/5}.$$

Approximate values are

$$\begin{aligned}
u_0 &\approx 6.000000,\\
u_1 &\approx -1.052786 - 3.440955i,\\
u_2 &\approx -1.947214 - 0.812299i,\\
u_3 &\approx -1.947214 + 0.812299i,\\
u_4 &\approx -1.052786 + 3.440955i.
\end{aligned}$$

11. $N = 5$ and

$$u_j = \frac{1}{5}\sum_{k=0}^{4}(\cos(k))e^{2\pi ijk/5}.$$

Approximate values are

$$\begin{aligned}
u_0 &\approx -0.103896,\\
u_1 &\approx 0.420513 + 0.294562i,\\
u_2 &\approx 0.131434 + 0.031205i,\\
u_3 &\approx 0.131434 - 0.031205i,\\
u_4 &\approx 0.420513 - 0.294562i.
\end{aligned}$$

12. $N = 6$ and

$$u_j = \frac{1}{6}\sum_{k=0}^{5}\ln(k+1)e^{2\pi ijk/6}.$$

Approximate values are

$$\begin{aligned}
u_0 &\approx 1.096542,\\
u_1 &\approx -0.249644 - 0.232302i,\\
u_2 &\approx -0.201697 - 0.084840i,\\
u_3 &\approx -0.193858,\\
u_4 &\approx -0.201697 + 0.084840i,\\
u_5 &\approx -0.249644 + 0.232302i.
\end{aligned}$$

For Problems 13 through 16 the complex Fourier coefficients of the function $f(t)$ having period p are calculated by

$$d_k = \frac{1}{p}\int_0^p f(t)e^{-2\pi ikt}\,dt, k = -3, -2, \cdots, 2, 3.$$

The DFT $N = 2^7 = 128$ is used to approximate these coefficients, using

$$f_k = \frac{1}{128}\sum_{j=0}^{127} f\left(\frac{jp}{128}\right)e^{-2\pi ijk/128}$$

for $k = -3, -2, -1, 0, 1, 2, 3$. These values were computed using MAPLE to nine decimal places and are given below rounded to six places.

k	d_k	f_k
-3	$-0.005177 + 0.075984i$	$0.000346 + 0.075849i$
-2	$-0.011816 + 0.115622i$	$-0.006293 + 0.115532i$
-1	$-0.051259 + 0.250780i$	$-0.045737 + 0.250753i$
0	0.454649	0.460171
1	$-0.051259 - 0.250798i$	$-0.045737 - 0.250753i$
2	$-0.011816 - 0.115622i$	$-0.006293 - 0.115532i$
3	$-0.005177 - 0.075984i$	$0.000346 - 0.075849i$

Table 14.1: Approximate values in Problem 13, Section 14.5.

k	d_k	f_k
-3	$0.007825 + 0.049165i$	$0.11551 + 0.049074i$
-2	$0.017079 + 0.071538i$	$0.020805 + 0.071478i$
-1	$0.058802 + 0.123155i$	$0.062528 + 0.123125i$
0	0.316738	0.320464
1	$0.058802 - 0.123155i$	$0.062528 - 0.123125i$
2	$0.017079 - 0.071538i$	$0.020804 - 0.071478i$
3	$0.007825 - 0.049165i$	$0.011551 - 0.049074i$

Table 14.2: Approximate values in Problem 14, Section 14.5.

13. $f(t) = \cos(t)$, $p = 2$, and

$$\begin{aligned} d_k &= \frac{1}{2}\int_0^2 \cos(t)e^{-i\pi kt}\,dt \\ &= -\frac{\sin(2)}{2(\pi^2k^2-1)} + \frac{ki(\cos(2)-1)}{2(\pi^2k^2-1)}i. \end{aligned}$$

DFT approximate values are given in Table 14.1.

14. $f(t) = e^{-t}, p = 3$ and

$$d_k = \frac{1}{3}\int_0^3 e^{-t}e^{-2\pi ikt/3}\,dt = \frac{3(1-e^{-3})}{9+4k^2\pi^2} - \frac{2k\pi(1-e^{-3})}{9+4k^2\pi^2}.$$

Table 14.2 lists DFT approximate values.

15. $f(t) = t^2$, $p = 1$ and

$$f_k = \int_0^1 t^2 e^{-2\pi ikt}\,dt = \frac{1}{2k^2\pi^2} + \frac{1}{2k\pi}i.$$

Table 14.3 lists DFT approximate values.

k	d_k	f_k
-3	$0.005629 - 0.053051i$	$0.001733 - 0.052956i$
-2	$0.012665 - 0.078577i$	$0.008769 - 0.079514i$
-1	$0.050661 - 0.159155i$	$0.046765 - 0.159123i$
0	0.333333	0.329437
1	$0.050661 + 0.159155i$	$0.046765 + 0.159123i$
2	$0.012665 + 0.079577i$	$0.008769 + 0.079514i$
3	$0.005629 + 0.053052i$	$0.001733 + 0.052956i$

Table 14.3: Approximate values in Problem 15, Section 14.5.

k	d_k	f_k
-3	$247.246215 - 515.579355i$	$201.215105 - 514.436038i$
-2	$452.586443 - 626.547636i$	$406.555000 - 625.785580i$
-1	$894.543813 - 612.101891i$	$848.512176 - 611.720909i$
0	1304.231619	1258.199915
1	$894.543813 + 612.1018911i$	$848.512177 + 611.720909i$
2	$452.586443 + 626.547636i$	$406.555000 + 625.785580i$
3	$247.246215 + 515.579355i$	

Table 14.4: Approximate values in Problem 16, Section 14.5.

16. $f(t) = te^{2t}$, $p = 4$, and

$$\begin{aligned} d_k &= \frac{1}{4}\int_0^4 te^{2t}e^{-2\pi ikt/2}\,dt \\ &= \frac{7e^8 + (16 - k^2\pi^2) + 16e^8k^2\pi^2}{(16 - k^2\pi^2)^2 + 64k^2\pi^2} + \frac{56e^8 - 2e^8k\pi(16 - k^2\pi^2) + 8k\pi}{(16 - k^2\pi^2)^2 + 64k^2\pi^2}i. \end{aligned}$$

DFT approximate values are given in Table 14.4.

14.6 Sampled Fourier Series

In Problems 1 - 6, the complex Fourier coefficients of $f(t)$, a function of period p, are computed using

$$d_n = \frac{1}{p}\int_0^p f(t)e^{-2k\pi it/p}\,dt.$$

The $10th$ partial sum of the series is formed and evaluated at t_0 to yield $S_{10}(t_0)$. Next, using $N = 128$, the DFT approximation is $S_{10}(t_0)$ requires the values $U_{n\,n=0}^{\,10}$ computed by

$$U_n = \sum_{j=0}^{127} f\left(\frac{jp}{128}\right)e^{-2\pi ijn/128}.$$

Then, with

$$V_n = U_n \text{ for } n = 0, 1, \cdots, 10, 118, 119, \cdots, 127$$

and

$$V_n = 0 \text{ for } 11 \leq n \leq 117,$$

we obtain the DFT approximation

$$w = \frac{1}{128}\sum_{k=0}^{127} V_k e^{2\pi i k t_0/p}.$$

The nonzero values if U_n (to six decimal places) are recorded below for each problem, followed by the DFT approximation w and the difference

$$\text{var} = |S_{10}(t_0) - w|$$

between the actual value and the DFT approximate value.

1. Compute $d_0 = 2$ and, for $n \neq 0$m

$$d_n = \frac{1}{2}\int_0^2 (1+t)e^{-n\pi i t}\,dt = \frac{1-2n\pi i}{n^2\pi^2}.$$

The complex Fourier expansion of $f(t)$ is

$$2 + \sum_{n=-\infty, n\neq 0}^{\infty} \frac{1-2n\pi i}{n^2\pi^2} e^{n\pi i t}.$$

The tenth partial sum at 1/8 is

$$S_{10}(1/8) \approx 1.020712.$$

For the DFT approximation we have Using these, compute

$$w \approx 1.055233 + 0.278759(10^{-9})i.$$

Finally,

$$\text{var} \approx |S_{10}(1/8) - w| \approx 0.034520.$$

Values of $\mathbf{U}_n$ are given in Table 14.5.

2. Compute

$$d_0 = \frac{1}{3}, d_n = \int_0^1 t^2 e^{-2\pi i n t}\,dt = \frac{2n\pi + (2n^2\pi^2 - 1)i}{n^3\pi^3}.$$

The complex Fourier series is

$$\frac{1}{3} + \sum_{n=-\infty, n\neq 0}^{\infty} \frac{2n\pi + (2n^2\pi^2-1)i}{n^3\pi^3} e^{2n\pi i t}.$$

$\mathbf{U}_0$	255		
$\mathbf{U}_1$	$-1 + 40.735481i$	$\mathbf{U}_{118}$	$-1 - 3.992224i$
$\mathbf{U}_2$	$-1 + 20.355468i$	$\mathbf{U}_{119}$	$-1 - 4.453202i$
$\mathbf{U}_3$	$-1 + 13.556669i$	$\mathbf{U}_{120}$	$-1 - 5.027339i$
$\mathbf{U}_4$	$-1 + 10.153170i$	$\mathbf{U}_{121}$	$-1 - 5.763142i$
$\mathbf{U}_5$	$-1 + 8.107786i$	$\mathbf{U}_{122}$	$-1 - 6.741452i$
$\mathbf{U}_6$	$-1 + 6.74152i$	$\mathbf{U}_{123}$	$-1 - 8.107786i$
$\mathbf{U}_7$	$-1 + 5.763142i$	$\mathbf{U}_{124}$	$-1 - 10.153170i$
$\mathbf{U}_8$	$-1 + 5.027339i$	$\mathbf{U}_{125}$	$-1 - 13.556670i$
$\mathbf{U}_9$	$-1 + 4.453202i$	$\mathbf{U}_{126}$	$-1 - 20.355468i$
$\mathbf{U}_{10}$	$-1 + 3.992224i$	$\mathbf{U}_{127}$	$-1 - 40.735484i$

Table 14.5: $\mathbf{U}_n$ values in Problem 1, Section 14.6.

$\mathbf{U}_0$	42.167969		
$\mathbf{U}_1$	$5.985858 + 20.367742i$	$\mathbf{U}_{118}$	$-0.433837 - 1.996112i$
$\mathbf{U}_2$	$1.122442 + 10.177734i$	$\mathbf{U}_{119}$	$-0.418629 - 2.226601i$
$\mathbf{U}_3$	$0.221810 + 6.778335i$	$\mathbf{U}_{120}$	$-0.397367 - 2.513670i$
$\mathbf{U}_4$	$-0.093411 + 5.076585i$	$\mathbf{U}_{121}$	$-0.366352 - 2.881571i$
$\mathbf{U}_5$	$-0.239312 + 4.053893i$	$\mathbf{U}_{122}$	$-0.318566 - 3.370726i$
$\mathbf{U}_6$	$-0.318566 + 3.370726i$	$\mathbf{U}_{123}$	$-0.239313 - 4.053893i$
$\mathbf{U}_7$	$-0.366352 + 2.881571i$	$\mathbf{U}_{124}$	$-0.093411 - 5.076585i$
$\mathbf{U}_8$	$-0.397367 + 2.513670i$	$\mathbf{U}_{125}$	$0.221810 - 6.678335i$
$\mathbf{U}_9$	$-0.418629 + 2.226601i$	$\mathbf{U}_{126}$	$1.122442 - 10.177734i$
$\mathbf{U}_{10}$	$-0.433837 + 1.996112i$	$\mathbf{U}_{127}$	$5.985857 - 20.367742i$

Table 14.6: $\mathbf{U}_n$ values in Problem 2, Section 14.6.

$\mathbf{U}_0$	58.901925		
$\mathbf{U}_1$	$-5.854287 - 32.096339i$	$\mathbf{U}_{118}$	$0.647851 + 2.829713i$
$\mathbf{U}_2$	$-0.805518 - 14.788044i$	$\mathbf{U}_{119}$	$0.633992 + 3.157208i$
$\mathbf{U}_3$	$0.044274 - 9.708611i$	$\mathbf{U}_{120}$	$0.614603 + 3.565443i$
$\mathbf{U}_4$	$0.336014 - 7.235154i$	$\mathbf{U}_{121}$	$0.586989 + 4.089267i$
$\mathbf{U}_5$	$0.470070 - 5.764387i$	$\mathbf{U}_{122}$	$0.542633 + 4.787014i$
$\mathbf{U}_6$	$0.542633 - 4.787014i$	$\mathbf{U}_{123}$	$0.470070 + 5.764387i$
$\mathbf{U}_7$	$0.586299 - 4.089267$	$\mathbf{U}_{124}$	$0.336014 + 7.235154i$
$\mathbf{U}_8$	$-3.565442i0.614603$	$\mathbf{U}_{125}$	$0.044274 + 9.708611i$
$\mathbf{U}_9$	$0.63391 - 3.157208i$	$\mathbf{U}_{126}$	$-0.805518 + 14.788044i$
$\mathbf{U}_{10}$	$0.647851 - 2.829712i$	$\mathbf{U}_{127}$	$-5.854287 + 32.096339i$

Table 14.7: $\mathbf{U}_n$ values in Problem 3, Section 14.6.

Then

$$S_{10}(1/2) \approx 0.2504564.$$

For the DFT approximation, compute From these we obtain

$$w \approx 0.246560 + 0.156250(10^{-9})i$$

and

$$\text{var} \approx 0.003896.$$

Approximate values for $\mathbf{U}_n$ are listed in Table 14.6.

3. Compute

$$d_0 = \frac{1}{2}\sin(2), d_n = \frac{-\sin(2) + n\pi(\cos(2) - 1)i}{2(n^2\pi^2 - 1)}.$$

The complex Fourier series of $f(t)$ is

$$\frac{1}{2}\sin(2) + \frac{1}{2}\sum_{n=-\infty, n\neq 0}^{\infty} \frac{-\sin(2) + n\pi(\cos(2) - 1)i}{2(n^2\pi^2 - 1)} e^{n\pi i t}.$$

Using this, compute

$$S_{10}(1/8) \approx 1.067161.$$

For the DFT approximation, compute Compute

$$w \approx 1.042757 - 0.267410(10^{-9})i$$

and

$$\text{var} \approx 0.024403.$$

Approximate values for $\mathbf{U}_n$ are given in Table 14.7.

$\mathbf{U}_0$	31.907298		
$\mathbf{U}_1$	$9.553181 - 14.227057i$	$\mathbf{U}_{118}$	$0.620232 + 1.951481i$
$\mathbf{U}_2$	$3.383468 - 9.071385i$	$\mathbf{U}_{119}$	$0.649821 + 2.174758i$
$\mathbf{U}_3$	$1.847063 - 6.366930i$	$\mathbf{U}_{120}$	$0.691097 + 2.451901i$
$\mathbf{U}_4$	$1.269469 - 4.860090i$	$\mathbf{U}_{121}$	$0.751110 + 2.805358i$
$\mathbf{U}_5$	$0.994545 - 3.915837i$	$\mathbf{U}_{122}$	$0.843127 + 3.271884i$
$\mathbf{U}_6$	$0.843127 - 3.271884i$	$\mathbf{U}_{123}$	$0.994545 + 3.915837i$
$\mathbf{U}_7$	$0.741110 - 2.805358i$	$\mathbf{U}_{124}$	$1.269469 + 4.860090i$
$\mathbf{U}_8$	$0.691097 - 2.451901i$	$\mathbf{U}_{125}$	$1.847063 + 6.366930i$
$\mathbf{U}_9$	$0.649821 - 2.174658i$	$\mathbf{U}_{126}$	$3.383468 + 9.071385i$
$\mathbf{U}_{10}$	$0.620232 - 1.951481i$	$\mathbf{U}_{127}$	$9.553181 + 14.227057i$

Table 14.8: $\mathbf{U}_n$ values in Problem 4, Section 14.6.

4. Compute
$$d_0 = e - 1, d_n = \frac{(e-1)(1+2n\pi i)}{1+4n^2\pi^2}.$$
The complex Fourier series of $f(t)$ is
$$\frac{e-1}{+}\sum_{n=-\infty,n\neq 0}^{\infty}\frac{(e-1)(1+2n\pi i)}{1+4n^2\pi^2}e^{2n\pi it}.$$
Then
$$S_{10}(1/4) \approx 0.827534 - 0.9(10^{-10})i.$$
For the DFT approximation, compute From these obtain the DFT approximation
$$w \approx 0.810504 - 0.954242(10^{-11})i$$
with
$$\text{var} \approx 0.017031.$$
Table 14.8 gives the approximate values for $\mathbf{U}_n$.

5. Compute
$$d_0 = \frac{1}{4}, d_n = \frac{3n\pi + (2n^2\pi^2 - 3)i}{4n^3\pi^3}.$$
The complex Fourier series is
$$\frac{1}{4} + \sum_{n=-\infty,n\neq 0}^{\infty}\frac{3n\pi + (2n^2\pi^2 - 3)i}{4n^3\pi^3}e^{2n\pi it}.$$
Then
$$S_{10}(1/4) \approx -0.000729.$$
For the DFT calculations, we need

$\mathbf{U}_0$	31.501953		
$\mathbf{U}_1$	$9.228787 + 17.271595i$	$\mathbf{U}_{118}$	$-0.400755 - 1.993017i$
$\mathbf{U}_2$	$1.933662 + 9.790716i$	$\mathbf{U}_{119}$	$-0.377943 - 2.222355i$
$\mathbf{U}_3$	$0.582715 + 6.663663i$	$\mathbf{U}_{120}$	$-0.346050 - 2.507623i$
$\mathbf{U}_4$	$0.109884 + 5.028208i$	$\mathbf{U}_{121}$	$-0.299528 - 2.872545i$
$\mathbf{U}_5$	$-0.108968 + 4.029124i$	$\mathbf{U}_{122}$	$-0.227849 - 3.356393i$
$\mathbf{U}_6$	$-0.227849 + 3.356393i$	$\mathbf{U}_{123}$	$-0.108968 - 4.029124i$
$\mathbf{U}_7$	$-0.299528 + 2.872544i$	$\mathbf{U}_{124}$	$0.109884 - 5.028208i$
$\mathbf{U}_8$	$-0.346050 + 2.507623i$	$\mathbf{U}_{125}$	$0.582715 - 6.663663i$
$\mathbf{U}_9$	$-0.377943 + 2.222355i$	$\mathbf{U}_{126}$	$1.933662 - 9.790715i$
$\mathbf{U}_{10}$	$-0.400755 + 1.993017i$	$\mathbf{U}_{127}$	$9.228787 - 17.271595i$

Table 14.9: $\mathbf{U}_n$ values in Problem 5, Section 14.6.

From these obtain

$$w \approx 0.003483 - 0.781250(10^{-10})i$$

and

$$\text{var} \approx 0.004212.$$

Table 14.9 lists the approximate values of $\mathbf{U}_n$.

6. The complex Fourier coefficients are $d_0 = \sin(1) - \cos(1)$ and, for $n \neq 0$,

$$d_n = \frac{\cos(1)(4n^2\pi^2 - 1) + \sin(1)(4n^2\pi^2 + 1)}{(4n^2\pi^2 - 1)^2} + \frac{4n\pi(1 - \cos(1)) - 2n\pi\sin(1) + 8n^2\pi^2}{(4n^2\pi^2 - 1)^2}i.$$

Compute

$$S_{10}(1/8) \approx 0.053390 - 0.6(10^{-10})i.$$

From these obtain

$$w \approx 0.149844 + 0.607562(10^{-9})$$

and

$$\text{var} \approx 0.096453.$$

Table 14.10 gives the approximate values for $\mathbf{U}_n$.

14.7 DFT Approximation of the Fourier Transform

1. With $f(t) = e^{-4t}$,

$$\hat{f}(\omega) = \int_0^\infty e^{-4t}e^{-i\omega t}\,dt = \frac{4 - i\omega}{\omega^2 + 16}.$$

$\mathbf{U}_0$	60.953531		
$\mathbf{U}_1$	$-94.509581 - 26.479226i$	$\mathbf{U}_{118}$	$1.064899 + 6.040217i$
$\mathbf{U}_2$	$-11.274203 - 30.060278i$	$\mathbf{U}_{119}$	$-.061000 + 6.737012i$
$\mathbf{U}_3$	$-3.690170 - 20.379819i$	$\mathbf{U}_{120}$	$0.815252 + 7.604519i$
$\mathbf{U}_4$	$-1.331661 - 15.319442i$	$\mathbf{U}_{121}$	$0.601640 + 8.715653i$
$\mathbf{U}_5$	$-0.286124 - 12.249522i$	$\mathbf{U}_{122}$	$0.270130 - 10.191613i$
$\mathbf{U}_6$	$0.270130 - 10.191613i$	$\mathbf{U}_{123}$	$-0.286124 + 12.249522i$
$\mathbf{U}_7$	$0.601640 - 8.715653i$	$\mathbf{U}_{124}$	$-1.331661 + 15.319442i$
$\mathbf{U}_8$	$0.815252 - 7.604519i$	$\mathbf{U}_{125}$	$-3.690170 + 20.379819i$
$\mathbf{U}_9$	$-0.961000 - 6.737012i$	$\mathbf{U}_{126}$	$-11.274203 + 30.060278i$
$\mathbf{U}_{10}$	$1.064899 - 6.040217i$	$\mathbf{U}_{127}$	$-0.945096 + 26.479226i$

Table 14.10: $\mathbf{U}_n$ values in Problem 6, Section 14.6.

Then

$$\hat{f}(4) = \frac{1}{8}(1 - i).$$

The DFT approximation to $\hat{f}(4)$ with $L = 3$ and $N = 512$ is

$$\frac{3\pi}{256} \sum_{j=0}^{511} f\left(\frac{3\pi j}{256}\right) e^{-3\pi i j/64} \approx 0.143860 - 0.124549i.$$

The error in the DFT approximation is approximately 0.018887.

2. We can directly compute

$$\hat{f}(1) = \int_0^{12\pi} t\cos(t)e^{-it}\,dt = 36\pi^2 + 3\pi i.$$

This is approximately $355.305785 + 9.9424777962i$.

For the DFT approximation, we have

$$\begin{aligned}\hat{f}(1) &\approx \frac{3\pi}{128} \sum_{j=0}^{511} f(3\pi j/128)e^{-3\pi i j/128} \\ &= 353.9178450 + 9.407739539i.\end{aligned}$$

3. With $ft) = te^{-2t}$, compute

$$\hat{f}(\omega) = \frac{4 - \omega^2}{(\omega^2 + 4)^2} - \frac{4\omega}{(\omega^2 + 4)^2} i.$$

Then

$$\hat{f}(12) \approx -0.006392 - 0.002191i.$$

The DFT approximation is

$$\frac{3\pi}{256}\sum_{j=0}^{511} f\left(\frac{3\pi j}{256}\right)e^{-9\pi ij/64} \approx -0.006506 - 0.002191i.$$

The error in the approximation is approximately 0.000114.

4. First compute

$$\hat{f}(4) = \int_0^\infty e^{-t}\cos(t)e^{-4it}\,dt = \frac{9}{130} - \frac{16}{65}i.$$

The DFT approximation gives

$$\begin{aligned}\hat{f}(4) &\approx \frac{\pi}{64}\sum_{j=0}^{511} f(\pi j/64)e^{-\pi ij/16}\\ &\approx 0.09397566230 - 0.2453501394i.\end{aligned}$$

To compare this with the exact value, write the decimal expansion

$$\hat{f}(4) = 0.06923076923 - 0.2461538642.$$

Chapter 15

Eigenfunction Expansions

15.1 General Eigenfunction Expansions

1. This problem is regular on $[0, L]$. The differential equation has characteristic equation
$$r^2 + \lambda^2 = 0,$$
with roots $r = \pm\sqrt{\lambda}$. We must take cases on λ.

Case 1. If $\lambda = 0$, the differential equation is $y'' = 0$, with general solution
$$y = a + bx$$
for constants a and b. Now $y(0) = 0 = a$ and $y'(L) = b = 0$, so the problem has only the trivial solution if $\lambda = 0$. Therefore 0 is not an eigenvalue of this problem.

Case 2. λ is positive, say $\lambda = \alpha^2$, with $\alpha > 0$. Then $\sqrt{\lambda} = \pm\alpha$, so the general solution of the differential equation is
$$y = c_1 e^{\alpha x} + c_2 e^{-\alpha x}.$$
Now $y(0) = c_1 + c_2 = 0$, so
$$y = c_1 e^{\alpha x} - c_1 e^{-\alpha x} = 2c_1 \sinh(\alpha x).$$
Next,
$$y'(L) = 2c_1\alpha\cosh(\alpha L) = 0.$$
But $\cosh(\alpha L) > 0$, and $\alpha > 0$, so $c_1 = 0$ and the problem has only the trivial solution for $\lambda > 0$. This problem has no positive eigenvalue.

Case 3. $\lambda < 0$, say $\lambda = -\alpha^2$, with $\alpha > 0$. Now the differential equation has the general solution
$$y = c_1\cos(\alpha x) + c_2\sin(\alpha x).$$

Now $y(0) = c_1 = 0$, so
$$y = c_2 \sin(\alpha x).$$
Next,
$$y'(L) = c_2 \alpha \cos(\alpha L) = 0.$$
To have a nontrivial solution we want to be able to choose $c_2 \neq 0$, so we must have $\cos(\alpha L) = 0$. Then αL is a positive zero of the cosine function,
$$\alpha L = \frac{(2n-1)\pi}{2},$$
in which n can be any positive integer. Then
$$\alpha = \sqrt{\lambda} = \frac{(2n-1)\pi}{2L}.$$
Since $\lambda = \alpha^2$, the eigenvalues of this problem, indexed by n, are
$$\lambda_n = \left(\frac{(2n-1)\pi}{2L}\right)^2$$
for $n = 1, 2, 3, \cdots$. Corresponding to each such eigenvalue we have the eigenfunction
$$\varphi_n(x) = \sin\left(\frac{(2n-1)\pi}{2L} x\right).$$
Of course, any nonzero constant multiple of this eigenfunction is also an eigenfunction.

Problems 2, 3 and 5 are solved by an analysis similar to that just done for Problem 2 and also in Example 15.2. We therefore omit the details for the solutions of these problems. Problems 6 through 10 are more involved and more details are provided.

2. The problem is regular on $[0, L]$. Then eigenvalues are
$$\lambda_0 = 0, \lambda_n = n^2 \text{ for } n = 1, 2, \cdots.$$
Corresponding eigenfunctions are
$$\varphi_n(x) = \cos(n\pi x) \text{ for } n = 0, 1, 2, \cdots.$$

3. The problem is regular on $[0, 4]$. Eigenvalues are
$$\lambda_n = \left[\left(n - \frac{1}{2}\right)\frac{\pi}{4}\right]^2$$
for $n = 1, 2, \cdots$. Corresponding eigenfunctions are
$$\varphi_n(x) = \cos((n - 1/2)\pi x/4).$$

4. The problem is periodic on $[0, \pi]$. Eigenvalues are $\lambda_0 = 0$ and
$$\lambda_n = 4n^2 \text{ for } n = 1, 2, \cdots .$$
Eigenfunctions are
$$\varphi_0(x) = a_n \cos(2nx) + b_n \sin(2nx)$$
for $n = 0, 1, 2, \cdots$, with a_n and b_n not both zero.

5. The problem is periodic on $[-3\pi, 3\pi]$. Eigenvalues are
$$\lambda_0 = 0 \text{ and } \lambda_n = \frac{n^2}{9}.$$
Eigenfunctions are
$$\varphi_n(x) = a_n \cos(nx/3) + b_n \sin(nx/3)$$
for $n = 0, 1, 2, \cdots$, with a_n and b_n not both zero.

6. The problem is regular on $[0, \pi]$. The eigenvalues are positive solutions of
$$\sin(\sqrt{\lambda}\pi) + 2\sqrt{\lambda}\cos(\sqrt{\lambda}\pi) = 0.$$
These are solutions of the equation transcendental
$$\tan(\sqrt{\lambda}\pi) = -2\sqrt{\lambda},$$
which cannot be solved algebraically. If we let $z = \sqrt{\lambda}$, then we need the roots of
$$\tan(\pi z) = -2z.$$
The graphs of $y = \tan(\pi z)$ and $y = -2z$ have infinitely many points of intersection with $z > 0$. The $z-$ coordinate of each such point of intersection is an eigenvalue. A numerical approximation technique must be used to produce some of the numerical values of these eigenvalues. The first four are
$$\lambda_1 \approx 0.48705, \lambda_2 \approx 2.54914, \lambda_3 \approx 6.56059, \lambda_4 \approx 12.56423.$$
Corresponding eigenfunctions are $\varphi_n(x) = \sin(\sqrt{\lambda_n}x)$.

7. This is a regular problem on $[0, 1]$. Eigenvalues are positive solutions of
$$\tan(\sqrt{\lambda}) = \frac{1}{2\sqrt{\lambda}}.$$
There are infinitely many such eigenvalues (examine graphs, a strategy suggested for Problem 6). The first four are
$$\lambda_1 \approx 0.42676, \lambda_2 \approx 10.8393, \lambda_3 \approx 40.4702, \lambda_4 \approx 89.8227.$$
Eigenfunctions are
$$\varphi_n(x) = 2\sqrt{\lambda_n}\cos(\sqrt{\lambda_n}x) + \sin(\sqrt{\lambda_n}x).$$

8. The problem is regular on $[0,1]$. The differential equation has characteristic equation
$$r^2 + 2r + (1+\lambda) = 0,$$
with roots
$$r = -1 \pm \sqrt{\lambda}i.$$
The general solution is
$$y(x) = c_1 e^x \cos(\sqrt{\lambda}x) + c_2 e^{-x}\sin(\sqrt{\lambda}x).$$
Now $y(0) = c_1 = 0$. Next,
$$y(1) = 0 = c_2 e^{-1}\sin(\sqrt{\lambda}).$$
This will be satisfied if we choose $\sqrt{\lambda}$ to be a zero of the sine function. For some nonzero integer n,
$$\sqrt{\lambda} = n\pi,$$
so
$$\lambda_n = n^2\pi^2$$
is an eigenvalue for each positive integer n. Corresponding eigenfunctions are
$$\varphi_n(x) = e^{-x}\sin(n\pi x)$$
or any nonzero constant multiples of this function.

9. The problem is regular on $[0,\pi]$. The differential equation can be written
$$y'' + 2y' + \lambda y = 0$$
and the characteristic equation has roots
$$-1 \pm \sqrt{1-\lambda}.$$
Consider cases on λ.
Case 1: $1 - \lambda = a^2 > 0$. The general solution is
$$y(x) = c_1 e^{(-1+a)x} + c_2 e^{(-1-a)x}.$$
Now
$$y(0) = c_1 + c_2 = 0$$
so $c_2 = -c_1$. Next
$$y(\pi) = c_1\left(e^{-\pi}e^{a\pi} - e^{-\pi}e^{-a\pi}\right).$$
Assuming that $c_1 \neq 0$ to avoid the trivial solution, this implies that
$$e^{a\pi} - e^{-a\pi} = 0,$$

so $e^{2a\pi} = 1$. But then $2a\pi = 0$, impossible since $a \neq 0$. Case 1 produces no eigenvalue for this problem.

Case 2: $1 - \lambda = 0$, so $\lambda = 1$. Now the general solution is

$$y(x) = c_1 e^{-x} + c_2 x e^{-x}.$$

Then $y(0) = c_1 = 0$, and then

$$y(\pi) = c_2 \pi e^{-\pi} = 0,$$

impossible unless $c_2 = 0$, resulting again in the trivial solution. This case yields no eigenvalue.

Case 3: $1 - \lambda = -a^2$, where $a > 0$. Now the general solution is

$$y = c_1 e^{-x}\cos(ax) + c_2 e^{-x}\sin(ax).$$

Then $y(0) = c_1 = 0$, and

$$y(\pi) = c_2 e^{-\pi}\sin(a\pi) = 0.$$

Again, to avoid the trivial solution, we must have $c_2 \neq 0$, so $\sin(a\pi) = 0$, so

$$a = \sqrt{\lambda - 1} = n,$$

a positive integer. Then $\lambda - 1 = n^2$, so the eigenvalues are

$$\lambda_n = 1 + n^2$$

for $n = 1, 2, \cdots$. Corresponding eigenfunctions are

$$\varphi_n(x) = e^{-x}\sin(nx).$$

10. The problem is regular on $[0, 8]$. Similar to the solution of Problem 9, eigenvalues are

$$\lambda_n = 8 + n^2\pi^2$$

and eigenfunctions are

$$\varphi_n(x) = e^{3x}\sin(n\pi x).$$

11. The eigenfunctions are $\varphi_n(x) = \sin(n\pi x/L)$. The coefficients in the eigenfunction expansion are

$$c_n = \frac{2}{L}\int_0^L (1 - \xi)\sin(n\pi\xi/L)\,d\xi = \frac{2}{n\pi}(1 + (-1)^n(L - 1))$$

for $n = 1, 2, \cdots$. The expansion is

$$1 - x = \sum_{n=1}^{\infty} \frac{2}{n\pi}(1 + (-1)^n(L - 1))\sin(n\pi x/L)$$

for $0 < x < L$. The fortieth partial sum of this series is compared to the function in Figure 15.1 for $L = 1$.

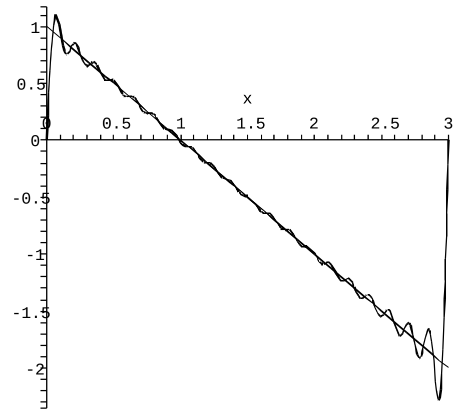

Figure 15.1: Partial sum in Problem 11, Section 15.1.

12. From Problem 1 with $L = \pi$ the eigenfunctions are

$$\varphi_n(x) = \sin((2n-1)x/2).$$

The coefficients in the expansion are

$$c_n = \frac{2}{\pi}|\xi| \sin((2n-1)\xi/2)\, d\xi = \frac{8}{\pi}\frac{(-1)^{n+1}}{(2n-1)^2}.$$

The expansion is

$$\sum_{n=1}^{\infty} \frac{8}{\pi}\frac{(-1)^{n+1}}{(2n-1)^2} \sin((2n-1)x/2)$$

for $0 < x < \pi$. Figure 15.2 shows the thirtieth partial sum of this expansion and the function itself.

13. From Problem 3 the eigenfunctions are

$$\varphi(x) = \cos((2n-1)\pi x/8).$$

The coefficients in the expansion are

$$\begin{aligned} c_n &= \frac{1}{2}\int_0^2 -\cos((2n-1)\pi\xi/8)\, d\xi + \frac{1}{2}\int_2^4 \cos((2n-1)\pi\xi/8)\, d\xi \\ &= \frac{4}{(2n-1)\pi}\left[(-1)^{n+1} + \sqrt{2}(\cos(n\pi/2) - \sin(n\pi/2))\right] \end{aligned}$$

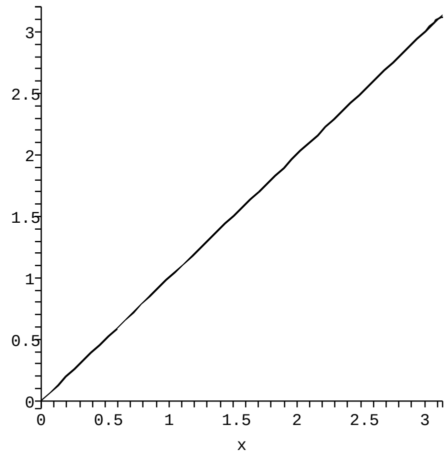

Figure 15.2: Partial sum in Problem 12, Section 15.1.

for $n = 1, 2, \cdots$. The expansion is

$$\sum_{n=1}^{\infty} \frac{4}{(2n-1)\pi} \left[(-1)^{n+1} + \sqrt{2}(\cos(n\pi/2) - \sin(n\pi/2))\right] \cos((2n-1)\pi x/8)$$

and this converges to

$$\begin{cases} -1 & \text{for } 0 < x < 2, \\ 0 & \text{for } x = 0, 2, 4, \\ 1 & \text{for } 2 < x < 4. \end{cases}$$

Figure 15.3 shows a graph of the function compared to the fortieth partial sum of this eigenfunction expansion.

14. The eigenfunctions are

$$\varphi_0(x) = 1, \varphi_n(x) = \cos(nx) \text{ for } n = 1, 2, \cdots.$$

The coefficients in the eigenfunction expansion are

$$c_0 = \frac{1}{\pi} \int_0^{\pi} \sin(2\xi)\, d\xi = 0,$$

$$c_2 = \frac{2}{\pi} \int_0^{\pi} \sin(2\xi)\cos(2\xi)\, d\xi = 0,$$

and, for $n = 1, 3, 4, \cdots$,

$$c_n = \frac{2}{\pi} \int_0^{\pi} \sin(2\xi)\cos(n\xi)\, d\xi = \frac{4}{\pi} \frac{((-1)^n - 1)}{n^2 - 4}.$$

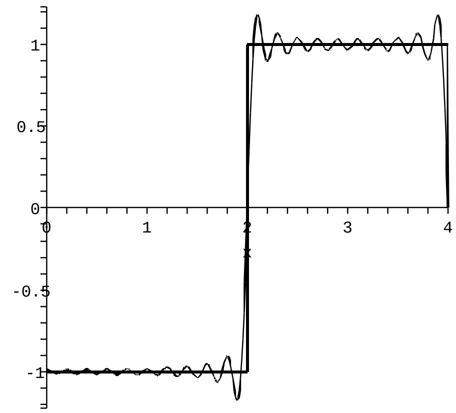

Figure 15.3: Partial sum in Problem 13, Section 15.1.

The eigenfunction expansion is

$$\frac{4}{\pi}\sum_{n=1,n\neq 2}^{\infty}\frac{((-1)^n-1)}{n^2-4}\cos(nx)=\sin(2x)$$

for $0<x<\pi$. Figure 15.4 shows a graph of $\sin(2x)$ compared to the thirtieth partial sum of this expansion.

15. The eigenfunctions are

$$\varphi_0(x)=1,\varphi_n(x)=a_n\cos(nx/3)+b_n\sin(nx/3)\text{ for }n=1,2,\cdots.$$

The coefficients in the eigenfunction expansion x^2 on $[-3\pi,3\pi]$ are

$$c_0=\frac{1}{6\pi}\int_{-3\pi}^{3\pi}\xi^2\,d\xi=3\pi^2,$$

$$a_n=\frac{1}{3\pi}\int_{-3\pi}^{3\pi}\xi^2\cos(n\xi/3)\,d\xi=\frac{36}{n^2}(-1)^n\text{ for }n=1,2,\cdots,$$

and

$$b_n=\frac{1}{3\pi}\int_{-3\pi}^{3\pi}\xi^2\sin(n\xi/3)\,d\xi=0\text{ for }n=1,2,\cdots.$$

The expansion is

$$3\pi^2+36\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos(nx/3)=x^2$$

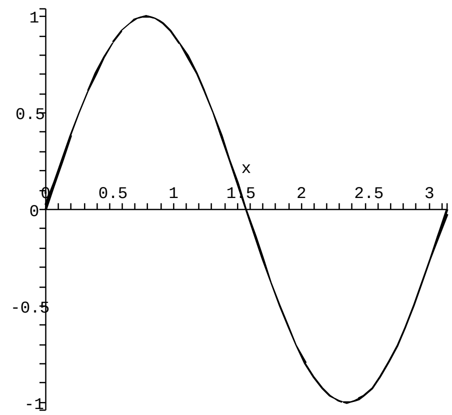

Figure 15.4: Partial sum in Problem 14, Section 15.1.

for $-3\pi < x < 3\pi$. Figure 15.5 is a graph of this function compared to the tenth partial sum of this expansion.

16. The eigenfunctions are $\varphi_n(x) = e^{-x}\sin(n\pi x)$ for $n = 1, 2, \cdots$. Notice that the Sturm-Liouville form of the differential equation is

$$(e^{2x}y')' + e^{2x}(1+\lambda)y = 0.$$

Therefore the weight function in this Sturm-Liouville problem is $p(x) = e^{2x}$. The coefficients in the eigenfunction expansion are

$$\begin{aligned} c_n & \frac{\int_{1/2}^1 e^x \sin(n\pi x)\,dx}{\int_0^1 \sin^2(n\pi x)\,dx} \\ &= \frac{2e^{1/2}(n\pi\cos(n\pi/2) - \sin(n\pi/2)) - 2en\pi(-1)^n}{1+n^2\pi^2}. \end{aligned}$$

The eigenfunction expansion is

$$\sum_{n=1}^{\infty} c_n e^{-x}\sin(n\pi x)$$

and this converges to

$$\begin{cases} 0 & \text{for } 0 < x < 1/2, \\ 1/2 & \text{for } x = 0, 1/2, 1, \\ 1 & \text{for } 1/2 < x < 1. \end{cases}$$

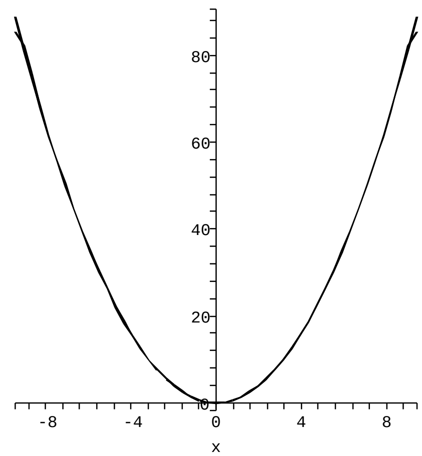

Figure 15.5: Partial sum in Problem 15, Section 15.1.

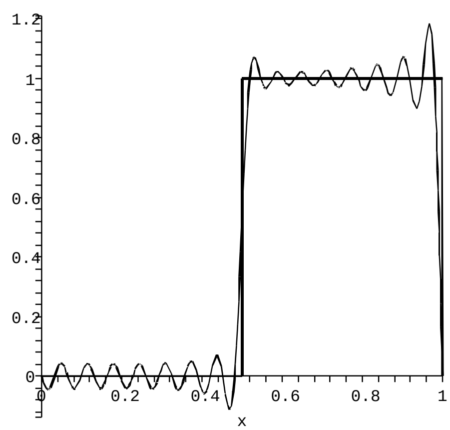

Figure 15.6: Thirtieth partial sum in Problem 16, Section 15.1.

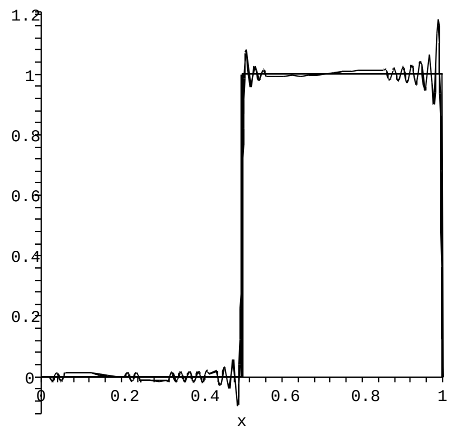

Figure 15.7: Ninetieth partial sum in Problem 16, Section 15.1.

Figure 15.6 shows a graph of $f(x)$ compared to the thirtieth partial sum of this expansion. This partial sum is not a very good fit to the function. Figure 15.7 shows a graph of the function and the ninetieth partial sum, a better fit. For improved accuracy we would have to take more terms in the partial sum.

17. Normalized eigenfunctions for Problem 3 are obtained by dividing each eigenfunction by its length, whose square is the dot product of this eigenfunction with itself.

$$\int_0^4 \cos^2\left(\frac{(2n-1)\pi x}{8}\right)\,dx = 2.$$

Therefore the normalized eigenfunctions are

$$\varphi_n(x) = \frac{1}{\sqrt{2}}\cos\left(\frac{(2n-1)\pi x}{8}\right),$$

for $n = 1, 2, 3, \cdots$. Now calculate the dot product

$$\varphi_n \cdot f = \frac{1}{\sqrt{2}}\int_0^4 x(4 - x\cos\left(\frac{(2n-1)\pi x}{8}\right)\,dx$$
$$= -\frac{256}{\sqrt{2}}\frac{4(-1)^n + (2n-1)\pi}{(2n-1)^3\pi^3}.$$

Since

$$f \cdot f = \int_0^4 x^2(4-x)^2\,dx = \frac{512}{15},$$

then Bessel's inequality yields (after some simplification),

$$\sum_{n=1}^{\infty}\left(\frac{4(-1)^n+(2n-1)\pi}{(2n-1)^3\pi^3}\right)^2 \leq \frac{512}{15}\frac{2}{(256)^2}=\frac{1}{960}.$$

18. Eigenfunctions are functions $\sin(\sqrt{\lambda}x)$, where λ are solutions of the transcendental equation

$$\sin(\sqrt{\lambda}\pi)+2\sqrt{\lambda}\cos(\sqrt{\lambda}\pi)=0.$$

Each λ is a positive solution of the equation

$$\tan(\sqrt{\lambda}\pi)=-2\sqrt{\lambda}.$$

We first need to normalize these eigenfunctions. Compute

$$\begin{aligned}\int_0^{\pi}\sin^2(\sqrt{\lambda}x)\,dx &= \frac{1}{2}\int_0^{\pi}(1-\cos(\sqrt{\lambda}x))\,dx\\ &= \frac{1}{2}\left[\pi-\frac{2\sin(\sqrt{\lambda}\pi)\cos(\sqrt{\lambda}\pi)}{2\sqrt{\lambda}}\right]\\ &= \frac{1}{2}\left(\pi+2\cos^2(\sqrt{\lambda}\pi)\right).\end{aligned}$$

The normalized eigenfunctions are

$$\varphi_n(x)=\left(\frac{2}{\pi+2\cos^2(\sqrt{\lambda_n}\pi)}\right)^{1/2}\sin(\sqrt{\lambda_n}x),$$

in which we have assigned subscripts to λ to indicate that this is the nth eigenfunction, associated with the nth eigenvalue. Now compute

$$\begin{aligned}\varphi_n\cdot f &= \sqrt{\frac{2}{\pi+2\cos^2(\sqrt{\lambda_n}\pi)}}\int_0^{\pi}e^{-x}\sin(\sqrt{\lambda_n}x)\,dx\\ &= \sqrt{\frac{2}{\pi+2\cos^2(\sqrt{\lambda_n}\pi)}}\left[\frac{e^{-\pi}}{1+\lambda_n}\left(-\sin(\sqrt{\lambda_n}\pi)-\sqrt{\lambda_n}\cos(\sqrt{\lambda_n}\pi)\right)+\frac{\sqrt{\lambda_n}}{1+\sqrt{\lambda_n}}\right]\\ &= \sqrt{\frac{2}{\pi+2\cos^2(\sqrt{\lambda_n}\pi)}}\frac{\lambda_n}{1+\lambda_n}(1+e^{-\pi}\cos(\sqrt{\lambda_n}\pi)).\end{aligned}$$

Further,

$$f\cdot f=\int_0^{\pi}e^{-2x}\,dx=\frac{1}{2}(1-e^{-2\pi})=e^{-\pi}\sinh(\pi).$$

Now Bessel's inequality gives us

$$\sum_{n=1}^{\infty}\frac{2\lambda_n^2}{(1+\lambda_n)^2(\pi+2\cos^2(\sqrt{\lambda_n}\pi))}\left(1+e^{-\pi}\cos(\sqrt{\lambda_n}\pi)\right)^2 \leq e^{-\pi}\sinh(\pi).$$

15.2 Legendre Polynomials

1. For this problem, use the recurrence relation to write $P_{n+1}(x)$ in terms of $P_n(x)$ and $P_{n-1}(x)$:

$$P_{n+1}(x) = \frac{2n+1}{n+1} x P_n(x) + \frac{n}{n+1} P_{n-1}(x)$$

for $n = 1, 2, \cdots$. Since we know $P_0(x)$ through $P_5(x)$, it is routine to derive the following:

$$\begin{aligned}
P_6(x) &= \frac{1}{16}(231x^6 - 315x^4 + 105x^2 - 5) \\
P_7(x) &= \frac{1}{16}(429x^7 - 693x^5 + 315x^3 - 35x) \\
P_8(x) &= \frac{1}{128}(6435x^8 - 12012x^6 + 6930x^4 - 1260x^2 + 35).
\end{aligned}$$

2. Using the Rodrigues formula, we obtain the following, which can be checked with the polynomials listed previously:

$$\begin{aligned}
P_2(x) &= \frac{1}{2^2 1!} \frac{d^2}{dx^2}((x^2-1)^2) \\
&= \frac{1}{8} \frac{d^2}{dx^2}(x^4 - 2x^2 + 1) = \frac{1}{2}(3x^2 - 1), \\
P_3(x) &= \frac{1}{2^3 3!} \frac{d^3}{dx^3}((x^2-1)^3) \\
&= \frac{1}{48} \frac{d^3}{dx^3}(x^6 - 3x^4 + 3x^2 - 1) = \frac{1}{2}(5x^3 - 3x), \\
P_4(x) &= \frac{1}{2^4 4!} \frac{d^4}{dx^4}((x^2-1)^4) \\
&= \frac{1}{384} \frac{d^4}{dx^4}(x^8 - 4x^6 + 6x^4 - 4x^2 + 1) = \frac{1}{8}(35x^4 - 30x^2 + 3), \\
P_5(x) &= \frac{1}{2^5 5!} \frac{d^5}{dx^5}((x^2-1)^5) \\
&= \frac{1}{3840} \frac{d^5}{dx^5}(1^{10} - 5x^8 + 10x^6 - 5x^2 + 1) \\
&= \frac{1}{8}(63x^5 - 70x^3 + 15x).
\end{aligned}$$

3. Using the given formula, obtain:

$$P_0(x) = (-1)^0 x^0 = 1,$$
$$P_1(x) = \frac{(-1)^0 2!}{2} x = x,$$
$$P_2(x) = (-1)^0 \frac{4!}{2!2!} x^2 + \frac{(-1)2!}{2^2} x^0 = \frac{1}{2}(3x^2 - 1);$$
$$P_3(x) = \frac{(-1)^0 6!}{2^3 3!3!} x^3 + \frac{(-1)4!}{2^3 2!} x = \frac{1}{2}(5x^3 - 3x),$$
$$P_4(x) = \frac{(-1)^0 8!}{2^4 4!4!} x^4 - \frac{6!}{2^4 3!2!} x^2 + \frac{4!}{2^4 2!2!} = \frac{1}{8}(35x^4 - 30x^2 + 3),$$
$$P_5(x) = \frac{(-1)^0 10!}{2^5 5!5!} x^5 - \frac{8!}{2^5 4!3!} x^3 + \frac{6!}{2^5 2!3!} x = \frac{1}{8}(63x^5 - 70x^3 + 15x).$$

4. Attempt to find a second solution of the form $Q_n(x) = z(x)P_n(x)$. Substitute this into Legendre's equation to obtain, after some manipulation,

$$z[(1-x^2)P_n'' - 2xP_n' + n(n+1)P_n] + z''(1-x^2)P_n + z'[2(1-x^2)P_n' - 2xP_n] = 0.$$

The first term is zero because P_n satisfies Legendre's differential equation. Therefore $z(x)$ satisfies

$$\frac{z''}{z'} - \frac{2x}{1-x^2} + 2\frac{P_n'}{P_n} = 0.$$

Integrate this to obtain

$$\ln|z'| + \ln|1-x^2| + 2\ln|P_n| = c.$$

Solve this for z' to obtain

$$z'(x) = \frac{K}{(1-x^2)(P_n(x))^2},$$

in which K is constant. Integrate again to obtain

$$z(x) = K\int \frac{1}{(1-x^2)(P_n(x))^2}\,dx.$$

We may choose $K = 1$ and obtain the second, linearly independent solution

$$Q_n(x) = P_n(x)\int \frac{1}{P_n(x)^2(1-x^2)}\,dx.$$

We will evaluate the first three of these second solutions. First,

$$Q_0(x) = \int \frac{1}{1-x^2}\,dx = \frac{1}{2}\int \left(\frac{1}{1+x} + \frac{1}{1-x}\right)dx$$
$$= \frac{1}{2}\ln\left|\frac{1+x}{1-x}\right| = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$$

if $-1 < x < 1$. Next,

$$\begin{aligned}
Q_1(x) &= x \int \frac{1}{x^2(1-x^2)}\,dx \\
&= x \int \left[\frac{1}{x^2} + \frac{1}{2}\left(\frac{1}{1+x} + \frac{1}{1-x}\right)\right] dx \\
&= -1 + \frac{x}{2}\ln\left(\frac{1+x}{1-x}\right).
\end{aligned}$$

Finally,

$$\begin{aligned}
Q_2(x) &= 2(3x^2-1) \int \frac{1}{(3x^2-1)^2(1-x^2)}\,dx \\
&= \frac{1}{4}(3x^2-1) \int \left[\frac{1}{x+1} - \frac{1}{x-1} + \frac{1}{(x+1/\sqrt{3})^2} + \frac{1}{(x-1/\sqrt{3})^2}\right] dx \\
&= \frac{1}{4}(3x^2-1)\ln\left(\frac{1+x}{1-x}\right) - \frac{3}{2}x.
\end{aligned}$$

5. From the diagram and the law of cosines,

$$R^2 = r^2 + d^2 - 2rd\cos(\theta)$$

so

$$\frac{R^2}{d^2} = 1 - 2\frac{r}{d}\cos(\theta) + \frac{r^2}{d^2}.$$

Then

$$\varphi(x,y,z) = \frac{1}{R} = \frac{1}{d}\frac{d}{R} = \frac{1}{d}\frac{1}{\sqrt{1 - 2\frac{r}{d}\cos(\theta) + \frac{r^2}{d^2}}}.$$

This concludes part (a). For (b), suppose $r/d < 1$. By comparing the result of (a) with the generating function for Legendre polynomials (with $x = \cos(\theta)$ and $t = r/d$), we have

$$\varphi(r) = \frac{1}{d}\sum_{n=0}^{\infty} P_n(\cos(\theta))\left(\frac{r^n}{d^n}\right),$$

which is equivalent to

$$\varphi(r) = \sum_{n=0}^{\infty} \frac{1}{d^{n+1}} P_n(\cos(\theta)) r^n.$$

For (c), suppose $r/d < 1$. Now write

$$\frac{R^2}{r^2} = 1 - 2\frac{d}{r}\cos(\theta) + \frac{d^2}{r^2}.$$

Then

$$\frac{r}{R} = \frac{1}{\sqrt{1 - 2\frac{d}{r}\cos(\theta) + \frac{d^2}{r^2}}}.$$

Again comparing with the generating function, we have

$$\varphi(r) = \frac{1}{r}\sum_{n=0}^{\infty} P_n(\cos(\theta))\left(\frac{d^n}{r^n}\right).$$

This is equivalent to

$$\varphi(r) = \frac{1}{r}\sum_{n=0}^{\infty} d^n P_n(\cos(\theta)) r^{-n}.$$

6. We know that, for $-1 < t < 1$,

$$\frac{1}{\sqrt{1 - 2xt + t^2}} = \sum_{n=0}^{\infty} P_n(x) t^n.$$

With $x = t = 1/2$ this gives us

$$\frac{1}{\sqrt{3/4}} = \sum_{n=0}^{\infty} P_n\left(\frac{1}{2}\right)\frac{1}{2^n}.$$

Then

$$\frac{2}{\sqrt{3}} = \sum_{n=0}^{\infty} \frac{1}{2^n} P_n\left(\frac{1}{2}\right).$$

We obtain the requested result by dividing both sides of this equation by 2.

7. One way to derive these results is to use the expression for $P_n(x)$ given in Problem 3. First,

$$\left[\frac{2n+1}{2}\right] = \left[n + \frac{1}{2}\right] = n,$$

so

$$P_{2n+1}(x) = \sum_{k=0}^{n} (-1)^k \frac{(4n+2-2k)!}{2^{2n+1} k!(2n+1-k)!(2n+1-2k)!} x^{2n+1-2k}.$$

Then $P_{2n+1}(0) = 0$, because there is a positive power of x in every term of $P_{2n+1}(x)$.

Next,

$$\left[\frac{2n}{2}\right] = n,$$

so

$$P_{2n}(x) = \sum_{k=0}^{\infty} (-1)^k \frac{(4n-2k)!}{2^{2n} k!(2n-k)!(2n-2k)!} x^{2n-2k}.$$

The constant term in this polynomial occurs when $k = n$, so

$$P_{2n}(0) = \frac{(-1)^n (2n)!}{2^{2n} (n!)^2}.$$

8. We can do these expansions by straightforward algebraic manipulation. However, we can also do this efficiently using matrices. Since the highest power of x occurring in the polynomials of (a) through (c) is 4, we need only use $P_0(x)$ through $P_4(x)$. Write

$$\begin{pmatrix} P_0(x) \\ P_1(x) \\ P_2(x) \\ P_3(x) \\ P_4(x) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -1/2 & 0 & 3/2 & 0 & 0 \\ 0 & -3/2 & 0 & 5/2 & 0 \\ 3/8 & 0 & -30/8 & 0 & 35/8 \end{pmatrix} \begin{pmatrix} 1 \\ x \\ x^2 \\ x^3 \\ x^4 \end{pmatrix}.$$

Invert the coefficient matrix to write

$$\begin{pmatrix} 1 \\ x \\ x^2 \\ x^3 \\ x^4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1/3 & 0 & 2/3 & 0 & 0 \\ 0 & 3/5 & 0 & 2/5 & 0 \\ 1/5 & 0 & 4/78 & 0 & 8/35 \end{pmatrix} \begin{pmatrix} P_0(x) \\ P_1(x) \\ P_2(x) \\ P_3(x) \\ P_4(x) \end{pmatrix}.$$

From these we read

$$\begin{aligned} 1 + 2x - x^2 &= \frac{2}{3} P_0(x) + 2P_1(x) - \frac{2}{3} P_2(x), \\ 2x + x^2 - 5x^3 &= \frac{1}{3} P_0(x) + 2P_1(x) + \frac{11}{3} P_2(x) - 2P_3(x), \\ 2 - x^2 + 4x^4 &= \frac{37}{15} P_0(x) + \frac{34}{21} P_2(x) + \frac{32}{35} P_4(x). \end{aligned}$$

In Problems 9 through 14 we have used MAPLE to compute Fourier-Legendre coefficients, which require integrals of the form $\int_{-1}^{1} f(x) P_n(x)\, dx$. This type of computation is reviewed in the MAPLE primer of the seventh edition of Advanced Engineering Mathematics. Recall that $P_n(x)$ is denoted in MAPLE as *LegendreP(n,x)*.

In some examples a "small" partial sum provides an approximation to the function with an error that is nearly undetectable in the scale of the graph. This is not to be expected in general, however, and sometimes many terms of a partial sum must be used to approximate a function with a partial sum of a Fourier-Legendre expansion.

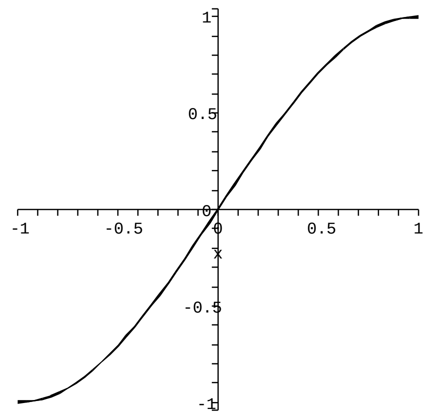

Figure 15.8: Partial sum in Problem 9, Section 15.2.

9. The function to be expanded is $f(x) = \sin(\pi x/2)$. Again approximating the integrals yielding the coefficients, we obtain

$$c_0 = c_2 = c_4 = 0, c_1 = 1.215854203, c_3 = -0.2248913308.$$

Figure 15.8 shows a graph of $\sin(\pi x/2)$ and this partial sum. These graphs are nearly identical in the scale of the graphics.

10. Write

$$e^{-x} = \sum_{n=0}^{\infty} c_n P_n(x)$$

for $-1 < x < 1$, where

$$c_n = \frac{2n+1}{2} \int_{-1}^{1} e^{-x} P_n(x)\, dx.$$

Numerical approximations of the first five coefficients are

$$c_0 = 1.1752101194, c_1 = -1.103638324, c_2 = 0.3578143500,$$
$$c_3 = -0.0704556300, c_4 = 0.00996502500.$$

Figure 15.9 compares a graph of e^{-x} with a graph of this partial sum $\sum_{n=0}^{4} c_n P_n(x)$.
Within the scale of the graph, e^{-x} and this partial sum are nearly indistinguishable.

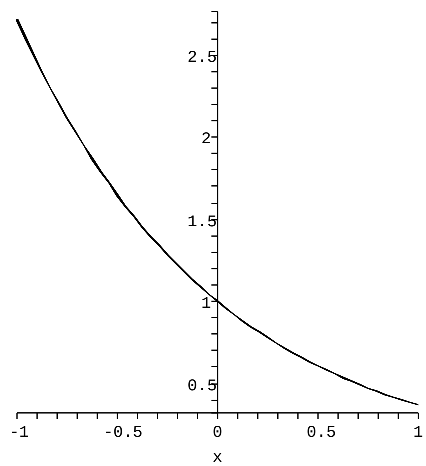

Figure 15.9: Partial sum in Problem 10, Section 15.2.

11. Compute

$$c_0 = 0.2726756433, c_1 = 0, c_2 = 0.4961198722,$$
$$c_3 = 0, c_4 = -0.06335726400.$$

Figure 15.10 shows a graph of this partial sum and the function.

12. The coefficients are approximately

$$c_0 = 0.841409850, c_1 = -0.9035060370, c_2 - 0.3101752600$$
$$c_3 = 0.06304606000, c_4 = 0.00909900000.$$

Figure 15.11 shows the function and this partial sum.

13. Compute

$$c_0 = 0, c_1 = 1.500000000, c_2 = 0,$$
$$c_3 = -0.8750000000, c_4 = 0.$$

Figure 15.12 shows a graph of this partial sum and the function. In this case, many more terms of the eigenfunction expansion are needed to approximate the function with any accuracy. Figure 15.13 shows the function and the fortieth partial sum of this expansion. This is a much better fit.

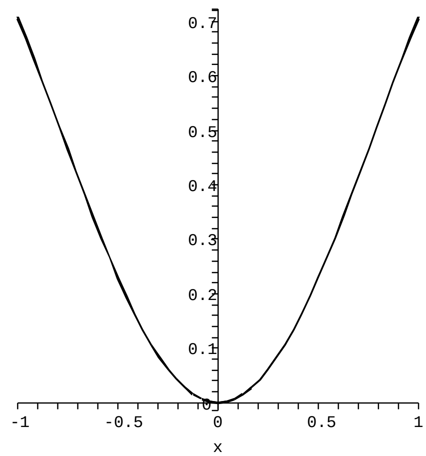

Figure 15.10: Partial sum in Problem 11, Section 15.2.

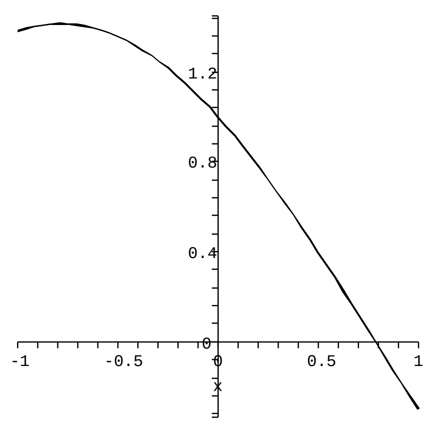

Figure 15.11: Partial sum in Problem 12, Section 15.2.

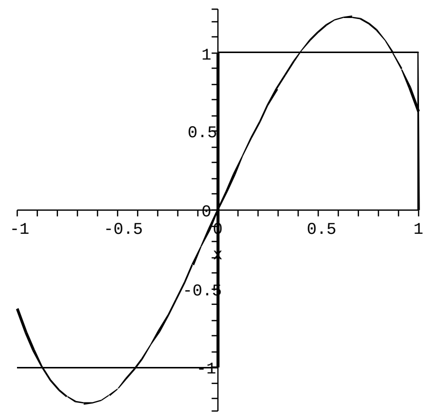

Figure 15.12: Fifth partial sum in Problem 13, Section 15.2.

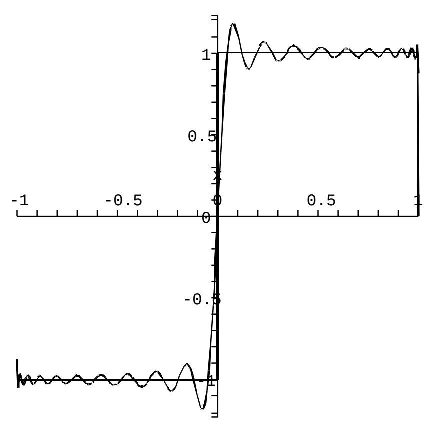

Figure 15.13: Fortieth partial sum in Problem 13, Section 15.2.

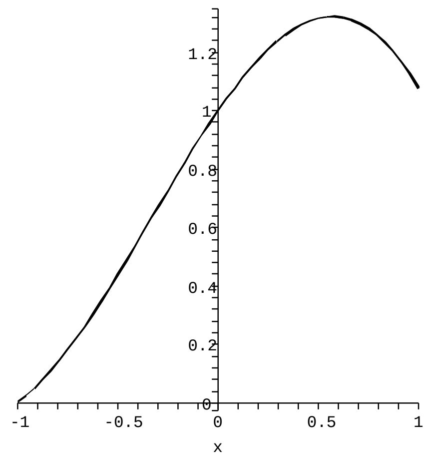

Figure 15.14: Partial sum in Problem 14, Section 15.2.

14. Compute

$$c_0 = 0.8411909850, c_1 = 0.7174008810, c_2 = -0.3101752600,$$
$$c_3 = -0.1820611100, c_4 = -0.00909900000.$$

Figure 15.14 shows a graph of this partial sum and the function.

15.3 Bessel Functions

Computations involving Bessel functions (values of Bessel functions at specific points, zeros, graphs, and integrals involving Bessel functions) require use of computational software. This is reviewed for MAPLE in the MAPLE primer of edition seven. Recall that $J_n(x)$ is denoted in MAPLE as BesselJ(n,x), and the kth zero of $J_n(x)$ is BesselJZeros(n,k). For a decimal value of this zero, use the evalf command.

1. Let $y = x^a J_\nu(bx^c)$. First compute

$$y' = ax^{a-1}J_\nu(bx^c) + x^a bcx^{c-1}J'_\nu(bx^c)$$

and

$$\begin{aligned} y'' &= a(a-1)x^{a-2}J_\nu(bx^c) \\ &\quad + [2ax^{a-1}bcx^{c-1} + x^a bc(c-1)x^{c-2}]J'_\nu(bx^c) \\ &\quad + x^a b^2c^2x^{2c-2}J''_\nu(bx^c). \end{aligned}$$

Substitute these into the differential equation and simplify to obtain

$$c^2x^{a-2}[(bx^c)^2J_\nu''(bx^c) + bx^cJ_\nu'(bx^c) + ((bx^c)^2 - \nu^2)J_\nu(bx^c)] = 0.$$

In Problems 2 - 9, the differential equation is solved by comparing it with the template equation (15.8), whose general solution we know. Solutions are for $x > 0$.

2. We need

$$2a = \frac{1}{3}, b^2c^2 = 1, 2c - 2 = 0, \text{ and } a^2 - \nu^2c^2 = \frac{7}{144}.$$

Then

$$a = \frac{1}{3}, c = b = 1, \nu = \frac{1}{4}$$

so the general solution of the differential equation is

$$y = c_1x^{1/3}J_{1/4}(x) + c_2x^{1/3}J_{-1/4}(x).$$

3. We need

$$1 - 2a = 1, b^2c^2 = 4, 2c - 2 = 2, a^2 - \nu^2c^2 = -\frac{4}{9},$$

so

$$a = 0, c = 2, b = 1, \nu = \frac{1}{3}.$$

The general solution is

$$y = c_1J_{1/3}x^2 + c_2J_{-1/3}(x^2).$$

For Problems 4 - 7, we will just give the values of a, b, c and ν and the general solution.

4. $a = 3, c = 4, b = 2, \nu = 1/2$, so

$$y = c_1x^3J_{1/2}(2x^4) + c_2x^3J_{-1/2}(2x^4).$$

5. $a = -1, c = 2, b = 2, \nu = 3/4$ and the general solution is

$$y = c_1\frac{1}{x}J_{3/4}(2x^2) + c_2\frac{1}{x}J_{-3/4}(2x^2).$$

6. $a = 2, c = 3, b = 1, \nu = 2/3$, so

$$y = c_1x^2J_{2/3}(x^3) + c_2x^2J_{-2/3}(x^3).$$

7. $a = 4, c = 3, b = 2, \nu = 3/4$, so

$$y = c_2x^4J_{3/4}(2x^3) + c_2x^4J_{-3/4}(2x^3).$$

8. $a = b = 0$, so this method produces only the trivial solution. However, observe that the differential equation is an Euler equation

$$x^2y'' + xy' - \frac{1}{16}y = 0,$$

with characteristic equation

$$r^2 - \frac{1}{16} = 0.$$

This has roots $\pm 1/4$. The Euler equation has the general solution

$$y = c_1x^{1/4} + c_2x^{-1/4}.$$

9. $a = -2, c = 3, b = 3, \nu = 1/2$, so the general solution is

$$y = c_1\frac{1}{x^2}J_{1/2}(3x^3) + c_2\frac{1}{x^3}J_{-1/2}(3x^3).$$

10. Differentiate $y = (1/bu)u'$ to obtain

$$y' = \frac{1}{b}\left(-\frac{1}{u^2}(u')^2 + \frac{1}{u}u''\right).$$

Substitute these into the given differential equation for y to obtain

$$\frac{1}{b}\left(-\frac{1}{u^2}(u')^2 + \frac{1}{u}u''\right) + b\left(\frac{1}{bu}u'\right)^2 = cx^m.$$

Simplify this equation to obtain

$$u'' - bcx^m = 0.$$

Compare this with equation (15.8), except now call the constants α, β and γ instead of a, b and c. We want

$$2\alpha - 1 = 0, 2\gamma - 2 = m, \beta^2\gamma^2 = -bc, \alpha^2 - \nu^2\gamma^2 = 0.$$

Then

$$\alpha = \frac{1}{2}, \gamma = \frac{m+2}{2}, \beta = \frac{2}{m+2}\sqrt{-bc}, \nu = \frac{1}{m+2}.$$

This gives us the general solution for u:

$$u(x) = c_2x^{1/2}J_{1/(m+2)}\left(\frac{2\sqrt{-bc}}{m+2}x^{(m+2)/2}\right) + c_2x^{1/2}J_{-1/(m+2)}\left(\frac{2\sqrt{-bc}}{m+2}x^{(m+2)/2}\right).$$

To complete the solution, substitute a solution for u into $y = u'/bu$. For example, if we take $c_1 = 1$ and $c_2 = 0$ and use this u, we obtain

$$y = \frac{1}{2bx} + \frac{J'_{1/(m+2)}\left(\frac{2\sqrt{-bc}}{m+2}x^{(m+2)/2}\right)}{J_{1/(m+2)}\left(\frac{2\sqrt{-bc}}{m+2}x^{(m+2)/2}\right)}.$$

11. With $z = x^{1/2}$, compute

$$\frac{dy}{dx} = \frac{1}{2}x^{-1/2}\frac{dy}{dz},$$
$$\frac{d^2y}{dx^2} = -\frac{1}{4}x^{-3/2}\frac{dy}{dz} + \frac{1}{4}x^{-1}\frac{d^2y}{dz^2}.$$

The transformed differential equation is

$$4z^4\left(\frac{1}{4}z^{-2}\frac{d^2y}{dz^2} - \frac{1}{4}z^{-3}\frac{dy}{dz}\right) + 4z^2\left(\frac{1}{2}z^{-1}\frac{dy}{dz}\right) + (z^2-9)y = 0.$$

Upon simplifying, this is

$$z^2y'' + zy' + (z^2-9)y = 0.$$

This fits the template (15.8) and has general solution

$$y(z) = c_1J_3(z) + c_2Y_{-3}(z).$$

Then

$$y(x) = c_1J_3(\sqrt{x}) + c_2Y_3(\sqrt{x}).$$

12. With $z = x^{3/2}$ the differential equation transforms to

$$4z^{4/3}\left[\frac{9}{4}z^{2/3}\frac{d^2y}{dz^2} + \frac{3}{4}z^{-1/3}\frac{dy}{dz}\right] + 4z^{2/3}\left[\frac{3}{2}z^{1/3}\frac{dy}{dz}\right] + (9z^2-16) = 0.$$

This simplifies to

$$z^2y'' + zy' + (z^2-4)y = 0,$$

with general solution

$$y(z) = c_1J_2(z) + c_2Y_2(z).$$

Then

$$y(x) = c_1J_2(x^{3/2}) + c_2Y_2(x^{3/2}).$$

13. With $z = 2x^{1/3}$, the transformed equation is

$$9\left(\frac{z}{2}\right)^6\left[\frac{4}{9}\left(\frac{z}{2}\right)^{-4}\frac{d^2y}{dz^2} - \frac{4}{9}\left(\frac{z}{2}\right)^{-5}\frac{dy}{dz}\right]$$
$$+9\left(\frac{z}{2}\right)^3\left[\frac{2}{3}\left(\frac{z}{2}\right)^{-2}\frac{dy}{dz}\right] + \left(4\left(\frac{z}{2}\right)^2 - 16\right)y = 0.$$

This simplifies to

$$z^2y'' + zy' + (z^2-16)y = 0,$$

with general solution

$$y(z) = c_1J_4(z) + c_2Y_4(z).$$

Then

$$y(x) = c_1J_4(2x^{1/3}) + c_2Y_4(2x^{1/3}).$$

14. Since $u = y/x^2$, then $y = x^2u$. The transformed equation is

$$9x^2[x^2u'' + 4xu' + 2u] - 27x[x^2u' + 2xu] + (9x^2 + 35)x^2u = 0.$$

Simplify this equation and divide by $9x^2$ to obtain

$$x^2u'' + xu' + (x^2 - 1/2)u = 0.$$

This has general solution

$$u(x) = c_1J_{1/3}(x) + c_2Y_{1/3}(x).$$

Then

$$y(x) = c_1x^2J_{1/3}(x) + c_2x^2Y_{1/3}(x).$$

15. With $u = x^{-2/3}y$, we have $y = x^{2/3}u$. The transformed equation is

$$\begin{aligned}&36x^2\left[x^{2/3}u'' + \frac{4}{3}x^{-1/3}u' - \frac{2}{9}x^{-4/3}u\right]\\&- 12x\left[x^{2/3}u' + \frac{2}{3}x^{-1/3}u\right] + (36x^2 + 7)x^{2/3}u = 0.\end{aligned}$$

Collect terms and divide by $36x^{2/3}$ to obtain

$$x^2u'' + xu' + (x^2 - 1/4)u = 0.$$

This has general solution

$$u(x) = c_1J_{1/2}(x) + c_2Y_{1/2}(x).$$

Then

$$y(x) = c_1x^{2/3}J_{1/2}(x) + c_2x^{2/3}Y_{1/2}(x).$$

16. We have $u = y\sqrt{x}$, so $y = x^{-1/2}u$. The differential equation transforms to

$$\begin{aligned}&4x^2\left[x^{-1/2}u'' - x^{-3/2}u' + \frac{3}{4}x^{-5/2}u\right]\\&+ 8x\left[x^{1/2}u' - \frac{1}{2}x^{-1/2}x^{-1/2}u\right] + (4x^2 - 35)u = 0.\end{aligned}$$

Collect terms and multiply by $\sqrt{x}/4$ to obtain

$$x^2u'' + xu' + (x^2 - 9)u = 0,$$

which has general solution

$$u(x) = c_1J_3(x) + c_2Y_3(x).$$

Then

$$y(x) = c_1x^{-1/2}J_3(x) + c_1x^{-1/2}Y_3(x).$$

17. Let α be a positive zero of J_0. Then $J_0(\alpha) = 0$. We want to show that

$$\int_0^1 J_1(\alpha x)\,dx = \frac{1}{\alpha}.$$

First, recall that $J_0'(x) = -J_1(x)$. Then

$$\int_0^\alpha J_1(s)\,ds = -J_0(s)\big]_0^\alpha = J_0(0) - J_0(\alpha) = 1,$$

since $J_0(0) = 1$. Now make the change of variables $s = \alpha x$ in the integral to obtain

$$\alpha \int_0^1 J_1(\alpha x)\,dx = 1,$$

and this is equivalent to what we want to show.

18. (a) Let $u(x) = J_0(\alpha x)$. Then

$$u' = \alpha J_0'(\alpha x) \text{ and } u'' = \alpha^2 J_0''(\alpha x).$$

Then

$$\begin{aligned} xu'' + u' + \alpha^2 xu &= \alpha^2 x J_0''(\alpha x) + \alpha J_0'(\alpha x) + \alpha^2 x J_0(\alpha x) \\ &= \alpha\left[\alpha x J_0''(\alpha x) + J_0'(\alpha x) + \alpha x J_0(\alpha x)\right] = 0, \end{aligned}$$

in which we have used Bessel's equation of order $\nu = 0$. If α is replaced by β and $v = J_0(\beta x)$, we obtain

$$xv'' + v' + \beta^2 xv = 0.$$

(b) Multiply the first equation by v and the second by u and then subtract the resulting first equation from the second to obtain

$$xuv'' - xvu'' + uv' - vu' + (\beta^2 - \alpha^2)xuv = 0.$$

We can write this equation as

$$(\beta^2 - \alpha^2)xuv = -[x(uv' - vu')]'.$$

(c) Finally, integrate both sides of this equation to obtain

$$(\beta^2 - \alpha^2)\int xJ_0(\alpha x)J_0(\beta x)\,dx = -x\left(J_0(\alpha x)\beta J_0'(\beta x) - \alpha J_0(\beta x)J_0'(\alpha x)\right),$$

and upon multiplying through by the -1 coefficient of x on the right, we obtain Lommel's integral.

19. By equation (15.20),

$$(x^n J_n(x))' = x^n J_{n-1}(x)$$

and integrating both sides yields

$$\int x^n J_{n-1}(x)\,dx = x^n J_n(x).$$

By equation (15.21),

$$(x^{-n} J_n(x))' = -x^{-n} J_{n+1}(x).$$

Again, by integrating, we obtain immediately that

$$\int x^{-n} J_{n+1}(x)\,dx = -x^{-n} J_n(x),$$

and this is equivalent to what we want to show.

20. These are immediate from Problem 19. First, we know from the first conclusion of Problem 19 that

$$\int s^n J_{n-1}(s)\,ds = s^n J_n(s).$$

Let $s = \alpha x$ to obtain

$$\alpha^n \int x^n J_{n-1}(\alpha x)\alpha\,dx = \alpha^n x^n J_n(\alpha x).$$

This yields the first conclusion of this problem. A similar calculation, using the second equation in Problem 19, yields the second equation in this problem.

21. Define

$$I_{n,k} = \int_0^1 (1-x^2)^k x^{n+1} J_n(\alpha x)\,dx.$$

For (a), begin with a result from Problem 19:

$$\int s^n J_{n-1}(s)\,ds = s^n J_n(s).$$

Replace n with $n+1$:

$$\int s^{n+1} J_n(s)\,ds = s^{n+1} J_{n+1}(s).$$

Then

$$\int_0^\alpha s^{n+1} J_n(s)\,ds = \left[s^{n+1} J_{n+1}(s)\right]_0^{\ \alpha} = \alpha^{n+1} J_{n+1}(\alpha).$$

Now let $s = \alpha x$ to obtain

$$\int_0^1 \alpha^{n+1} x^{n+1} J_n(\alpha x)\alpha, dx = \alpha^{n+1} J_{n+1}(\alpha).$$

Then

$$\int_0^1 x^{n+1} J_n(\alpha x)\, dx = \frac{1}{\alpha} J_{n+1}(\alpha).$$

But

$$I_{n,0} = \int_0^1 x^{n+1} J_n(\alpha x)\, dx.$$

This proves that

$$I_{n,0} = \frac{1}{\alpha} J_{n+1}(\alpha).$$

Now use the first integral in Problem 20, with $n+1$ in place of n, to write

$$x^{n+1} J_n(\alpha x) = \frac{d}{dx}\left(\frac{1}{\alpha} x^{n+1} J_{n+1}(\alpha x)\right).$$

Substitute this into the definition of $I_{n,k}$ to write

$$I_{n,k} = \int_0^1 (1-x^2)^k \frac{d}{dx}\left(\frac{1}{\alpha} x^{n+1} J_{n+1}(\alpha x)\right) dx.$$

This completes part (b). Now, for (c), integrate the expression of (b) by parts:

$$\begin{aligned}
I_{n,k} &= \int_0^1 (1-x^2)^k \frac{d}{dx}\left(\frac{1}{\alpha} x^{n+1} J_{n+1}(\alpha x)\right) dx \\
&= (1-x^2)^k \frac{x^{n+1}}{\alpha} J_{n+1}(\alpha x)\Big]_0^1 \\
&- \int_0^1 \frac{1}{\alpha} x^{n+1} J_{n+1}(\alpha x) k (1-x^2)^{k-1}(-2x)\, dx \\
&= \frac{2k}{\alpha} \int_0^1 (1-x^2)^{k-1} x^{n+2} J_{n+1}(\alpha x)\, dx \\
&= \frac{2k}{\alpha} I_{n+1,k-1}.
\end{aligned}$$

This relates $I_{n,k}$ to the value of this integral when n is increased by 1 and k decreased by 1. In particular, if we carry out k repetitions of this operation, eventually increasing n by k, and decreasing k by k, we obtain

$$\begin{aligned}
I_{n,k} &= \frac{2k}{\alpha} I_{n+1,k-1} \\
&= \frac{2k}{\alpha}\left[\frac{2(k-1)}{\alpha} I_{n+2,k-2}\right] \\
&= \frac{2^2 k(k-1)}{\alpha^2} I_{n-2,k-2} \\
&= \frac{2^2 k(k-1)}{\alpha^2}\left[\frac{2(k-2)}{\alpha} I_{n+3,k-2}\right] \\
&= \frac{2^3 k(k-1)(k-2)}{\alpha^3} I_{n+3,k-3} \\
&= \cdots = \frac{2^k k!}{\alpha^k} I_{n+k,0}.
\end{aligned}$$

Since k is a positive integer, we can write $k! = \Gamma(k+1)$, as in the statement of the problem. This gives us the result of part (d).

Finally, for part (e), combine the conclusions of parts (a) and (d) to write

$$\int_0^1 (1-x^2)^k x^{n+1} J_n(\alpha x)\,dx = \frac{2^k \Gamma(k+1)}{\alpha^{k+1}} J_{n+k+1}(\alpha).$$

To obtain the result of part (f), write the last equation as

$$J_{n+k+1}(\alpha) = \frac{\alpha^{k+1}}{2^k \Gamma(k+1)} \int_0^1 (1-x^2)^k x^{n+1} J_n(\alpha x)\,dx.$$

The rest is just notation to obtain a different perspective. Rewrite the last equation by writing x in place of α and t in place of x to obtain

$$J_{n+k+1}(x) = \frac{x^{k+1}}{2^k \Gamma(k+1)} \int_0^1 t^{n+1}(1-t^2)^k J_n(xt)\,dt.$$

Finally, for (g), let $m - n = k + 1$ to write

$$J_m(x) = \frac{2x^{m-n}}{2^{m-n}\Gamma(m-n)} \int_0^1 t^{n+1}(1-t^2)^{m-n-1} J_n(xt)\,dt.$$

It is important to observe that these results do not require that k be an integer, since $k!$ has been replaced by $\Gamma(k+1)$, which is well defined if $k+1 > 0$. In the conclusions derived in this problem, it is enough to have $n > -1$, $k > -1$ and, in (g), $m > n > -1$.

22. Use the given expression for $J_{-1/2}(xt)$ with $n = -1/2$ and $m > -1/2$ in part (g) of Problem 21 to write

$$\begin{aligned}
J_m(x) &= \frac{2x^{m+1/2}}{2^{m+1/2}\Gamma(m+1/2)} \int_0^1 t^{1/2}(1-t^2)^{m-1/2}\left(\frac{2}{\pi x t}\right)^{1/2} \cos(xt)\,dt \\
&= \frac{x^m}{2^{m-1}\Gamma(m+1/2)} \int_0^1 (1-t^2)^{m-1/2} \cos(xt)\,dt.
\end{aligned}$$

This is the requested result with m used in place of n in order to make use of the preceding problem's conclusion.

23. Start with the following result from Problem 22:

$$J_m(x) = \frac{x^m}{2^{m-1}\Gamma(m+1/2)} \int_0^1 (1-t^2)^{m-1/2} \cos(xt)\, dt.$$

Now make the change of variables $t = \sin(\theta)$ in the integral. When $t = 0$, $\theta = 0$ and when $t = 1$, $\theta = \pi/2$. Further, $dt = \cos(t)\, dt$ and

$$1 - t^2 = 1 - \sin^2(t) = \cos^2(t).$$

Then

$$\begin{aligned} J_m(x) &= \frac{x^m}{2^{m-1}\Gamma(m+1/2)} \int_0^{\pi/2} (\cos^2(t))^{m-1/2} \cos(x\sin(\theta)) \cos(t)\, dt \\ &= \frac{x^m}{2^{m-1}\Gamma(m+1/2)} \int_0^{\pi/2} \cos^{2m}(\theta) \cos(x\sin(\theta))\, d\theta. \end{aligned}$$

In Problems 24 through 29, we want a Fourier-Bessel expansion

$$\sum_{n=1}^{\infty} c_n J_1(j_n x),$$

where j_n is the nth positive zero of $J_1(x)$ and

$$c_n = \frac{2\int_0^1 x f(x) J_1(j_n x)\, dx}{J_2(j_n)^2}.$$

24. With $f(x) = e^{-x}$, the fifth partial sum (Figure 15.15) of the Fourier-Bessel function bears little resemblance to the function. Figure 15.16 shows the thirty-fifth partial sum of this expansion.

25. With $f(x) = x$, then nth Fourier-Bessel coefficient for expanding in a series of eigenfunctions $J_1(j_n x)$ is

$$c_n = \frac{2}{J_2(j_n)^2} \int_0^1 x J_1(j_n x)\, dx.$$

Figure 15.17 shows the fifth partial sum, compared to the function in this expansion. Clearly this fifth partial sum does not approximate the function at all well. Figure 15.18 shows the function and the thirty-fifth partial sum, suggesting convergence of the Fourier-Bessel expansion to the function as more terms are included in the expansion.

26. Figures 15.19 and 15.20 show the fifth and thirty-fifth partial sums of this Fourier-Bessel expansion.

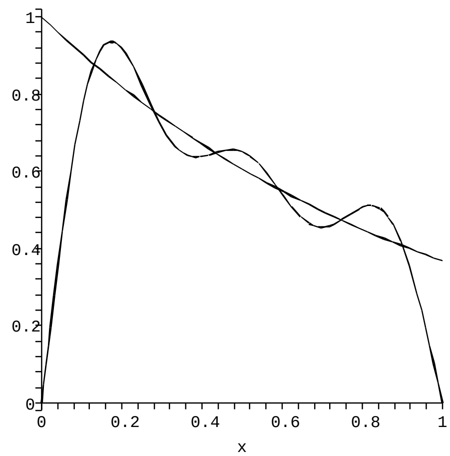

Figure 15.15: Fifth partial sum in Problem 24, Section 15.3.

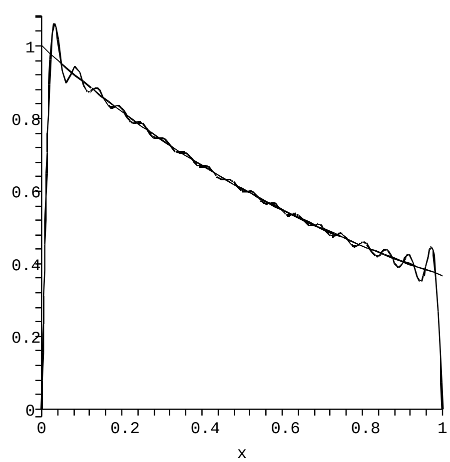

Figure 15.16: Thirty-fifth partial sum in Problem 24, Section 15.3.

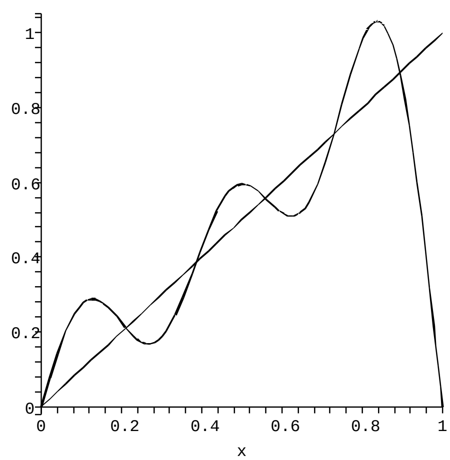

Figure 15.17: Fifth partial sum in Problem 25, Section 15.3.

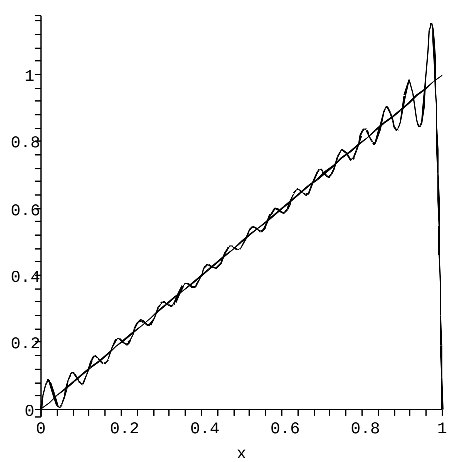

Figure 15.18: Thirty-fifth partial sum in Problem 25, Section 15.3.

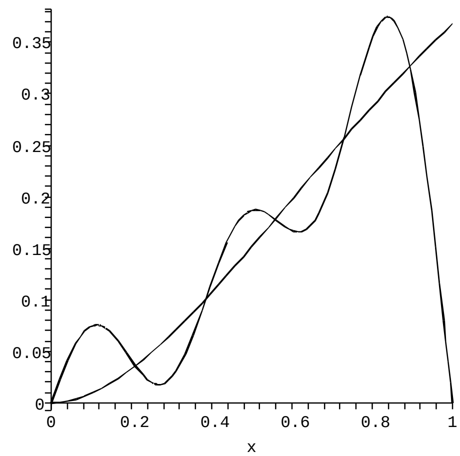

Figure 15.19: Fifth partial sum in Problem 26, Section 15.3.

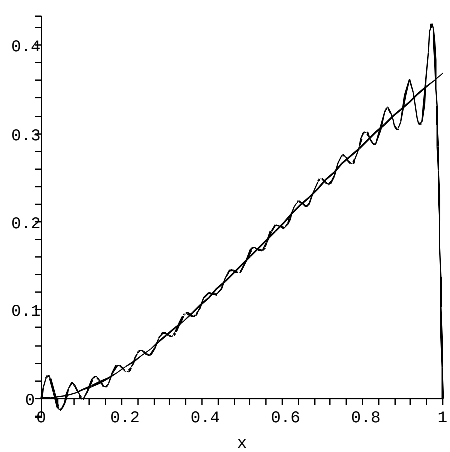

Figure 15.20: Thirty-fifth partial sum in Problem 26, Section 15.3.

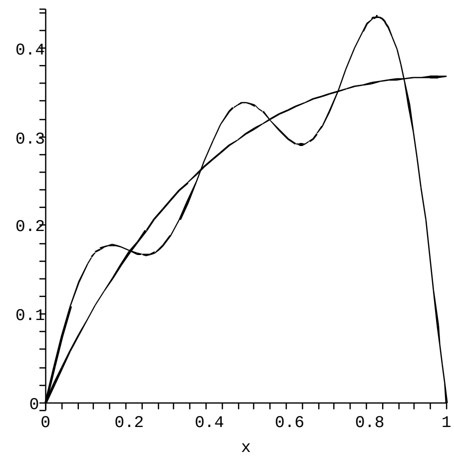

Figure 15.21: Fifth partial sum in Problem 27, Section 15.3.

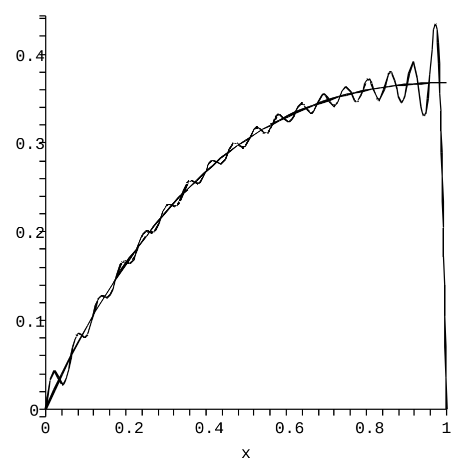

Figure 15.22: Thirty-fifth partial sum in Problem 27, Section 15.3.

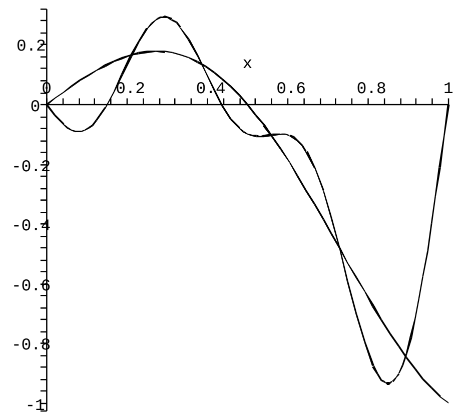

Figure 15.23: Fifth partial sum in Problem 28, Section 15.3.

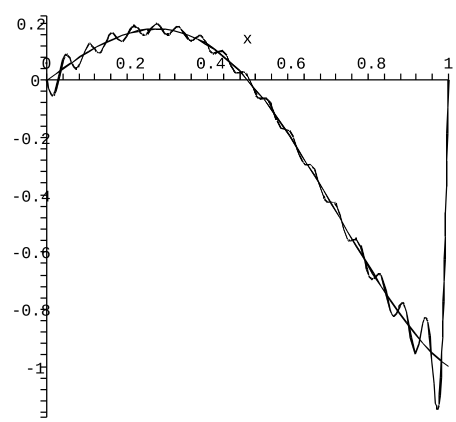

Figure 15.24: Thirty-fifth partial sum in Problem 28, Section 15.3.

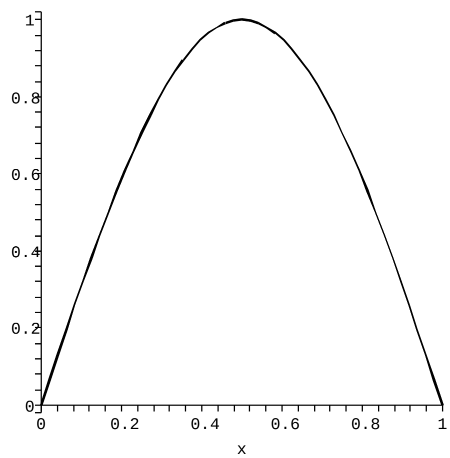

Figure 15.25: Fifth partial sum in Problem 29, Section 15.3.

27. Figures 15.21 and 15.22 show the fifth and thirty-fifth partial sums of this Fourier-Bessel expansion of $f(x) = xe^{-x}$.

28. Figures 15.23 and 15.24 show the fifth and thirty-fifth partial sums of this Fourier-Bessel expansion of $f(x) = x\cos(\pi x)$.

29. Figure 15.25 shows the fifth partial sum of the Fourier-Bessel expansion of $f(x) = \sin(\pi x)$. This partial sum appears to be a good approximation to the function.

For Problems 30 through 35, we expand the functions of Problems 24 through 29, respectively, except now we use a Fourier-Bessel expansion in terms of Bessel functions of the first kind of order 2. This series will have the form

$$\sum_{n=1}^{\infty} c_n J_2(j_n x),$$

where j_n is the nth positive zero of $J_2(x)$ and

$$c_n = \frac{2\int_0^1 xf(x)J_2(j_n x)\,dx}{J_3(j_n)^2}.$$

For each problem, we graph the fifth and the twenty-fifth partial sum, compared to the function.

30. The fifth and twenty-fifth partial sums are given in Figures 15.26 and 15.27, respectively.

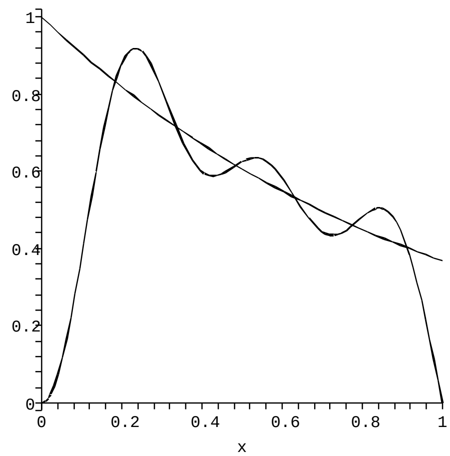

Figure 15.26: Fifth partial sum in Problem 30, Section 15.3.

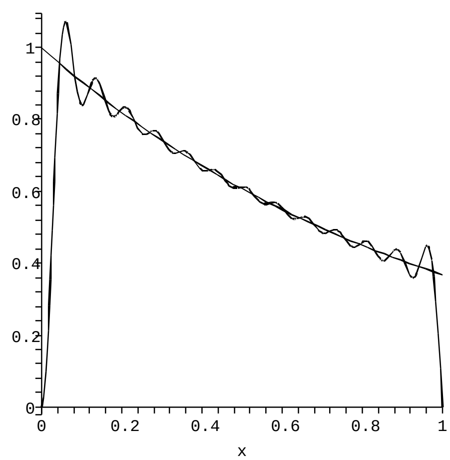

Figure 15.27: Twenty-fifth partial sum in Problem 30, Section 15.3.

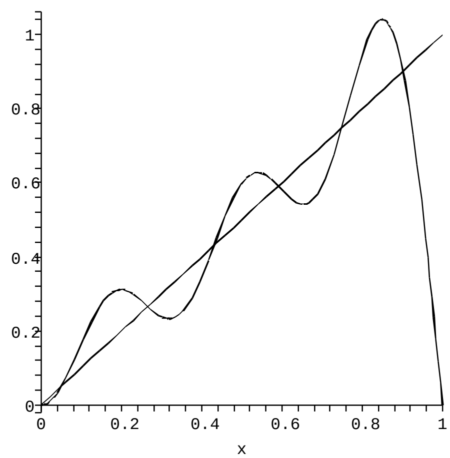

Figure 15.28: Fifth partial sum in Problem 31, Section 15.3.

31. Figures 15.28 and 15.29 show the fifth and twenty-fifth partial sums of this expansion.

32. Figures 15.30 and 15.31 show the fifth and twenty-fifth partial sums.

33. The fifth and twenty-fifth partial sums are shown in Figures 15.32 and 15.33.

34. The fifth and twenty-fifth partial sums are given in Figures 15.34 and 15.35.

35. The fifth and twenty-fifth partial sums are shown in Figures 15.36 and 15.37.

36. Make the change of variables $t = u^2$ in the integral defining $\Gamma(1/2)$ to obtain

$$\begin{aligned}\Gamma(1/2) &= \int_0^\infty t^{-1/2}e^{-t}\,dt \\ &= \int_0^\infty \frac{1}{u}e^{-u^2}2u\,du \\ &= 2\int_0^\infty e^{-u^2}\,du = 2\left(\frac{\sqrt{\pi}}{2}\right) = \sqrt{\pi}.\end{aligned}$$

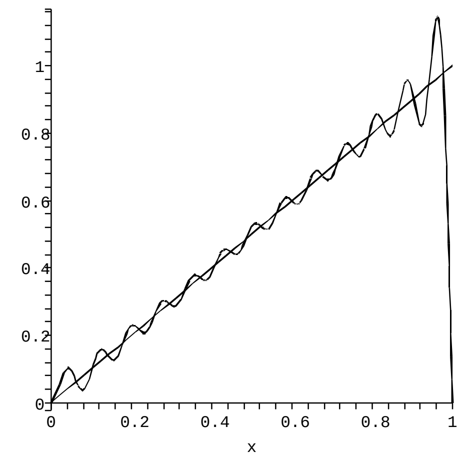

Figure 15.29: Twenty-fifth partial sum in Problem 31, Section 15.3.

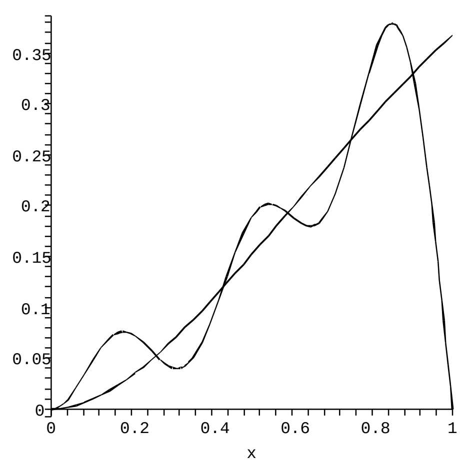

Figure 15.30: Fifth partial sum in Problem 32, Section 15.3.

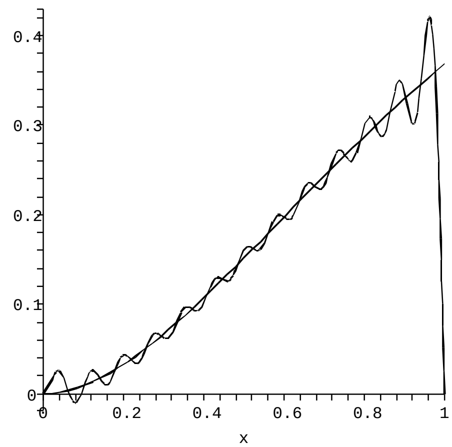

Figure 15.31: Twenty-fifth partial sum in Problem 32, Section 15.3.

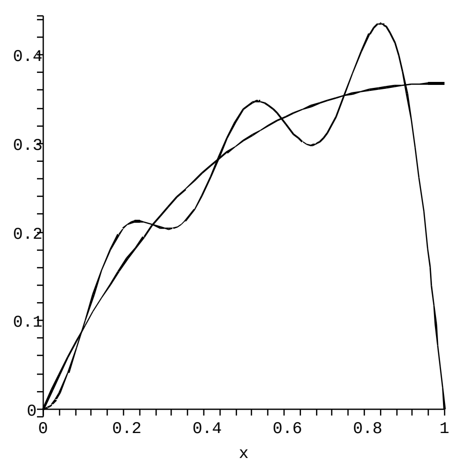

Figure 15.32: Fifth partial sum in Problem 33, Section 15.3.

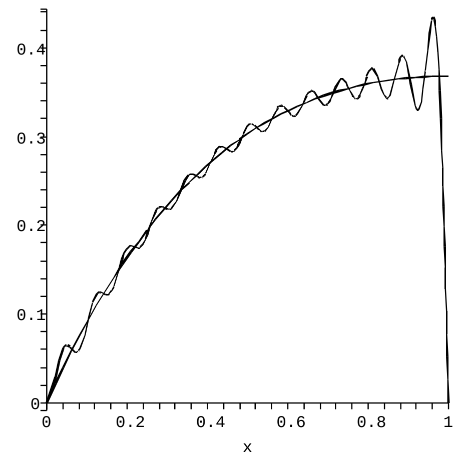

Figure 15.33: Twenty-fifth partial sum in Problem 33, Section 15.3.

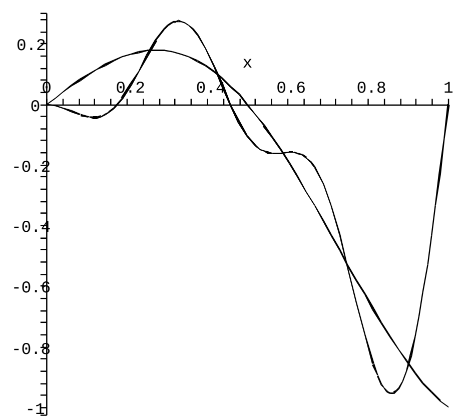

Figure 15.34: Fifth partial sum in Problem 34, Section 15.3.

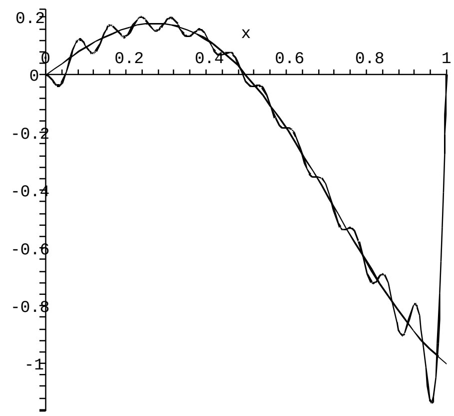

Figure 15.35: Twenty-fifth partial sum in Problem 34, Section 15.3.

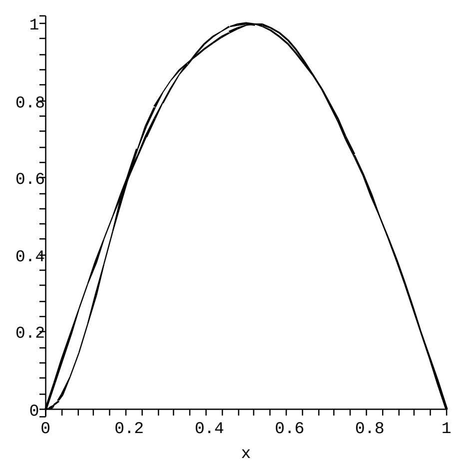

Figure 15.36: Fifth partial sum in Problem 35, Section 15.3.

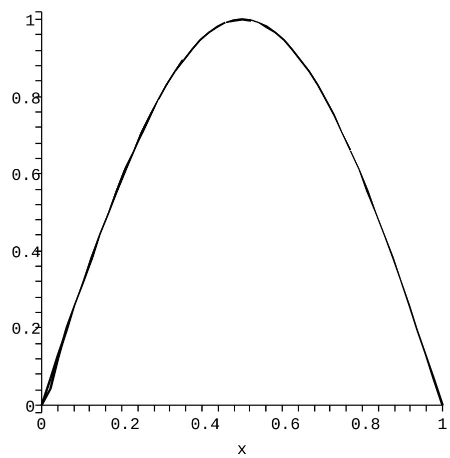

Figure 15.37: Twenty-fifth partial sum in Problem 35, Section 15.3.

37. Let $t = ry$ in the integral defining the gamma function to write

$$\begin{aligned}\Gamma(x) &= \int_0^\infty t^{x-1}e^{-t}\,dt \\ &= \int_0^\infty (ry)^{x-1}e^{-ry}r\,dy \\ &= r^x\int_0^\infty y^{x-1}e^{-ry}\,dy\end{aligned}$$

and this is the integral to be derived with the variable of integration denoted y instead of t.

38. Let $t = y^2$ in the definition of the gamma function to obtain

$$\begin{aligned}\Gamma(x) &= \int_0^\infty t^{x-1}e^{-t}\,dt \\ &= \int_0^\infty y^{2x-2}e^{-y^2}2y\,dy \\ &= \int_0^\infty y^{2x-1}e^{-y^2}\,dy.\end{aligned}$$

This is the conclusion we want to derive, with the variable of integration denoted y instead of t.

39. Let $t = u/(1+u)$ in the definition of the beta function. Then $u \to \infty$ as

$t \to 1$, and

$$dt = \frac{1}{(1+u)^2}\,du.$$

We obtain

$$\begin{aligned} B(x,y) &= \int_0^1 t^{x-1}(1-t)^{y-1}\,dt \\ &= \int_0^\infty \left(\frac{u}{1+u}\right)^{x-1}\left(1-\frac{u}{1+u}\right)^{y-1}\frac{1}{(1+u)^2}\,du \\ &= \int_0^\infty \frac{u^{x-1}}{(1+u)^{x+y}}\,du, \end{aligned}$$

and this is what we wanted to show.

40. Let x and y be positive numbers. We want to show that

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

This can be done by a clever use of double integrals, but here is a proof using the convolution operation of the Laplace transform. First, it is routine to check that

$$\mathcal{L}[t^{x-1}](s) = \frac{\Gamma(x)}{s^x}.$$

This is consistent with the result obtained in Chapter Three for the case that $x = n$, an integer greater than 1, since in that case $\Gamma(x) = (x-1)!$. From this, we have

$$\mathcal{L}^{-1}\left[\frac{1}{s^x}\right] = \frac{t^{x-1}}{\Gamma(x)}.$$

Using this, we will derive the result by computing an inverse Laplace transform, first directly, then using the convolution theorem:

$$\begin{aligned} \mathcal{L}^{-1}\left[\frac{1}{s^{x+y}}\right](t) &= \frac{t^{x+y-1}}{\Gamma(x+y)} \\ &= \mathcal{L}^{-1}\left[\frac{1}{s^x}\right] * \mathcal{L}^{-1}\left[\frac{1}{s^y}\right] \\ &= \int_0^t \frac{u^{x-1}}{\Gamma(x)}\frac{1}{\Gamma(y)}(t-u)^{y-1}\,du \\ &= \frac{1}{\Gamma(x)\Gamma(y)}\int_0^t u^{x-1}(t-u)^{y-1}\,du \\ &= \frac{1}{\Gamma(x)\Gamma(y)}\int_0^t u^{x-1}\left(1-\frac{u}{t}\right)^{y-1} t^{y-1}\,du. \end{aligned}$$

Now make the change of variables $w = u/t$ in the last integral to continue the last equation:

$$\begin{aligned}
&\frac{1}{\Gamma(x)\Gamma(y)}\int_0^1 (wt)^{x-1}(1-w)^{y-1}t^{y-1}\,dw \\
&= \frac{1}{\Gamma(x)\Gamma(y)}\int_0^1 t^{x+y-1}w^{x-1}(1-w)^{y-1}\,dw \\
&= \frac{t^{x+y-1}}{\Gamma(x)\Gamma(y)}B(x,y).
\end{aligned}$$

Looking at the beginning and end of this string of equalities, we have shown that

$$\frac{t^{x+y-1}}{\Gamma(x+y)} = \frac{t^{x+y-1}}{\Gamma(x)\Gamma(y)}B(x,y).$$

Upon dividing out t^{x+y-1} we obtain the expression to be proved.

Chapter 16

The Wave Equation

16.1 Derivation of the Equation

1. Compute
$$\frac{\partial^2 y}{\partial t^2} = -\frac{n^2\pi^2 c^2}{L^2}\sin(n\pi x/L)\cos(n\pi ct/L)$$
and
$$\frac{\partial^2 y}{\partial x^2} = -\frac{n^2\pi^2}{L^2}\sin(n\pi x/L)\cos(n\pi ct/L).$$
Therefore
$$\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2}.$$

2. Compute the partial derivatives
$$\frac{\partial^2 z}{\partial t^2} = -(n^2+m^2)c^2\sin(nx)\cos(my)\cos(\sqrt{n^2+m^2}ct),$$
$$\frac{\partial^2 z}{\partial x^2} = -n^2\sin(nx)\cos(my)\cos(\sqrt{n^2+m^2}ct),$$
$$\frac{\partial^2 z}{\partial y^2} = -m^2\sin(nx)\cos(my)\cos(\sqrt{n^2+m^2}ct).$$
Therefore
$$\frac{\partial^2 y}{\partial t^2} = c^2\left(\frac{\partial^2 y}{\partial x^2} + \frac{\partial^2 y}{\partial y^2}\right).$$

3. Chain-rule differentiations yield
$$\frac{\partial^2 y}{\partial t^2} = \frac{1}{2}(c^2 f''(x+ct) + c^2 f''(x-ct))$$
and
$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{2}(f''(x+ct) + f''(x-ct)).$$

Then

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}.$$

4. It is a routine differentiation to verify that $y(x,t)$ satisfies the one-dimensional wave equation. For the boundary conditions,

$$y(0,t) = y(2\pi,t) = \frac{1}{c}\sin(ct)$$

for $t > 0$. For the initial conditions, $y(x,0) = \sin(x)$ and

$$\frac{\partial y}{\partial t}(x,0) = \left. -c\sin(x)\sin(ct) + \cos(x)\cos(ct)\right|_{t=0} = \cos(x).$$

5. Let $z(x,y,t)$ be the vertical displacement of the point of the membrane located at (x,y) at time $t > 0$. Then $z(x,y,t)$ satisfies the two-dimensional wave equation

$$\frac{\partial^2 z}{\partial t^2} = c^2\left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\right).$$

Because the membrane occupies the region $0 \le x \le a, 0 \le y \le b$ and is fastened at all the points of its rectangular boundary, we have the boundary conditions

$$z(0,y,t) = z(a,y,t) = z(x,0,t) = z(x,b,t) = 0$$

for $0 < x < a$, $0 < y < b$ and $t > 0$. Finally, the initial conditions are

$$z(x,y,0) = f(x,y), \frac{\partial z}{\partial t}(x,y,0) = g(x,y).$$

6. Let $u(x,t)$ be the transverse displacement at time t of the point of the string located at x. Then

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2} - k\left(\frac{\partial u}{\partial t}\right)^2 \text{ for } 0 < x < L, t < 0.$$

The boundary conditions are

$$u(0,t) = u(L,t) = 0 \text{ for } t > 0$$

and the initial conditions are

$$u(x,0) = f(x), \frac{\partial u}{\partial t}(x,0) = 0 \text{ for } 0 < x < L.$$

16.2 Wave Motion on an Interval

For each of Problems 1 through 8, separation of variables in the wave equation with the fixed end conditions at $x = 0$ and $x = L$ yields the general solution

$$y(x,t) = \sum_{n=1}^{\infty}\left[a_n \cos(n\pi ct/L) + \frac{L}{n\pi c} b_n \sin(n\pi ct/L)\right]\sin(n\pi x/L),$$

where

$$a_n = \frac{2}{L}\int_0^L f(\xi)\sin(n\pi\xi/L)\,d\xi$$

for $n = 1, 2, \cdots$ and

$$b_n = \frac{2}{L}\int_0^L g(\xi)\sin(n\pi\xi/L)\,d\xi$$

for $n = 1, 2, \cdots$. f is the initial position function and g is the initial velocity function.

To write the solution in specific instances, we need only identify c and L and compute the coefficients for the particular initial position and velocity functions.

1. $L = 2$, $f(x) = 0$ and $g(x) = 2x(1 - H(x - 1))$, with H the Heaviside function. Then

$$a_n = 0 \text{ and } b_n = \frac{4}{n^2\pi^2}\left[2\sin(n\pi/2) - n\pi\cos(n\pi/2)\right]$$

for $n = 1, 2, \cdots$. Then

$$y(x,t) = \sum_{n=1}^{\infty}\frac{8}{n^3\pi^3 c}\left[2\sin(n\pi/2) - n\pi\cos(n\pi/2)\right]\sin(n\pi x/2)\sin(n\pi ct/2).$$

Figure 16.1 shows wave profiles increasing in amplitude at times $t = 1/10, 1/3$ and $1/2$, with the wave at $t = 1/2$ achieving its highest point near $x = 1.3$.

2. $c = 3$ and $L = 4$, $f(x) = 2\sin(\pi x)$ and $g(x) = 0$. Each $a_n = 0$ if $n \neq 4$, $a_4 = 2$, and $b_n = 0$, so the solution is

$$y(x,t) = 2\sin(\pi x)\cos(3\pi t).$$

Figure 16.2 is a graph of the solution with $c = 1$, at times $t = 0.1$ (highest near the origin), and $t = 0.7$.

3. The solution is

$$y(x,t) = \frac{108}{\pi^4}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^4}\sin((2n-1)\pi x/3)\sin((2(2n-1)\pi t/3)).$$

Figure 16.3 shows the solution waves increasing in amplitude over times $t = 0.1, 0.3$ and 0.7.

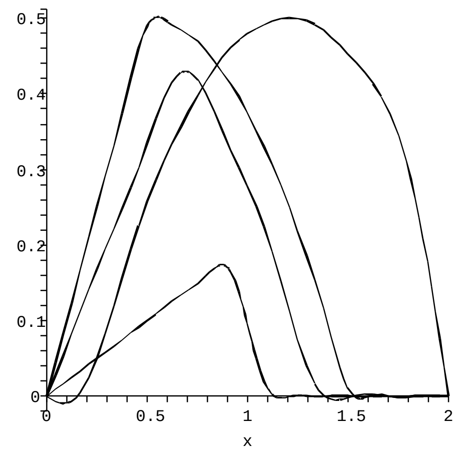

Figure 16.1: Waves in Problem 1, Section 16.2.

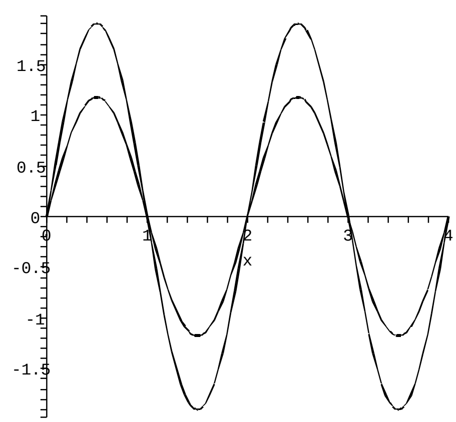

Figure 16.2: Waves in Problem 2, Section 16.2.

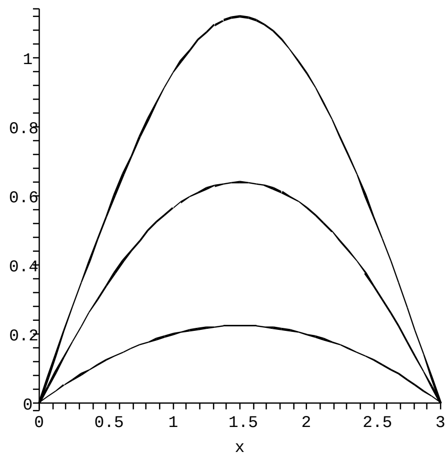

Figure 16.3: Waves in Problem 3, Section 16.2.

4. The solution is

$$y(x,t) = \sin(x)\cos(3t) + \frac{4}{3\pi}\sum_{n-1}^{\infty} \frac{1}{(2n-1)^2}\sin((2n-1)x)\sin(3(2n-1)t).$$

Figure 16.4 shows the wave profile at $t = 0.4$ (highest wave), $t = 0.7$ (lowest), and $t = 0.9$ (middle wave).

5. The solution is

$$y(x,t) = \frac{24}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^2}\sin((2n-1)x/2)\cos((2n-1)\sqrt{2}t).$$

Figure 16.5 shows the waves moving downward through times $t = 0.3$, 0.5, 0.9 and 1.4.

6. The solution is

$$y(x,t) = \frac{5}{\pi^3}\sum_{n=1}^{\infty} \frac{1}{n^3}\sin(n\pi x/5)\left[5\sin(4n\pi/5) + n\pi\cos(4n\pi/5)\right]\sin(2n\pi t/5)$$

Figure 16.6 shows the waves moving to the left through times $t = 0.3$, 0.7 and 1.2.

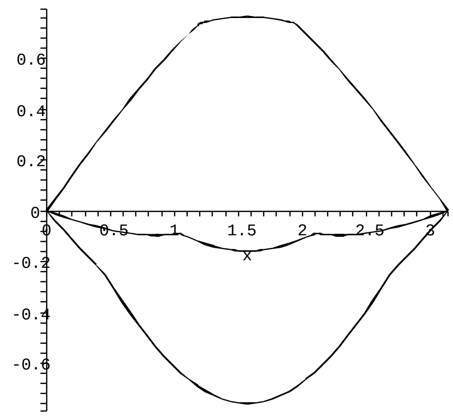

Figure 16.4: Waves in Problem 4, Section 16.2.

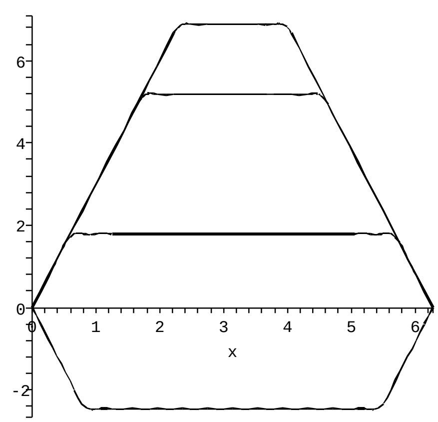

Figure 16.5: Profiles of the solution in Problem 5, Section 16.2.

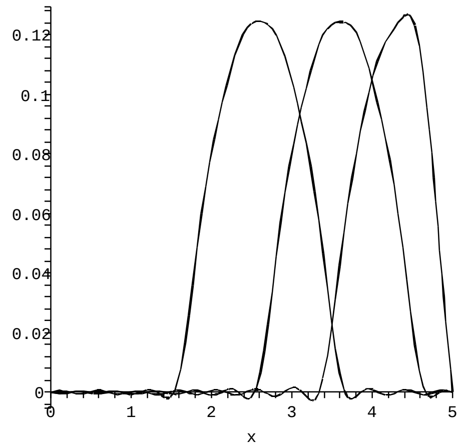

Figure 16.6: Profiles of the solution in Problem 6, Section 16.2.

7. The solution is

$$y(x,t) = -\frac{32}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3}\sin((2n-1)\pi x/2)\cos(3(2n-1)\pi t/2)$$
$$+\frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\left[\cos(n\pi/4)-\cos(n\pi/2)\right]\sin(n\pi x/2)\sin(3n\pi t/2)$$

Waves for this solution are shown in Figure 16.7, increasing in amplitude for times $t = 0.3$, 0.4 and 0.5.

8. The solution is

$$y(x,t) = \sin(2x)\cos(10t) + \frac{2}{5}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin(nx)\sin(5nt)$$

In Figure 16.8, the lowest wave (near the origin) is at $t = 0.3$, then the higher one at $t = 0.5$, and the middle wave at 0.7.

9. The differential equation is not separable, due to the $2x$ forcing term. Let $y(x,t) = Y(x,t) - h(x)$ and choose h to obtain a problem for Y that we have solved. Substitute y into the partial differential equation to obtain

$$\frac{\partial^2 Y}{\partial t^2} = 3\left(\frac{\partial^2 Y}{\partial x^2} - h''\right) + 2x.$$

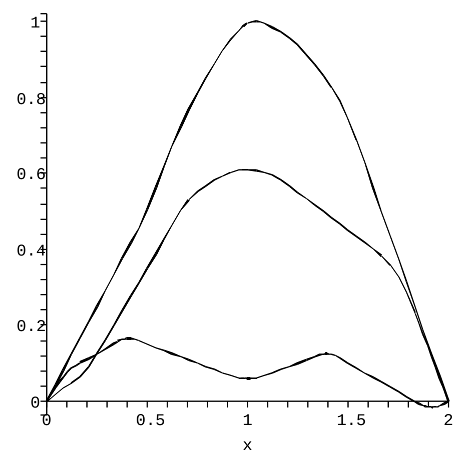

Figure 16.7: Profiles of the solution in Problem 7, Section 16.2.

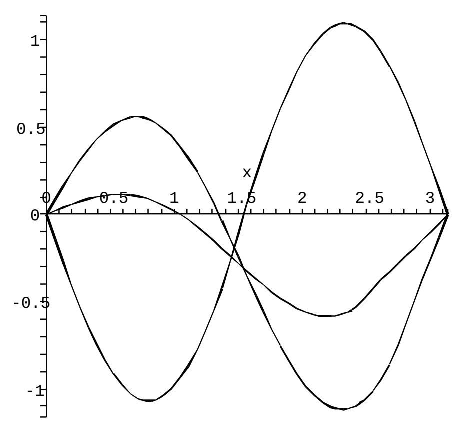

Figure 16.8: Profiles of the solution in Problem 8, Section 16.2.

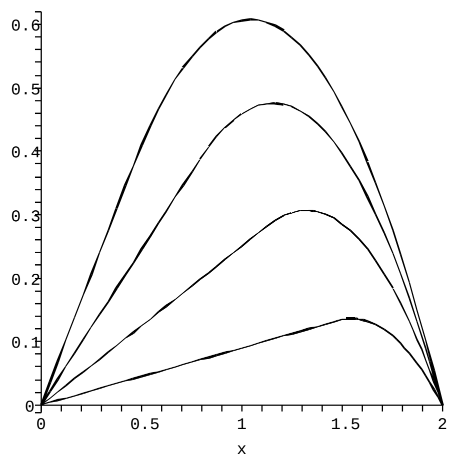

Figure 16.9: Profiles of the solution in Problem 9, Section 16.2.

This is the standard wave equation for Y if $3h''(x) = 2x$. For homogeneous boundary conditions at 0 and 2, we need $h(0) = h(2) = 0$. Solve for $h(x)$ by two integrations to obtain

$$h(x) = \frac{1}{9}(x^3 - 4x).$$

Then $Y(x,t)$ satisfies the standard problem

$$\begin{aligned} \frac{\partial^2 Y}{\partial t^2} &= 3\frac{\partial^2 Y}{\partial x^2}, \\ Y(0,t) &= Y(2,t)0, \\ Y(x,0) &= h(x) = \frac{1}{9}(x^3 - 4x), \frac{\partial Y}{\partial t}(x,0) = 0. \end{aligned}$$

Write the solution $Y(x,t)$ and then

$$\begin{aligned} y(x,t) &= -h(x) + Y(x,t) \\ &= -\frac{1}{9}(x^3 - 4x) + \frac{32}{3\pi^3}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}\sin(n\pi x/3)\cos(\sqrt{3}n\pi t/2). \end{aligned}$$

The waves increase in amplitude in Figure 16.9 through times $t = 0.3$, 0.5, 0.7 and 1.4.

10. Follow the ideas of Example 16.7 and the solution to Problem 9. Let $y(x,t) = Y(x,t) - h(x)$ and choose h to obtain a standard problem that

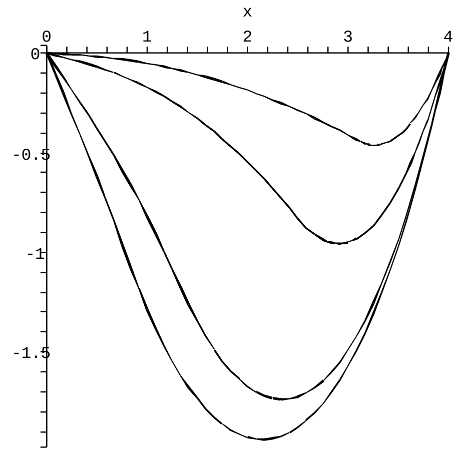

Figure 16.10: Profiles of the solution in Problem 10, Section 16.2.

we have solved for Y. Substituting $y(x,t)$ into the wave equation and the boundary and initial conditions, we obtain

$$h(x) = \frac{1}{108}(64x - x^4)$$

Y must satisfy the wave equation with $c = 3$ and the conditions

$$Y(0,t) = Y(4,t) = 0, Y(x,0) = h(x), \frac{\partial Y}{\partial t}(x,0) = 0.$$

Solve this standard problem for Y and obtain

$$\begin{aligned} y(x,t) &= \frac{1}{108}(x^4 - 64x) \\ &- \frac{512}{9\pi^5}\sum_{n=1}^{\infty} \frac{2(1-(-1)^n) + n^2\pi^2(-1)^n}{n^5} \sin(n\pi x/4)\cos(3n\pi t/4). \end{aligned}$$

Waves move downward in Figure 16.10 through times $t = 0.3$, 0.5, 0.7 and 1.6.

11. Let $y(x,t) = Y(x,t) - h(x)$. Substitute $y(x,t)$ into the wave equation and use the boundary conditions to obtain a simpler problem for Y (that is, one we have already solved). This occurs if

$$-h''(x) - \cos(x) = 0, h(0) = h(2\pi) = 0.$$

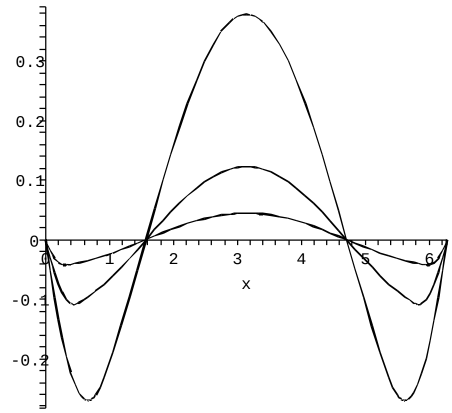

Figure 16.11: Profiles of the solution in Problem 11, Section 16.2.

Then $h(x) = \cos(x) - 1$ and $Y(x,t)$ satisfies the wave equation with $c = 1$ and the conditions

$$Y(0,t) = Y(2\pi,t) = 0, Y(x,0) = \cos(x) - 1, \frac{\partial Y}{\partial t}(x,0) = 0.$$

Solve this familiar problem for Y to obtain

$$\begin{aligned} y(x,t) &= 1 - \cos(x) \\ &+ \frac{16}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)((2n-1)^2-4)}\sin((2n-1)x/2)\cos((2n-1)t/2). \end{aligned}$$

Graphs of solutions in Figure 16.11 move upward (nearest the origin) through times $t = 0.3$, 0.5 and 0.9.

12. Substitute $u(x,t) = X(x)T(t)$ into the fourth order partial differential equation to obtain, after separating variables,

$$a^2X^{(4)} - \lambda X = 0, T'' + a^2\lambda T = 0.$$

The boundary conditions on $u(x,t)$ impose the conditions

$$X''(0) = X''(\pi) = X^{(3)}(0) = X^{(3)}(\pi) = 0.$$

Now consider cases on λ.

Case 1: $\lambda = 0$

Then $X(x) = A + Bx + Cx^2 + Dx^3$ and the boundary conditions in turn require that $C = 0$, $6D\pi = 0$ (so $D = 0$), and A and B are arbitrary constants. Thus $\lambda = 0$ is an eigenvalue with eigenfunction $X_0(x) = A + Bx$. In this case $T_0(t) = \alpha + \beta t$.

Case 2: $\lambda > 0$

Let $\lambda = \alpha^4$, with α real. The general solution for X is

$$X(x) = A\cos(\alpha x) + B\sin(\alpha x) + C\cosh(\alpha x) + D\sinh(\alpha x).$$

The boundary conditions give us four equations:

$$\begin{aligned}
&-A + C = 0\\
&-A\cos(\alpha\pi) - B\sin(\alpha\pi) + C\cosh(\alpha\pi) + D\sinh(\alpha\pi) = 0\\
&-B + D = 0\\
&A\sin(\alpha\pi) + B\cos(\alpha\pi) + C\sinh(\alpha\pi) + D\cosh(\alpha\pi) = 0.
\end{aligned}$$

From the first and third equations, $A = C$ and $B = D$. The second and fourth equations become

$$\begin{aligned}
C(\cosh(\alpha\pi) - \cos(\alpha\pi)) + D(\sinh(\alpha\pi) - \sin(\alpha\pi)) = 0\\
C(\sinh(\alpha\pi) + \sin(\alpha\pi)) + D(\cosh(\alpha\pi) - \cos(\alpha\pi)) = 0.
\end{aligned}$$

The determinant of this homogeneous, 2×2 system is

$$2(1 - \cosh(\alpha\pi)\cos(\alpha\pi)).$$

For this system to have a nontrivial solution, α must be chosen so that this determinant is zero. Thus in this case we need

$$\cos(\alpha\pi)\cosh(\alpha\pi) = 1.$$

This equation has infinitely many positive solutions, which we label in increasing order $\alpha_1 < \alpha_2 < \cdots$. The eigenvalues obtained in this case are $\lambda_n = \alpha_n^4$. Corresponding eigenfunctions are

$$X_n(x) = r_n(\cos(\alpha_n x) + \cosh(\alpha_n x)) + \sin(\alpha_n x) + \sinh(\alpha_n x),$$

where

$$r_n = \frac{\sin(\alpha_n\pi) - \sinh(\alpha_n\pi)}{\cosh(\alpha_n\pi) - \cos(\alpha_n\pi)}.$$

For each α_n, we obtain

$$T_n(t) = A_n\cos(a\alpha^2 t) + B_n\sin(a\alpha^2 t).$$

Case 3: $\lambda < 0$

Let $\lambda = -4\alpha^4$, for α positive. The roots of the characteristic equation $r^4 + 4\alpha^4 = 0$ are $(1+i)\alpha$, $(1-i)\alpha$, $(-1+i)\alpha$ and $(-1-i)\alpha$. Then

$$X(x) = e^{\alpha x}(A\cos(\alpha x) + B\sin(\alpha x)) + e^{-\alpha x}(C\cos(\alpha x) + D\sin(\alpha x)).$$

The boundary conditions yield four equations for the coefficients:

$$\begin{aligned}
&B - D = 0\\
&A - B - C - D = 0\\
&- Ae^{\alpha\pi}\sin(\alpha\pi) + Be^{\alpha\pi}\cos(\alpha\pi) + Ce^{-\alpha\pi}\sin(\alpha\pi) - De^{-\alpha\pi}\cos(\alpha\pi) = 0\\
&- Ae^{\alpha\pi}(\cos(\alpha\pi) + \sin(\alpha\pi)) + Be^{\alpha\pi}(\cos(\alpha\pi) - \sin(\alpha\pi))\\
&- Ce^{-\alpha\pi}(\cos(\alpha\pi) - \sin(\alpha\pi)) - De^{-\alpha\pi}(\cos(\alpha\pi) + \sin(\alpha\pi)) = 0.
\end{aligned}$$

The determinant of this 4×4 system is $\cosh(2\alpha\pi) - \cos(2\alpha\pi)$, which is zero only if $\alpha = 0$. Thus this case yields no nontrivial solutions, and the problem has no negative eigenvalues.

Thus far we have solved parts (a) and (b) of this problem. Finally consider part (c). The given boundary conditions imply that

$$X(0) = X(\pi) = X''(0) = X''(\pi) = 0.$$

Coordinate these conditions with the conclusions of the above analysis for part (b). There are three cases.

$\lambda = 0$

As noted above, $X(x) = A + Bx + Cx^2 + Dx^3$ in this case. The boundary conditions give us

$$X(0) = A = 0, X''(0) = 2C = 0, X''(\pi) = 6D\pi = 0, X(\pi) = B\pi = 0.$$

This yields only the trivial solution, so 0 is not an eigenvalue of this problem.

$\lambda > 0$

Using the previous analysis, we have the general form

$$X(x) = A\cos(\alpha x) + B\sin(\alpha x) + C\cosh(\alpha x) + D\sinh(\alpha x).$$

Now $X(0) = 0$ gives us $A + C = 0$ and $X''(0) = 0$ gives $-A + C = 0$, so $A = C = 0$. Then

$$X(\pi) = B\sin(\alpha\pi) + D\sinh(\alpha\pi)$$

and

$$X''(\pi) = -B\sin(\alpha\pi) + D\sinh(\alpha\pi) = 0.$$

This 2×2 system has determinant $\sinh^2(\alpha\pi)$, and this is zero exactly when $\alpha = n = 1, 2, \cdots$. This gives eigenvalues $\lambda_n = n^4$ and eigenfunctions $X_n(x) = \sin(nx)$. We also obtain $T_n(t) = A_n\cos(an^2t) + B_n\sin(an^2t)$.

$\lambda < 0$

Using the previous analysis in this case, we have

$$X(x) = e^{\alpha x}(A\cos(\alpha x) + B\sin(\alpha x)) + e^{-\alpha x}(C\cos(\alpha x) + D\sin(\alpha x)).$$

Now $X(0) = 0$ gives us $A + C = 0$ and $X''(0) = 0$ gives us $B - D = 0$. Then $C = -A$ and $D = B$. Substitute these into the equations and use the two boundary conditions at π to obtain

$$\begin{aligned} &A\cos(\alpha\pi)\sinh(\alpha\pi) + D\sin(\alpha\pi)\cosh(\alpha\pi) = 0 \\ &-A\sin(\alpha\pi)\cosh(\alpha\pi) + B\cos(\alpha\pi)\sinh(\alpha\pi) = 0. \end{aligned}$$

The determinant of the coefficients of this system is

$$\cosh^2(\alpha\pi)\sinh^2(\alpha\pi) + \sin^2(\alpha\pi)\cosh^2(\alpha\pi)$$

and this is nonzero if $\alpha > 0$. This case yields no nontrivial solutions, so this problem has no negative eigenvalue.

13. Separation of variables gives us

$$X'' + \lambda X = 0, X(0) = X(L) = 0,$$

and

$$T'' + AT' + (B + c^2\lambda)T = 0, T'(0) = 0.$$

Eigenvalues and eigenfunctions for X are

$$\lambda_n = \frac{n^2\pi^2}{L^2}, X_n(t) = \sin(n\pi x/L).$$

With these eigenvalues, the characteristic equation of the differential equation for T is

$$r^2 + Ar + (B + c^2n^2\pi^2/L^2) = 0,$$

with roots

$$r = \frac{-A}{2} \pm \frac{1}{2}\sqrt{A^2 - 4\left(B + \frac{c^2n^2\pi^2}{L^2}\right)}.$$

The given condition $A^2L^2 < 4(BL^2 + c^2\pi^2)$ ensures that these roots are complex. Let

$$r_n^2 = 4(BL^2 + c^2n^2\pi^2) - A^2L^2.$$

The roots are then

$$r = -\frac{A}{2} \pm \frac{r_n}{2L}i.$$

Then

$$T_n(t) = e^{-At/2}\left[a_n\cos(r_nt/2L) + b_n\sin(r_nt/2L)\right].$$

Then $T'(0) = 0$ gives $-Aa_n/2 + b_n r_n/2L = 0$, hence

$$b_n = \frac{AL}{r_n} a_n.$$

By superposition,

$$u(x,t) = e^{-At/2} \sum_{n=1}^{\infty} a_n \sin(n\pi x/L) \left[\cos(r_n t/2L) + \frac{AL}{r_n} \sin(r_n t/2L)\right].$$

To satisfy $u(x,0) = f(x)$, choose

$$a_n = \frac{2}{L} \int_0^L f(\xi) \sin(n\pi\xi/L)\, d\xi.$$

14. Let $y(x,t) = Y(x,t) + \psi(x)$. Substitute this into the wave equation to obtain

$$\frac{\partial^2 Y}{\partial t^2} = 9\left(\frac{\partial^2 Y}{\partial t^2} + \psi''(x)\right) + 5x^3.$$

This is simplified if $\psi(x)$ is chosen so that

$$\psi''(x) + \frac{5}{9}x^3 = 0.$$

Further,

$$y(0,t) = Y(0,t) + \psi(0) = 0 \text{ and } y(4,t) = Y(4,t) + \psi(4) = 0$$

is simplified if $\psi(0) = \psi(4) = 0$. Integrate to solve for $\psi(x)$, obtaining

$$\psi(x) = \frac{1}{36}x(256 - x^4).$$

The problem for Y is

$$\begin{aligned} \frac{\partial^2 Y}{\partial t^2} &= 9\frac{\partial^2 Y}{\partial x^2}, \\ Y(0,t) &= Y(4,t) = 0, \\ Y(x,0) &= \cos(\pi x) - \psi(x), \frac{\partial Y}{\partial t}(x,0) = 0. \end{aligned}$$

Then

$$y(x,t) = Y(x,t) + \psi(x) = \sum_{n=1}^{\infty} a_n \sin(n\pi x/4)\cos(3n\pi t/4) + \frac{1}{36}x(256 - x^4),$$

where

$$a_n = \frac{2}{4}\int_0^4 \left(\cos(\pi x) - \frac{1}{36}x(256 - x^4)\right) \sin(n\pi x/4)\, dx$$

$$= \begin{cases} \frac{2n(1-(-1)^n)}{\pi(n^2-16)} + \frac{10240(-1)^n(n^2\pi^2-6)}{9n^5\pi^5} & \text{for } n \neq 4, \\ \frac{10(16\pi^2-6)}{9\pi^5} & \text{for } n = 4. \end{cases}$$

Therefore the solution of the forced problem is

$$y(x,t) = \sum_{n=1, n\neq 4}^{\infty} \left[\frac{2n(1-(-1)^n)}{\pi(n^2-16)} + \frac{10240(-1)^n(n^2\pi^2-6)}{9n^5\pi^5}\right] \sin(n\pi x/4)\cos(3n\pi t/4)$$
$$+ \frac{10(16\pi^2-6)}{9\pi^5}\sin(\pi x)\cos(3\pi t) + \frac{1}{36}x(256-x^4).$$

Without the forcing term, the problem has a solution of the form

$$y(x,t) = \sum_{n=1}^{\infty} \alpha_n \sin(n\pi x/4)\cos(3n\pi t/4)$$

where

$$\alpha_n = \frac{2}{4}\int_0^4 \cos(\pi x)\sin(n\pi x/4)\,dx$$
$$= \begin{cases} \frac{2n(1-(-1)^n)}{\pi(n^2-16)} & \text{for } n \neq 4, \\ 0 & \text{for } n = 4. \end{cases}$$

Thus the unforced solution is

$$y(x,t) = \sum_{n=1, n\neq 4}^{\infty} \frac{2n(1-(-1)^n)}{\pi(n^2-16)} \sin(n\pi x/4)\cos(3n\pi t/4).$$

Figure 16.12 compares the forced wave (upper graph) with the unforced solution at $t = 0.4$. Figure 16.13 does this for $t = 0.8$, and Figure 16.14 for $t = 1.4$.

15. Set $y(x,t) = Y(x,t) + \psi(x)$. To simplify the problem for $Y(x,t)$, choose $\psi(x)$ to satisfy

$$\psi''(x) = -\frac{1}{9}\cos(\pi x), \psi(0) = \psi(4) = 0.$$

By integrating we find that

$$\psi(x) = \frac{1}{9\pi^2}(\cos(\pi x) - 1).$$

The solution $Y(x,t)$ has the form

$$Y(x,t) = \sum_{n=1}^{\infty} a_n \sin(n\pi x/4)\cos(3n\pi t/4),$$

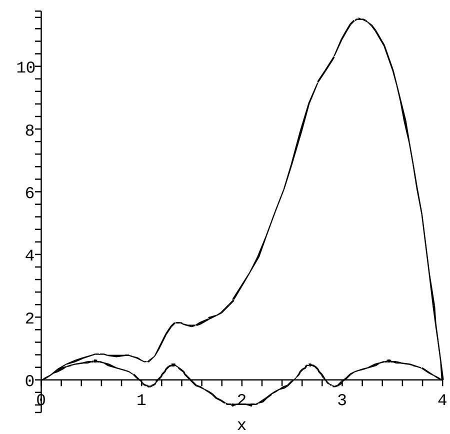

Figure 16.12: Forced and unforced motion in Problem 14, Section 16.2, at $t = 0.4$.

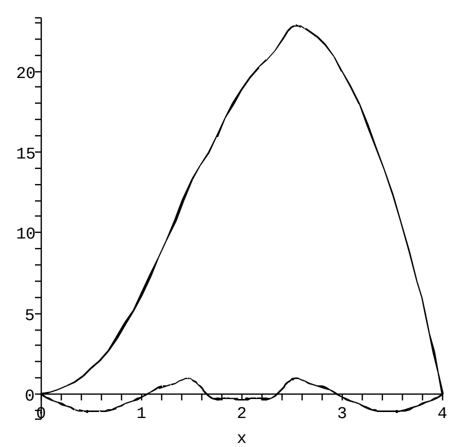

Figure 16.13: Forced and unforced motion in Problem 14, Section 16.2, at $t = 0.8$.

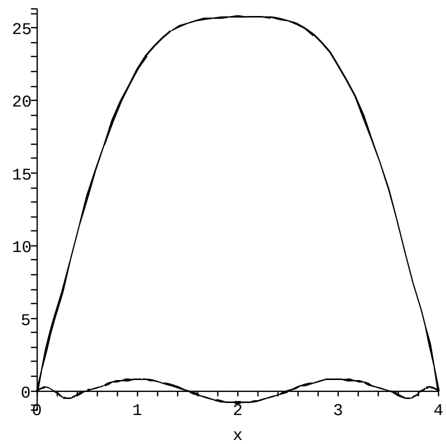

Figure 16.14: Forced and unforced motion in Problem 14, Section 16.2, at $t = 1.4$.

where

$$a_n = \frac{2}{4}\int_0^4 \left[x(4-x) - \frac{1}{9\pi^2}(\cos(\pi x) - 1)\right] \sin(n\pi x/4)\,dx$$

$$= \begin{cases} \frac{-32(1-(-1)^n)(288-17n^2)}{9n^3\pi^3(n^2-16)} & \text{for } n \neq 4, \\ 0 & \text{for } n = 4. \end{cases}$$

The solution for the forced motion is

$$y(x,t) = \sum_{n=1}^{\infty} a_n \sin(n\pi x/4)\cos(3n\pi t/4) + \frac{1}{9\pi^2}(\cos(\pi x) - 1).$$

Without the forcing term, the solution has the form

$$y(x,t) = \sum_{n=1}^{\infty} \alpha_n \sin(n\pi x/4)\cos(3n\pi t/4),$$

where

$$\alpha_n = \frac{1}{2}\int_0^4 x(4-x)\sin(n\pi x/4)\,dx = \frac{64(1-(-1)^n)}{n^3\pi^3}.$$

Thus the solution for the unforced motion is

$$y(x,t) = \sum_{n=1}^{\infty} \frac{64(1-(-1)^n)}{n^3\pi^3}\sin(n\pi x/4)\cos(3n\pi t/4).$$

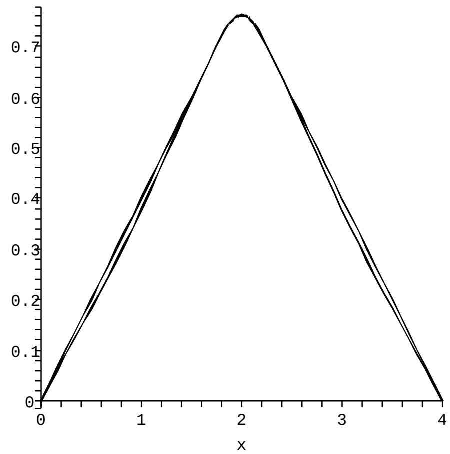

Figure 16.15: Forced and unforced motion in Problem 15, Section 16.2, at $t = 0.6$.

Forced and unforced solutions at $t = 0.6$, 1 and 1.4 are shown, respectively, in Figures 16.15, 16.16 and 16.17. In this example the forced motion is very similar to the unforced motion.

16. Let $y(x,t) = Y(x,t) + \psi(x)$ and obtain a simpler problem for $Y(x,t)$ by choosing $\psi(x)$ to satisfy

$$\psi''(x) = \frac{1}{9}e^{-x}, \psi(0) = \psi(4) = 0.$$

Then

$$\psi(x) = \frac{1}{36}(4e^{-x} + (1 - e^{-4})x - 4).$$

The solution for Y has the form

$$Y(x,t) = \sum_{n=1}^{\infty} a_n \sin(n\pi x/4)\cos(3n\pi t/4),$$

where, for $n \neq 4$,

$$\begin{aligned} a_n &= \frac{1}{2}\int_0^4 (\sin(\pi x) - \psi(x))\,dx \\ &= \frac{32(1 - (-1)^n e^{-4})(n^2 - 16)}{9n\pi(16n^2 - 16n^2\pi^2 + n^4\pi^2 - 256)} \end{aligned}$$

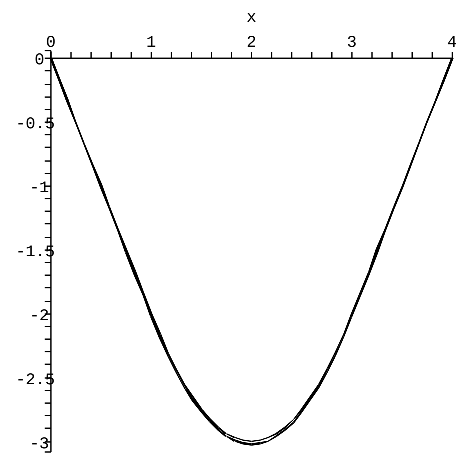

Figure 16.16: Forced and unforced motion in Problem 15, Section 16.2, at $t = 1$.

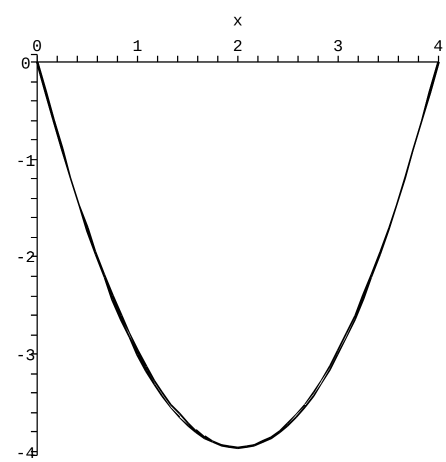

Figure 16.17: Forced and unforced motion in Problem 15, Section 16.2, at $t = 1.4$.

and, for $n = 4$,

$$a_4 = \frac{1 - e^{-4} + 18\pi + 18\pi^3}{18\pi(1 + \pi^2)}.$$

The solution for the forced motion is

$$y(x,t) = \sum_{n=1, n\neq 4}^{\infty} a_n \sin(n\pi x/4)\cos(3n\pi t/4)$$
$$+ a_4 \sin(\pi x)\cos(3\pi t) + \frac{1}{36}(4e^{-x} + (1 - e^{-4})x - 4).$$

Without the forcing term, the solution has the form

$$y(x,t) = \sum_{n=1}^{\infty} \alpha_n \sin(n\pi x/4)\cos(3n\pi t/4),$$

where

$$\alpha_n = \frac{1}{2}\int_0^4 \sin(\pi x)\sin(n\pi x/4)\,dx = \begin{cases} 0 & \text{for } n \neq 4, \\ 1 & \text{for } n = 1. \end{cases}$$

Thus the unforced motion is described by

$$y(x,t) = \sin(\pi x)\cos(3\pi t).$$

Forced and unforced solutions at shown in Figure 16.18 for $t = 0.6$, in Figure 16.19 for $t = 1$ and in Figure 16.20 for $t = 1.4$.

16.3 Wave Motion in an Infinite Medium

In each of Problems 1 through 6, the Fourier integral on $-\infty < x < \infty$ yields a solution of the wave equation with initial condition $f(x)$ and initial velocity $g(x)$, and having the form

$$y(x) = \int_0^{\infty} ((a_\omega \cos(\omega x) + b_\omega \sin(\omega x))\ dx$$
$$+ \int_0^{\infty} (\alpha_\omega \cos(\omega x) + \beta_\omega \sin(\omega x))\ d\omega,$$

where

$$a_\omega = \frac{1}{\pi}\int_{-\infty}^{\infty} f(\xi)\cos(\omega\xi)\,d\xi,$$
$$b_\omega = \frac{1}{\pi}\int_{-\infty}^{\infty} f(\xi)\sin(\omega\xi)\,d\xi,$$
$$\alpha_\omega = \frac{1}{\pi\omega c}\int_{-\infty}^{\infty} g(\xi)\cos(\omega\xi)\,d\xi,$$
$$\beta_\omega = \frac{1}{\pi\omega c}\int_{-\infty}^{\infty} f(\xi)\sin(\omega\xi)\,d\xi.$$

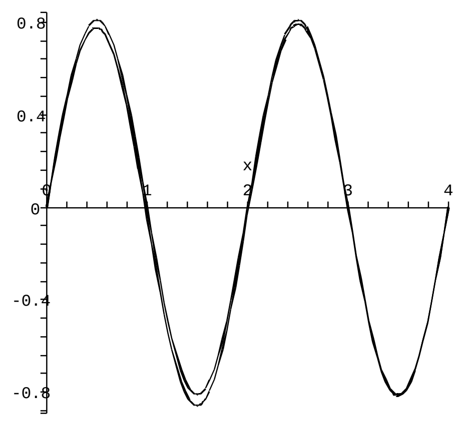

Figure 16.18: Forced and unforced motion in Problem 16, Section 16.2, at $t = 0.6$.

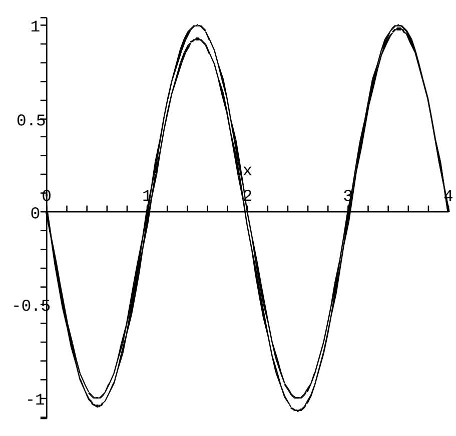

Figure 16.19: Forced and unforced motion in Problem 16, Section 16.2, at $t = 1$.

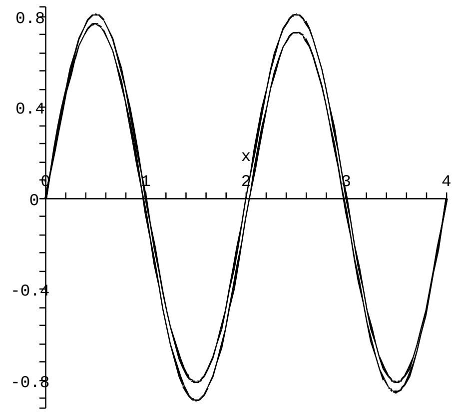

Figure 16.20: Forced and unforced motion in Problem 16, Section 16.2, at $t = 1.4$.

1. Compute

$$a_\omega = \frac{1}{\pi}\int_{-\infty}^{\infty} e^{-5|\xi|}\cos(\omega\xi)\,d\xi = \frac{10}{(25+\omega^2)\pi}.$$

Immediately $b_\omega = 0$ because $e^{-5|\omega|}\sin(\omega\omega)$ is an odd function. With the zero initial velocity condition, these are all the coefficients and the solution is

$$y(x,t) = \frac{10}{\pi}\int_0^{\infty}\left(\frac{1}{25+\omega^2}\right)\cos(\omega x)\cos(12\omega t)\,d\omega.$$

If we use the Fourier transform in x, take the transform of the wave equation to obtain

$$\hat{y}'' + 144\omega^2\hat{y} = 0;$$
$$\hat{y}(\omega,0) = \int_{-\infty}^{\infty} e^{-5|\xi|}e^{-i\omega\xi}\,d\xi = \frac{10}{25+\omega^2},$$
$$\hat{y}'(\omega,0) = 0.$$

The solution of this problem is

$$\hat{y}(\omega,t) = \frac{10}{25+\omega^2}\cos(12\omega t).$$

Invert this to obtain the solution

$$y(x,t) = \mathrm{Re}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{10}{25+\omega^2}\cos(12\omega t)e^{i\omega x}\,d\omega\right].$$

Since $e^{i\omega x} = \cos(\omega x) + i\sin(\omega x)$, it is easy to extract the real part of this integral and verify that the solution obtained by using the transform agrees with that obtained using the Fourier integral.

2. Compute the Fourier integral coefficients of $f(x)$:

$$\frac{1}{\pi}\int_0^8 (8-\xi)\cos(\omega\xi)\,d\xi = \frac{1-\cos(8\omega)}{\pi\omega^2}$$

and

$$\frac{1}{\pi}\int_0^8 (8-\xi)\sin(\omega\xi)\,d\xi = \frac{8\omega-\sin(8\omega)}{\pi\omega^2}.$$

The solution is

$$y(x,t) = \int_0^\infty \left[\left(\frac{1-\cos(8\omega)}{\pi\omega^2}\right)\cos(\omega x) + \left(\frac{8\omega-\sin(8\omega)}{\pi\omega^2}\right)\sin(\omega x)\right]\cos(8\omega t)\,d\omega.$$

To solve this problem using the Fourier transform, apply the transform to the initial-boundary value problem to obtain:

$$\begin{aligned}
&\hat{y}'' + 64\omega^2\hat{y} = 0;\\
&\hat{y}(\omega,0) = \int_0^8 (8-\xi)e^{-i\omega\xi}\,d\xi = \frac{1-8\omega i - e^{-8\omega i}}{\omega^2};\\
&\hat{y}'(\omega,0) = 0.
\end{aligned}$$

This problem has solution

$$\hat{y}(\omega,t) = \frac{1-8\omega i - e^{-8\omega i}}{\omega^2}\cos(8\omega t).$$

Invert and take the real part to obtain the solution

$$y(x,t) = \mathrm{Re}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1-8\omega i - e^{-8\omega i}}{\omega^2}\cos(8\omega t)e^{i\omega x}\,d\omega\right).$$

3. For the Fourier integral solution, calculate the coefficients

$$\frac{1}{4\pi\omega}\int_{-\pi}^{\pi}\sin(\xi)\cos(\omega\xi)\,d\xi = 0$$

and

$$\frac{1}{4\pi\omega}\int_{-\pi}^{\pi}\sin(\xi)\sin(\omega\xi)\,d\xi = -\frac{\sin(\pi\omega)}{2\pi\omega(\omega^2-1)}.$$

The solution is

$$y(x,t) = \int_0^\infty \left(-\frac{\sin(\pi\omega)}{2\pi\omega(\omega^2-1)}\right)\sin(\omega x)\sin(4\omega t)\,d\omega.$$

To apply the Fourier transform, first transform the initial-boundary value problem to obtain

$$\begin{aligned}
&\hat{y}'' + 16\omega^2\hat{y} = 0;\\
&\hat{y}(\omega, 0) = 0;\\
&\hat{y}(\omega, 0) = \int_{-\pi}^{\pi} \sin(\xi)e^{-i\omega\xi}\,d\xi = \frac{2i\sin(\pi\omega)}{\omega^2 - 1}.
\end{aligned}$$

The solution of this transformed problem is

$$\hat{y}(\omega, t) = \frac{2i\sin(\pi\omega)}{4\omega(\omega^2 - 1)}\sin(4\omega t).$$

Invert this to obtain the solution

$$y(x, t) = \mathrm{Re}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{i\sin(\pi\omega)}{2\omega(\omega^2 - 1)}\sin(4\omega t)e^{i\omega x}\,d\omega\right).$$

4. For the Fourier integral solution, compute

$$\frac{1}{\pi}\int_{-2}^{2}(2 - |\xi|)\cos(\omega\xi)\,d\xi = \frac{2}{\pi\omega^2}(1 - \cos(2\omega))$$

and

$$\frac{1}{\pi}\int_{-2}^{2}(2 - |\xi|)\sin(\omega\xi) = 0.$$

The solution is

$$y(x, t) = \int_0^{\infty}\left(\frac{2}{\pi\omega^2}(2 - \cos(2\omega))\right)\cos(\omega x)\cos(\omega t)\,d\omega.$$

For a solution by Fourier transform, first transform the initial-boundary value problem to

$$\begin{aligned}
&\hat{y}'' + \omega^2\hat{y} = 0;\\
&\hat{y}(\omega, 0) = \int_{-2}^{2}(2 - |\xi|)e^{-i\omega\xi}\,d\xi = \frac{2}{\omega^2}(1 - \cos(2\omega));\\
&\hat{y}'(\omega, 0) = 0.
\end{aligned}$$

This problem has solution

$$\hat{y}(\omega, t) = \frac{2}{\omega^2}(1 - \cos(2\omega))\cos(\omega t).$$

Inverting this gives the solution

$$y(x, t) = \mathrm{Re}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2}{\omega^2}(1 - \cos(2\omega))\cos(\omega t)e^{i\omega x}\,d\omega\right).$$

5. Compute the coefficients

$$a_\omega = \frac{1}{3\pi\omega}\int_1^\infty e^{-2\xi}\cos(\omega\xi)\,d\xi = \frac{e^{-2}}{3\pi\omega}\frac{2\cos(\omega) - \omega\sin(\omega)}{4+\omega^2}$$

and

$$b_\omega = \frac{1}{3\pi\omega}\int_1^\infty e^{-2\xi}\sin(\omega\xi)\,d\xi = \frac{e^{-2}}{3\pi\omega}\frac{2\sin(\omega) + \omega\cos(\omega)}{4+\omega^2}.$$

The solution is

$$y(x,t) = \int_0^\infty (a_\omega\cos(\omega x) + b_\omega\sin(\omega x))\sin(3\omega t).$$

To obtain the solution using the Fourier transform, first transform the problem to obtain

$$\hat{y}'' + 9\omega^2\hat{y} = 0;$$
$$\hat{y}(\omega,0) = 0;$$
$$\hat{y}'(\omega,0) = \mathcal{F}(e^{-2x}H(x-1)) = \frac{(2-i\omega)e^{-(2+i\omega)}}{4+\omega^2}.$$

This problem has solution

$$\hat{y}(\omega,t) = \frac{(2-i\omega)e^{-(2+i\omega)}}{3\omega(4+\omega^2)}\sin(3\omega t).$$

Invert this to obtain the solution

$$y(x,t) = \mathrm{Re}\left(\frac{1}{2\pi}\int_{-\infty}^\infty \frac{(2-i\omega)e^{-(2+i\omega)}}{3\omega(4+\omega^2)}\sin(3\omega t)e^{i\omega x}\,d\omega\right).$$

6. Compute the coefficients

$$\frac{1}{2\pi\omega}\int_{-2}^2 g(\xi)\cos(\omega\xi)\,d\xi = 0$$

and

$$\frac{1}{2\pi\omega}\int_{-2}^2 g(\xi)\sin(\omega\xi)\,d\xi = \frac{1-\cos(2\omega)}{\pi\omega^2}.$$

The solution is

$$y(x,t) = \int_0^\infty \left(\frac{1-\cos(2\omega)}{\pi\omega^2}\right)\sin(\omega x)\sin(2\omega t)\,d\omega.$$

To solve the problem using the Fourier transform, first obtain the transformed problem

$$\hat{y}'' + 4\omega^2\hat{y} = 0;$$
$$\hat{y}(\omega,0) = 0;$$
$$\hat{y}'(\omega,0) = \int_{-2}^2 g(\xi)e^{-i\omega\xi}\,d\xi = \frac{2(1-\cos(2\omega))}{\omega}.$$

This problem has the solution

$$\hat{y}(\omega, t) = \frac{1 - \cos(2\omega)}{\omega^2} \sin(2\omega t).$$

Invert this to obtain the solution

$$y(x,t) = \operatorname{Re}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1-\cos(2\omega)}{\omega^2}\sin(2\omega t)e^{i\omega x}\,d\omega\right).$$

16.4 Wave Motion in a Semi-Infinite Medium

For each of these problems, separation of variables and the Fourier sine integral yields a solution of the form

$$y(x,t) = \int_0^{\infty} \sin(\omega x)(a_\omega \cos(c\omega t) + b_\omega \sin(\omega c t))\,d\omega,$$

where

$$a_\omega = \frac{2}{\pi}\int_0^{\infty} f(\xi)\sin(\omega\xi)\,d\xi$$

and

$$b_\omega = \frac{2}{\pi c\omega}\int_0^{\infty} g(\xi)\sin(\omega\xi)\,d\xi.$$

1. For a Fourier sine integral solution, calculate

$$a_\omega = \frac{2}{\pi}\int_0^1 \xi(1-\xi)\sin(\omega\xi)\,d\xi = \frac{2}{\pi}\left[\frac{2}{\omega^3}(1-\cos(\omega)) - \frac{\sin(\omega)}{\omega^2}\right].$$

and $b_\omega = 0$. The solution is

$$y(x,t) = \frac{2}{\pi}\int_0^{\infty}\left[\frac{2}{\omega^3}(1-\cos(\omega)) - \frac{\sin(\omega)}{\omega^2}\right]\sin(\omega x)\cos(3\omega t)\,d\omega.$$

To solve the problem using the Fourier sine transform, first take the transform of the initial-boundary value problem to obtain

$$\hat{y}_S'' + 9\omega^2\hat{y}_S = 0;$$
$$\hat{y}_S(\omega, 0) = \int_0^1 \xi(1-\xi)\sin(\omega\xi)\,d\xi = \frac{2(1-\cos(\omega)) - \omega\sin(\omega)}{\omega^3};$$
$$\hat{y}_S'(\omega, 0) = 0.$$

The solution of this transformed problem is

$$\hat{y}_S(\omega, t) = \left[\frac{2(1-\cos(\omega)) - \omega\sin(\omega)}{\omega^3}\right]\cos(3\omega t).$$

Invert this to obtain the solution

$$y(x,t) = \frac{2}{\pi}\int_0^{\infty}\left[\frac{2(1-\cos(\omega)) - \omega\sin(\omega)}{\omega^3}\right]\sin(\omega x)\cos(3\omega t)\,d\omega.$$

2. For the Fourier sine integral solution, compute $a_\omega = 0$ and
$$b_\omega = \frac{2}{3\pi\omega}\int_4^{11} 2\sin(\omega\xi)\,d\xi = \frac{4(\cos(4\omega)-\cos(11\omega))}{3\pi\omega^2}.$$
The solution is
$$y(x,t) = \frac{4}{3\pi}\int_0^\infty \frac{\cos(4\omega)-\cos(11\omega)}{\omega^2}\sin(\omega x)\sin(3\omega t)\,d\omega.$$
To solve using the Fourier sine transform, first transform the problem, obtaining
$$\hat{y}_S'' + 9\omega^2\hat{y}_S = 0;$$
$$\hat{y}_S(\omega,0) = 0;$$
$$\hat{y}_S'(\omega,0) = \int_4^{11} 2\sin(\omega\xi)\,d\xi = \frac{2(\cos(4\omega)-\cos(11\omega))}{\omega}.$$
The solution of this transformed problem is
$$\hat{y}_S(\omega,t) = \frac{2(\cos(4\omega)-\cos(11\omega))}{3\omega^2}\sin(3\omega t).$$
Invert this to obtain the solution
$$y(x,t) = \frac{4}{3\pi}\int_0^\infty \frac{\cos(4\omega)-\cos(11\omega)}{\omega^2}\sin(\omega)\sin(3\omega t)\,d\omega.$$

3. To solve the problem using the Fourier sine integral, compute $a_\omega = 0$ and
$$b_\omega = \frac{2}{2\pi\omega}\int_{\pi/2}^{5\pi/2} \cos(\xi)\sin(\omega\xi), d\xi = \frac{\sin(\omega\pi/2)-\sin(5\omega\pi/2)}{\pi\omega(\omega^2-1)}.$$
This gives us the solution
$$y(x,t) = \int_0^\infty \frac{\sin(\omega\pi/2)-\sin(5\omega\pi/2)}{\pi\omega(\omega^2-1)}\sin(\omega x)\sin(2\omega t)\,d\omega.$$
For the Fourier sine transform solution, transform the problem to obtain
$$\hat{y}_S'' + 4\omega^2\hat{y}_S = 0;$$
$$\hat{y}_S(\omega,0) = 0;$$
$$\hat{y}_S'(\omega,0) = \int_{\pi/2}^{5\pi/2} \cos(\xi)\sin(\omega\xi)\,d\xi = \frac{\sin(\omega\pi/2)-\sin(5\omega\pi/2)}{\omega^2-1}.$$
The solution of the transformed problem is
$$\hat{y}_S(\omega,t) = \frac{\sin(\omega\pi/2)-\sin(5\omega\pi/2)}{2\omega(\omega^2-1)}\sin(2\omega t).$$
Invert this to obtain the solution
$$y(x,t) = \frac{2}{\pi}\int_0^\infty \frac{\sin(\omega\pi/2)-\sin(5\omega\pi/2)}{2\omega(\omega^2-1)}\sin(\omega x)\sin(2\omega t)\,d\omega.$$

4. Compute $b_\omega = 0$ and

$$a_\omega = \frac{2}{\pi}\int_0^\infty -2e^{-\xi}\sin(\omega\xi)\,d\xi = -\frac{4\omega}{\pi(1+\omega^2)}.$$

This yields the solution

$$y(x,t) = -\frac{4}{\pi}\int_0^\infty \frac{\omega}{1+\omega^2}\sin(\omega x)\cos(6\omega t)\,d\omega.$$

For the solution by Fourier sine transform, first transform the problem to obtain

$$\begin{aligned}
&\hat{y}_S'' + 36\omega^2\hat{y}_S = 0;\\
&\hat{y}_S(\omega,0) = \int_0^\infty -2e^{-\xi}\sin(\omega\xi)\,d\xi = -\frac{2\omega}{1+\omega^2};\\
&\hat{y}_S'(\omega,0) = 0.
\end{aligned}$$

This problem has the solution

$$\hat{y}_S(\omega,t) = -\frac{2\omega}{1+\omega^2}\cos(6\omega t).$$

Invert this to obtain the solution

$$y(x,t) = -\frac{4}{\pi}\int_0^\infty \frac{\omega}{1+\omega^2}\sin(\omega x)\cos(6\omega t)\,d\omega.$$

5. To use the Fourier sine integral, compute $a_\omega = 0$ and

$$\begin{aligned}
b_\omega &= \frac{2}{14\pi\omega}\int_0^3 \xi^2(3-\xi)\sin(\omega\xi)\,d\xi\\
&= \frac{3}{7\pi\omega^5}(2\sin(3\omega) - 4\omega\cos(3\omega) - 3\omega^2\sin(3\omega) - 2\omega).
\end{aligned}$$

This yields the solution

$$y(x,t) = \int_0^\infty b_\omega \sin(\omega x)\sin(14\omega t)\,d\omega.$$

To use the Fourier sine transform, first transform the problem to obtain

$$\begin{aligned}
\hat{y}_S'' + 196\omega^2\hat{y}_S &= 0;\\
\hat{y}_S(\omega,0) &= 0;\\
\hat{y}_S'(\omega,0) &= \int_0^3 \xi^2(3-\xi)\sin(\omega\xi)\,d\xi\\
&= \frac{3}{\omega^4}\left(2\sin(3\omega) - 4\omega\cos(3\omega) - 3\omega^2\sin(3\omega) - 2\omega\right).
\end{aligned}$$

This transformed problem has the solution

$$\hat{y}_S(\omega,t) = \frac{3}{14\omega^5}(2\sin(3\omega) - 4\omega\cos(3\omega) - 3\omega^2\sin(3\omega) - 2\omega)\sin(14\omega t).$$

Invert this to obtain the solution

$$y(x,t) =$$
$$\frac{2}{\pi}\int_0^\infty \frac{3}{14\omega^5}(2\sin(3\omega) - 4\omega\cos(3\omega) - 3\omega^2\sin(3\omega) - 2\omega)\sin(\omega x)\sin(14\omega t)\,d\omega.$$

16.5 Laplace Transform Techniques

1. Apply the Laplace transform (with respect to t) of the partial differential equation to obtain

$$s^2Y(x,s) = c^2Y''(x,s) + \frac{K}{s}.$$

Here primes denote differentiation with respect to x, and the initial conditions have been inserted through the operational formula for the transform of $\partial^2 y/\partial t^2$. Write this equation as

$$Y'' - \frac{s^2}{c^2}Y = -\frac{K}{c^2 s}.$$

Think of this as a linear second-order differential equation in x, with s carried along as a parameter. The general solution is

$$Y(x,s) = c_1e^{sx/c} + c_2e^{-sx/c} + \frac{K}{s^3}.$$

Here c_1 and c_2 are "constant" in the sense of having no dependence on x, but they may be functions of s. Now

$$Y(0,s) = [y(0,t)](s) = F(s) = c_1 + c_2 + \frac{K}{s^3}.$$

We want $\lim_{x\to\infty} y(x,t) = 0$, so $\lim_{s\to\infty} Y(x,s) = 0$, hence $c_1 = 0$. Therefore

$$c_2 = F(s) - \frac{K}{s^3}.$$

Then

$$Y(x,s) = \left(F(s) - \frac{K}{s^3}\right)e^{-sx/c} + \frac{K}{s^3}.$$

The solution is obtained by applying the inverse transform (in s) to the last equation. Recalling equation (3.6) for the inverse Laplace transform of a function of the form $e^{-as}F(s)$, we obtain

$$y(x,t) = \left[f\left(t - \frac{x}{c}\right) - \frac{K}{2}\left(t - \frac{x}{c}\right)^2\right]H\left(t - \frac{x}{c}\right) + \frac{1}{2}Kt^2,$$

in which H is the Heaviside function.

2. Apply the transform to the partial differential equation, with respect to t, using the initial conditions, to obtain

$$9s^2Y(x,s) + Y''(x,s) - 6sY'(x,s) = 0.$$

Then

$$Y'' - 6sY' + 9s^2Y = 0.$$

This has characteristic equation

$$r^2 - 6sr + 9s^2 = (r-3s)^2 = 0,$$

with repeated roots $3s$, so the general solution is

$$Y(x,s) = c_1e^{3sx} + c_2xe^{3xs}.$$

Now

$$\mathcal{L}[y(0,t)](s) = Y(0,s) = c_1$$

so

$$Y(x,s) = c_2xe^{3xs}.$$

Next,

$$\mathcal{L}[y(2,t)](s) = F(s) = 2c_2e^{6s}.$$

Then

$$c_2 = \frac{1}{2}e^{-6s}F(s),$$

and

$$Y(x,s) = \frac{1}{2}e^{-6s}F(s)xe^{3xs} = \frac{1}{2}xF(s)e^{(3x-6)s}.$$

The solution is

$$y(x,t) = \frac{1}{2}xf(t-(6-3x))H(t-(6-3x)).$$

3. From the partial differential equation and the initial conditions,

$$s^2Y(x,s) = c^2Y'' - \frac{A}{s^2}.$$

Then

$$Y'' - \frac{s^2}{c^2}Y = \frac{A}{s^2}.$$

This has general solution

$$Y(x,s) = c_1e^{sx/c} + c_2e^{-sx/c} - \frac{A}{s^4}.$$

Because $\lim_{x\to\infty} y(x,t) = 0$, we must also have

$$\lim_{s\to\infty} Y(x,s) = 0.$$

This requires that $c_1 = 0$, so

$$Y(x,s) = c_2 e^{-sx/c} - \frac{A}{s^4}.$$

Next, $y(0,t) = 0$, so

$$Y(0,s) = c_2 - \frac{A}{s^4},$$

so

$$c_2 = \frac{A}{s^4}.$$

Finally we have

$$Y(x,s) = \frac{A}{s^4} e^{-sx/c} - \frac{A}{s^4}.$$

Then

$$y(x,t) = \frac{A}{6}\left(t - \frac{x}{c}\right)^3 H\left(t - \frac{x}{c}\right) - \frac{A}{6}t^3.$$

4. From the partial differential equation and initial conditions we have

$$s^2 Y(x,s) = c^2 Y''(x,s).$$

Then

$$y'' - \frac{s^2}{c^2}Y = 0,$$

with general solution

$$Y(x,s) = c_1 e^{sx/c} + c_2 e^{-sx/c}.$$

Since $\lim_{x\to\infty} y(x,t) = 0$, then

$$\lim_{x\to\infty} Y(x,s) = 0$$

and we must choose $c_1 = 0$. Then

$$Y(x,s) = c_2 e^{-sx/c}.$$

Since $y(0,t) = f(t)$, then

$$Y(0,s) = c_2 = F(s)$$

so

$$Y(x,s) = e^{-sx/c}F(s).$$

The solution is

$$y(x,t) = f\left(t - \frac{x}{c}\right) H\left(t - \frac{x}{c}\right).$$

5. Transforming the partial differential equation yields

$$s^2 Y(x,s) = c^2 Y''(x,s) - \frac{Ax}{s^2}.$$

Then

$$Y'' - \frac{s^2}{c^2} Y = \frac{Ax}{c^2 s^2}.$$

This has general solution

$$Y(x,s) = c_1 e^{sx/c} + c_2 e^{-sx/c} - \frac{Ax}{s^4}.$$

Now $c_1 = 0$ from the condition that $\lim_{x\to\infty} y(x,t) = 0$. Then

$$Y(x,s) = c_2 e^{-sx/c} - \frac{Ax}{s^4}.$$

Then

$$\mathcal{L}[y(0,t)](s) = Y(0,s) = c_2 = F(s),$$

so

$$Y(x,s) = e^{-sx/c} F(s) - \frac{Ax}{s^4}.$$

Invert this to obtain the solution

$$y(x,t) = f\left(t - \frac{x}{c}\right) H\left(t - \frac{x}{c}\right) - \frac{1}{6} A x t^4.$$

16.6 d'Alembert's Solution

1. With $c = 1$, characteristics are $x - t = k_1$ and $x + t = k_2$. The solution by d'Alembert's formula is

$$\begin{aligned} u(x,t) &= \frac{1}{2}(f(x-t) + f(x+t)) - \frac{1}{2}\int_{x-t}^{x+t} \xi \, d\xi \\ &= \frac{1}{2}\left((x-t)^2 + (x+t)^2\right) - \left[\frac{\xi^2}{4}\right]_{x-t}^{x+t} \\ &= x^2 - xt + t^2. \end{aligned}$$

2. With $c = 4$ the characteristics are $x - 4t = k_1$ and $x + 4t = k_2$. The solution is

$$\begin{aligned} u(x,t) = &\frac{1}{2}\left((x-4t)^2 - 2(x-4t) + (x+4t)^2 - 2(x+4t)\right) \\ &+ \frac{1}{8}\int_{x-4t}^{x+4t} \cos(\xi)\, d\xi \\ &= x^2 + 16t^2 - 2x + \frac{1}{4}\cos(x)\sin(4t). \end{aligned}$$

For Problems 3 through 6 we provide only the solution, omitting details.

3.

$$\begin{aligned} u(x,t) &= \frac{1}{2}(\cos(\pi(x-7t)) + \cos(\pi(x+7t))) \\ &\quad + t - x^2 t - \frac{49}{3}t^3 \\ &= \frac{1}{2}\cos(\pi x)\cos(7\pi t) + t - x^2 t - \frac{49}{3}t^3 \end{aligned}$$

4.

$$\begin{aligned} u(x,t) &= \frac{1}{2}(\sin(2(x-5t)) + \sin(2(x+5t)) + x^3 t + 25xt^3 \\ &= \sin(2x)\cos(10t) + x^3 t + 25xt^3 \end{aligned}$$

5.

$$u(x,t) = \frac{1}{2}\left(e^{x-14t} + e^{x+14t}\right) + xt = e^x \cosh(14t) + xt$$

6.

$$u(x,t) = x^2 + 144t^2 - 5x + 3t$$

7.

$$\begin{aligned} u(x,t) &= \frac{1}{2}(f(x-4t) + f(x+4t)) + \frac{1}{8}\int_{x-4t}^{x+4t} e^{-\xi}\, d\xi \\ &\quad + \frac{1}{8}\int_0^t \int_{x-4t+4\eta}^{x+4t-4\eta} (\xi + \eta)\, d\xi\, d\eta \\ &= x + \frac{1}{4}e^{-x}\sinh(4t) + \frac{1}{2}xt^2 + \frac{1}{6}t^3 \end{aligned}$$

8.

$$\begin{aligned} u(x,t) &= \frac{1}{2}(f(x-2t) + f(x+2t)) + \frac{1}{4}\int_{x-2t}^{x+2t} 2\xi\, d\xi \\ &\quad + \frac{1}{4}\int_0^t \int_{x-2t+2\eta}^{x+2t-2\eta} 2\xi\eta\, d\xi\, d\eta \\ &= \frac{1}{2}(\sin(x-2t) + \sin(x+2t)) + 2xt + \frac{1}{3}xt^3 \end{aligned}$$

9.

$$\begin{aligned} u(x,t) &= \frac{1}{2}(f(x-8t) + f(x+8t)) + \frac{1}{16}\int_{x-8t}^{x+8t} \cos(2\xi)\, d\xi \\ &\quad + \frac{1}{16}\int_0^t \int_{x-8t+8\eta}^{x+8t-8\eta} \eta\cos(\xi)\, d\xi\, d\eta \\ &= x^2 + 64t^2 - x + \frac{1}{32}(\sin(2(x+8t)) - \sin(2(x-8t))) + \frac{1}{12}xt^4 \end{aligned}$$

10.

$$\begin{aligned}
u(x,t) &= \frac{1}{2}(f(x-4t)+f(x+4t)) + \frac{1}{8}\int_{x-4t}^{x+4t} \xi e^{-\xi}\,d\xi \\
&+ \frac{1}{8}\int_0^t \int_{x-4t+4\eta}^{x+4t-4\eta} \xi\sin(\eta)\,d\xi\,d\eta \\
&= x^2 + 16t^2 + \frac{1}{2}xe^{-x}\sinh(4t) + \frac{1}{4}e^{-x}\sinh(4t) \\
&- te^{-x}\cosh(4t) - x\sin(t) + xt
\end{aligned}$$

11.

$$\begin{aligned}
u(x,t) &= \frac{1}{2}(f(x-3t)+f(x+3t)) + \frac{1}{6}\int_{x-3t}^{x+3t} d\xi \\
&+ \int_0^t \int_{x-3t+3\eta}^{x+3t-3\eta} 3\xi\eta^3\,d\xi\,d\eta \\
&= \frac{1}{2}(\cosh(x-3t)+\cosh(x+3t)) + t + \frac{9}{10}xt^5
\end{aligned}$$

12.

$$\begin{aligned}
u(x,t) &= \frac{1}{2}(f(x-7t)+f(x+7t)) + \frac{1}{14}\int_{x-7t}^{x+7t} \sin(\xi)\,d\xi \\
&+ \frac{1}{14}\int_0^t \int_{x-7t+7\eta}^{x+7t-7\eta} (\xi-\cos(\eta))\,d\xi\,d\eta \\
&= 1 + x - \frac{1}{14}(\cos(x-7t)-\cos(x+7t)) + \frac{1}{2}xt^2 + \cos(t)
\end{aligned}$$

For each of Problems 13 - 16, we give graphs of the wave position at selected times.

13. In Figures 16.21 through 16.25, the wave is shown at times $t = 1/2$, 1, 2, 3 and 4.

14. Figures 16.26 through 16.30 show the wave profile at times $t = 1/2$, 2/3, 7/8, 1.2 and 3.

15. Figures 16.31 through 16.34 show graphs of the solution at times $t = 1/2$, $t = 0.9$, $t = 1.3$ and $t = 1.8$.

16. Figures 16.35 through 16.38 show wave positions at times $t = 1/2$, $t = 0.7$, $t = 0.9$ and $t = 1.3$.

17. Figures 16.39 through 16.41 show wave positions at times $t = 1, 1.4, 1.7$.

18. Figures 16.42 through 16.45 show the wave at times $t = 0.7, 1.4$, $1.7, 2.2$.

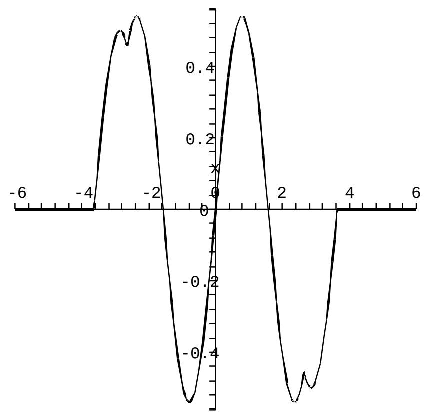

Figure 16.21: Wave position in Problem 13, Section 16.6, at $t = 1/2$.

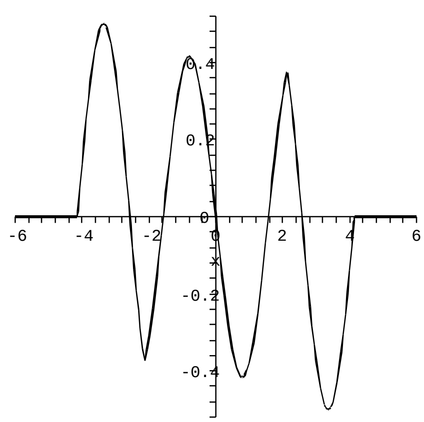

Figure 16.22: Problem 13, Section 16.6, $t = 1$.

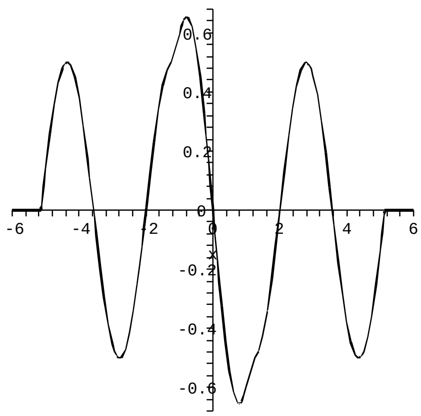

Figure 16.23: Problem 13, Section 16.6, $t = 2$.

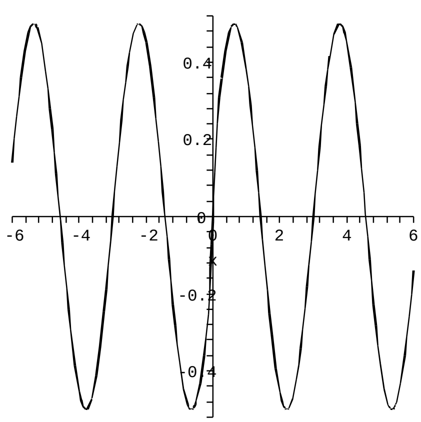

Figure 16.24: Problem 13, Section 16.6, $t = 3$.

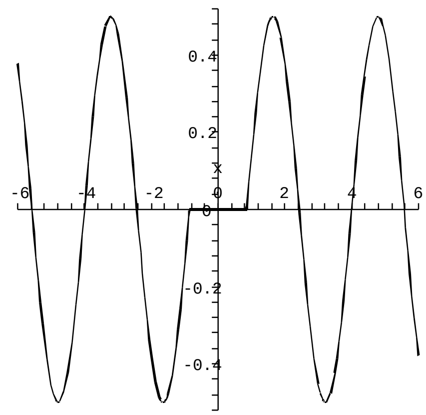

Figure 16.25: Problem 13, Section 16.6, $t = 4$.

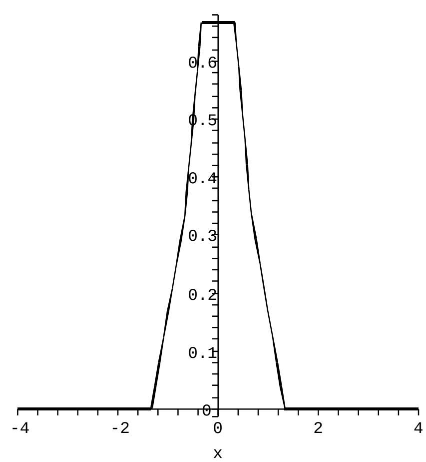

Figure 16.26: Problem 14, Section 16.6, $t = 1/3$.

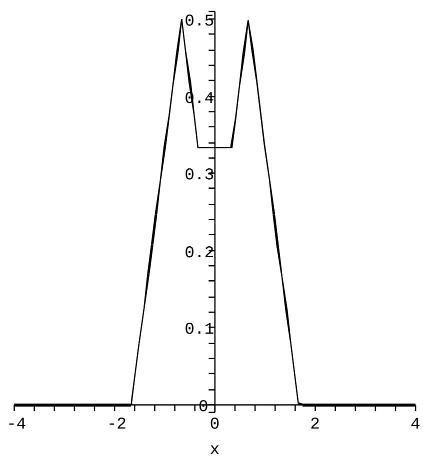

Figure 16.27: Problem 14, Section 16.6, at $t = 2/3$.

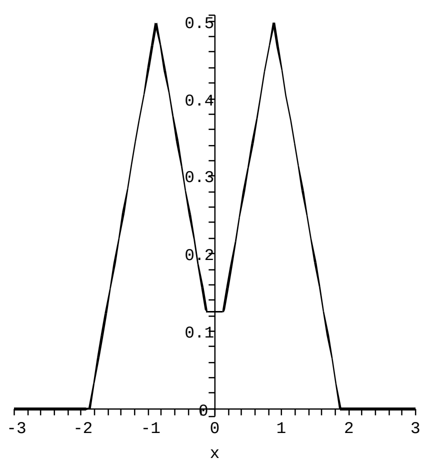

Figure 16.28: Problem 14, Section 16.6, at $t = 7/8$.

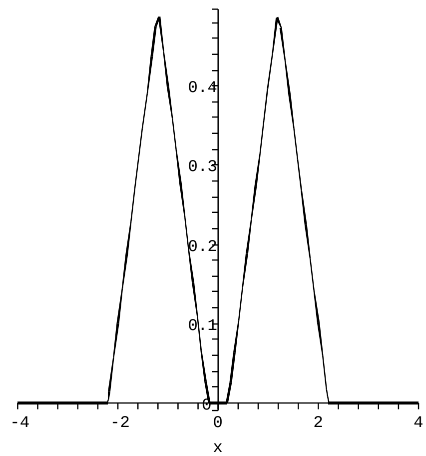

Figure 16.29: Problem 14, Section 16.6, at $t = 1.2$.

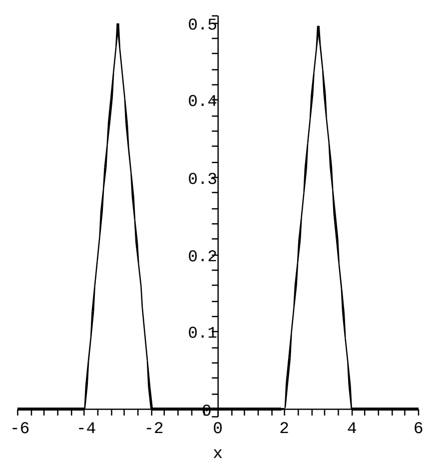

Figure 16.30: Problem 14, Section 16.6, at $t = 3$.

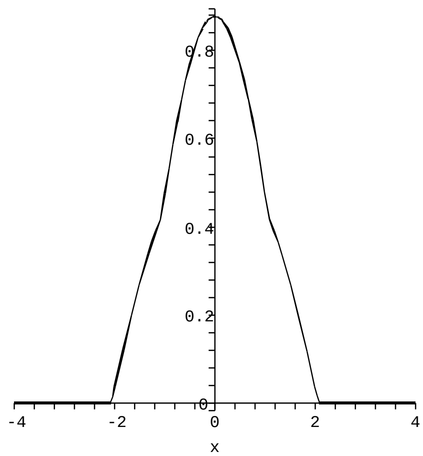

Figure 16.31: Problem 15, Section 16.6, at $t = 1/2$.

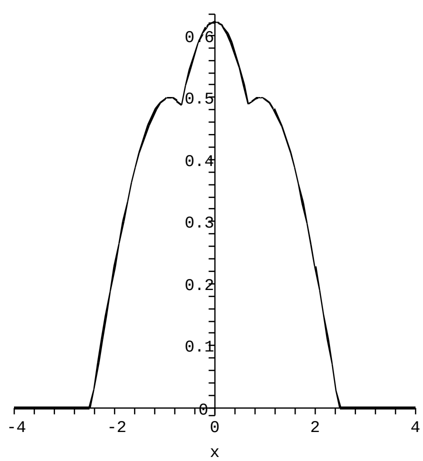

Figure 16.32: Problem 15, Section 16.6, at $t = 0.9$.

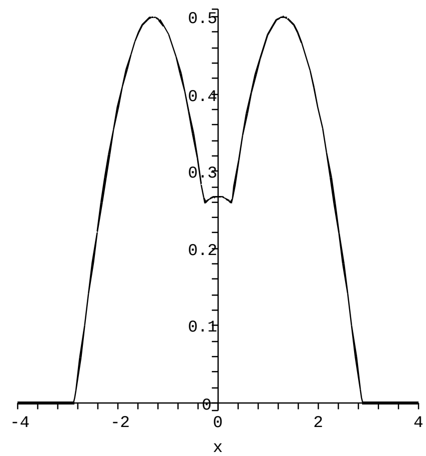

Figure 16.33: Problem 15, Section 16.6, at $t = 1.3$.

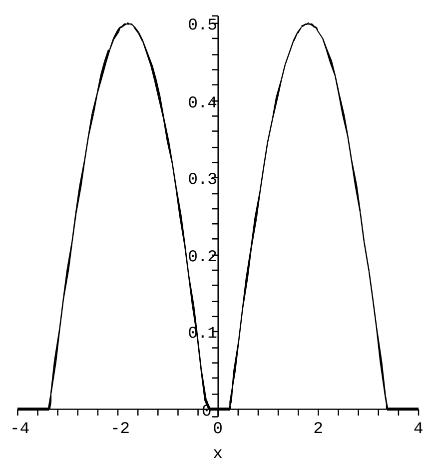

Figure 16.34: Problem 15, Section 16.6, at $t = 1.8$.

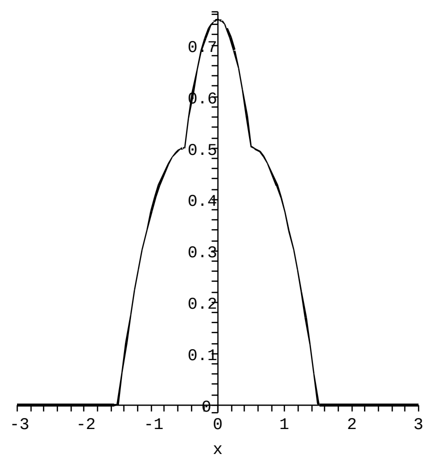

Figure 16.35: Problem 16, Section 16.6, at $t = 1/2$.

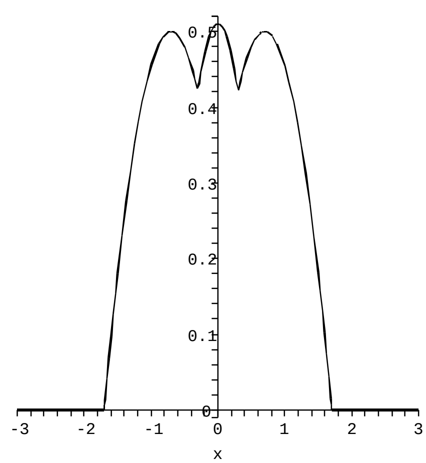

Figure 16.36: Problem 16, Section 16.6, at $t = 0.7$.

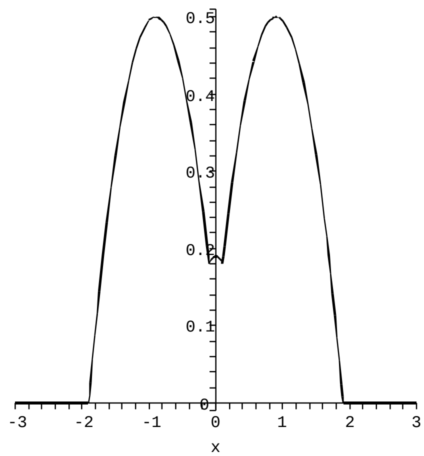

Figure 16.37: Problem 16, Section 16.6, at $t = 0.9$.

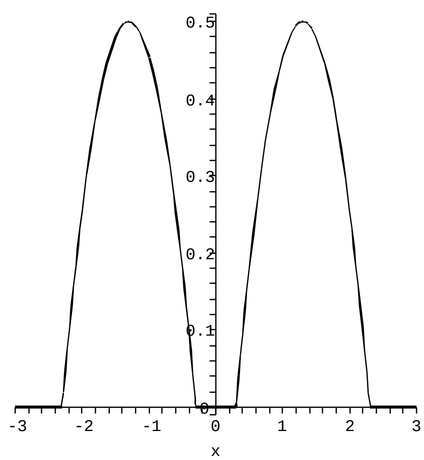

Figure 16.38: Problem 16, Section 16.6, at $t = 1.3$.

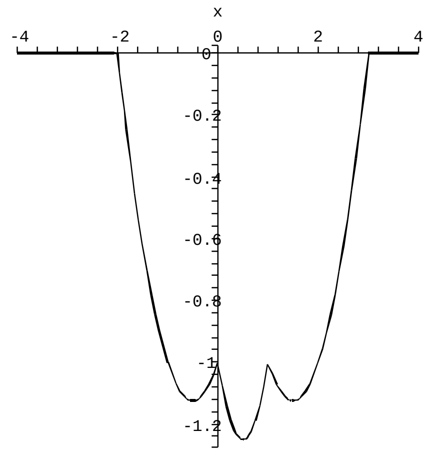

Figure 16.39: Problem 17, Section 16.6, at $t = 1$.

16.7 Vibrations in a Circular Membrane I

In each of these problems, the solution has the form

$$z(r,t) = \sum_{n=1}^{\infty} c_n J_0(j_n r)\cos(j_n t),$$

where j_n is the nth zero of $J_0(x)$. For a given initial displacement $f(r)$, the coefficients are

$$z(r,t) = \frac{2}{J_1(j_n)^2}\int_0^1 sf(s)J_0(j_n s)$$

for $n = 1, 2, \cdots$.

1. For $f(r) = 1 - r$, these coefficients are approximately

   $$a_1 = 0.78542, a_2 = 0.06869, a_3 = 0.05311, a_4 = 0.01736, a_5 = 0.01698.$$

   Figure 16.46 shows the displacement at times $t = 0.05, 0.25,\ 0.5, 0.75$ and 1.25.

2. With $f(r) = 1 - r^2$ the coefficients are approximately

   $$a_1 = 1.10802, a_1 = -0.13978, a_3 = 0.04548, a_4 = -0.02099, a_5 = 0.011637.$$

   Figure 16.47 shows the displacement at times $t = 0.05, 0.25,\ 0.5, 0.75$ ad 1.25.

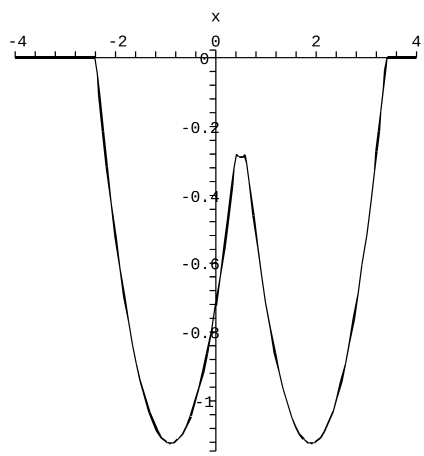

Figure 16.40: Problem 17, Section 16.6, at $t = 1.4$.

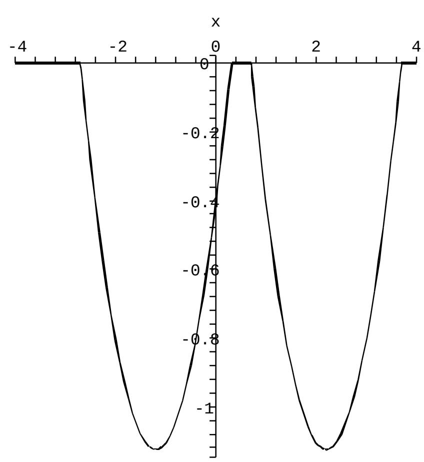

Figure 16.41: Problem 17, Section 16.6, at $t = 1.7$.

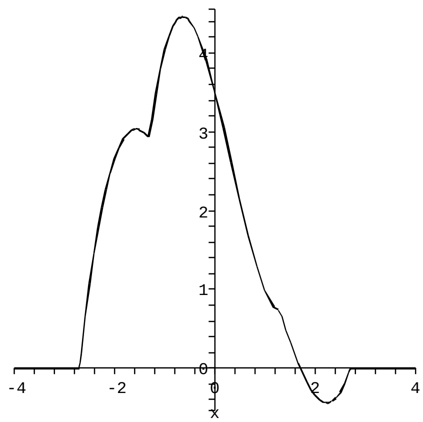

Figure 16.42: Problem 18, Section 16.6, at $t = 0.7$.

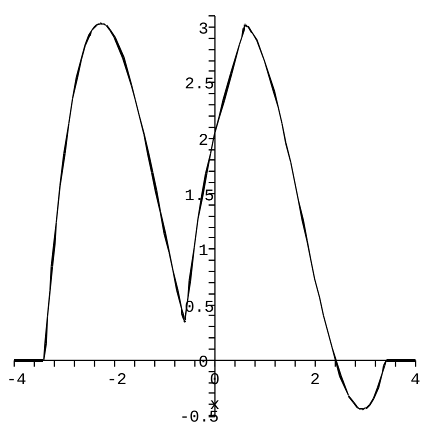

Figure 16.43: Problem 18, Section 16.6, at $t = 1.4$.

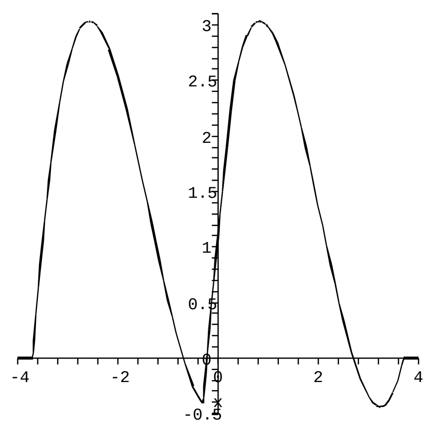

Figure 16.44: Problem 18, Section 16.6, at $t = 1.7$.

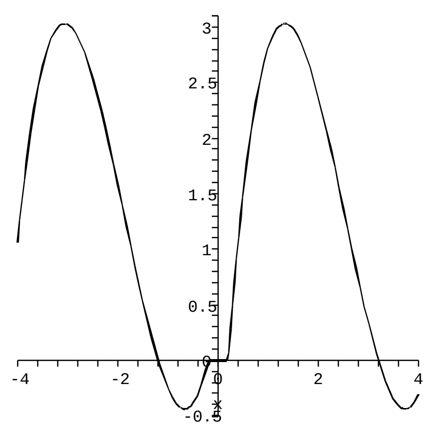

Figure 16.45: Problem 18, Section 16.6, at $t = 2.2$.

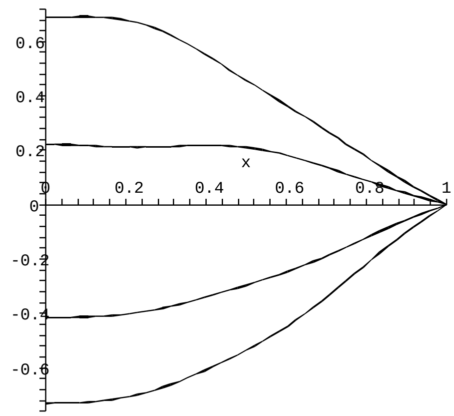

Figure 16.46: Solution positions in Problem 1, Section 16.7.

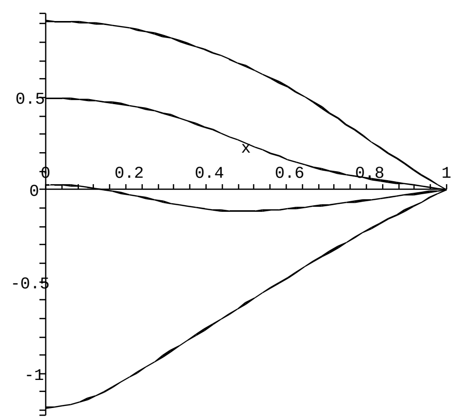

Figure 16.47: Solution positions in Problem 2, Section 16.7.

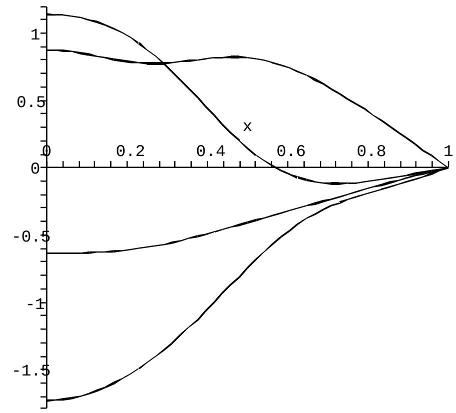

Figure 16.48: Solution positions in Problem 3, Section 16.7.

3. For $f(r) = \sin(\pi r)$, the coefficients are approximately

   $a_1 = 1.25335, a_2 = -0.80469, a_3 = -0.11615, a_4 = -0.09814, a_5 = -0.03740.$

   Figure 16.48 shows the displacement at times $t = 0.05, 0.25,\ 0.5, 0.75$ ad 1.25.

16.8 Vibrations in a Circular Membrane II

1. With zero initial velocity the solution will have the appearance

$$z(r, \theta, t) = \sum_{n=0}^{\infty}\sum_{k=1}^{\infty}[a_{nk}\cos(n\theta) + b_{nk}\sin(n\theta)]J_n\left(\frac{j_{nk}}{2}r\right)\cos(j_{nk}t).$$

   We need to choose the coefficients to satisfy the initial condition that

$$z(r, \theta, 0) = f(r, \theta) = (4 - r^2)\sin^2(\theta).$$

   Putting $t = 0$ into the series, we need

$$(4 - r^2)\sin^{(}\theta) = \sum_{n=0}^{\infty}\sum_{k=1}^{\infty}[a_{nk}\cos(n\theta) + b_{nk}\sin(n\theta)]J_n\left(\frac{j_{nk}}{2}r\right).$$

   Write $\sin(\theta) = (1 - \cos(2\theta))/2$ and exploit the simple nature of the θ dependence in $f(r, \theta)$ to conclude, by matching coefficients of the $\cos(n\theta)$

terms for $n = 0$ and $n = 2$, that

$$\frac{4-r^2}{2} = \frac{1}{2}\alpha_0(r) = \sum_{k=1}^{\infty} a_{0k} J_0\left(\frac{k_{0k}}{2} r\right)$$

and

$$-\frac{4-r^2}{2} = \alpha_2(r) = \sum_{k=1}^{\infty} a_{2k} J_2\left(\frac{j_{2k}}{2} r\right).$$

Further, $\alpha_n(r) = 0$ for $n \neq 2$ and $\beta_n(r) = 0$ for $n \geq 0$, from which it follows that

$$a_{nk} = 0 \text{ for } n \neq 0, n \neq 2, k \geq 1$$

and

$$b_{nk} = 0 \text{ for } n \geq 0, k \geq 1.$$

Finally, using the orthogonality of the Bessel functions $J_0(j_{0k}r/2)$ for $k = 1, 2, \cdots$, and $J_2(j_{0k}r/2)$, we can calculate the coefficients as

$$a_{0k} = \frac{2}{[J_1(j_{0k})]^2} \int_0^1 \xi(1-\xi^2) J_0(j_{0k}\xi)\, d\xi \text{ for } k \geq 1$$

and

$$a_{2k} = \frac{4}{[J_3(j_{2k})]^2} \int_0^1 \xi(\xi^2-1) J_2(j_{2k}\xi)\, d\xi \text{ for } k \geq 1.$$

Using MAPLE, we can carry out numerical approximations of coefficients in the solution. Some of the terms are

$$\begin{aligned} z(r,\theta,t) \approx\ & 1.108022 J_0(1.202413r)\cos(2.404826t) - 0.139778 J_0(2.760039r)\cos(5.520078t) \\ & + 0.045476 J_0(4.326864r)\cos(8.653728t) + \cdots \\ & - 2.976777 J_2(2.567811r)\cos(5.135622t)\cos(2\theta) \\ & - 1.434294 J_2(4.208622r)\cos(8.417244t)\cos(2\theta) \\ & - 1.140494 J_2(5.809921r)\cos(11.619841t)\cos(2\theta) + \cdots . \end{aligned}$$

2. Evaluate the solution at $r = 0$ to obtain

$$z(0,\theta,t) = \sum_{n=0}^{\infty}\sum_{k=1}^{\infty} [a_{nk}\cos(n\theta) + b_{nk}\sin(n\theta)] J_n(0) \cos(j_{nk}at/R).$$

We want to show that this is zero for all $t \geq 0$. Now, $J_n(0) = 0$ if $n \geq 1$, and $J_0(0) = 1$, so this problem reduces to showing that, for all $t \geq 0$,

$$z(0,\theta,t) = \sum_{k=1}^{\infty} a_{ok} \cos\left(\frac{z_{0k}}{R} at\right) = 0.$$

But, the coefficients in this series are

$$a_{0k} = \frac{1}{2\pi \int_0^R r J_0^2(z_{0k}r/R)\, dr} \int_0^R r J_0\left(\frac{z_{ok}}{R} r\right) \int_{-\pi}^{\pi} f(r,\theta)\, d\theta\, dr.$$

But if $f(r,\theta)$ is an odd function in θ, then

$$\int_{-\pi}^{\pi} f(r,\theta)\,d\theta = 0$$

hence $a_{0k} = 0$ for $k = 1, 2, \cdots$, and therefore $z(0,\theta,t) = 0$ for all times, as we wanted to show.

16.9 Vibrations in a Rectangular Membrane

1. Separate variables in the wave equation by setting $z(x,y,t) = X(x)Y(y)T(t)$ to obtain

$$\frac{T''}{T} - \frac{X''}{X} = \frac{Y''}{Y} = -\alpha,$$

in which α is the separation constant. Separate again to get

$$\frac{T''}{T} + \alpha = \frac{X''}{X} = -\lambda.$$

This gives us the separated problems for X, Y and Z:

$$\begin{aligned} X'' + \lambda X = 0; X(0) = X(2\pi) = 0, \\ Y'' + \alpha Y = 0; Y(0) = Y(2\pi) = 0, \\ T'' + (\alpha+\lambda)T = 0; T'(0) = 0. \end{aligned}$$

The eigenvalues and eigenfunctions are, respectively,

$$\begin{aligned} \lambda_n &= n^2/4, X_n(x) = \sin(nx/2), \\ \alpha_m &= m^2/4, Y_m(y) = \sin(my/2), \\ T_{nm}(t) &= \cos(\sqrt{n^2+m^2}t/2). \end{aligned}$$

The solution has the form

$$z(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} c_{nm}\sin(nx/2)\sin(my/2)\cos(\sqrt{n^2+m^2}t/2).$$

We need

$$z(x,y,0) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} c_{nm}\sin(nx/2)\sin(my/2) = x^2\sin(y).$$

Choose the coefficients

$$\begin{aligned} c_{nm} &= \frac{1}{\pi^2}\int_0^{2\pi}\int_0^{2\pi} \xi^2\sin(\eta)\sin(n\xi/2)\sin(m\eta/2)\,d\xi\,d\eta \\ &= \frac{1}{\pi^2}\int_0^{2\pi} \xi^2\sin(n\xi/2)\,d\xi \int_0^{2\pi}\sin(\eta)\sin(m\eta/2)\,d\eta \\ &= -\frac{8}{\pi n^3}\left(2(1-(-1)^n) + n^2\pi^2(-1)^n\right). \end{aligned}$$

The solution is

$$z(x,y,t)$$
$$= \sum_{n=1}^{\infty} \frac{-8}{\pi} \left(\frac{1}{n^3}(2(1-(-1)^n) + n^2\pi^2(-1)^n) \right) \sin(nx/2)\sin(y)\cos(\sqrt{4+n^2}t/2).$$

This is a single sum because the integrals

$$\int_0^{2\pi} \sin(\eta)\sin(m\eta/2)\,d\eta$$

are zero except for $m = 2$.

2. After separating variables, we find that $X_n(x) = \sin(nx)$ and $Y_n(x) = \sin(my)$. Solutions for T have the form

$$T_{nm}(t) = a_{nm}\cos(3\sqrt{n^2+m^2}t) + b_{nm}\sin(3\sqrt{n^2+m^2}t).$$

Thus attempt a solution

$$z(x,y,t) =$$
$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \sin(nx)\sin(my)(a_{nm}\cos(3\sqrt{n^2+m^2}t)$$
$$+ b_{nm}\sin(3\sqrt{n^2+m^2}t)).$$

To satisfy the initial condition $z(x,y,0) = 0$, choose each $a_n = 0$. Now we need to choose the coefficients b_{nm} so that

$$\frac{\partial z}{\partial t}(x,y,0) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} 3b_{nm}\sqrt{n^2+m^2}\sin(nx)\sin(my) = xy.$$

Then

$$b_{nm} = \frac{1}{3\sqrt{n^2+m^2}}\frac{2}{\pi}\int_0^{\pi} x\sin(nx)\,dx \left(\frac{2}{\pi}\right)\int_0^{\pi} y\sin(my)\,dy$$
$$= \frac{4}{3\pi^2\sqrt{n^2+m^2}}\left(\frac{\pi(-1)^{n+1}}{n}\right)\left(\frac{\pi(-1)^{m+1}}{m}\right)$$
$$= \frac{4(-1)^{n+m}}{3nm\sqrt{n^2+m^2}}.$$

The solution is

$$z(x,y,t) =$$
$$\frac{4}{3}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{(-1)^{n+m}}{nm\sqrt{n^2+m^2}}\sin(nx)\sin(my)\sin(3\sqrt{n^2+m^2}t).$$

3. Separation of variables gives us the eigenfunctions $X_n(x) = \sin(nx/2)$ and $Y_m(y) = \sin(my/2)$, and we find that

$$T_{nm}(t) = a_{nm}\cos(\sqrt{n^2+m^2}t) + b_{nm}\sin(\sqrt{n^2+m^2}t).$$

The solution has the form

$$z(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(a_{nm}\cos(\sqrt{n^2+m^2}t) + b_{nm}\sin(\sqrt{n^2+m^2}t))\sin(nx/2)\sin(my/2).$$

The condition that $z(x,y,0) = 0$ is satisfied if all $a_{nm} = 0$. Thus the solution has the form

$$z(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} b_{nm}\sin(nx/2)\sin(my/2)\sin(\sqrt{n^2+m^2}t).$$

Now we need to choose the coefficients b_{nm} so that

$$\frac{\partial z}{\partial t}(x,y,0) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} b_{nm}\sqrt{n^2+m^2}\sin(nx/2)\sin(my/2) = 1.$$

Then

$$\begin{aligned} b_{nm} &= \frac{1}{\sqrt{n^2+m^2}}\frac{1}{\pi}\int_0^{2\pi}\sin(nx/2)\,dx\,\frac{1}{\pi}\int_0^{2\pi}\sin(my/2)\,dy \\ &= \frac{1}{\pi\sqrt{n^2+m^2}}\left(\frac{2(1-(-1)^n)}{n}\right)\left(\frac{2(1-(-1)^m)}{m}\right). \end{aligned}$$

Notice that $b_{nm} = 0$ if either n or m is even. Thus in the double summation we need only retain the terms in which both n abd m are odd. We can therefore write the solution

$$z(x,y,t) = + \frac{16}{\pi^2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} c_{nm}\sin((2n-1)x/2)\sin((2m-1)y/2)\sin(\sqrt{\alpha_{nm}}t),$$

where

$$c_{nm} = \frac{1}{(2n-1)(2m-1)\sqrt{\alpha_{nm}}}$$

and

$$\alpha_{nm} = (2n-1)^2 + (2m-1)^2.$$

Chapter 17

The Heat Equation

17.1 Initial and Boundary Conditions

1. Let $u(x,t)$ be the temperature at time t of the cross section at x. Then u satisfies
$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \text{ for } t > 0, 0 < x < L.$$
The boundary conditions are
$$u(0,t) = 0, \frac{\partial u}{\partial x}(L,t) = 0 \text{ for } t > 0.$$
The initial condition is
$$u(x,0) = f(x) \text{ for } 0 < x < L.$$

2. $u(x,t)$ satisfies the conditions
$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \text{ for } t > 0, 0 < x < L,$$
$$u(0,t) = \alpha(t), u(L,t) = \beta(t) \text{ for } t > 0,$$
$$u(x,0) = f(x) \text{ for } 0 < x < L.$$

3. $u(x,t)$ satisfies the conditions
$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \text{ for } t > 0, 0 < x < L,$$
with
$$\frac{\partial u}{\partial x}(0,t) = 0, u(L,t) = \beta(t) \text{ for } t > 0,$$
$$u(x,0) = f(x) \text{ for } 0 < x < L.$$

17.2 The Heat Equation on $[0, L]$

For the first three problems, separation of variables and the given boundary conditions $u(0,t) = u(L,t) = 0$ yield the eigenvalues and eigenfunctions

$$\lambda_n = \frac{n^2\pi^2}{L^2}, X_n(x) = \sin(n\pi x/L).$$

The corresponding time solutions are

$$T_n(t) = e^{-kn^2\pi^2 t/L^2}.$$

Solutions have the form

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin(n\pi x/L) e^{-kn^2\pi^2 t/L^2}.$$

The coefficients are determined by

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} c_n \sin(n\pi x/L),$$

hence

$$c_n = \frac{2}{L}\int_0^{\infty} f(\xi)\sin(n\pi\xi/L)\,d\xi.$$

1. With $f(x) = x(L-x)$,

$$c_n = \frac{2}{L}\int_0^L \xi(L-\xi)\sin(n\pi\xi/L)\,d\xi = \frac{4L^2}{n^3\pi^3}(1-(-1)^n).$$

Note that $c_{2n} = 0$ because $1-(-1)^{2n} = 0$. We therefore retain only the odd indices in the solution:

$$u(x,t) = \frac{8L^2}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3}\sin((2n-1)\pi x/L)e^{-k(2n-1)^2\pi^2 t/L^2}.$$

Figure 17.1 shows the temperature function (decreasing) at times $t = 0.2$, 0.4, 0.7 and 1.5.

2. With $k = 4$ and $f(x) = x^2(L-x)$, compute

$$c_n = \frac{2}{L}\int_0^L \xi^2(L-\xi)\sin(n\pi\xi/L)\,d\xi = -\frac{4L^3}{\pi^3}\left(\frac{1+2(-1)^n}{n^3}\right).$$

The solution is

$$u(x,t) = -\frac{4L^3}{\pi^3}\sum_{n=1}^{\infty}\left(\frac{1+2(-1)^n}{n^3}\right)\sin(n\pi x/L)e^{-4n^2\pi^2 t/L^2}.$$

Figure 17.2 shows this temperature function at times $t = 0.2$, 0.4 and 1.3.

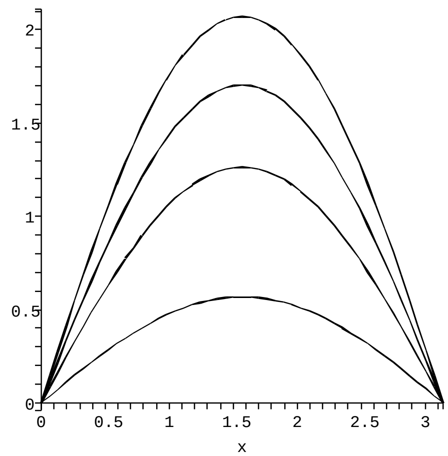

Figure 17.1: Temperature distribution in Problem 1, Section 17.2.

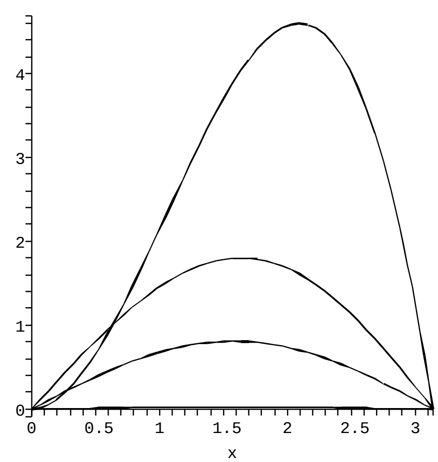

Figure 17.2: Problem 2, Section 17.2.

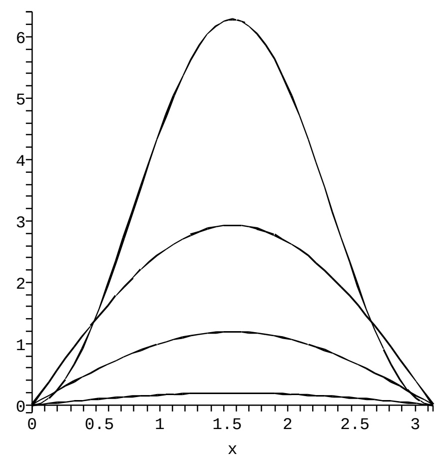

Figure 17.3: Problem 3, Section 17.2.

3. The coefficients are given by

$$c_n = \frac{2}{L}\int_0^L L(1-\cos(2\pi\xi/L))\sin(n\pi\xi/L)\,d\xi$$
$$= \begin{cases} \frac{8L((-1)^n-1)}{n\pi(n^2-4)} & \text{for } n \neq 2, \\ 0 & \text{for } n = 2. \end{cases}$$

In addition, since $(-1)^n - 1 = 0$ if n is even, we have

$$c_4 = c_6 = \cdots = c_{\text{even}} = 0.$$

Therefore the solution is

$$u(x,t) = \\ -\frac{16L}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)((2n-1)^2-4)}\sin((2n-1)\pi x/L)e^{-3(2n-1)^2\pi^2 t/L^2}.$$

Figure 17.3 shows the temperature function at times $t = 0.2$, 0.5 and 1.1.

In Problems 4 through 7, separation of variables and the insulated end conditions $\partial u/\partial x(0,t) = \partial u/\partial x(L,t) = 0$ yield the eigenvalue $\lambda_0 = 1$ with eigenfunction $X_0(x) = 1$, and eigenvalues and eigenfunctions

$$\lambda_n = \frac{n^2\pi^2}{L^2}, X_n(x) = \cos(n\pi x/L).$$

The associated time functions are

$$T_n(t) = e^{-kn^2\pi^2 t/L^2}.$$

The solution has the form

$$u(x,t) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n \cos(n\pi x/L) e^{-kn^2\pi^2 t/L^2},$$

where

$$c_n = \frac{2}{L}\int_0^L f(\xi)\cos(n\pi\xi/L)\,d\xi \text{ for } n = 0, 1, 2, \cdots .$$

4. Compute

$$\begin{aligned} c_0 &= \frac{2}{L}\int_0^L f(\xi)\cos(n\pi\xi/L)\,d\xi \\ &= \frac{2}{\pi}\int_0^\pi \sin(\xi)\,d\xi = \frac{4}{\pi}, \text{ and} \\ c_n &= \frac{2}{\pi}\int_0^\pi \sin(\xi)\cos(n\xi)\,d\xi \\ &= \begin{cases} 0 & \text{for } n = 1, 3, 5, \cdots, \\ -\frac{4}{\pi}\left(\frac{1}{n^2-1}\right) & \text{for } n = 2, 4, \cdots . \end{cases} \end{aligned}$$

The solution is

$$u(x,t) = \frac{2}{\pi} - \frac{4}{\pi}\left(\frac{1}{4n^2-1}\right)\cos(2nx)e^{-4n^2 t}.$$

Figure 17.4 shows the temperature function at times $t = 0.2$, 0.4 and 0.7.

5. Compute

$$c_0 = \frac{1}{\pi}\int_0^{2\pi} \xi(2\pi - \xi)\,d\xi = \frac{4\pi^2}{3},$$

and

$$c_n = \frac{1}{\pi}\int_0^{2\pi} \xi(2\pi - \xi)\cos(nx)\,d\xi = -\frac{4}{n^2} \text{ for } n = 1, 2, \cdots .$$

The solution is

$$u(x,t) = \frac{2\pi^2}{3} - 4\sum_{n=1}^{\infty} \frac{1}{n^2}\cos(nx)e^{-4n^2 t}.$$

Figure 17.5 shows the solution at times $t = 0.2$, 0.4 and 0.7.

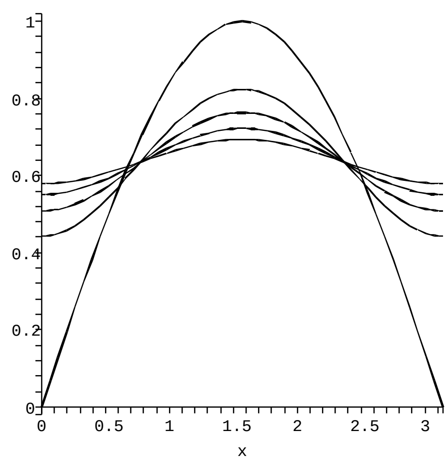

Figure 17.4: Problem 4, Section 17.2.

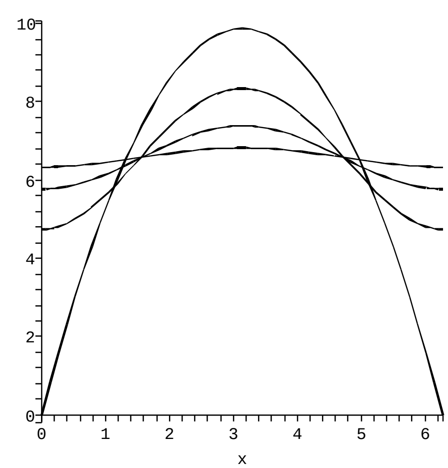

Figure 17.5: Problem 5, Section 17.2.

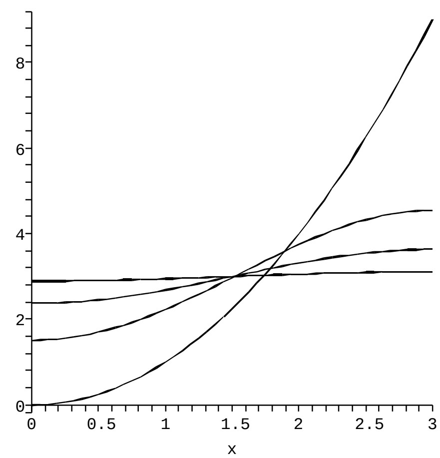

Figure 17.6: Problem 6, Section 17.2.

6. Compute

$$c_0 = \frac{2}{3}\int_0^3 \xi^2\, d\xi = 6$$

and, for $n = 1, 2, \cdots$,

$$c_n = \frac{2}{3}\int_0^3 \xi^2 \cos(n\pi\xi/3)\, d\xi = \frac{36(-1)^n}{n^2\pi^2}.$$

The solution is

$$u(x,t) = 3 + \frac{36}{\pi^2}\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}\cos(n\pi x/3)e^{-4n^2\pi^2 t/9}.$$

The solution is shown in Figure 17.6 at times $t = 0.2$, 0.4 and 0.8.

7. Compute

$$c_0 = \frac{2}{6}\int_0^6 e^{-\xi}\, d\xi = \frac{1}{3}\left(1 - e^{-6}\right),$$

and, for $n = 1, 2, \cdots$,

$$\begin{aligned} c_n &= \frac{1}{3}\int_0^6 e^{-\xi}\cos(n\pi\xi/6)\, d\xi \\ &= \frac{12}{36 + n^2\pi^2}\left(1 - (-1)^n e^{-6}\right). \end{aligned}$$

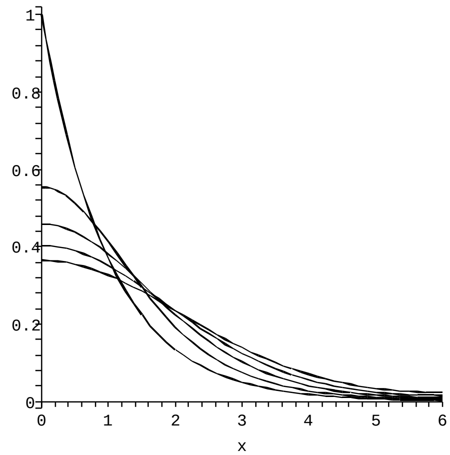

Figure 17.7: Problem 7, Section 17.2.

The solution is

$$u(x,t) = \frac{1}{6}\left(1-e^{-6}\right)+\sum_{n=1}^{\infty}\frac{12}{36+n^2\pi^2}\left(1-(-1)^n e^{-6}\right)\cos(n\pi x/6)e^{-n^2\pi^2 t/18}.$$

Figure 17.7 shows the temperature function at $t = 0.2$, 0.4 and 0.8.

8. The initial-boundary value problem is

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2},$$
$$\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(L,t) = 0,$$
$$u(x,0) = B.$$

The coefficients in the series solution are

$$c_0 = \frac{2}{L}\int_0^L B\,d\xi = 2B$$

and

$$c_n = \frac{2}{L}\int_0^L B\cos(n\pi\xi/L)\,d\xi = 0$$

for $n = 1, 2, \cdots$. The solution is

$$u(x,t) = B.$$

This is consistent with intuition.

9. The initial-boundary value problem for the temperature function is

$$\begin{aligned}\frac{\partial u}{\partial t} &= k\frac{\partial^2 u}{\partial x^2},\\ u(0,t) &= \frac{\partial u}{\partial x}(L,t) = 0,\\ u(x,0) &= B.\end{aligned}$$

Separate variables in the heat equation by putting $u(x,t) = X(x)T(t)$. We obtain the two problems:

$$X'' + \lambda X = 0, X(0) = 0, X'(L) = 0$$

and

$$T' + \lambda kT = 0.$$

The problem for $X(x)$ is routine and we obtain the eigenvalues and eigenfunctions

$$\lambda_n = \frac{(2n-1)^2\pi^2}{4L^2} \text{ and } X_n(x) = \sin((2n-1)\pi x/2L).$$

Then

$$T_n(t) = e^{-k(2n-1)^2\pi^2 kt/4L^2}.$$

By superposition, the solution has the form

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin((2n-1)\pi x/2L)e^{-k(2n-1)^2\pi^2 t/4L^2}.$$

The coefficients are given by

$$c_n = \frac{2}{L}\int_0^L B\sin((2n-1)\pi\xi/2L)\,d\xi = \frac{4B}{(2n-1)\pi}.$$

The solution is

$$u(x,t) = \frac{4B}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\sin((2n-1)\pi x/2L)e^{-k(2n-1)^2\pi^2 t/4L^2}.$$

10. The initial-boundary value problem for $u(x,t)$ is

$$\begin{aligned}\frac{\partial u}{\partial t} &= 9\frac{\partial^2 u}{\partial x^2},\\ u(0,t) &= \frac{\partial u}{\partial x}(L,t) = 0,\\ u(x,0) &= x^2.\end{aligned}$$

From Problem 9 with $k = 9$ and $L = 2$, the solution is

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin((2n-1)\pi x/4) e^{-9(2n-1)^2\pi^2 t/16},$$

where

$$\begin{aligned} c_n &= \int_0^2 \xi^2 \sin((2n-1)\pi\xi/4)\, d\xi \\ &= -\frac{64}{\pi^3}\left(\frac{2+(-1)^n(2n-1)\pi}{(2n-1)^3}\right) \end{aligned}$$

for $n = 1, 2, \cdots$.

For given x in $[0,2]$, $\lim_{t\to\infty} u(x,t) = 0$.

11. Let $u(x,t) = e^{\alpha x+\beta t} v(x,t)$ to transform the given problem. Substitute this into the heat equation and divide out the common exponential factor to obtain

$$\beta v + \frac{\partial v}{\partial t} = k\left(\alpha^2 v + 2\alpha\frac{\partial v}{\partial x} + \frac{\partial^2 v}{\partial x^2} + A\alpha v + A\frac{\partial v}{\partial x} + Bv\right).$$

The idea is to choose α and β to obtain a standard heat equation for v. To do this, we must eliminate terms containing v or $\partial v/\partial x$. Thus choose

$$\begin{aligned} 2\alpha + A &= 0, \\ k(\alpha^2 + A\alpha + B) - \beta &= 0. \end{aligned}$$

Then

$$\alpha = -\frac{A}{2} \text{ and } \beta = k\left(B - \frac{A^2}{4}\right).$$

With these choices, v satisfies

$$\begin{aligned} \frac{\partial v}{\partial t} &= k\frac{\partial^2 v}{\partial x^2} \\ v(0,t) &= v(L,t) = 0 \\ v(x,0) &= e^{-\alpha x} u(x,0). \end{aligned}$$

12. Follow the method suggested in Problem 11 with $A = 4$ and $B = 2$ and $k = 1$. Choose $\alpha = \beta = -2$ to define the transformation

$$u(x,t) = e^{-2x-2t} v(x,t).$$

The transformed problem for v is

$$\begin{aligned} \frac{\partial v}{\partial t} &= \frac{\partial^2 v}{\partial x^2}, \\ v(0,t) &= v(\pi,t) = 0, \\ v(x,0) &= e^{2x} u(x,0) = x(\pi - x)e^{2x}. \end{aligned}$$

This is a standard problem for v that we have solved previously. The solution has the form

$$v(x,t) = \sum_{n=1}^{\infty} c_n \sin(nx) e^{-n^2 t},$$

where

$$\begin{aligned} c_n &= \frac{2}{\pi} \int_0^{\pi} \xi(\pi - \xi) e^{2\xi} \sin(n\xi)\, d\xi \\ &= \frac{4n}{\pi(n^2+1)}((-1)^n e^{2\pi}(12 + 8\pi - n^2(1+2\pi)) - (12 + 8\pi - n^2(1-2\pi))). \end{aligned}$$

The solution for the original problem is

$$u(x,t) = e^{-2x-2t} \sum_{n=1}^{\infty} c_n \sin(nx) e^{-n^2 t}.$$

13. Follow the idea of Problem 11 with $A = 6$, $B = 0$ and $k = 1$. Then $\alpha = -3$ and $\beta = -9$ to make the transformation

$$u(x,t) = e^{-3x-9t} v(x,t).$$

Then v satisfies the standard problem

$$\begin{aligned} \frac{\partial v}{\partial t} &= \frac{\partial^2 v}{\partial x^2}, \\ v(0,t) &= v(4,t) = 0, \\ v(x,0) &= e^{3x} u(x,0) = e^{3x}. \end{aligned}$$

This problem has a solution of the form

$$v(x,y) = \sum_{n=1}^{\infty} c_n \sin(n\pi x/4) e^{-n^2\pi^2 t/16},$$

where

$$\begin{aligned} c_n &= \frac{1}{2} \int_0^4 e^{3\xi} \sin(n\pi\xi/4)\, d\xi \\ &= \frac{2n\pi}{144 + n^2\pi^2}(1 - e^{12}(-1)^n). \end{aligned}$$

The solution for the original problem for u is

$$u(x,t) = e^{-3x-9t} \sum_{n=1}^{\infty} c_n \sin(n\pi x/4) e^{-n^2\pi^2 t/16}.$$

Figure 17.8 shows the solution at times $t = 0.2$, 0.4, 0.7 and 1.1.

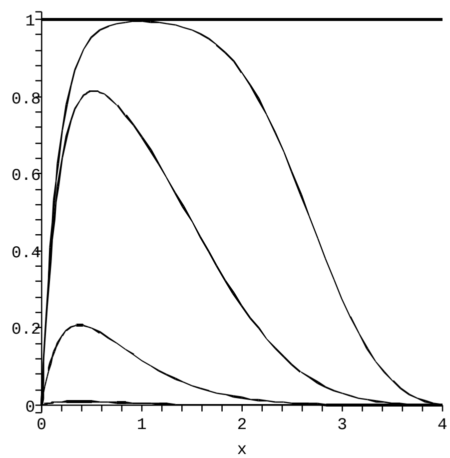

Figure 17.8: Problem 13, Section 17.2.

14. From Problem 11 with $A = -6$, $B = 0$ and $k = 1$, choose $\alpha = 3$ and $\beta = -9$ to define the transformation $u(x,t) = e^{3x-9t}v(x,t)$. Then $v(x,t)$ satisfies

$$\begin{aligned}\frac{\partial v}{\partial t} &= \frac{\partial^2 v}{\partial x^2},\\ v(0,t) &= v(\pi,t) = 0,\\ v(x,0) &= e^{-3x}u(x,0) = x(\pi - x)e^{-3x}.\end{aligned}$$

This problem has the solution

$$v(x,t) = \sum_{n=1}^{\infty} c_n \sin(nx)e^{-n^2 t},$$

where

$$\begin{aligned}c_n &= \frac{2}{\pi}\int_0^{\pi} e^{-3\xi}\xi(\pi-\xi)\sin(n\xi)\,d\xi\\ &= \frac{4n}{\pi(n^2+9)^3}(1-(-1)^n e^{-3\pi})(3\pi(n^2+9)+n^2-27).\end{aligned}$$

Then

$$u(x,t) = e^{3x-9t}v(x,t).$$

Figure 17.9 shows the solution at times $t = 0.2$, 0.4 and 0.6.

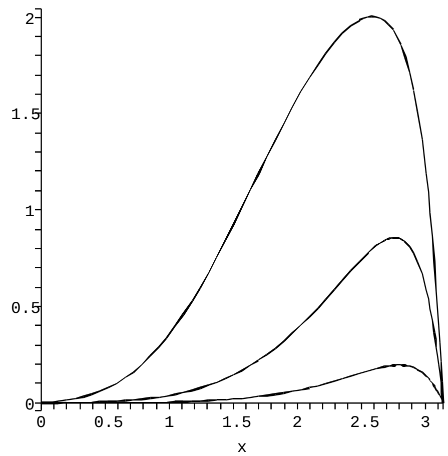

Figure 17.9: Problem 14, Section 17.2.

15. If we attempt a separation of variables in this problem, we find that this method fails because of the nonhomogeneous boundary conditions $u(0,t) = 2$ and $u(1,t) = 5$. We address this issue by transforming the problem. Let $u(x,t) = v(x,t) + L(x)$, where the idea is to choose $L(x)$ to obtain a problem for v that we know how to solve. Substituting u into the given initial-boundary value problem, we obtain a problem for v:

$$\frac{\partial v}{\partial t} = 16\frac{\partial^2 v}{\partial x^2} + 16L''(x),$$
$$v(0,t) + L(0) = 2, v(1,t) + L(1) = 5,$$
$$v(x,0) + L(x) = x^2.$$

To simplify the partial differential equation, make $L''(x) = 0$. To make the boundary conditions homogeneous, also choose L so that $L(0) = 2$ and $L(1) = 5$. Thus, we want

$$L''(x) = 0; L(0) = 2, L(1) = 5.$$

Routine integrations yield $L(x) = 3x + 2$. Now the problem for $v(x,t)$ is standard:

$$\frac{\partial v}{\partial t} = 16\frac{\partial^2 v}{\partial x^2}$$
$$v(0,t) = 0, v(1,t) = 0,$$
$$v(x,0) = x^2 - 3x - 2.$$

This problem has the solution

$$v(x,t) = \sum_{n=1}^{\infty} c_n \sin(n\pi x) e^{-16n^2\pi^2 t},$$

where

$$\begin{aligned} c_n &= 2\int_0^1 (\xi^2 - 3\xi - 2)\sin(n\pi\xi)\,d\xi \\ &= \frac{4}{n^3\pi^3}\left((-1)^n(1+2n^2\pi^2) - (1+n^2\pi^2)\right). \end{aligned}$$

The original problem has the solution

$$u(x,t) = 3x + 2 + \sum_{n=1}^{\infty} c_n \sin(n\pi x) e^{-16n^2\pi^2 t}.$$

16. The nonhomogeneous boundary condition $u(0,t) = T$ prevents separation of variables from solving this problem. However, we want to preserve the condition $u(L,t) = 0$. Thus let $u(x,t) = v(x,t) + h(x)$, where $h''(x) = 0$ and $h(0) = T, h(L) = 0$. Routine integration gives us

$$h(x) = T - \frac{T}{L}x.$$

Now the problem for v is

$$\begin{aligned} \frac{\partial v}{\partial t} &= k\frac{\partial^2 v}{\partial x^2} \\ v(0,t) &= 0, v(L,t) = 0, \\ v(x,0) &= x(L-x) - T + \frac{T}{L}x = \frac{1}{L}(Lx - T)(L-x). \end{aligned}$$

This has the solution

$$v(x,t) = \sum_{n=1}^{\infty} c_n \sin(n\pi x/L) e^{-kn^2\pi^2 t/L^2},$$

where

$$\begin{aligned} c_n &= \frac{2}{L}\int_0^L \frac{1}{L}(L\xi - T)(L-\xi)\sin(n\pi\xi/L)\,d\xi \\ &= \frac{2}{n^3\pi^3}(2L^2(1-(-1)^n) - n^2\pi^2 T). \end{aligned}$$

With this solution for $v(x,t)$, then

$$u(x,t) = v(x,t) + T - \frac{T}{L}x.$$

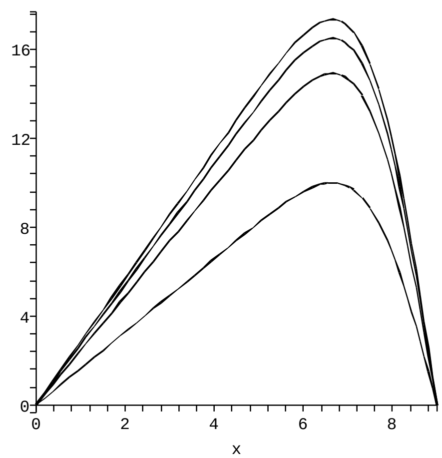

Figure 17.10: Problem 17, Section 17.2, with $t = 0.2$.

17. Let $u(x,t) = e^{-\alpha t}w(x,t)$ and substitute into the heat equation, choosing α to eliminate the $-Aw$ term. This requires that

$$-\alpha w e^{-\alpha t} + \frac{\partial w}{\partial t}e^{-\alpha t} = 4\frac{\partial^2 w}{\partial x^2}e^{-\alpha t} - Awe^{-\alpha t}.$$

Thus choose $\alpha = A$. Then w satisfies

$$\frac{\partial w}{\partial t} = 4\frac{\partial^2 w}{\partial x^2},$$
$$w(0,t) = w(9,t) = 0,$$
$$w(x,0) = 3x.$$

By separation of variables we obtain the solution

$$w(x,t) = \frac{54}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin(n\pi x/9)e^{-4n^2\pi^2 t/81}.$$

The solution of the problem for u is

$$u(x,t) = e^{-At}w(x,t).$$

The diagrams show the solution at various times for $A = 1/4$, $1/2$, 1 and 3. Figure 17.10 is for $t = 0.2$, Figure 17.11 for $t = 0.7$, and Figure 17.12 for $t = 1.4$.

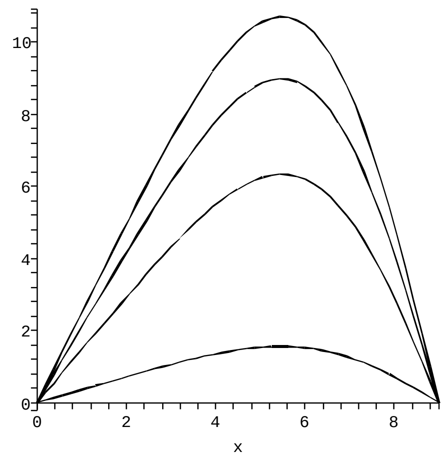

Figure 17.11: Problem 17, Section 17.2, for $t = 0.7$.

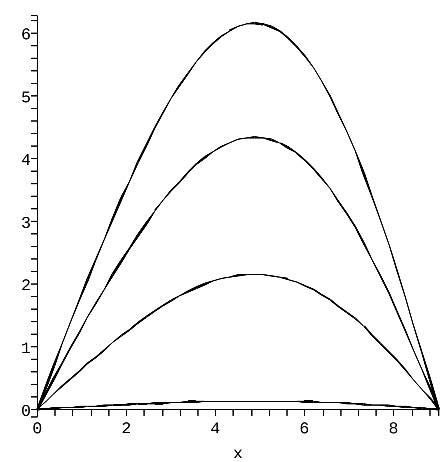

Figure 17.12: Problem 17, Section 17.2, at $t = 1.4$.

18. Let $u(x,t) = v(x,t) + h(x)$. Substitute this into the problem for u and choose h to obtain homogeneous boundary conditions. This gives us

$$h(x) = T\left(1 - \frac{x}{L}\right).$$

The problem for v is

$$\begin{aligned} \frac{\partial v}{\partial t} &= 9\frac{\partial^2 v}{\partial x^2}, \\ v(0,t) &= v(L,t) = 0, \\ v(x,0) &= -T\left(1 - \frac{x}{L}\right). \end{aligned}$$

By separation of variables we obtain the solution

$$v(x,t) = \sum_{n=1}^{\infty} a_n \sin(n\pi x/L) e^{-9n^2\pi^2 t/L^2},$$

where

$$a_n = \frac{2}{L}\int_0^L -T\left(1 - \frac{\xi}{L}\right)\sin(n\pi\xi/L)\,d\xi = -\frac{2T}{n\pi}.$$

Then

$$u(x,t) = T\left(1 - \frac{x}{L}\right) - \frac{2T}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin(n\pi x/L)e^{-9n^2\pi^2 t/L^2}.$$

In each of Problems 19 through 23, obtain a solution of the form

$$u(x,t) = \sum_{n=1}^{\infty} T_n(t)\sin(n\pi x/L) + \sum_{n=1}^{\infty} b_n \sin(n\pi x/L)e^{-kn^2\pi^2 t/L^2},$$

where

$$b_n = \frac{2}{L}\int_0^L f(\xi)\sin(n\pi\xi/L)\,d\xi$$

for $n = 1, 2, \cdots$ and $T_n(t)$ is the solution of

$$T_n'(t) + k\frac{n^2\pi^2}{L^2}T_n(t) = B_n(t); T_n(0) = b_n,$$

with

$$B_n(t) = \frac{2}{L}\int_0^L F(\xi,t)\sin(n\pi\xi/L)\,d\xi$$

for $n = 1, 2, \cdots$.

Note that the second term in this solution for $u(x,t)$ is the solution to the problem without the forcing term.

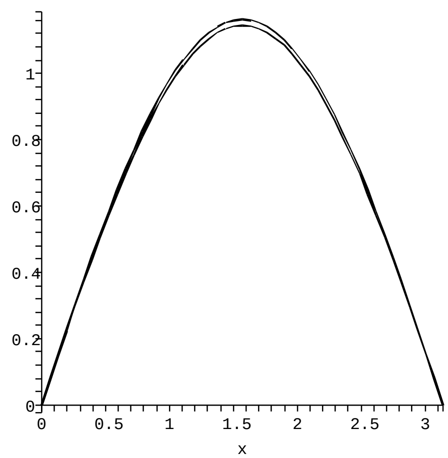

Figure 17.13: Problem 19, Section 17.2, at $t = 0.2$, with and without the source term.

19. With $k = 4$, $L = \pi$, $f(x) = x(\pi - x)$ and $F(x,t) = t$, compute

$$B_n(t) = \frac{2}{\pi}\int_0^{\pi} t\sin(n\xi)\,d\xi = \frac{2t}{n\pi}(1-(-1)^n),$$

$$b_n = \frac{4}{\pi n^3}(1-(-1)^n),$$

and

$$T_n(t) = \frac{1}{8\pi n^5}(1-(-1)^n)(-1+4n^2t+e^{-4n^2t}).$$

The solution is

$$\begin{aligned} u(x,t) = &\sum_{n=1}^{\infty}\frac{1}{8\pi n^5}(1-(-1)^n)(-1+4n^2t+e^{-4n^2t})\sin(nx) \\ &+\sum_{n=1}^{\infty}\frac{4}{\pi n^3}(1-(-1)^n)\sin(nx)e^{-4n^2t}. \end{aligned}$$

Figure 17.13 shows the solution with and without the source term, at time $t = 0.2$. Figure 17.14 is at $t = 0.5$, and Figure 17.15 at $t = 1.1$.

20. Compute

$$B_n(t) = \frac{1}{2}\int_0^4 \xi\sin(t)\sin(n\pi\xi/4)\,d\xi = \frac{8(-1)^{n+1}}{n\pi}\sin(t),$$

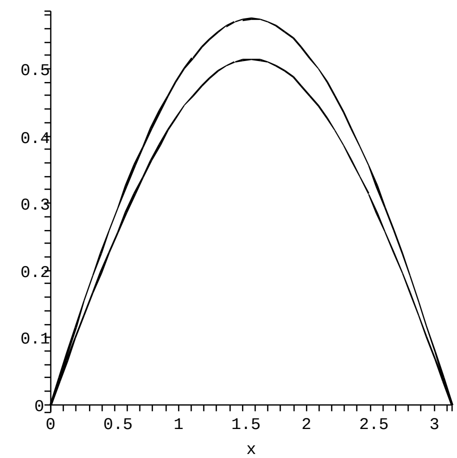

Figure 17.14: Problem 19, Section 17.2, at $t = 0.5$, with and without source term.

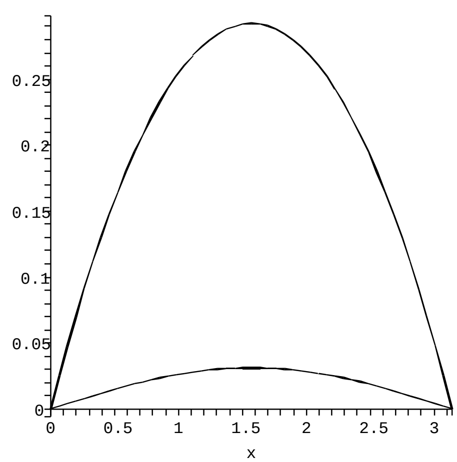

Figure 17.15: Problem 19, Section 17.2, at $t = 1.1$, with and without source term.

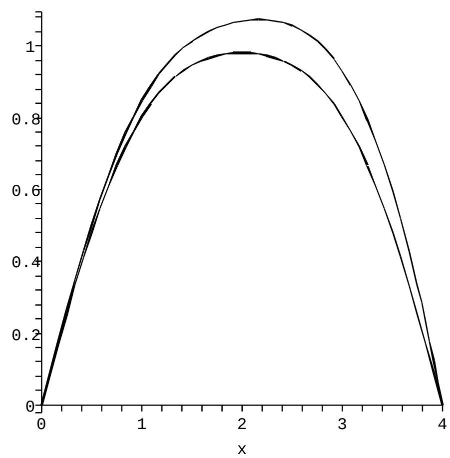

Figure 17.16: Problem 20, Section 17.2, at $t = 0.3$.

$$b_n = \frac{1}{2}\int_0^4 \sin(n\pi\xi/4)\,d\xi = \frac{2}{n\pi}(1-(-1)^n),$$

and

$$T_n(t) = \frac{128(-1)^n}{n\pi(n^4\pi^4+256)}(16\cos(t) - n^2\pi^2\sin(t) - 16e^{-n^2\pi^2 t/16}).$$

The solution is

$$u(x,t) = \frac{128(-1)^n}{n\pi(n^4\pi^4+256)}(16\cos(t) - n^2\pi^2\sin(t) - 16e^{-n^2\pi^2 t/16})\sin(n\pi x/4)$$
$$+\sum_{n=1}^{\infty}\frac{2}{n\pi}(1-(-1)^n)\sin(n\pi x/4)e^{-n^2\pi^2 t/16}.$$

Figures 17.16 through 17.20 show the solution with and without the source term, at times $t = 0.3$, 0.7, 1.8, 3.9 and 4.6, respectively.

21. Compute

$$\begin{aligned} B_n(t) &= \frac{2}{5}\int_0^5 t\cos(\xi)\sin(n\pi\xi/5)\,d\xi \\ &= \frac{2t}{n^2\pi^2-25}((-1)^{n+1}(5+n\pi)+n\pi), \end{aligned}$$

$$b_n = \frac{2}{5}\int_0^5 \xi^2(5-\xi)\sin(n\pi\xi/5)\,d\xi = \frac{500}{n^3\pi^3}((-1)^{n+1}-1),$$

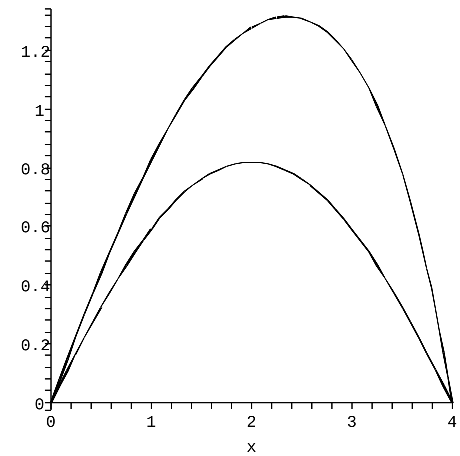

Figure 17.17: Problem 20, Section 17.2, at $t = 0.7$.

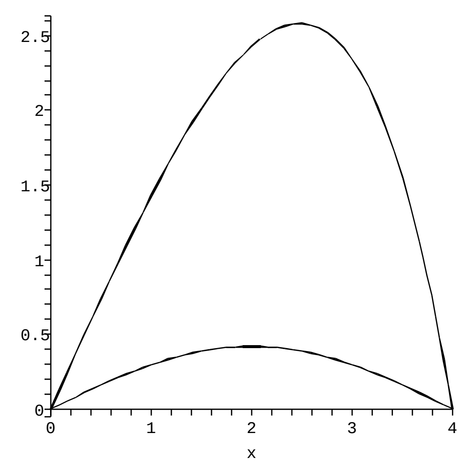

Figure 17.18: Problem 20, Section 17.2, at $t = 1.8$.

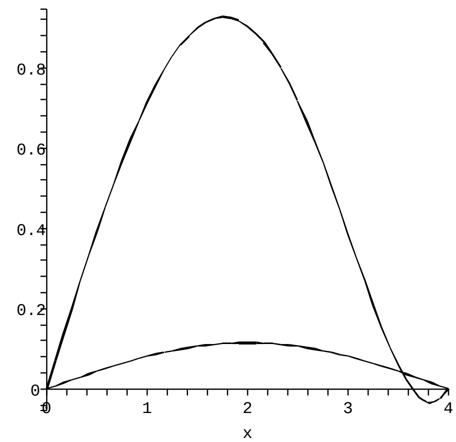

Figure 17.19: Problem 20, Section 17.2, at $t = 3.9$.

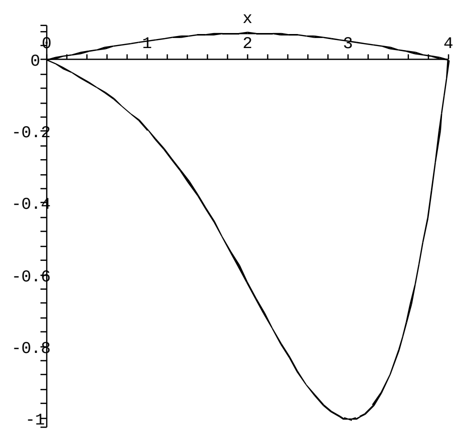

Figure 17.20: Problem 20, Section 17.2, at $t = 4.63$.

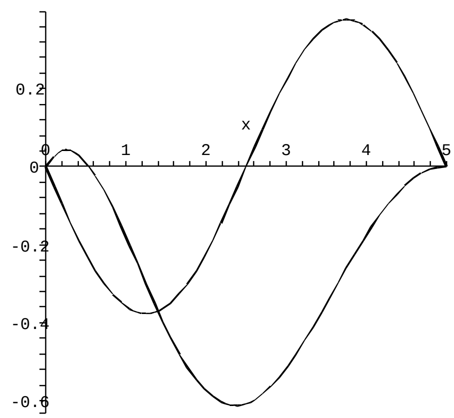

Figure 17.21: Problem 21, Section 17.2, with and without source term, at $t = 1.5$.

and

$$T_n(t) = \frac{50(1-\cos(5)(-1)^n)}{n^3\pi^3(n^2\pi^2-25)}(n^2\pi^2 t - 25 + 25e^{-n^2\pi^2 t/25}).$$

The solution is

$$u(x,t) = \sum_{n=1}^{\infty} \frac{50(1-\cos(5)(-1)^n)}{n^3\pi^3(n^2\pi^2-25)}(n^2\pi^2 t - 25 + 25e^{-n^2\pi^2 t/25})\sin(n\pi x/5)$$
$$+ \sum_{n=1}^{\infty} \frac{500}{n^3\pi^3}((-1)^{n+1}-1)\sin(n\pi x/5)e^{-n^2\pi^2 t/25}.$$

Figures 17.21, 17.22 and 17.23 show the solution, with and without source term, at times $t = 1.5$, 2.5 and 2.9, respectively.

22. Compute

$$B_n(t) = \int_0^1 K\sin(n\pi\xi/2)\,d\xi = \frac{2K}{n\pi}(1-\cos(n\pi/2)),$$

$$b_1 = 1 \text{ and } b_n = 0 \text{ for } n \neq 1,$$

$$T_n(t) = \frac{2K}{n^3\pi^3}(1-\cos(n\pi/2))(1-e^{-n^2\pi^2 t}),$$

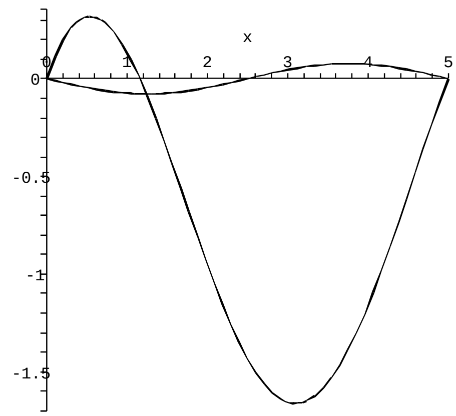

Figure 17.22: Problem 22, Section 17.2, at $t = 2.5$.

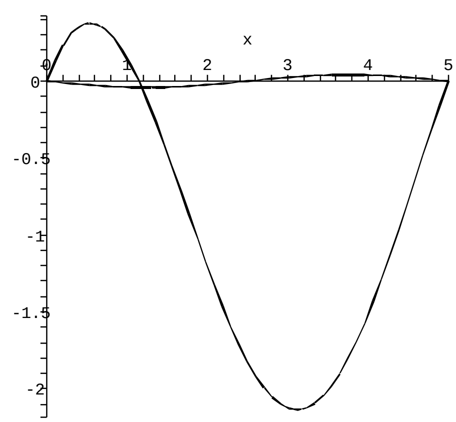

Figure 17.23: Problem 21, Section 17.2, at $t = 2.9$.

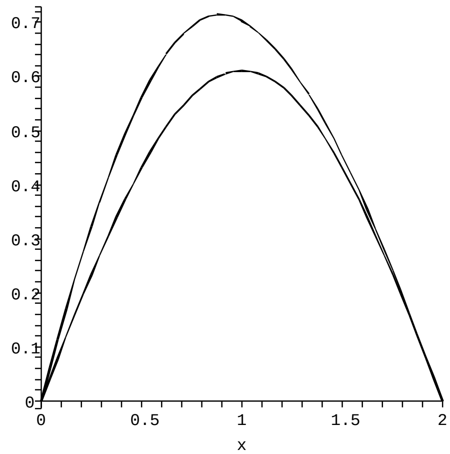

Figure 17.24: Problem 22, Section 17.2, with and without source term, at $t = 0.05$.

and the solution is

$$u(x,t) = \sum_{n=1}^{\infty} \frac{2K}{n^3\pi^3}(1-\cos(n\pi/2))(1-e^{-n^2\pi^2 t})\sin(n\pi x/2) + \sin(\pi x/2)e^{-\pi^2 t}.$$

Figures 17.24 through 17.27 show the solution, with and without source terms, at times $t = 0.05$, 0.1, 0.2 and 0.4, respectively. $K = 4$ is used in the graphs.

23. Compute

$$B_n(t) = \frac{2}{3}\int_0^3 \xi t \sin(n\pi\xi/3)\,d\xi = \frac{6t}{n\pi}(-1)^{n+1},$$

$$b_n = \frac{2}{3}\int_0^3 K\sin(n\pi\xi/3)\,d\xi = \frac{2K}{n\pi}(1-(-1)^n),$$

$$T_n(t) = \frac{27(-1)^{n+1}}{128n^5\pi^5}(16n^2\pi^2 - 9 + 9e^{-16n^2\pi^2 t/9}),$$

and the solution is

$$u(x,t) = \sum_{n=1}^{\infty} \frac{27(-1)^{n+1}}{128n^5\pi^5}(16n^2\pi^2 - 9 + 9e^{-16n^2\pi^2 t/9})\sin(n\pi x/3) + \sum_{n=1}^{\infty}\frac{2K}{n\pi}(1-(-1)^n)\sin(n\pi x/3)e^{-16n^2\pi^2 t/9}.$$

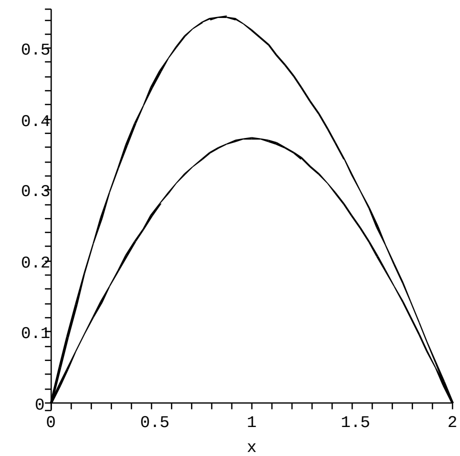

Figure 17.25: Problem 22, Section 17.2, at $t = 0.1$.

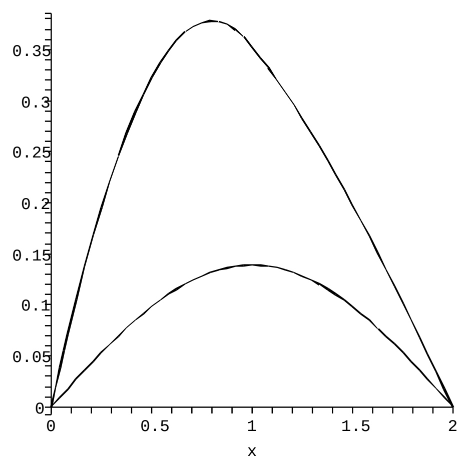

Figure 17.26: Problem 22, Section 17.2, at $t = 0.2$.

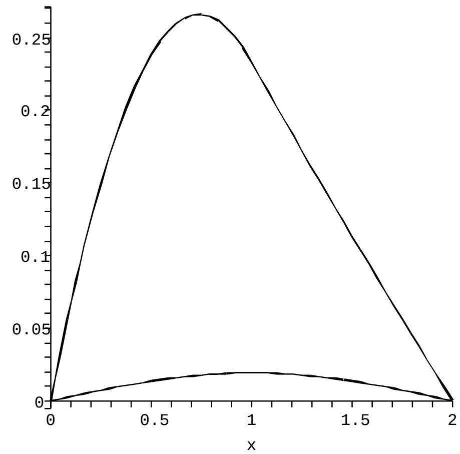

Figure 17.27: Problem 22, Section 17.2, at $t = 0.4$.

Figures 17.28, 17.29 and 17.30 show the solution, with and without source term, at times $t = 0.1$, 0.2 and 0.3, respectively. $K = 4$ is used in these graphs.

17.3 Solutions in an Infinite Medium

In each of Problems 1 through 4, separation of variables and the requirement of a bounded solution yield a solution of the form

$$u(x,t) = \int_0^\infty (a_\omega \cos(\omega x) + b_\omega \sin(\omega x))e^{-\omega^2 kt}\,d\omega,$$

where

$$a_\omega = \frac{1}{\pi}\int_{-\infty}^{\infty} f(\xi)\cos(\omega\xi)\,d\xi \text{ and } b_\omega = \frac{1}{\pi}\int_{-\infty}^{\infty} f(\xi)\sin(\omega\xi)\,d\xi.$$

To write the solution of this problem using the Fourier transform, first transform the problem to obtain

$$\frac{d\hat{u}}{dt} + k\omega^2\hat{u} = 0; \hat{u}(\omega, 0) = \hat{f}(\omega).$$

The solution of this transformed problem is

$$\hat{u}(\omega, t) = \hat{f}(\omega)e^{-k\omega^2 t}.$$

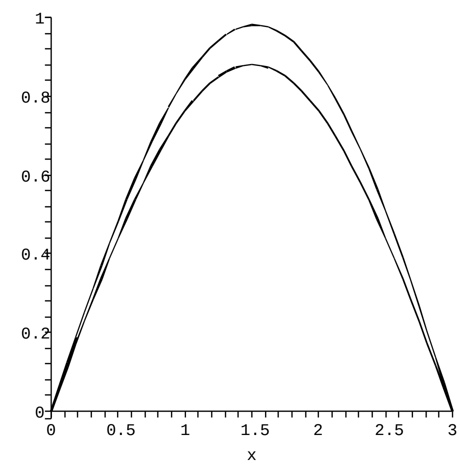

Figure 17.28: Problem 23, Section 17.2, with and without source term, at $t = 0.1$.

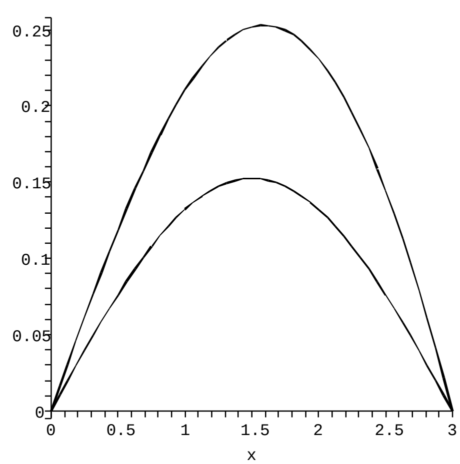

Figure 17.29: Problem 23, Section 17.2, at $t = 0.2$.

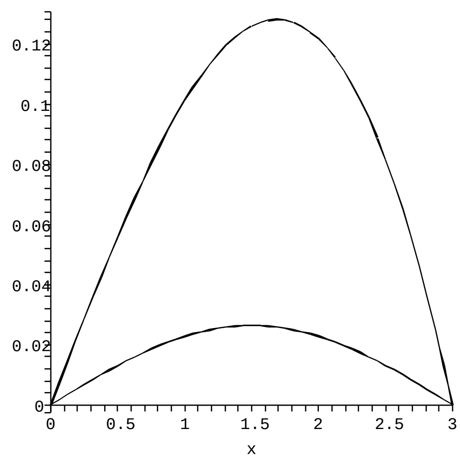

Figure 17.30: Problem 23, Section 17.2, at $t = 0.3$.

Recover the solution $u(x, t)$ of the original problem by taking the Fourier transform of this solution of the transformed problem. Use the result that

$$\mathcal{F}^{-1}\left[e^{-k\omega^2 t}\right] = \frac{1}{2\sqrt{\pi k t}} e^{-x^2/4kt},$$

together with the convolution theorem, to obtain

$$u(x, t) = \mathcal{F}^{-1}(\hat{f}(\omega)e^{-k\omega^2 t}) = \frac{1}{2\sqrt{\pi k t}} \int_{-\infty}^{\infty} f(\xi) e^{-(x-\xi)^2/4kt}\, d\xi.$$

1. Compute

$$a_\omega = \frac{1}{\pi}\int_{-\infty}^{\infty} e^{-4|\xi|}\cos(\omega\xi)\, d\xi = \frac{8}{\pi}\frac{1}{16+\omega^2} \text{ and } b_\omega = 0.$$

The solution is

$$u(x, t) = \frac{8}{\pi}\int_0^{\infty} \frac{1}{16+\omega^2}\cos(\omega x) e^{-\omega^2 kt}\, d\omega.$$

Using the Fourier transform we obtain the form of the solution

$$u(x, t) = \frac{1}{2\sqrt{\pi k t}} \int_{-\infty}^{\infty} e^{-4|\xi|} e^{-(x-\xi)^2/4kt}\, d\xi.$$

2. Compute
$$a_\omega = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin(\xi)\cos(\omega\xi)\,d\xi = 0,$$
and
$$b_\omega = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin(\xi)\sin(\omega\xi)\,d\xi = \frac{2\sin(\omega\pi)}{\pi(\omega^2-1)}.$$
The solution is
$$u(x,t) = \frac{2}{\pi}\int_0^\infty \left[\frac{\sin(\omega\pi)}{\omega^2-1}\right]\sin(\omega x)e^{-\omega^2 kt}\,d\omega.$$

For the solution by Fourier transform, first write
$$f(x) = \sin(x)(H(x+\pi) - H(x-\pi))$$
to obtain
$$\begin{aligned} u(x,t) &= \frac{1}{2\sqrt{\pi kt}}\int_{-\infty}^{\infty} \sin(\xi)(H(\xi+\pi) - H(\xi-\pi))e^{-(x-\xi)^2/4kt}\,d\xi \\ &= \frac{1}{2\sqrt{\pi kt}}\int_{-\pi}^{\pi} \sin(\xi)e^{-(x-\xi)^2/4kt}\,d\xi. \end{aligned}$$

3. Compute
$$a_\omega = \frac{1}{\pi}\int_0^4 \xi\cos(\omega\xi)\,d\xi = \frac{1}{\pi}\frac{4\omega\sin(4\omega)+\cos(4\omega)-1}{\omega^2}$$
and
$$b_\omega = \frac{1}{\pi}\int_0^4 \xi\sin(\omega\xi)\,d\xi = \frac{1}{\pi}\frac{\sin(4\omega)-4\omega\cos(\omega)}{\omega^2}.$$
The solution is
$$u(x,t) = \int_0^\infty (a_\omega\cos(\omega x) + b_\omega\sin(\omega x))e^{-\omega^2 kt}\,d\omega.$$

If we want to solve the problem using the Fourier transform, write
$$f(x) = x(H(x) - H(x-4))$$
to obtain
$$\begin{aligned} u(x,t) &= \frac{1}{2\sqrt{\pi kt}}\int_{-\infty}^{\infty} \xi(H(\xi) - H(\xi-4))e^{-(x-\xi)^2/4kt}\,d\xi \\ &= \frac{1}{2\sqrt{\pi kt}}\int_0^4 \xi e^{-(x-\xi)^2/4kt}\,d\xi. \end{aligned}$$

4. Compute
$$a_\omega = \frac{1}{\pi}\int_{-1}^{1} e^{-\xi}\cos(\omega\xi)\,d\xi = \frac{2}{\pi}\frac{\cos(\omega)\sinh(1)+\omega\sin(\omega)\cosh(1)}{\omega^2+1}$$
and
$$b_\omega = \frac{1}{\pi}\int_{-1}^{1} e^{-\xi}\sin(\omega\xi)\,d\xi = \frac{2}{\pi}\frac{\omega\cos(\omega)\sinh(1)-\sin(\omega)\cosh(1)}{\omega^2+1}.$$
The solution is
$$u(x,t) = \frac{2}{\pi}\int_0^\infty (a_\omega\cos(\omega x)+b_\omega\sin(\omega x))e^{-\omega^2 kt}\,d\omega.$$
To use the Fourier transform, write $f(x) = e^{-x}(H(x+1)-H(x-1))$ to obtain
$$\begin{aligned} u(x,t) &= \frac{1}{2\sqrt{\pi kt}}\int_{-\infty}^{\infty} e^{-\xi}(H(\xi+1)-H(\xi-1))e^{-(x-\xi)^2/4kt}\,d\xi \\ &= \frac{1}{2\sqrt{\pi kt}}\int_{-1}^{1} e^{-\xi}e^{-(x-\xi)^2/4kt}\,d\xi. \end{aligned}$$

In Problems 5 through 8 the problems are stated on the half line and are solved by separation of variables.

5. Compute
$$b_\omega = \frac{2}{\pi}\int_0^\infty e^{-\alpha\xi}\sin(\omega\xi)\,d\xi = \frac{2}{\pi}\frac{\omega}{\omega^2+\alpha^2}.$$
The solution is
$$u(x,t) = \frac{2}{\pi}\int_0^\infty \left(\frac{\omega}{\omega^2+\alpha^2}\right)\sin(\omega x)e^{-k\omega^2 t}\,d\omega.$$

6. Compute
$$b_\omega = \frac{2}{\pi}\int_0^\infty \xi e^{-\alpha\xi}\sin(\omega\xi)\,d\xi = \frac{2}{\pi}\frac{2\alpha\omega}{(\alpha^2+\omega^2)^2}.$$
The solution is
$$u(x,t) = \frac{2}{\pi}\int_0^\infty \left(\frac{2\alpha\omega}{(\alpha^2+\omega^2)^2}\right)\sin(\omega x)e^{-k\omega^2 t}\,d\omega.$$

7. The coefficients are
$$b_\omega = \frac{2}{\pi}\int_0^h \sin(\omega\xi)\,d\xi = \frac{2}{\pi}\frac{1-\cos(h\omega)}{\omega}.$$
The solution is
$$u(x,t) = \frac{2}{\pi}\int_0^\infty \left(\frac{1-\cos(h\omega)}{\omega}\right)\sin(\omega x)e^{-k\omega^2 t}\,d\omega.$$

8. Compute

$$b_\omega = \frac{2}{\pi}\int_0^2 \xi\sin(\omega\xi)\,d\xi = \frac{2}{\pi}\frac{\sin(2\omega) - 2\omega\cos(2\omega)}{\omega^2}.$$

The solution is

$$u(x,t) = \frac{2}{\pi}\int_0^\infty \left(\frac{\sin(2\omega) - 2\omega\cos(2\omega)}{\omega^2}\right)\sin(\omega x)e^{-k\omega^2 t}\,d\omega.$$

In each of Problems 9 and 10, we use a Fourier transform on the half-line to solve the problem.

9. Apply the Fourier sine transform with respect to x to the given problem to obtain:

$$\hat{U}_S' + \omega^2\hat{U}_S + t\hat{U}_S = 0,$$
$$\hat{U}_S(\omega,0) = \mathcal{F}(xe^{-x}) = \frac{2\omega}{(1+\omega^2)^2}.$$

This problem has solution

$$\hat{U}_S(\omega,t) = \frac{2\omega}{(1+\omega^2)^2}e^{-(\omega^2 t + t^2/2)}.$$

Now use the inversion formula to obtain the solution

$$u(x,t) = \frac{4}{\pi}\int_0^\infty \frac{\omega}{(1+\omega^2)^2}e^{-\omega^2 t - t^2/2}\sin(\omega x)\,d\omega.$$

10. Apply the Fourier cosine transform to the problem

$$\hat{U}_C' + (1+\omega^2)\hat{U}_C = -f(t); \hat{U}_C(\omega,0) = 0.$$

This has solution

$$\hat{U}_C(\omega,t) = -e^{-(1+\omega^2)t}\int_0^t f(\tau)e^{(1+\omega^2)\tau}\,d\tau = -f(t) * e^{-(1+\omega^2)t}.$$

Invert this to obtain

$$u(x,t) = -\frac{2}{\pi}\int_0^\infty f(t) * e^{-(1+\omega^2)t}\cos(\omega x)\,d\omega.$$

11. Let

$$F(x) = \int_0^\infty e^{-\zeta^2}\cos(x\zeta)\,d\zeta.$$

Think of this as a function of x. Differentiate under the integral sign to obtain

$$F'(x) = \int_0^\infty -\zeta e^{-\zeta^2}\sin(x\zeta)\,d\zeta.$$

Integrate by parts to obtain

$$F'(x) = \left[\frac{1}{2}e^{-\zeta^2}\sin(x\zeta)\right]_0^\infty - \frac{1}{2}\int_0^\infty \frac{1}{2}e^{-\zeta^2}x\cos(x\zeta)\,d\zeta.$$

Now observe that

$$F'(x) = -\frac{x}{2}F(x).$$

This is a separable first order differential equation. Write it as

$$\frac{F'(x)}{F(x)} = -\frac{x}{2}$$

and integrate to obtain

$$\ln|F(x)| = -\frac{1}{4}x^2 + c.$$

Then

$$F(x) = ke^{-x^2/4},$$

where $k = e^c$ is a constant to be determined. But,

$$F(0) = k = \int_0^\infty e^{-\zeta^2}\,d\zeta = \frac{1}{2}\sqrt{\pi},$$

an integral that is well known (for example, it is widely used in statistics). Therefore

$$F(x) = \frac{\sqrt{\pi}}{2}e^{-x^2/4}.$$

Upon letting $x = \alpha/\beta$, we have

$$F(\alpha/\beta) = \int_0^\infty e^{-\zeta^2}\cos\left(\frac{\alpha}{\beta}\zeta\right)\,d\zeta = \frac{\sqrt{\pi}}{2}e^{-\alpha^2/4\beta^2}.$$

Finally, since $e^{-\zeta^2}\cos(x\zeta$ is an even function in ζ, then

$$\int_{-\infty}^\infty e^{-\zeta^2}\cos\left(\frac{\alpha}{\beta}\zeta\right)\,d\zeta = \sqrt{\pi}e^{-\alpha^2/4\beta^2}.$$

17.4 Laplace Transform Techniques

1. Apply the Laplace transform (in t) to the partial differential equation, using the initial condition, to write

$$sU(x,s) - u(x,0) = kU''(x,s),$$

or, since $u(x,0) = 0$,

$$U''(x,s) - \frac{s}{k}U(x,s) = 0.$$

This has general solution

$$U(x,s) = c_1 e^{\sqrt{s/k}x} + c_2 e^{-\sqrt{s/k}x}.$$

Now $u(0,t) = 0$, so

$$U(0,s) = c_1 + c_2 = 0$$

so $c_2 = -c_1$. Then

$$U(x,s) = c_1\left(e^{\sqrt{s/k}x} - e^{-\sqrt{s/k}x}\right) = c\sinh\left(\sqrt{\frac{s}{k}}x\right).$$

Next, $u(L,t) = T_0$, so $U(L,s) = T_0/s$, so

$$c\sinh\left(\sqrt{\frac{s}{k}}L\right) = \frac{T_0}{s}$$

so

$$U(x,s) = \frac{T_0}{s}\frac{\sinh(\sqrt{s/k}x)}{\sinh(\sqrt{s/k}L)}.$$

The solution $u(x,t)$ is the inverse transform of $U(x,t)$. To compute this inverse, let $\alpha = \sqrt{s/k}$ and write

$$\begin{aligned}\frac{\sinh(\sqrt{s/k}x)}{s\sinh(\sqrt{s/k}L)} &= \frac{e^{\alpha x} - e^{-\alpha x}}{s(e^{\alpha L} - e^{-\alpha L})}\\ &= \frac{e^{\alpha(x-L)} - e^{-\alpha(x+L)}}{s(1-e^{-2\alpha L})}.\end{aligned}$$

Now essentially duplicate the calculation done in the section (with cosh in place of sinh) to obtain the solution

$$u(x,t) = T_0\sum_{n=0}^{\infty}\left(\operatorname{erfc}\left(\frac{(2n+1)L - x}{2\sqrt{kt}}\right) - \operatorname{erfc}\left(\frac{(2n+1)L + x}{2\sqrt{kt}}\right)\right).$$

2. Take the Laplace transform (in t) of the heat equation and use the initial condition to obtain

$$U'' - \frac{s}{k}U = 0,$$

with general solution

$$U(x,s) = c_1 e^{\sqrt{s/k}x} + c_2 e^{-\sqrt{s/k}x}.$$

Now $\lim_{x\to\infty} u(x,t) = 0$, so $\lim_{x\to\infty} U(x,s) = 0$, forcing $c_1 = 0$. We can therefore write

$$U(x,s) = ce^{-\sqrt{s/k}x}.$$

Next, $u(0,t) = t^2$, so

$$U(0,s) = \frac{2}{s^3} = c$$

so

$$U(x,s) = \frac{2}{s^3} e^{-\sqrt{s/k}x}.$$

Write this as

$$U(x,s) = \frac{2}{s^2}\left(\frac{1}{s}e^{-\sqrt{s/k}x}\right).$$

By writing $U(x,s)$ in this way, we are able to take the inverse transform of both factors. Now use the convolution theorem to write the solution

$$u(x,t) = 2t * \operatorname{erfc}\left(\frac{x}{2\sqrt{kt}}\right).$$

3. Take the transform, with respect to t, of the heat equation to obtain

$$sU(x,s) - e^{-x} = kU''(x,s).$$

Then

$$U'' - \frac{s}{k}U = -\frac{1}{k}e^{-x}.$$

This is a linear, second-order, nonhomogeneous differential equation. The general solution of the associated homogeneous equation is

$$U_h(x,s) = c_1 e^{\sqrt{s/k}x} + c_2 e^{-\sqrt{s/k}x}.$$

Use undetermined coefficients to find a particular solution of the nonhomogeneous differential equation. Substitute $U_p(x,s) = Ae^{-x}$ into the differential equation to obtain

$$A - \frac{s}{k}A = -\frac{1}{k},$$

so

$$A = \frac{1}{s-k}$$

and we obtain

$$U(x,s) = U_h(x,s) + U_p(x,s) = c_1 e^{\sqrt{s/k}x} + c_2 e^{-\sqrt{s/k}x} + \frac{1}{s-k}e^{-x}.$$

Now $\lim_{x\to\infty} u(x,t) = 0$, so $c_1 = 0$. Then

$$U(x,s) = U_h(x,s) + U_p(x,s) = ce^{-\sqrt{s/k}x} + \frac{1}{s-k}e^{-x},$$

in which we wrote c for c_2. Since $u(0,t) = 0$, then

$$U(0,s) = c + \frac{1}{s-k}.$$

Then $c = -1/(s-k)$, so

$$U(x,s) = -\frac{1}{s-k}e^{-\sqrt{s/k}x} + \frac{1}{s-k}e^{-x}.$$

Since

$$\mathcal{L}^{-1}\left[\frac{1}{s-k}\right](t) = e^{kt}$$

then we have, using the convolution theorem,

$$u(x,t) = -e^{kt} * \mathcal{L}^{-1}\left[e^{-\sqrt{s/k}x}\right](t) + e^{kt}e^{-x}.$$

By consulting a table, we find that

$$\mathcal{L}^{-1}\left[e^{-(x/\sqrt{k})s}\right](t) = \frac{x}{2\sqrt{\pi k t^3}}e^{-x^2/4kt}.$$

Therefore, somewhat more explicitly,

$$u(x,t) = -e^{kt} * \frac{x}{2\sqrt{\pi k t^3}}e^{-x^2/4kt} + e^{kt-x}.$$

4. Take the transform of the heat equation to obtain

$$U''(x,s) - \frac{s}{k}U(x,s) = -\frac{1}{k}.$$

This has general solution

$$U(x,s) = c_1 e^{\sqrt{s/k}x} + c_2 e^{-\sqrt{s/k}x} + \frac{1}{s}.$$

Now use the boundary conditions. First,

$$U(0,s) = c_1 + c_2 + \frac{1}{s} = 0.$$

Next,

$$U(L,s) = c_1 e^{\sqrt{s/k}L} + c_2 e^{-\sqrt{s/k}L} + \frac{1}{s} = 0.$$

Solve these to obtain

$$c_1 = -\frac{1}{s}\frac{(1 - e^{-\sqrt{s/k}L})}{e^{\sqrt{s/k}L} - e^{-\sqrt{s/k}L}},$$

$$c_2 = -\frac{1}{s}\frac{(e^{\sqrt{s/k}L} - 1)}{e^{\sqrt{s/k}L} - e^{-\sqrt{s/k}L}}.$$

By carrying out manipulations like those done in the text, and using the geometric series, we can write

$$U(x,s) = 2\sum_{n=1}^{\infty}(-1)^n \frac{1}{s}e^{\sqrt{s/k}(x-nL)} + \frac{1}{s}.$$

Invert this series term by term to write the solution

$$u(x,t) = 2\sum_{n=1}^{\infty}(-1)^n \operatorname{erfc}\left(\frac{nL-x}{2\sqrt{kt}}\right).$$

17.5 Heat Conduction in an Infinite Cylinder

In these problems the solution has the form

$$u(r,t) = \sum_{n=1}^{\infty} a_n J_0(j_n r/R) e^{j_n^2 kt/R^2},$$

where

$$a_n = \frac{2}{J_1(j_n)^2}\int_0^1 \xi f(R\xi) J_0(j_n\xi)\, d\xi,$$

with j_n the nth positive zero of $J_0(x)$.

1. With $R = 1$ and $f(r) = r$, the first five coefficients are approximately

   $$a_1 = 0.8175, a_2 = -1.1335, a_3 = 0.7983, a_4 = -0.7470, a_5 = 0.6315.$$

   The first five terms of the series solution are approximately

   $$\begin{aligned} u(r,t) \approx\ & 0.8175 J_0(2.40483r)e^{-5.7832t} - 1.1335 J_0(5.5201r)e^{-30.5588t} \\ & + 0.7983 J_0(8.6537r)e^{-74.8791t} - 0.74701 J_0(11.7914r)e^{-139.0402t} \\ & + 0.6316 J_0(14.9309r)e^{-222.9324t}. \end{aligned}$$

   Figure 17.31 shows the sum of these five terms for times $t = 0.001$, 0.025, 0.1, 0.3 and 0.5.

2. The coefficients are

   $$a_n = \frac{2}{J_1(j_n)^2}\int_0^1 \xi e^{3\xi} J_0(j_n\xi)\, d\xi.$$

   The first five coefficients are approximately

   $$a_1 = 9.1181, a_2 = -15.3926, a_3 = 14.6004, a_4 = -13.5432, a_5 = 12.3173.$$

   The first five terms of the solution are approximately

   $$\begin{aligned} u(r,t) \approx\ & 9.1181 J_0(0.8016r)e^{-10.2812t} - 15.3926 J_0(1.8400r)e^{-54.1711t} \\ & + 14.6004 J_0(2.8846r)e^{-133.1325t} - 13.5432 J_0(3.3905r)e^{-247.1827t} \\ & + 12.3173 J_0(4.9770r)e^{-396.3241t}. \end{aligned}$$

   Figure 17.32 shows the sum of these five terms for times $t = 0.0025$, 0.001, 0.005, 0.01 and 0.2.

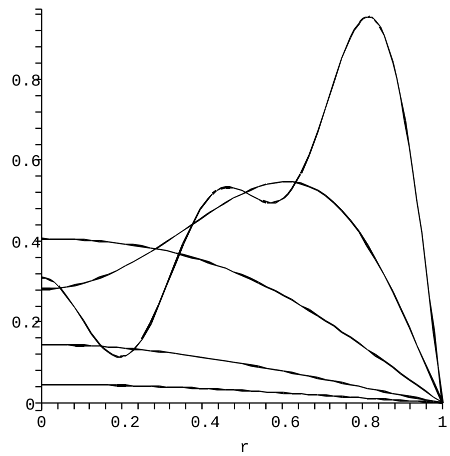

Figure 17.31: Problem 1, Section 17.4.

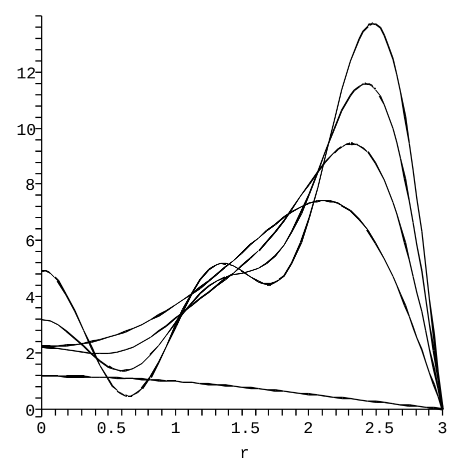

Figure 17.32: Problem 2, Section 17.4.

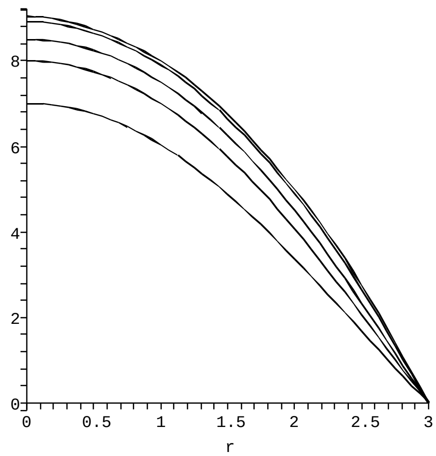

Figure 17.33: Problem 3, Section 17.4.

3. With $R = 3$ and $f(r) = 1 - r^2$, the first five coefficients are approximately

$$a_1 = 9.9722, a_2 = -1.2580, a_3 = 0.4093, a_4 = -0.1889, a_5 = 0.1047.$$

The first five terms of the solution are approximately

$$\begin{aligned} u(r,t) &\approx 9.9722J_0(0.8016r)e^{-0.3213t} - 1.2580J_0(1.8400r)e^{-1.6929t} \\ &+ 0.4093J_0(2.8846r)e^{-4.1604t} - 0.1889J_0(3.9305r)e^{-7.7245t} \\ &+ 0.1047J_0(4.9770r)e^{-12.3851t}. \end{aligned}$$

Figure 17.33 shows the sum of these five terms for times $t = 0.001$, 0.05, 0.25, 0.5 and 1.

17.6 Heat Conduction in a Rectangular Plate

1. Attempt a solution of the form $u(x, y, t) = X(x)Y(y)T(t)$. Substitution of this into the heat equation and separation of variables, coupled with the boundary conditions, yields the separated equations:

$$X'' + \lambda X = 0; X(0) = X(L) = 0,$$
$$Y'' + \mu Y = 0; Y(0) = Y(K) = 0,$$
$$T' + k(\lambda + \mu)T = 0.$$

The eigenvalues and eigenfunctions for the problems in X and Y are

$$\lambda_n = \frac{n^2\pi^2}{L^2}, X_n(x) = \sin(n\pi x/L),$$

and

$$\mu_m = \frac{m^2\pi^2}{K^2}, Y_m(y) = \sin(m\pi y/K).$$

The solution has the form of a double superposition

$$u(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} c_{nm}\sin(n\pi x/L)\sin(m\pi y/K)e^{-k\alpha_{nm}t},$$

in which

$$\alpha_{nm} = \frac{n^2\pi^2}{L^2} + \frac{m^2\pi^2}{K^2}.$$

The coefficients must be chosen so that

$$u(x,y,0) = f(x,y) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} c_{nm}\sin(n\pi x/L)\sin(m\pi y/K).$$

Thus choose

$$c_{nm} = \frac{4}{LK}\int_0^L\int_0^K f(\xi,\eta)\sin(n\pi\xi/L)\sin(m\pi\eta/K)\,d\xi\,d\eta.$$

2. With the given constants and initial position function, the coefficients are

$$\begin{aligned} c_{nm} &= \frac{2}{3}\int_0^2 \xi^2(2-\xi)\sin(n\pi\xi/2)\,d\xi\int_0^3 \sin(\eta)(3-\eta)\sin(m\pi\eta/3)\,d\eta \\ &= \frac{2}{3}\left(\frac{-32}{n^3\pi^3}(1+2(-1)^n)\right)\left(\frac{54m\pi}{(m^2\pi^2-9)^2}(1-(-1)^m m\pi\cos(3))\right). \end{aligned}$$

The solution is

$$u(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} c_{nm}\sin(n\pi x/2)\sin(m\pi y/3)e^{-4\alpha_{nm}\pi^2 t},$$

in which

$$\alpha_{nm} = \frac{n^2}{4} + \frac{m^2}{9}.$$

3. The coefficients in the series solution are

$$c_{nm} = \frac{4}{\pi^2}\int_0^\pi \sin(\xi)\sin(n\xi)\,d\xi\int_0^\pi \cos(\eta/2)\sin(m\eta)\,d\eta.$$

Now

$$\int_0^\pi \sin(\eta)\sin(n\eta)\,d\eta = \begin{cases} 0 & \text{for } n \neq 1, \\ \pi/2 & \text{for } n = 1, \end{cases}$$

then the only nonzero coefficients are

$$c_{1m} = \frac{2}{\pi}\frac{4m}{4m^2-1}.$$

The solution is

$$u(x,y,t) = \frac{8}{\pi}\sin(x)\sum_{m=1}^{\infty}\left(\frac{m}{4m^2-1}\right)\sin(my)e^{-(1+m^2)t}.$$

Chapter 18

The Potential Equation

18.1 Laplace's Equation

1. If f and g are harmonic on D, then $f_{xx} + f_{yy} = 0$ and $g_{xx} + g_{yy} = 0$ on D. For any numbers α and β,

$$\begin{aligned}&(\alpha f + \beta g)_{xx} + (\alpha f + \beta g)_{yy}\\&= \alpha(f_{xx} + f_{yy}) + \beta(g_{xx} + g_{yy}) = 0,\end{aligned}$$

so $\alpha f + \beta g$ is harmonic on D.

2. (a)

$$(x^3 - 3xy^2)_{xx} + (x^3 - 3xy^2)_{yy} = 6x - 6x = 0$$

(b)

$$(3x^2y - y^3)_{xx} + (3x^2y - y^3)_{yy} = 6y - 6y = 0$$

(c)

$$\begin{aligned}&(x^4 - 6x^2y^2 - y^4)_{xx} + (x^4 - 6x^2y^2 - y^4)_{yy}\\&= (12x^2 - 12y^2) + (-12x^2 + 12y^2) = 0.\end{aligned}$$

(d)

$$(4x^3y - 4xy^3)_{xx} + (4x^3y - 4xy^3)_{yy} = 24xy - 24xy = 0$$

(e)

$$\begin{aligned}&(\sin(x)(e^y + e^{-y})_{xx} + (\sin(x)(e^y + e^{-y})_{yy}\\&= -\sin(x)(e^y + e^{-y}) + \sin(x)(e^y + e^{-y}) = 0\end{aligned}$$

(f)

$$\begin{aligned}&\cos(x)(e^y - e^{-y})_{xx} + \cos(x)(e^y - e^{-y})_{yy}\\&= -\cos(x)(e^y - e^{-y}) + \cos(x)(e^y - e^{-y}) = 0\end{aligned}$$

(g)

$$(e^{-x}\cos(y))_{xx} + (e^{-x}\cos(y))_{yy} = e^{-x}\cos(y) - e^{-x}\cos(y) = 0$$

(h) Let $f(x,y) = \ln(x^2+y^2)$ for $(x,y) \neq (0,0)$. Then

$$f_x = \frac{2x}{x^2+y^2}, f_{xx} = \frac{2y^2-2x^2}{(x^2+y^2)^2},$$

and

$$f_y = \frac{2y}{x^2+y^2}, f_{yy} = \frac{2x^2-2y^2}{(x^2+y^2)^2}$$

so $f_{xx} + f_{yy} = 0$ on the plane with the origin removed.

18.2 Dirichlet Problem for a Rectangle

1. Substitute $u(x,y) = X(x)Y(y)$ into Laplace's equation and use the boundary conditions to obtain

$$X'' + \lambda X = 0; X(0) = X(1) = 0$$

and

$$Y'' - \lambda Y = 0; Y(\pi) = 0.$$

The regular Sturm-Liouville problem for X has been solved in connection with the heat and wave equations. The eigenvalues and eigenfunctions are

$$\lambda_n = n^2\pi^2, X_n(x) = \sin(n\pi x).$$

The problem for Y has solutions that are constant multiples of hyperbolic sines of the form $\sinh(n\pi(\pi - y))$. For each n, we have functions

$$u_n(x,y) = a_n \sin(n\pi x)\sinh(n\pi(\pi - y))$$

that are harmonic and satisfy the homogeneous boundary conditions on the edges $x = 0$, $x = 1$ and $y = \pi$. To satisfy a boundary condition $u(x,0) = f(x)$, we would normally have to use a superposition

$$u(x,y) = \sum_{n=1}^{\infty} a_n \sin(n\pi x)\sinh(n\pi(\pi - y)).$$

However, in this simple problem in which $u(x,0) = \sin(\pi x)$, we observe that we can get by with $n = 1$ and choose a_1 so that

$$u(x,0) = a_1 \sin(\pi x)\sinh(\pi^2) = \sin(\pi x).$$

Thus choose $a_n = 0$ for $n = 2, 3, \cdots$, and

$$a_1 = \frac{1}{\sinh(\pi^2)}$$

to obtain the solution

$$u(x,y) = \frac{1}{\sinh(\pi^2)}\sin(\pi x)\sinh(\pi(\pi - y)).$$

2. The homogeneous boundary conditions on the edges $y = 0$ and $y = 2$, with separation of variables, yield the problem

$$Y'' + \lambda Y = 0; Y(0) = Y(2) = 0.$$

The gives us eigenvalues and eigenfunctions

$$\lambda_n = \frac{n^2\pi^2}{4}, Y_n(y) = \sin(n\pi y/2).$$

The problem for X is

$$X'' - \frac{n^2\pi^2}{4}X = 0; X(3) = 0,$$

with solutions that are of the form $X_n(x) = b_n \sinh(n\pi(3-x)/2)$. Attempt a solution of the form

$$u(x,y) = \sum_{n=1}^{\infty} b_n \sinh(n\pi(3-x)/2)\sin(n\pi y/2).$$

We have $u(3,y) = 0$. We need

$$u(0,y) = y(2-y) = \sum_{n=1}^{\infty} b_n \sinh(3n\pi/2)\sin(n\pi y/2).$$

This is a Fourier sine expansion of $y(2-y)$ on $[0,2]$, so choose

$$\begin{aligned} b_n &= \frac{1}{\sinh(3n\pi/2)}\int_0^2 \eta(2-\eta)\sin(n\pi\eta/2)\,d\eta \\ &= \frac{16}{\sinh(3n\pi/2)}\frac{1-(-1)^n}{n^3\pi^3}. \end{aligned}$$

The solution is

$$u(x,y) = \frac{16}{\pi^3}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n^3}\frac{\sinh(n\pi(3-x)/2)}{\sinh(3n\pi/2)}\sin(n\pi y/2).$$

3. Separate the variables and use the homogeneous boundary conditions to derive the general form of the solution:

$$u(x,y) = \sum_{n=1}^{\infty} a_n \frac{\sinh(n\pi y)}{\sinh(4n\pi)}\sin(n\pi x).$$

The coefficients must be chosen so that

$$u(x,4) = \sum_{n=1}^{\infty} a_n \sin(n\pi x) = x\cos(\pi x/2).$$

This is a Fourier sine expansion of $x\cos(\pi x/2)$ on $[0,1]$, so choose

$$a_n = 2\int_0^1 \xi\cos(\pi\xi/2)\sin(n\pi\xi)\,d\xi = \frac{32n(-1)^{n+1}}{\pi^2(4n^2-1)^2}.$$

4. Because two sides have nonhomogeneous boundary conditions, separate this problem into two problems, on each of which the boundary conditions are homogeneous on three sides of the square. Write

$$u(x,y) = v(x,y) + w(x,y)$$

where

$$\nabla^2 v = 0; v(x,0) = v(x,\pi) = v(\pi,y) = 0, v(0,y) = \sin(y)$$

and

$$\nabla^2 w = 0; w(0,y) = w(\pi,y) = 0 = w(x,\pi) = 0, w(x,0) = x(\pi - x).$$

The problem for v has a solution of the form

$$v(x,y) = \sum_{n=1}^{\infty} a_n \sin(ny)\frac{\sinh(n(\pi - x))}{\sinh(n\pi)}.$$

We need

$$v(0,y) = \sin(y) = \sum_{n=1}^{\infty} a_n \sin(ny).$$

Thus $a_1 = 0$ and $a_n = 0$ for $n = 2, 3, \cdots$. The solution for v is

$$v(x,y) = \sin(y)\frac{\sinh(\pi - x)}{\sinh(\pi)}.$$

The solution for w has the form

$$w(x,y) = \sum_{n=1}^{\infty} b_n \sin(nx)\frac{\sinh(n(\pi - y))}{\sinh(n\pi)}.$$

We need

$$w(x,0) = x(\pi - x) = \sum_{n=1}^{\infty} b_n \sin(nx).$$

Thus choose

$$b_n = \frac{2}{\pi}\int_0^\pi \xi(\pi-\xi)\sin(n\xi)\,d\xi$$
$$= \frac{4}{n^3\pi}(1-(-1)^n).$$

Then

$$w(x,y) = \frac{4}{\pi}\sum_{n=1}^\infty \frac{(1-(-1)^n)}{n^3}\sin(x)\frac{\sinh(n(\pi-y))}{\sinh(n\pi)}.$$

5. There are nonhomogeneous boundary conditions on two edges, so write $u(x,y) = v(x,y) + w(x,y)$, where

$$\nabla^2 v = 0; v(0,y) = v(\pi,y) = v(x,0) = 0, v(x,\pi) = x\sin(\pi x),$$

and

$$\nabla^2 w = 0; w(x,0) = w(x,\pi) = w(0,y) = 0, w(2,y) = \sin(y).$$

These are defined on $0 < x < 2$, $0 < y < \pi$. The solution for w has the form

$$w(x,y) = \sum_{n=1}^\infty b_n \sin(ny)\frac{\sinh(nx)}{\sinh(2n)}.$$

We need

$$w(2,y) = \sum_{n=1}^\infty b_n\sin(ny) = \sin(y)$$

so choose $b_1 = 1$ and all other $b_n = 0$. Then

$$w(x,y) = \sin(y)\frac{\sinh(x)}{\sinh(2)}.$$

We find that v has the form

$$v(x,y) = \sum_{n=1}^\infty a_n \sin(n\pi x/2)\frac{\sinh(n\pi y/2)}{\sinh(n\pi^2/2)}.$$

We need

$$v(x,\pi) = x\sin(\pi x) = \sum_{n=1}^\infty a_n\sin(n\pi x/2).$$

This is the sine expansion of $x\sin(\pi x)$ on $[0,2]$, so choose

$$a_n = \int_0^2 \xi\sin(\pi\xi)\sin(n\pi\xi/2)\,d\xi$$
$$= \begin{cases} \frac{16n}{\pi^2((n^2-4)^2)}((-1)^n - 1) & \text{for } n = 1,3,4,\cdots, \\ 1 & \text{for } n = 2. \end{cases}$$

Then

$$v(x,y) = \sin(\pi x)\frac{\sinh(\pi y)}{\sinh(\pi^2)} + \frac{16}{\pi^2}\sum_{n=1,n\neq 2}^{\infty}\frac{n}{(n^2-4)^2}((-1)^n-1)\sin(n\pi x/2)\frac{\sinh(n\pi y/2)}{\sinh(n\pi^2/2)}.$$

6. Separation of variables and the homogeneous boundary conditions on the sides $y = 0$, $y = b$ and $x = 0$ yield a solution of the form

$$u(x,y) = \sum_{n=1}^{\infty} a_n \sin((2n-1)\pi y/2b)\frac{\sinh((2n-1)\pi x/2b)}{\sinh((2n-1)\pi a/2b)}.$$

We need

$$u(a,y) = g(y) = \sum_{n=1}^{\infty} a_n \sin((2n-1)\pi y/2b).$$

Then

$$a_n = \frac{2}{b}\int_0^b g(\eta)\sin((2n-1)\pi\eta/2b)\,d\eta.$$

7. Separation of variables and the homogeneous boundary conditions on the sides $x = 0$, $x = a$ and $y = 0$ give us a solution of the form

$$u(x,y) = \sum_{n=1}^{\infty} c_n \sin((2n-1)\pi x/2a)\frac{\sinh((2n-1)\pi y/2a)}{\sinh((2n-1)\pi b/2a)}.$$

We need

$$u(x,b) = f(x) = \sum_{n1}^{\infty} a_n \sin((2n-1)\pi x/2a),$$

so choose

$$c_n = \frac{2}{a}\int_0^a f(\xi)\sin((2n-1)\pi\xi/2a)\,d\xi.$$

8. There are homogeneous boundary conditions on the sides $x = 0$, $x = a$ and $y = 0$, so the solution has the form

$$u(x,y) = \sum_{n=1}^{\infty}\sin(n\pi x/a)\frac{\sinh(n\pi y/a)}{\sinh(n\pi b/a)}.$$

Then

$$u(x,b) = x(x-a)^2 = \sum_{n=1}^{\infty} a_n \sin(n\pi x/a).$$

Then

$$a_n = \frac{2}{a}\int_0^a \xi(\xi-a)^2 \sin(n\pi\xi/a)\,d\xi$$
$$= \frac{4}{n^3\pi^3}(1+2n\pi(-1)^n)$$

for $n = 1, 2, \cdots$. The solution is

$$u(x,y) = \frac{4}{\pi^3}\sum_{n=1}^{\infty}\frac{1+2n\pi(-1)^n}{n^3}\sin(n\pi x/a)\frac{\sinh(n\pi y/a)}{\sinh(n\pi b/a)}.$$

9. Decompose the problem into two problems, in each of which the boundary data is homogeneous on three sides. Let $u(x,y) = v(x,y) + w(x,y)$, where

$$\nabla^2 v = 0; v(x,0) = v(x,1) = v(4,y) = 0, v(0,y) = \sin(\pi y)$$

and

$$\nabla^2 w = 0; w(x,0) = w(x,1) = w(0,y) = 0, w(4,y) = y(1-y).$$

These problems are defined on $0 < x < 4$, $0 < y < 1$. The solution for v has the form

$$v(x,y) = \sum_{n=1}^{\infty}\sin(n\pi y)\frac{\sinh(n\pi(4-x))}{\sinh(4n\pi)}.$$

Then

$$v(0,y) = \sin(\pi y) = \sum_{n=1}^{\infty} a_n \sin(n\pi y),$$

so $a_1 = 1$ and, for $n = 2, 3, \cdots$, $a_n = 0$. Then

$$v(x,y) = \sin(\pi y)\frac{\sinh(\pi(4-x))}{\sinh(4\pi)}.$$

The solution for w has the form

$$w(x,y) = \sum_{n=1}^{\infty} b_n \sin(n\pi y)\sinh(n\pi x).$$

Then

$$w(4,y) = y(1-y) = \sum_{n=1}^{\infty} b_n \sinh(4n\pi)\sin(n\pi y),$$

so

$$b_n = \frac{2}{\sinh(4n\pi)}\int_0^1 \xi(1-\xi)\sin(n\pi\xi)\,d\xi$$
$$= \frac{4(1-(-1)^n)}{n^3\pi^3\sinh(4n\pi)}.$$

18.3 Dirichlet Problem for a Disk

In each of Problems 1 through 8, the solution has the form

$$u(r,\theta) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(\frac{r}{R}\right)^n (a_n\cos(n\theta) + b_n\sin(n\theta)),$$

where

$$a_n = \frac{1}{\pi R^n}\int_{-\pi}^{\pi} f(\xi)\cos(n\xi)\,d\xi$$

for $n = 0, 1, 2, \cdots$ and

$$b_n = \frac{1}{\pi R^n}\int_{-\pi}^{\pi} f(\xi)\sin(n\xi)\,d\xi$$

for $n = 1, 2, \cdots$.

1. With $f(\theta) = 1$ we can easily match coefficients to get $a_0 = 2$ and, for $n = 1, 2, \cdots$, $a_n = b_n = 0$. The solution is $u(r,\theta) = 1$.

2. Again, we can match coefficients, getting all coefficients to be zero except $a_4 = 8/3^4$. The solution is

$$u(r,\theta) = 8\left(\frac{r}{3}\right)^4\cos(4\theta).$$

3. Calculate

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi}(\xi^2 - \xi)\,d\xi = \frac{2\pi^2}{3},$$
$$a_n = \frac{1}{2^n\pi}\int_{-\pi}^{\pi}(\xi^2 - \xi)\cos(n\xi)\,d\xi = \frac{4(-1)^n}{n^2 2^n},$$
$$n_n = \frac{1}{2^n\pi}\int_{-\pi}^{\pi}(\xi^2 - \xi)\sin(n\xi)\,d\xi = \frac{2(-1)^n}{n2^n}.$$

The solution is

$$u(r,\theta) = \frac{\pi^2}{3} + 2\sum_{n=1}^{\infty}\left(\frac{r}{2}\right)^n\frac{(-1)^n}{n^2}(2\cos(n\theta) + n\sin(n\theta)).$$

4. Compute

$$a_n = \frac{1}{5^n\pi}\int_{-\pi}^{\pi}\xi\cos(\xi)\sin(n\xi)\,d\xi = 0 \text{ for } n = 0, 1, 2, \cdots,$$
$$b_n = \frac{1}{5^n\pi}\int_{-\pi}^{\pi}\xi\cos(\xi)\sin(n\xi)\,d\xi = \frac{2(-1)^n n}{(n^2-1)5^n} \text{ for } n = 2, 3, \cdots,$$
$$b_1 = \frac{1}{5\pi}\int_{-\infty}^{\infty}\xi\cos(\xi)\sin(\xi)\,d\xi = -\frac{1}{10}.$$

The solution is

$$u(r,\theta) = \frac{-r}{10}\sin(\theta) + 2\sum_{n=2}^{\infty}\frac{(-1)^n n}{n^2-1}\left(\frac{r}{5}\right)^n \sin(n\theta).$$

5. Compute

$$\begin{aligned}
a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} e^{-\xi}\,d\xi = \frac{2\sinh(\pi)}{\pi},\\
a_n &= \frac{1}{4^n\pi}\int_{-\pi}^{\pi} e^{-\xi}\cos(n\xi)\,d\xi = \frac{2\sinh(\pi)}{\pi}\frac{(-1)^n}{4^n(n^2+1)} \text{ for } n \geq 1\\
b_n &= \frac{1}{4^n\pi}\int_{-\pi}^{\pi} e^{-\xi}\sin(n\xi)\,d\xi = \frac{2\sinh(\pi)}{\pi}\frac{n(-1)^n}{4^n(n^2+1)} \text{ for } n \geq 1.
\end{aligned}$$

The solution is

$$u(r,\theta) = \frac{\sinh(\pi)}{\pi} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2+1}\left(\frac{r}{4}\right)^n \sinh(\pi)(\cos(n\theta) + n\sin(n\theta)).$$

6. Write $\sin^2(\theta) = (1-\cos(2\theta))/2$ and we can identify the only nonzero coefficients as $a_0 = -1/2$ and $a_2 = -1$. The solution is

$$u(r,\theta) = \frac{1}{2} - \frac{r}{2}\cos(2\theta).$$

7. Compute

$$\begin{aligned}
a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi}(1-\xi^2)\,d\xi = 2 - \frac{2}{3}\pi^2,\\
a_n &= \frac{1}{8^n\pi}\int_{-\pi}^{\pi}(1-\xi^2)\cos(n\xi)\,d\xi = \frac{4(-1)^{n+1}}{8^n n^2}, n \geq 1,\\
b_n &= \frac{1}{8^n\pi}\int_{-\pi}^{\pi}(1-\xi^2)\sin(n\xi)\,d\xi = 0, n \geq 1.
\end{aligned}$$

The solution is

$$u(r,\theta) = 1 - \frac{1}{3}\pi^2 + \sum_{n=1}^{\infty}\frac{4(-1)^{n+1}}{n^2}\left(\frac{r}{8}\right)^n \cos(n\theta).$$

8. Compute the coefficients

$$\begin{aligned}
a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} \xi e^{2\xi}\,d\xi = \frac{2\cosh(2\pi)-\sinh(2\pi)}{2\pi},\\
a_n &= \frac{1}{\pi 4^n}\int_{-\pi}^{\pi} \xi e^{2\xi}\cos(n\xi)\,d\xi\\
&= \frac{2(-1)^n}{\pi(n^2+4)^2 4^n}(\cosh(2\pi)(8\pi+2\pi n^2)+\sinh(2\pi)(n^2+4)), n \geq 1\\
b_n &= \frac{1}{\pi 4^n}\int_{-\pi}^{\pi} \xi e^{2\xi}\sin(n\xi)\,d\xi\\
&= \frac{2(-1)^n}{\pi(n^2+4)^2 4^n}(\cosh(2\pi)(4n\pi+n^3\pi^3)-4n\sinh(2\pi)), n \geq 1.
\end{aligned}$$

With these coefficients the solution is

$$\begin{aligned}
u(r,\theta) &= \frac{2\cosh(2\pi)-\sinh(2\pi)}{4\pi}\\
&+ \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{(n^2+4)^2}\left(\frac{r}{4}\right)^n(\cosh(2\pi)(8\pi+2\pi n^2)+\sinh(2\pi)(n^2+4))\cos(n\theta)\\
&+ \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{(n^2+4)^2}\left(\frac{r}{4}\right)^n(\cosh(2\pi)(4n\pi+n^3\pi^3)-4n\sinh(2\pi))\sin(n\theta).
\end{aligned}$$

9. Letting $U(r,\theta) = u(r\cos(\theta, r\sin(\theta))$, this Dirichlet problem in polar coordinates is

$$\nabla^2 U(r,\theta) = 0, U(4,\theta) = 16\cos^2(\theta),$$

for $-\pi \leq \theta \leq \pi$ and $0 \leq r < 3$. Write $16\cos^2(\theta) = 8(1+\cos(2\theta))$ to recognize that

$$\frac{1}{2}a_0 = 8, a_2(4^2) = 8,$$

and all other $a_n = 0$. The solution is

$$U(r,\theta) = 8 + 8\left(\frac{r}{4}\right)^2\cos(2\theta).$$

Convert this solution back to rectangular coordinates using $x = r\cos(\theta)$ and $y = r\sin(\theta)$ and the identity $\cos(2\theta) = 2\cos^2(\theta) - 1$ to obtain

$$u(x,y) = 8 + \frac{1}{2}(x^2 - y^2).$$

10. The Dirichlet problem in polar coordinates is

$$\nabla^2 U(r,\theta) = 0, U(3,\theta) = 3(\cos(\theta) - \sin(\theta)),$$

for $0 \le r < 3$, $-\pi \le \theta \le \pi$. Identify $3a_1 = 3$ and $3b_1 = 3$, with all other coefficients zero. Then

$$U(r,\theta) = r(\cos(\theta) - \sin(\theta)).$$

In rectangular coordinates, the original problem has the solution

$$u(x,y) = x - y.$$

11. In polar coordinates the problem is

$$\nabla^2 U(r,\theta) = 0, U(2,\theta) = 4(\cos^2(\theta) - \sin^2(\theta)) = 4\cos(2\theta),$$

for $0 \le r < 2$ and $-\pi \le \theta \le \pi$. Identify $4 = a_2(2^2)$, with all other coefficients zero, to obtain

$$u(r,\theta) = r^2\cos(2\theta).$$

In rectangular coordinates, the solution is

$$u(x,y) = x^2 - y^2.$$

12. In polar coordinates we have the problem

$$\nabla^2 U(r,\theta) = 0, U(5,\theta) = 25\sin(\theta)\cos(\theta)$$

for $0 \le r < 5$ and $-\pi \le \theta \le \pi$. Write this as

$$U(5,\theta) = \frac{25}{2}\sin(2\theta)$$

to identify $a_2(5^2) = 25/2$, with all other coefficients zero. The solution is

$$U(r,\theta) = \frac{1}{2}r^2\sin(2\theta).$$

To convert this solution to rectangular coordinates, use

$$\frac{1}{2}r^2\sin(2\theta) = r\sin(\theta)r\cos(\theta)$$

to obtain

$$u(x,y) = xy.$$

18.4 Poisson's Integral Formula

1. From Poisson's integral formula with $R = 1$ and $f(\theta) = \theta$ we obtain the integral

$$u(r,\theta) = \frac{1-r^2}{2\pi}\int_{-\pi}^{\pi}\frac{\xi}{1+r^2-2r\cos(\xi-\theta)}\,d\xi.$$

The requested numerical values are

$$u(1/2,\pi) \approx 0, u(3/4,\pi/3) \approx 0.882613, u(0.2,\pi/4) \approx 0.2465422.$$

2. Use Poisson's integral formula with $R = 4$ and $f(\theta) = \sin(4\theta)$ to obtain

$$u(r, \theta) = \frac{16 - r^2}{2\pi} \int_{-\pi}^{\pi} \frac{\sin(4\xi)}{16 + r^2 - 8r\cos(\xi - \theta)} \, d\xi.$$

   The requested numerical values are

$$u(1, \pi/6) \approx 0.0033829, u(3, 7\pi/2) \approx 0.30997(10^{-12}),$$
$$u(1, \pi/4) \approx 0.4105(10^{-12}), u(2.5, \pi/12) \approx 0.132145.$$

3. With $R = 15$ and $f(\theta) = \theta^3 - \theta$, we obtain

$$u(4, \pi) \approx 0.837758(10)^{-12}, u(12, 3\pi/2) \approx -2.571176, u(8, \pi/4) \approx 0.59705,$$

$$u(7, 0) \approx -0.628310(10^{-11}).$$

4. With $R = 6$ and $f(\theta) = e^{-\theta}$, we obtain

$$u(5.5, 3\pi/5) \approx 0.409013, u(4, 2\pi/7) \approx 1.174463, u(1, \pi)$$
$$\approx 4.333381, u(4, \pi/4) \approx 1.209883.$$

5. First observe that $u(r, \theta) = r^n \sin(n\theta)$ is harmonic on the disk $r \leq 1$, hence is the solution of the Dirichlet problem

$$\nabla^2(r, \theta) = 0 \text{ for } 0 \leq r < 1, -\pi \leq \theta < \pi,$$
$$u(1, \theta) = \sin(n\theta) \text{ for } -\pi \leq \theta < \pi.$$

   By the Poisson integral formula, this unique solution must be given by

$$r^n \sin(n\theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1 - r^2}{1 + r^2 - 2r\cos(\xi - \theta)} \sin(n\xi) \, d\xi.$$

18.5 Dirichlet Problem for Unbounded Regions

1. Use the integral formula for the solution on the upper half plane to obtain

$$\begin{aligned}
u(x, y) &= \frac{y}{\pi} \left[\int_{-4}^{0} \frac{-1}{y^2 + (\xi - x)^2} \, d\xi + \int_0^4 \frac{1}{y^2 + (\xi - x)^2} \, d\xi \right] \\
&= \frac{y}{\pi} \int_0^4 \left[\frac{-1}{y^2 + (\xi + x)^2} + \frac{1}{y^2 + (\xi - x)^2} \right] d\xi \\
&= \frac{1}{\pi} \left[2\arctan\left(\frac{x}{y}\right) - \arctan\left(\frac{4 + x}{y}\right) + \arctan\left(\frac{4 - x}{y}\right) \right]
\end{aligned}$$

   for $-\infty < x < \infty, y > 0$.

2. From the integral formula for the upper half plane, the solution is

$$u(x,y) = \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{e^{-|\xi|}}{y^2 + (\xi - x)^2}\, d\xi$$

for $-\infty < x < \infty, y > 0$.

3. From the formula derived for the solution in the right quarter plane, an integral solution of this problem is

$$u(x,y) = \frac{2}{\pi}\int_0^{\infty} \left[\frac{1}{y^2 + (t-x)^2} - \frac{1}{y^2 + (t+x)^2}\right] e^{-\xi}\cos(\xi)\, d\xi.$$

4. First separate variables by setting $u(x,y) = X(x)Y(y)$. We obtain

$$X'' - \omega^2 Y = 0, Y'' + \omega^2 Y = 0.$$

Use the condition $u(x,0) = X(x)Y(0) = 0$ and the condition that $X(x)$ remains bounded as $x \to \infty$ to obtain

$$X(x)Y(y) = B_\omega e^{-\omega x}\sin(\omega y)$$

for each $\omega > 0$. Now attempt a superposition

$$u(x,y) = \int_0^{\infty} e^{-\omega x}\sin(\omega y)\, d\omega.$$

Now

$$u(x,0) = g(y) = \int_0^{\infty} B_\omega \sin(\omega y)\, d\omega.$$

This is the Fourier sine expansion of $g(y)$, hence choose

$$B_\omega = \frac{2}{\pi}\int_0^{\infty} g(\eta)\sin(\omega\eta)\, d\eta.$$

Given g, this yields an integral formula for the solution $u(x,y)$.

We can also approach this problem using the Fourier sine transform in y. Let the Fourier sine transform of $u(x,y)$ be $\hat{u}_S(x,\omega)$. Transform the differential equation to obtain and boundary condition to obtain

$$\hat{u}_S'' - \omega^2\hat{u}_S = 0; \hat{u}_S(0,\omega) = \hat{g}_S(\omega).$$

We also require that $\hat{u}_S(x,\omega)$ remains bounded as x increases. This problem for the transformed function has solution

$$\hat{u}_S(x,\omega) = \hat{g}_S(\omega)e^{-\omega x}.$$

Invert this to obtain

$$u(x,y) = \frac{2}{\pi}\int_0^{\infty} \hat{g}_S(\omega)\sin(\omega y)e^{-\omega x}\, d\omega.$$

To see that this is the same solution obtained by separation of variables, replace $\hat{g}_S(\omega)$ by its integral from the definition of the sine transform to obtain

$$u(x,y) = \frac{2}{\pi}\int_0^\infty \left(\int_0^\infty g(\xi)\sin(\xi\omega)\,d\xi\right)\sin(\omega y)e^{-\omega x}\,d\omega.$$

5. To solve this problem, split it into two problems, in each of which there is a single nonhomogeneous boundary condition on one edge. One of the problems thus formed is exactly Problem 4. For the other, exchange x and y and the names of f and g. Finally, add these two solutions to obtain

$$u(x,y) = \frac{2}{\pi}\int_0^\infty \left(\int_0^\infty f(\xi)\sin(\omega\xi)\,d\xi\right)\sin(\omega x)e^{-\omega y}\,d\omega$$
$$+\frac{2}{\pi}\int_0^\infty \left(\int_0^\infty g(\xi)\sin(\omega\xi)\,d\xi\right)\sin(\omega y)e^{-\omega x}\,d\omega.$$

6. If $u(x,y)$ is harmonic on the upper half plane and $u(x,0)=f(x)$ along the real axis, then for $y<0$ define $v(x,y)=u(x,-y)$. It is routine to check that

$$\nabla^2 v = 0; v(x,0) = u(x,0) = f(x).$$

This defines a problem for v on the upper half plane. This problem for v has solution

$$v(x,y) = -\frac{y}{\pi}\int_{-\infty}^\infty \frac{f(\xi)}{y^2+(\xi-x)^2}\,d\xi.$$

Since $u(x,y)=v(x,-y)$, this solution for v on the upper half plane yields a solution for u on the lower half plane.

7. Because of the homogeneous boundary condition along $y=0$ and the particular function specified along the $x=0$ edge, we are led to try a Fourier sine transform in y. Let $\hat{u}_S(x,\omega)$ be the Fourier sine transform of $u(x,y)$ in y. The transformed problem is

$$\hat{u}_S'' - \omega^2\hat{u}_S = 0; \hat{u}_S(0,\omega) = \frac{1}{1+\omega^2}.$$

This problem is easily solved to obtain

$$\hat{u}_S(x,\omega) = \frac{e^{-\omega x}}{1+\omega^2}.$$

Invert this to obtain the solution

$$u(x,y) = \frac{2}{\pi}\int_0^\infty \frac{e^{-\omega x}}{1+\omega^2}\sin(\omega y)\,d\omega.$$

8. We will use the Fourier transform in x. Transform the problem to obtain

$$\hat{u}(\omega, y)'' - \omega^2 \hat{u}(\omega, y) = 0; \hat{u}(\omega, 0) = \frac{1}{a + i\omega}.$$

The solution of this problem is

$$\hat{u}(\omega, y) = A_\omega e^{-\omega y} + B_\omega e^{\omega y}.$$

Here $-\infty < x < \infty$, and we require that $\hat{u}(\omega, y)$ be bounded. Thus write

$$\hat{u}(\omega, y) = C_\omega e^{-|\omega| y}$$

and choose

$$C_\omega = \frac{1}{a + i\omega} = \frac{a - i\omega}{a^2 + \omega^2}.$$

Now invert the transform to obtain

$$u(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{e^{-|\omega| y}}{a^2 + \omega^2} e^{i\omega x} \, d\omega.$$

To simplify this expression, break the integral into integrals over $(-\infty, 0]$ and $[0, \infty)$. Make the change of variable $\omega = -\eta$ in the first integral, rename $\omega = \eta$ in the second, and recombine the integrals to obtain the solution

$$u(x, y) = \frac{1}{\pi} \int_0^{\infty} \frac{e^{-\eta y}}{a^2 + \eta^2} (a \cos(\eta x) + \eta \sin(\eta x)) \, d\eta.$$

9. The solution for the upper half plane is

$$\begin{aligned} u(x, y) &= \frac{y}{\pi} \int_4^8 \frac{A}{y^2 + (\xi - x)^2} \, d\xi \\ &= \frac{A}{\pi} \left[\arctan\left(\frac{8 - x}{y}\right) - \arctan\left(\frac{4 - x}{y}\right) \right]. \end{aligned}$$

10. This is a Dirichlet problem on the rectangle $0 \le x \le \pi$, $0 \le y \le 2$. Separate variables to obtain the differential equations

$$X'' - \lambda X = 0, Y'' + \lambda Y = 0$$

where $u(x, y) = X(x)Y(y)$. We also have the boundary conditions

$$X(0) = 0, Y'(0) = Y(2) = 0.$$

The eigenvalues and eigenfunctions of this problem for Y are

$$\lambda_n = \left(\frac{(2n - 1)\pi}{4}\right)^2, Y_n(y) = \cos((2n - 1)\pi y/4).$$

Then

$$X_n(x) = \sinh((2n-1)\pi x/4).$$

This suggests a solution of the form

$$u(x,y) = \sum_{n=1}^{\infty} c_n \sinh((2n-1)\pi x/4)\cos((2n-1)\pi y/4).$$

This is an eigenfunction expansion in terms of the eigenfunctions of a regular Sturm-Liouvlle problem, and we compute the coefficients to obtain

$$c_n = \frac{8}{\sinh((2n-1)\pi^2/4)}.$$

11. The solution for the right half plane can be obtained from the integral formula for the upper half plane by interchanging x and y. We obtain

$$\begin{aligned} u(x,y) &= \frac{x}{\pi}\int_{-1}^{1} \frac{1}{x^2+(\eta-y)^2}\,d\eta \\ &= \frac{1}{\pi}\left[\arctan\left(\frac{1-y}{x}\right)+\arctan\left(\frac{1+y}{x}\right)\right] \end{aligned}$$

for $x>0$ and $-\infty<y<\infty$.

12. With the zero function values given on the left edge, we use a Fourier sine transform in x. Let $\hat{u}_S(\omega,y)$ be the transformed of $u(x,y)$. Transform the problem to obtain

$$\hat{u}_S'' - \omega^2\hat{u}_S = 0; \hat{u}_S(\omega,0)=0, \hat{u}_S(\omega,1)=\hat{f}_S(\omega).$$

This problem has solution

$$\hat{u}_S(\omega,y) = \frac{1}{\sinh(\omega)}\hat{f}_S(\omega)\sinh(\omega y).$$

Invert this to obtain the solution

$$u(x,y) = \frac{2}{\pi}\int_0^{\infty} \hat{f}_S(\omega)\frac{\sinh(\omega y)}{\sinh(\omega)}\sin(\omega x)\,d\omega.$$

18.6 A Dirichlet Problem for a Cube

1. Let $u(x,y,z) = X(x)Y(y)Z(z)$ and separate variables. The boundary conditions give us

$$X(0)=X(1)=Y(0)=Y(1)=Z(0)=0.$$

We obtain solutions of the form

$$u_{nm}(x,y,z) = \sin(n\pi x)\sin(m\pi y)\sinh(\pi\sqrt{n^2+m^2}z).$$

Use a superposition

$$u(x,y,z) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} c_{nm}\sin(n\pi x)\sin(m\pi y)\sinh(\pi\sqrt{n^2+m^2}z).$$

We need

$$u(x,y,1) = xy = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} c_{nm}\sin(n\pi x)\sin(m\pi y)\sinh(\pi\sqrt{n^2+m^2}).$$

As we have done with wave and heat equations in two space variables, choose

$$\begin{aligned} c_{nm} &= \frac{4}{\sinh(\pi\sqrt{n^2+m^2})}\int_0^1 \xi\sin(n\pi\xi)\,d\xi \int_0^1 \eta\sin(m\pi\eta)\,d\eta \\ &= \frac{4(-1)^{n+m}}{nm\pi^2\sinh(\sqrt{n^2+m^2}\pi)}. \end{aligned}$$

The solution is

$$u(x,y,z) =$$
$$\frac{4}{\pi^2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{(-1)^{n+m}}{nm\sinh(\sqrt{n^2+m^2}\pi)}\sin(n\pi x)\sin(m\pi y)\sinh(\sqrt{n^2+m^2}\pi z).$$

2. Two separations of variables and the homogeneous boundary conditions give us a solution of the form

$$u(x,y,z) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} c_{nm}\sin(ny/2)\sin(m\pi z)\sinh(\sqrt{n^2+4m^2\pi^2}x/2),$$

where

$$\begin{aligned} c_{nm} &= \frac{1}{\sinh(\sqrt{n^2+4m^2\pi^2}\pi)}\frac{1}{\pi}\int_0^{2\pi} 2\sin(n\eta/2)\,d\eta \int_0^1 2\sin(m\pi\xi)\,d\xi \\ &= \frac{4}{\pi\sinh(\sqrt{n^2+4m^2\pi^2}\pi)}(1-(-1)^n)(1-(-1)^m). \end{aligned}$$

3. Write the solution as the sum of solutions of two simpler problems:

$$\begin{aligned} &\nabla^2 w = 0, \\ &w(0,y,z) = w(1,y,z) = w(x,0,z) = w(x,2\pi,z) = w(x,y,0) = 0, \\ &w(x,y,\pi) = 1, \end{aligned}$$

and

$$\begin{aligned} &\nabla^2 v = 0, \\ &v(0,y,z) = v(1,y,z) = v(x,y,0) = v(x,y,\pi)v(x,0,z) = 0, \\ &v(x,2\pi,z) = 1. \end{aligned}$$

Each of these problems is solved by a straightforward separation of variables. For the problem in w, we obtain

$$w(x,y,z)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}a_{nm}\sin(n\pi x)\sin(my/2)\sinh(\sqrt{4n^2\pi^2+m^2}z/2),$$

in which

$$\begin{aligned}a_{nm}&=\frac{1}{\sinh(\sqrt{4n^2\pi^2+m^2}\pi/2)}\int_0^1 2\sin(n\pi\xi)\,d\xi\int_0^{2\pi}\frac{1}{\pi}\sin(m\eta/2)\,d\eta\\&=\frac{4}{\sinh(\sqrt{4n^2\pi^2+m^2}\pi/2)}\left(\frac{1-(-1)^n}{n\pi}\right)\left(\frac{1-(-1)^m}{m\pi}\right).\end{aligned}$$

Next, we obtain

$$v(x,y,z)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}b_{nm}\sin(n\pi x)\sin(mz)\sinh(\sqrt{n^2\pi^2+m^2}y),$$

where

$$\begin{aligned}b_{nm}&=\frac{1}{\sinh(2\sqrt{n^2\pi^2+m^2}\pi)}(2)\int_0^1\sin(n\pi\xi)\,d\xi\frac{2}{\pi}\int_0^{\pi}\sin(m\eta)\,d\eta\\&=\frac{8}{\sinh(2\sqrt{n^2\pi^2+m^2}\pi)}\left(\frac{1-(-1)^n}{n\pi}\right)\left(\frac{1-(-1)^m}{m\pi}\right).\end{aligned}$$

4. The solution u of this problem is a sum of the solutions of the following two simpler problems:

$$\begin{aligned}&\nabla^2 v=0,\\&v(0,y,z)=0, v(1,y,z)=\sin(\pi y)\sin(z),\\&v(x,0,z)=v(x,2,z)=0,\\&v(x,y,0)=v(x,y,\pi)=0,\end{aligned}$$

and

$$\begin{aligned}&\nabla^2 w=0,\\&w(0,y,z)=w(1,y,z)=0,\\&w(x,0,z)=w(x,2,z)=0,\\&w(x,y,0)=x^2(1-x)y(2-y), w(x,y,\pi)=0.\end{aligned}$$

Each of these problems can be solved by separation of variables. We obtain solutions of the form

$$v(x,y,z)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}a_{nm}\sinh(\sqrt{m^2+(n\pi/2)^2}x)\sin(n\pi y/2)\sin(mz)$$

and

$$w(x,y,z) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} b_{nm}\sin(n\pi x)\sin(m\pi y/2)\sinh(\sqrt{(n\pi)^2+(m\pi/2)^2}(\pi - z)).$$

The coefficients a_{nm} are easily obtained by inspection. Observe that $a_{21}\sqrt{\pi^2+1} = 1$ and all other $a_{nm} = 0$. The coefficients b_{nm} require the usual integrations, and we obtain

$$b_{nm} =$$
$$\frac{2}{\sinh(\pi\sqrt{4n^2+m^2}/2)}\int_0^1 \xi^2(1-\xi)\sin(n\pi\xi)\,d\xi \int_0^2 \eta(2-\eta)\sin(m\pi\eta/2)\,d\eta$$
$$= -\frac{64}{\pi^6}(1+2(-1)^n)(1-(-1)^m).$$

18.7 Steady-State Heat Equation for a Sphere

In Problems 1 through 4, the solution has the form

$$u(\rho,\varphi) = \sum_{n=0}^{\infty}\frac{2n+1}{2}\left(\int_{-1}^{1} f(\arccos(\xi))P_n(\xi)\,d\xi\right)\left(\frac{\rho}{R}\right)^n P_n(\cos(\varphi)).$$

In the following, we approximate the required integrals for numerical values of the coefficients of the first six terms in this series solution. In some cases, integrals can be seen to be zero by exploiting even-odd properties of Legendre polynomials and possibly of the function f.

1. For $f(\varphi) = A\varphi^2$, the integrals to be approximated are

$$I_n = \int_{-1}^{1}(\arccos(\xi))^2 P_n(\xi)\,d\xi$$

for $n = 0, 1, \cdots, 5$. We will insert A into the series after these integrals are computed. We have

$$I_0 \approx 5.86960441, I_1 \approx -2.46740110, I_2 \approx 0.4444444,$$
$$I_3 \approx -1.154212688, I_4 \approx 0.09111111, I_5 \approx -0.03855314.$$

The first six terms of the approximated series solution are

$$u(\rho,\varphi) \approx A\left[\frac{1}{2}(5.86960441) - \frac{3}{2}(2.46740)\frac{\rho}{R}\cos(\varphi)\right.$$
$$\frac{5}{4}(0.44444)\left(\frac{\rho}{R}\right)^2(3\cos^2(\varphi)-1) - \frac{7}{4}(0.154212)\left(\frac{\rho}{R}\right)^3(5\cos^3(\varphi)-3\cos(\varphi))$$
$$+\frac{9}{16}(0.071111)\left(\frac{\rho}{R}\right)^4(35\cos^4(\varphi)-30\cos^3(\varphi)+3)$$
$$-\frac{11}{16}(0.03855314)\left(\frac{\rho}{R}\right)^5(63\cos^5(\varphi)-70\cos^3(\varphi)+15\cos(\varphi)) + \cdots.$$

2. For $f(\varphi) = \sin(\varphi)$, use the identity
$$\sin(\arccos(x)) = \sqrt{1-x^2}.$$
Note also that $I_1 = I_3 = I_5 = 0$ because $P_n(x)$ is odd if n is odd. We obtain the approximate expansion
$$\begin{aligned} u(\rho,\varphi) \approx{}& \frac{1}{2}(1.570796327) - \frac{5}{2}(0.196349541)\left(\frac{\rho}{R}\right)^2 (3\cos^2(\varphi) - 1) \\ &- \frac{9}{2}(0.024543693)\left(\frac{\rho}{R}\right)^4 (35\cos^4(\varphi) - 30\cos^2(\varphi) + 3) + \cdots. \end{aligned}$$

3. The first six terms of the approximation are
$$\begin{aligned} u(\rho,\varphi) ={}& \frac{1}{2}12.15672076 - \frac{3}{2}(6.573472)\left(\frac{\rho}{R}\right)\cos(\varphi) \\ &+ \frac{5}{4}(2.094395)\left(\frac{\rho}{R}\right)^2 (3\cos^2(\varphi) - 1) \\ &- \frac{7}{4}(0.6869585)\left(\frac{\rho}{R}\right)^3 (5\cos^3(\varphi) - 3\cos(\varphi) \\ &+ \frac{9}{16}(-.33510322)\left(\frac{\rho}{R}\right)^3 (35\cos^4(\varphi) - 30\cos^3(\varphi) + 3) \\ &- \frac{11}{16}(0.17787555)\left(\frac{\rho}{R}\right)^5 (63\cos^5(\varphi) - 70\cos^3(\varphi) + 15\cos(\varphi)) + \cdots. \end{aligned}$$

4. With $f(\varphi) = 2 - \varphi^2$, we can use the orthogonality of the Legendre polynomials on $[-1,1]$ to simplify the calculations. Since $P_0(x)$ and $P_n(x)$ are orthogonal, then $\int_{-1}^{1} 2P_n(x)\,dx = 0$ for $n = 1, 2, \cdots$. Then, for $n \geq 1$,
$$\int_{-1}^{1} (2-\cos(x))^2 P_n(x)\,dx = \int_{-1}^{1} (\arccos(x))^2 P_n(x)\,dx,$$
and these integrals were approximated in Problem 2. After carrying out the needed calculations, we obtain the approximate solution
$$\begin{aligned} u(\rho,\varphi) \approx{}& -\frac{1}{2}(1.86960441) + \frac{3}{2}\left(\frac{\rho}{R}\right)\cos(\varphi) - \frac{5}{4}(0.44444)\left(\frac{\rho}{R}\right)^2 (3\cos^2(\varphi) - 1) \\ &+ \frac{7}{4}(0.154212)\left(\frac{\rho}{R}\right)^3 (5\cos^5(\varphi) - 3\cos(\varphi)) \\ &- \frac{9}{16}(0071111)\left(\frac{\rho}{R}\right)^4 (35\cos^4(\varphi) - 30\cos^2(\varphi) + 3) \\ &+ \frac{11}{16}(0.03855314)\left(\frac{\rho}{R}\right)^5 (63\cos^5(\varphi) - 70\cos^3(\varphi) + 15\cos(\varphi)) + \cdots. \end{aligned}$$

5. The problem to be solved is
$$\begin{aligned} \frac{\partial^2 u}{\partial\rho^2} + \frac{2}{\rho}\frac{\partial u}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial\varphi^2} + \frac{\cot(\varphi)}{\rho^2}\frac{\partial u}{\partial\varphi} &= 0, \\ u(R_1,\varphi) &= T_1, \\ u(R_2,\varphi) &= 0. \end{aligned}$$

Here $-\pi/2 \leq \varphi \leq \pi/2$ and $R_1 \leq \rho \leq R_2$. We will solve this problem by separation of variables. Let $u(\rho, \varphi) = F(\rho)\Phi(\varphi)$ to obtain

$$F'' + \frac{2}{\rho}F' - \frac{\lambda}{\rho^2}F = 0$$

and

$$\Phi'' + \cot(\varphi)\Phi' + \lambda\Phi = 0$$

with λ the separation constant. The bounded solution for $\Phi(\varphi)$ on $[-\pi/2, \pi/2]$ is

$$\Phi_n(\varphi) = P_n(\cos(\varphi))$$

and this is the nth eigenfunction corresponding to the eigenvalue $\lambda_n = n(n+1)$ of Legendre's differential equation. Solutions for $F(\rho$ for $n = 0, 1, 2, \cdots$ have the form

$$F_n(\rho) = a_n\rho^n + b_n\rho^{-n-1}.$$

Attempt a superposition

$$u(\rho, \varphi) = \sum_{n=0}^{\infty}(a_n\rho^n + b_n\rho^{-n-1})P_n(\cos(\varphi)).$$

The condition specified at $\rho = R_1$ requires that

$$u(R_1, \varphi) = T_1 = \sum_{n=0}^{\infty}(a_nR_1^n + b_nR_1^{-n-1})P_n(\cos(\varphi)).$$

The condition at $\rho = R_2$ requires that

$$u(R_2, \varphi) = 0 = \sum_{n=0}^{\infty}(a_nR_2^n + b_nR_2^{-n-1})P_n\cos(\varphi).$$

From the orthogonality of the Legendre polynomials $P_n(x)$ on $[-1, 1]$, we conclude that

$$a_0 + \frac{1}{R_1}b_0 = T_1, a_0 + \frac{1}{R_2}b_0 = 0,$$

and, for $n = 1, 2, \cdots$,

$$a_nR_1^n + \frac{1}{R_1^{n+1}}b_n = 0, a_nR_2^n + \frac{1}{R_2^{n+1}}b_n = 0.$$

Solve these equations for the coefficients to obtain

$$a_0 = \frac{T_1R_1}{R_1 - R_2}, b_0 = -\frac{T_1R_1R_2}{R_1 - R_2},$$

and

$$a_n = b_n = 0 \text{ for } n = 1, 2, \cdots .$$

The solution is

$$u(\rho, \varphi) = \frac{T_1R_1}{R_1 - R_2}\left[\frac{R_2}{\rho} - 1\right].$$

18.8 The Neumann Problem

1. First,

$$\int_0^1 4\cos(\pi x)\,dx = 0,$$

so a solution may exist. (If this integral had been nonzero, there could not have been a solution). From the boundary conditions on the opposite edges $x = 0$ and $x = 1$, we find by separation of variables that there will be a solution of the form

$$u(x, y) = c_0 + \sum_{n=1}^{\infty}[c_n \cosh(n\pi y) + d_n \cosh(n\pi(1-y))]\cos(n\pi y).$$

The boundary condition at $y = 1$ becomes

$$\frac{\partial u}{\partial y}(x, 1) = 0 = \sum_{n=1}^{\infty} n\pi c_n \sin(n\pi)\cos(n\pi x).$$

Therefore $c_n = 0$ for $n \geq 1$. On the edge $y = 0$,

$$\frac{\partial u}{\partial y}(x, 0) = 4\cos(\pi x) = \sum_{n=1}^{\infty} -n\pi d_n \sinh(n\pi)\cos(n\pi x).$$

Then $d_n = 0$ if $n \geq 2$, and

$$d_1 = -\frac{4}{\pi\sinh(\pi)}.$$

A solution is given by

$$u(x, y) = c_0 - \frac{4}{\pi\sinh(\pi)}\cosh(\pi(1-y))\cos(\pi x).$$

Since c_0 is arbitrary, this solution is not unique.

2. First check that

$$\int_0^{\pi}\left(y - \frac{y}{2}\right)\,dy = 0,$$

so it is worthwhile to look for a solution. From the zero boundary conditions on the opposite edges $y = 0$ and $y = \pi$ we expect to see a solution of the form

$$u(x, y) = c_0 + \sum_{n=1}^{\infty}[c_n \cosh(nx) + d_n \cosh(n(1-x))]\cos(ny).$$

The boundary conditions on edges $x = 0$ and $x = 1$ give us

$$\frac{\partial u}{\partial x}(0, y) = y - \frac{\pi}{2} = \sum_{n=1}^{\infty} -nd_n \sinh(n)\cos(ny)$$

and

$$\frac{\partial u}{\partial x}(1, y) = \cos(y) = \sum_{n=1}^{\infty} nc_n \sinh(n) \cos(ny).$$

The coefficients are given by

$$c_n = \frac{1}{n \sinh(n)} \frac{2}{\pi} \int_0^{\pi} \cos(y) \cos(ny)\, dy = \begin{cases} 0 & \text{for } n \neq 1, \\ 1/\sinh(1) & \text{for } n = 1 \end{cases}$$

and

$$d_n = \frac{-1}{n \sinh(n)} \frac{2}{\pi} \int_0^{\pi} \left(y - \frac{\pi}{2}\right) dy = \frac{2(1 - (-1)^n)}{\pi n^3 \sinh(n)}$$

for $n \geq 1$. We have the solution

$$\begin{aligned} u(x, y) &= c_0 + \frac{\cosh(x)}{\sinh(1)} \\ &\quad + \sum_{n=1}^{\infty} \frac{2((-1)^n - 1)}{\pi n^2 \sinh(n)} \cosh(n(1 - x)) \cos(ny). \end{aligned}$$

3. A solution may exist because $\int_0^{\pi} \cos(3x)\, dx = 0$. From the zero boundary conditions on edges $x = 0$ and $x = \pi$, separation of variables will yield a solution of the form

$$u(x, y) = c_0 + \sum_{n=1}^{\infty} [c_n \cosh(ny) + d_n \cosh(n(\pi - y))] \cos(nx).$$

Now

$$\frac{\partial u}{\partial y}(x, 0) = \cos(3x) = \sum_{n=1}^{\infty} -nd_n \sinh(n\pi) \cos(nx),$$

so

$$c_3 = -\frac{1}{3 \sinh(3\pi)} \text{ and } d_n = 0 \text{ for } n \neq 3.$$

The boundary condition at $y = \pi$ gives us

$$\frac{\partial u}{\partial y}(x, \pi) = 6x - 3\pi = \sum_{n=1}^{\infty} nc_n \sinh(n\pi) \cos(nx),$$

so

$$\begin{aligned} c_n &= \frac{1}{n \sinh(n\pi)} \frac{2}{\pi} \int_0^{\pi} \int_0^{\pi} (6x - 3\pi) \cos(nx)\, dx \\ &= \frac{1}{n \sinh(n\pi)} \frac{12}{n^2 \pi} ((-1)^n - 1) \end{aligned}$$

for $n = 1, 2, 3, \cdots$. The solution is

$$u(x, y) = c_0 - \frac{\cosh(3(\pi - y))}{3\sinh(3\pi)}\cos(3x) + \sum_{n=1}^{\infty} \frac{12(-1)^n - 1)}{n^3\pi\sinh(n\pi)}\cos(ny)\cos(nx).$$

4. Write $u(x, y) = X(x)Y(y)$ and separate the variables in Laplace's equation, then use the boundary conditions to obtain

$$X'' - \lambda X = 0, X'(0) = X'(\pi) = 0,$$
$$Y'' + \lambda Y = 0.$$

The solutions for X are $X_0(x) = 1$ and $X_n(x) = \cos(nx)$ for $n = 1, 2, \cdots$. We find that

$$Y_n(y) = c_n \cosh(ny) + d_n \cosh(n(\pi - y))$$

for $n = 1, 2, \cdots$. Thus we seek a solution of the form

$$u(x, y) = c_0 + \sum_{n=1}^{\infty}[c_n \cosh(ny) + d_n \cosh(n(\pi - y))]\cos(nx).$$

At the edge $y = \pi$ we have

$$u(x, \pi) = 0 = c_0 + \sum_{n=1}^{\infty}[c_n \cosh(n\pi) + d_n]\cos(nx).$$

Along the edge $y = 0$ we find

$$u(x, 0) = f(x) = c_0 + \sum_{n=1}^{\infty}[c_n + d_n \cosh(n\pi)]\cos(nx).$$

Thus, for a solution, the coefficients must be chosen to satisfy the equations

$$c_0 = 0,$$
$$c_n \cosh(n\pi) + d_n = 0,$$
$$c_n + d_n \cosh(n\pi) = \frac{2}{\pi}\int_0^{\pi} f(x)\cos(nx)\,dx$$

for $n = 1, 2, \cdots$. Now, the determinant of the matrix of coefficients of this system (for $n \geq 1$) is

$$\begin{vmatrix} \cosh(n\pi) & 1 \\ 1 & \cosh(n\pi) \end{vmatrix} = \cosh^2(n\pi) - 1 = \sinh^2(n\pi) \neq 0.$$

Thus there is a unique solution of these algebraic equations for the $c_n{}'s$ and $d_n{}'s$, for each positive integer n, so the problem for u has a unique solution.

5. With $u(x,y) = X(x)Y(y)$ we obtain

$$X'' - \lambda X = 0; Y'' + \lambda Y = 0, Y(0) = Y(1) = 0.$$

Then

$$Y_n(y) = \sin(n\pi y), X_n(x) = c_n \cosh(n\pi x) + d_n \cosh(n\pi(x-1)).$$

A solution will have the form

$$u(x,y) = \sum_{n=1}^{\infty} [c_n \cosh(n\pi x) + d_n \cosh(n\pi(1-x))] \sin(n\pi y).$$

The boundary conditions at $x = 1$ and $x = 0$ give us, respectively,

$$\frac{\partial u}{\partial x}(1,y) = 0 = \sum_{n=1}^{\infty} n\pi c_n \sinh(n\pi) \sin(n\pi y)$$

and

$$\frac{\partial u}{\partial x}(0,y) = 3y^2 - 2y = \sum_{n=1}^{\infty} -n\pi \sinh(n\pi) \sin(n\pi y).$$

Then each $c_n = 0$ and

$$\begin{aligned} d_n &= \frac{-2}{n\pi \sinh(n\pi)} \int_0^1 (3y^2 - 2y) \sin(n\pi y)\, dy \\ &= \frac{2}{n^4 \pi^4 \sinh(n\pi)} [n^2 \pi^2 (-1)^n + 6(1 - (-1)^n)] \end{aligned}$$

for $n = 1, 2, \cdots$. This yields the solution

$$u(x,y) = \sum_{n=1}^{\infty} \frac{2}{n^4 \pi^4 \sinh(n\pi)} [n^2\pi^2(-1)^n + 6(1-(-1)^n)] \cosh(n\pi(1-x)) \sin(n\pi y).$$

6. Since $\int_{-\pi}^{\pi} \sin(3\theta)\, d\theta = 0$, a solution is possible. Any such solution will have the form

$$u(r,\theta) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} [a_n r^n \cos(n\theta) + b_n r^n \sin(n\theta)].$$

The boundary condition on $r = R$ gives us

$$\begin{aligned} \frac{\partial u}{\partial r}(R,\theta) &= \sin(3\theta) \\ &= \sum_{n=1}^{\infty} [n a_n R^{n-1} \cos(n\theta) + n b_n R^{n-1} \sin(n\theta)]. \end{aligned}$$

By inspection, we see that we can choose each $a_n = 0$, $b_n = 0$ for $n \neq 3$, and $3b_3 R^2 = 1$. Thus we have the solution

$$u(r,\theta) = \frac{1}{2}a_0 + \frac{R}{3}\left(\frac{r}{R}\right)^3 \sin(3\theta).$$

7. Check that $\int_{-\pi}^{\pi} \cos(2\theta)\, d\theta = 0$, so there may be a solution. Any such solution must have the form

$$r(r,\theta) = \frac{1}{2}a_0 + \sum_{=1}^{\infty}[a_n r^n \cos(n\theta) + b_n r^n \sin(n\theta)].$$

From the boundary condition on $r = R$ we have

$$\begin{aligned}\frac{\partial u}{\partial r}(R,\theta) &= \cos(2\theta) \\ &= \sum_{n=1}^{\infty}[na_n R^{n-1}\cos(n\theta) + nb_n R^{n-1}\sin(n\theta)].\end{aligned}$$

As in the preceding problem, observe that we can choose each $b_n = 0$, $a_n = 0$ for $n \neq 2$, and $2a_2R = 1$. The solution is

$$u(r,\theta) = \frac{1}{2}a_0 + \frac{R}{2}\left(\frac{r}{R}\right)^2 \cos(2\theta).$$

8. First observe that $\int_{-\infty}^{\infty} xe^{-|x|}\, dx = 0$, so there may be a solution. Using the general solution developed in Section 18.8.3, we immediately write the solution

$$u(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \ln(y^2 + (\xi - y)^2)\xi e^{-|\xi|}\, d\xi + c.$$

9. Since $\int_{-\infty}^{\infty} e^{-|x|}\sin(x)\, dx = 0$, the necessary condition for a solution to exist is satisfied. Write the solution

$$u(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \ln(y^2 + (\xi - y)^2)e^{-|\xi|}\sin(\xi)\, d\xi.$$

10. A solution of the Dirichlet problem for the lower half-plane was obtained in Problem 6 of Section 18.5. Using that result and the technique discussed for solving a Neumann problem in the upper half-plane, we can write the solution for the lower half-plane as

$$u(x,y) = -\frac{1}{2\pi}\int_{-\infty}^{\infty} \ln(y^2 + (\xi - x)^2)f(\xi)\, d\xi + c.$$

Notice that we can also observe that the sign of the outer normal derivative changes along the real axis as we move from the upper half-plane to the lower half-plane, accounting for the negative sign in the integral.

11. We will apply a Fourier cosine transform (with respect to x) to this problem. Let

$$U_C(\omega,y) = \mathcal{F}_C[u(x,y)](\omega,y).$$

Using the operational formula for the cosine transform, we obtain

$$-\omega^2 U_C - \frac{\partial u}{\partial x}(0,y) + U_C'' = 0,$$

in which primes denote differentiation with respect to y. Since $u_x(0, y) = 0$, we obtain

$$U_C'' - \omega^2 U_C = 0,$$

with the general solution

$$U_C(\omega, y) = a_\omega e^{\omega y} + b_\omega e^{-\omega y}.$$

To have bounded solutions for $y > 0$, choose each $a_\omega = 0$, so

$$U_C(\omega, y) = b_\omega e^{-\omega y}.$$

Now invert the cosine transform, obtaining

$$u(x, y) = \int_0^\infty a_\omega \cos(\omega x) e^{-\omega y}\, d\omega.$$

From this, calculate

$$\frac{\partial u}{\partial y}(x, 0) = \int_0^\infty -\omega a_\omega \cos(\omega x)\, d\omega = f(x)$$

to complete the solution by setting

$$a_\omega = -\frac{2}{\pi\omega} \int_0^\infty f(\xi) \cos(\omega\xi)\, d\xi.$$

12. Because of the boundary condition at $x = 0$, we will use a Fourier sine transform on x. This gives the transformed problem

$$\omega^2 U_S(\omega, y) + \omega u(0, y) + U_S''(\omega, y) = 0.$$

Putting $u(0, y) = 0$ and solving the resulting differential equation for bounded solutions, we obtain

$$U_S(\omega, y) = b_\omega e^{-\omega y}.$$

Now calculate

$$U_S'(\omega, 0) = -\omega b_\omega = \hat{f}_S(\omega)$$

to obtain

$$b_\omega = -\frac{2}{\pi\omega} \int_0^\infty f(\xi) \sin(\omega\xi)\, d\xi.$$

The solution is

$$u(x, y) = \int_0^\infty b_\omega e^{-\omega y} \sin(\omega x)\, d\omega.$$

Chapter 19

Complex Numbers and Functions

19.1 Geometry and Arithmetic of Complex Numbers

1.
$$(3-4i)(6+2i) = (18+8) + (-24+6)i = 26-28i$$

2.
$$i(6-2i) + |1-i| = 6i + 2 + \sqrt{1+1} = 2 + \sqrt{2} + 6i$$

3.
$$\frac{2+i}{4-7i} = \frac{(2+i)(4+7i)}{(4-7i)(4+7i)} = \frac{1}{65}(1+18i)$$

4.
$$\frac{(2+i)-(3-4i)}{(5-i)(3+i)} = \frac{(-1+5i)(16-2i)}{(16+2i)(16-2i)} = \frac{-3+41i}{130}$$

5.
$$(17-6i)\overline{-3-12i} = (17-6i)(-3+12i) = 4+228i$$

6.
$$\left|\frac{3i}{-4+8i}\right| = \frac{|3i|}{|-4+8i|} = \frac{3}{\sqrt{80}} = \frac{3}{4\sqrt{5}}$$

7.
$$i^3 - 4i^2 + 2 = -i + 4 + 2 = 6 - i$$

8.
$$(3+i)^3 = 27 + 3(3^2)i + 3(3)i^2 + i^3 = 27 + 27i - 9 - i = 18 + 26i$$

9.

$$\left(\frac{-6+2i}{1-8i}\right)^2 = \left[\frac{(-6+2i)(1+8i)}{(1-8i)(1+8i)}\right]^2$$
$$= \frac{(-22-46i)^2}{65^2} = \frac{-1632+2024i}{4225}$$

10.

$$(-1-8i)(2i)(4-i) = (-3-8i)(2+8i) = 58-40i$$

In each of Problems 11 - 16, n denotes an arbitrary integer.

11.

$$|3i| = 3, \arg(3i) = \frac{\pi}{2} + 2n\pi$$

12.

$$|-2+2i| = \sqrt{8} = 2\sqrt{2} \text{ and } \arg(-2+2i) = \frac{3\pi}{4} + 2n\pi$$

13.

$$|-3+2i| = \sqrt{13} \text{ and } \arg(-3+2i) = -\arctan(2/3) + (2n+1)\pi$$

14.

$$|8+i| = \sqrt{65} \text{ and } \arg(8+i) = \arctan(1/8) + 2n\pi$$

15.

$$|-4| = 4 \text{ and } \arg(-4) = (2n+1)\pi$$

16.

$$|3-4i| = \sqrt{25} = 5 \text{ and } \arg(3-4i) = -\arctan(4/3) + 2n\pi$$

17. Since $|-2+2i| = 2\sqrt{2}$ and $3\pi/4$ is an argument of $2+2i$, the polar form is

$$z = 2\sqrt{2}e^{3\pi i/4}.$$

Here there is no point in adding $2n\pi$ to the argument to obtain all arguments, since $e^{2n\pi i} = 1$ for any integer n.

18. $|-7i| = 7$ and an argument of $-7i$ is $3\pi/2$ (or, just as good, $-\pi/2$). The polar form is

$$-7i = 7e^{3\pi i/2}.$$

We could also write

$$-7i = 7e^{-\pi i/2}.$$

19. $|5-2i| = \sqrt{29}$ and an argument of $5-2i$ is $-\arctan(2/5)$, so the polar form of $5-2i$ is

$$5-2i = \sqrt{29}e^{-\arctan(2/5)i}.$$

20. $|-4-i| = \sqrt{17}$ and an argument of $-4-i$ is $\pi + \arctan(1/4)$, so the polar form is
$$-4 - i = \sqrt{17}e^{(\pi+\arctan(1/4))i}.$$

21. $|8+i| = \sqrt{65}$ and an argument of $8+i$ is $\arctan(1/8)$, so the polar form is
$$8 + i = \sqrt{65}e^{\arctan(1/8)i}.$$

22. $|-12+3i| = \sqrt{153}$ and an argument of $-12+3i$ is $-\arctan(3/12)$, so the polar form is
$$-12 + 3i = \sqrt{153}e^{-\arctan(1/4)i}.$$

23. Since $i^2 = -1$, we have
$$\begin{aligned} i^{4n} &= (i^2)^{2n} = ((-1)^2)^n = 1, \\ i^{4n+1} &= i^{4n}i = i, \\ i^{4n+2} &= i^{4n}i^2 = i^2 = -1, \\ i^{4n+3} &= i^{4n}i^3 = i^{4n}i^2 i = -i. \end{aligned}$$

24. Since $(a+ib)^2 = a^2 - b^2 + 2abi$, then
$$\mathrm{Re}((a+ib)^2) = a^2 - b^2 \text{ and } \mathrm{Im}((a+ib)^2) = 2ab.$$

25. Suppose first that z, w, u form the vertices of a triangle, labeled in the clockwise order around the triangle. The sides of this triangle are vectors represented by the complex numbers $w-z$, $u-w$ and $z-u$. This triangle is equilateral if and only if
$$|w - z| = |u - w| = |z - u|$$
and each of the vector sides can be rotated by $\theta = 2\pi/3$ radians clockwise to align with the next side. This occurs exactly when
$$(u - w) = (w - z)e^{-2\pi i/3} \text{ and } (z - u) = (u - w)e^{-2\pi i/3}.$$
Dividing these equations gives us
$$\frac{u - w}{z - u} = \frac{w - z}{u - w}.$$
Then
$$(u - w)(u - w) = (w - z)(z - u),$$
or, equivalently,
$$w^2 - 2wu + u^2 = zw + uz - uv - z^2.$$
Finally, rearrange this equation to obtain
$$z^2 + w^2 + u^2 = zw + zu + wu.$$

26. Let $z = x + iy$. Then

$$\begin{aligned} z^2 = (\overline{z})^2 &\text{ if and only if} \\ (x+iy)^2 = (x-iy)^2 &\text{ if and only if} \\ x^2 - y^2 + 2ixy = x^2 - y^2 - 2ixy &\text{ if and only if} \\ 2ixy = -2ixy &\text{ if and only if} \\ 2ixy = 0. & \end{aligned}$$

Now, $2xy = 0$ can occur in only two ways. Either $x = 0$, in which case z is pure imaginary, or $y = 0$, in which case z is real.

27. Suppose first that $|z| = 1$. Then

$$|\overline{z}| = |z\overline{z}| = 1.$$

Then
$$\left|\frac{z-w}{1-\overline{z}w}\right| = \left|\frac{z-w}{z\overline{z}-\overline{z}w}\right| = \frac{|z-w|}{|\overline{z}||z-w|} = \frac{1}{|\overline{z}|} = 1.$$
If $|w| = 1$, argue as follows:

$$\left|\frac{z-w}{1-\overline{z}w}\right| = \left|\frac{z-w}{\overline{w}w-\overline{z}w}\right| = \frac{1}{w}\left|\frac{z-w}{\overline{w}-\overline{z}}\right| = 1.$$

because
$$|z-w| = |\overline{w-z}| = |\overline{w}-\overline{z}|.$$

28. Compute

$$\begin{aligned} &|z+w|^2 + |z-w|^2 \\ &= (z+w)(\overline{z}+\overline{w}) + (z-w)(\overline{z}-\overline{w}) \\ &= z\overline{z} + z\overline{w} + w\overline{z} + w\bar(w) + z\bar(z) + z\overline{w} - w\overline{z} - z\overline{w} \\ &= 2z\overline{z} + 2w\overline{w} \\ &= 2\left(|z|^2 + |w|^2\right). \end{aligned}$$

29. M consists of all $x + iy$ with $y < 7$. This is the half-plane lying below the horizontal line $y = 7$. M is open and the boundary points are all complex numbers $x + 7i$ on the "edge" of M. None of the boundary points of M belong to M.

30. S consists of all points outside the circle of radius 2 about the origin. This set is open, and the boundary points are all the points on the circle $|z| = 2$. No boundary points of S are in S.

31. U consists of all points $x + iy$ with $1 < x \le 3$. This is the vertical strip between the lines $x = 1$ and $x = 3$, including points on the line $x = 3$, but not those on $x = 1$. The boundary points are the points on these vertical

lines. these are all points $1 + iy$ and $3 + iy$. Only the boundary points $3 + iy$ are in U. The boundary points $1 + iy$ are not in U. U is not open, since it contains some boundary points. U is not closed, because U does not contain all of its boundary points.

32. V consists of all points $x + iy$ with $2 < x \leq 3$ and $-1 < y < 1$. These are points inside the rectangle having vertices $(2, 1)$, $(3, 1)$, $(2, -1)$ and $(3, -1)$. The boundary points are the points on the edges of this rectangle. Of these points, only the points $3 + iy$ with $-1 < y < 1$ are in V. V is not open because V contains some of its boundary points. V is not closed because V does not contain all of its boundary points.

33. W consists of all points $x+iy$ with $x > y^2$. These are points "enclosed by" the parabola $x = y^2$. Boundary points are the points on this parabola, which are complex numbers y^2+iy. Since W contains no boundary points, W is open. Since W does not contain all of its boundary points, W is not closed.

34. The boundary points of R are all points of the form $0+(1/m)i$ and $(1/n)+0i)$, that is, all pure imaginary points $z = i/m$ and all real points $z = 1/n$. Since none of these boundary points are in R, R is open. Since there are boundary points of R not in R, R is not closed.

19.2 Complex Functions

1. Write
$$f(z) = z - i = x + iy - i = x + (y - 1)i$$
so
$$u(x, y) = x, v(x, y) = y - 1.$$
The Cauchy-Riemann equations are satisfied because
$$\frac{\partial u}{\partial x} = 1 = \frac{\partial v}{\partial y}$$
and
$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = 0.$$
Since u, v and their first partial derivatives are continuous for all $x + iy$, f is differentiable for all z.

2.
$$f(z) = z^2 - iz = x^2 - y^2 + 2ixy - ix + y = x^2 - y^2 + y + i(2xy - x).$$
Then
$$u(x, y) = x^2 - y^2 + y, v(x, y) = 2xy - x.$$

Then

$$\frac{\partial u}{\partial x} = 2x = \frac{\partial v}{\partial y}$$

and

$$\frac{\partial u}{\partial y} = -2y + 1 = -\frac{\partial v}{\partial x}.$$

Since u, v and their first partial derivatives are continuous for all z, f is differentiable for all z.

3. $f(z) = |x + iy| = \sqrt{x^2 + y^2}$, so

$$u(x, y) = \sqrt{x^2 + y^2}, v(x, y) = 0.$$

If x and y are not both zero, then the partial derivatives are

$$\frac{\partial u}{\partial x} = \frac{x}{\sqrt{x^2 + y^2}}, \frac{\partial u}{\partial y} = \frac{y}{\sqrt{x^2 + y^2}},$$
$$\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = 0.$$

The Cauchy-Riemann equations are not satisfied at any point with both $x \neq 0$ and $y \neq 0$. The only point left to check is $z = 0$, where $x = y = 0$. Now the above expressions for the partial derivatives of v are still valid, but those for the partial derivatives of u are not, and we must fall back on the definition of the partial derivatives:

$$\begin{aligned}\frac{\partial u}{\partial x}(0, 0) &= \lim_{h \to 0} \frac{u(h, 0) - u(0, 0)}{h} \\ &= \lim_{h \to 0} \frac{\sqrt{h^2}}{h} \\ &= \lim_{h \to 0} \frac{|h|}{h}.\end{aligned}$$

This limit does not exist, because

$$\frac{|h|}{h} = \begin{cases} 1 & \text{if } h > 0, \\ -1 & \text{if } h < 0. \end{cases}$$

Similarly, $(\partial u/\partial y)(0, 0)$ does not exist. Thus the Cauchy-Riemann equations fail at every z, and this function is not differentiable anywhere.

4. $f(z)$ is defined for all nonzero z. For $z \neq 0$,

$$\begin{aligned}f(z) &= 2 + \frac{1}{z} = 2 + \frac{1}{x + iy} \\ &= 2 + \frac{x - iy}{x^2 + y^2} \\ &= 2 + \frac{x}{x^2 + y^2} - \frac{y}{x^2 + y^2} i.\end{aligned}$$

Therefore the real and imaginary parts of $f(z)$ are

$$u(x,y) = 2 + \frac{x}{x^2+y^2} \text{ and } v(x,y) = -\frac{y}{x^2+y^2}.$$

For $z \neq 0$, compute the partial derivatives

$$\begin{aligned}
\frac{\partial u}{\partial x} &= \frac{y^2-x^2}{(x^2+y^2)^2}, \\
\frac{\partial u}{\partial y} &= \frac{-2xy}{(x^2+y^2)^2}, \\
\frac{\partial v}{\partial x} &= \frac{2xy}{(x^2+y^2)^2}, \\
\frac{\partial v}{\partial y} &= \frac{y^2-x^2}{(x^2+y^2)^2}.
\end{aligned}$$

The Cauchy-Riemann equations hold at all nonzero z. Further, u, v, and their first partial derivatives are continuous for nonzero z, so f is differentiable at all nonzero z.

5.

$$f(z) = i|z|^2 = (x^2+y^2)i$$

so

$$u(x,y) = 0, v(x,y) = x^2+y^2.$$

Compute

$$\begin{aligned}
\frac{\partial u}{\partial x} &= \frac{\partial u}{\partial y} = 0, \\
\frac{\partial v}{\partial x} &= 2x, \frac{\partial v}{\partial y} = 2y.
\end{aligned}$$

The Cauchy-Riemann equations hold only at $z = 0$, so f is certainly not differentiable if $z \neq 0$. To determine if f is differentiable at 0, consider

$$\lim_{h\to 0} \frac{f(h)-f(0)}{h} = \lim_{h\to 0} \frac{i|h|^2}{h} = \lim_{h\to 0} i\left(\frac{|h|}{h}\right)|h| = 0$$

because $||h|/h| = 1$. Therefore $f'(0) = 0$.

6. $f(z) = x + iy + y = (x+y) + iy$, so

$$u(x,y) = x+y, v(x,y) = y.$$

Then

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = \frac{\partial v}{\partial y} = 1, \frac{\partial v}{\partial x} = 0.$$

The Cauchy-Riemann equations do not hold at any z, so f is not differentiable anywhere.

7.
$$f(z) = \frac{x+iy}{x} = 1 + \frac{y}{x}i,$$
so
$$u(x,y) = 1, v(x,y) = \frac{y}{x}$$
for $x \neq 0$. Then
$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = 0, \frac{\partial v}{\partial x} = -y/x^2, \frac{\partial v}{\partial y} = 1/x.$$
The Cauchy-Riemann equations do not hold at any point at which the function is defined, so f is not differentiable anywhere.

8. Write $f(z) = u(x,y) + iv(x,y)$. Then
$$\begin{aligned} f(z) &= (x+iy)^3 - 8(x+iy) + 2 \\ &= z^3 - 8x + 2 = u(x,y) + iv(x,y), \end{aligned}$$
so
$$u(x,y) = x^3 - 3xy^2 - 8x + 2, v(x,y) = 3x^2y - y^3 - 8y.$$
Then, at all $z = x + iy$,
$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2 - 8 = \frac{\partial v}{\partial y}$$
and
$$\frac{\partial u}{\partial y} = -6xy = -\frac{\partial v}{\partial x}.$$
The Cauchy-Riemann equations hold for all z. Since u, v and its first partial derivatives are continuous everywhere, f is differentiable for all z.

9.
$$f(z) = (\overline{z})^2 = (x - iy)^2 = x^2 - y^2 - 2xyi.$$
Then
$$u(x,y) = x^2 - y^2, v(x,y) = -2xy.$$
Then
$$\begin{aligned} \frac{\partial u}{\partial x} &= 2x, \frac{\partial u}{\partial y} = -2y, \\ \frac{\partial v}{\partial x} &= -2y, \frac{\partial v}{\partial y} = -2x. \end{aligned}$$
The Cauchy-Riemann equations hold only at $z = 0$. But
$$\begin{aligned} \lim_{h\to 0} \frac{f(h) - f(0)}{h} &= \lim_{h\to 0} \frac{(\overline{h})^2}{h} \\ &= \lim_{h\to 0} \left(\frac{\overline{h}}{h}\right)\overline{h} \\ &= 0 \end{aligned}$$
because $\overline{h}/h$ has magnitude 1, and $\overline{h} \to 0$ as $h \to 0$. Therefore $f'(0) = 0$.

10.
$$f(z) = iz + |z| = ix - y + \sqrt{x^2+y^2}.$$

Then
$$u(x,y) = \sqrt{x^2+y^2} - y, v(x,y) = x.$$

Now
$$\begin{aligned}\frac{\partial u}{\partial x} &= \frac{x}{\sqrt{x^2+y^2}},\\ \frac{\partial u}{\partial y} &= -1 + \frac{y}{\sqrt{x^2+y^2}},\end{aligned}$$

$$\frac{\partial v}{\partial x} = 1, \frac{\partial v}{\partial y} = 0.$$

The Cauchy-Riemann equations are not satisfied at any z, so f is not differentiable anywhere.

11.
$$f(z) = -4z + \frac{1}{z} = -4x - 4iy + \frac{1}{x+iy} = -4x - 4yi + \frac{x-iy}{x^2+y^2}$$

for $z \neq 0$. Then
$$u(x,y) = -4x + \frac{x}{x^2+y^2}, v(x,y) = -4y - \frac{y}{x^2+y^2}.$$

Compute
$$\frac{\partial u}{\partial x} = -4 + \frac{y^2-x^2}{(x^2+y^2)^2}, \frac{\partial u}{\partial y} = \frac{-2xy}{(x2+y^2)^2},$$

$$\frac{\partial v}{\partial x} = \frac{2xy}{(x^2+y^2)^2}, \frac{\partial v}{\partial y} = -4 + \frac{y^2-x^2}{(x^2+y^2)^2}.$$

The Cauchy-Riemann equations hold for all $z \neq 0$, so f is differentiable at all z at which it is defined ($z \neq 0$).

12.
$$\begin{aligned}f(z) = \frac{z-i}{z+i} &= \frac{x+i(y-1)}{x+i(y+1)}\\ &= \frac{x^2+y^2-1-2xi}{x^2+(y+1)^2}.\end{aligned}$$

Then
$$u(x,y) = \frac{x^2+y^2-1}{x^2+(y-1)^2} \text{ and } v(x,y)\frac{-2x}{x^2+(y+1)^2}.$$

Compute

$$\begin{aligned}\frac{\partial u}{\partial x} &= \frac{4x(y+1)}{(x^2+(y+1)^2)^2},\\ \frac{\partial u}{\partial y} &= \frac{2(y+1)^2-2x^2}{(x^2+(y+1)^2)^2},\\ \frac{\partial v}{\partial x} &= \frac{2x^2-2(y+1)^2}{(x^2+(y+1)^2)^2},\\ \frac{\partial v}{\partial y} &= \frac{4x(y+1)}{(x^2+(y+1)^2)^2}.\end{aligned}$$

f is not defined at $z=-i$. For all other z, the Cauchy-Riemann equations are satisfied, and u, v, and their first partial derivatives are continuous. Therefore f is differentiable for all $z \neq i$.

13. Let $z_n = x_n + iy_n$ and $z_0 = x_0 + iy_0$. Write $f(z) = u(x,y) + iv(x,y)$. Since u and v are continuous at (x_0, y_0), then

$$f(z_n) = u(x_n, y_n) + uv(x_n, y_n) \to x(x_0, y_0) + iv(x_0, y_0) = f(z_0).$$

19.3 The Exponential and Trigonometric Functions

1.

$$e^i = e^{0+i} = e^0(\cos(1) + i\sin(1)) = \cos(1) + i\sin(1).$$

2. There are several ways we can proceed. Perhaps the easiest is to use the fact that

$$\sin(x+iy) = \sin(x)\cosh(y) + i\cos(x)\sinh(y).$$

Then

$$\sin(1-4i) = \sin(1)\cosh(4) - i\cos(1)\sinh(4).$$

Here we have also used the fact that $\cosh(-4) = \cosh(4)$ and $\sinh(-4) = -\sinh(4)$.

Another approach is to begin with the definition of $\sin(z)$ and use Euler's formula:

$$\begin{aligned}
\sin(1-4i) &= \frac{1}{2i}\left(e^{i(1-4i)} - e^{-i(1-4i)}\right)\\
&= \frac{1}{2i}\left(e^{4+i} - e^{-4-i}\right)\\
&= \frac{1}{2i}\left(e^{4}e^{i} - e^{-4}e^{-i}\right)\\
&= \frac{1}{2i}\left(e^{4}(\cos(1)+i\sin(1)) - e^{-4}(\cos(1)-i\sin(1))\right)\\
&= \frac{1}{2i}(e^{4}-e^{-4})\cos(1) + \frac{1}{2}(e^{4}+e^{-4})\sin(1)\\
&= \cosh(4)\sin(1) + \frac{1}{i}\sinh(4)\cos(1)\\
&= \sin(1)\cosh(4) - i\cos(1)\sinh(4).
\end{aligned}$$

3. Use the fact that

$$\cos(x+iy) = \cos(x)\cosh(y) - i\sin(x)\sinh(y)$$

to write

$$\cos(3+2i) = \cos(3)\cosh(2) - i\sin(3)\sinh(2).$$

4. Observe that the trigonometric and hyperbolic functions are related by

$$\sin(z) = -i\sinh(iz) \text{ and } \cos(z) = \cosh(iz).$$

Then

$$\begin{aligned}
\tan(3i) &= \frac{\sin(3i)}{\cos(3i)} = \frac{-i\sinh(-3)}{\cosh(-3)}\\
&= \frac{i\sinh(3)}{\cosh(3)} = i\tanh(3).
\end{aligned}$$

5.

$$e^{5+2i} = e^{5}(\cos(2)+i\sin(2))$$

6.

$$\begin{aligned}
\cot(1-\pi i/4) &= \frac{\cos(1-\pi i/4)}{\sin(1-\pi i/4)}\\
&= \frac{\cos(1)\cosh(\pi/4) + i\sin(1)\sinh(\pi/4)}{\sin(1)\cosh(\pi/4) - i\cos(1)\sinh(\pi/4)}.
\end{aligned}$$

Multiply the numerator and denominator of this quotient by the conjugate of the denominator to obtain

$$\begin{aligned}
&\cot(1-\pi i/4)\\
&= \frac{(\cos(1)\cosh(\pi/4)+i\sin(1)\sinh(\pi/4))(\sin(1)\cosh(\pi/4)+i\cos(1)\sinh(\pi/4))}{\sin^2(1)\cosh^2(\pi/4)+\cos^2(1)\sinh^2(\pi/4)}\\
&= \frac{\sin(1)\cos(1)(\cosh^2(\pi/4)-\sinh^2(\pi/4))+i\sinh(\pi/4)\cosh(\pi/4)(\sin^2(1)+\cos^2(1))}{\sin^2(1)\cosh^2(\pi/4)+\cos^2(1)\sinh^2(\pi/4)}\\
&= \frac{\sin(1)\cos(1)+i\sinh(\pi/4)\cosh(\pi/4)}{\sin^2(1)\cosh^2(\pi/4)+(1-\sin^2(1))\sinh^2(\pi/4)}\\
&= \frac{\sin(1)\cos(1)+i\sinh(\pi/4)\cosh(\pi/4)}{\sin^2(1)\cosh^2(\pi/4)+(1-\sin^2(1))\sinh^2(\pi/4)}\\
&= \frac{\sin(1)\cos(1)+i\sinh(\pi/4)\cosh(\pi/4)}{\sin^2(1)\sinh^2(\pi/4)}.
\end{aligned}$$

7.

$$\begin{aligned}
\sin^2(1+i) &= \frac{1}{2}[1-\cos(2(1+i))]\\
&= \frac{1}{2}[1-\cos(2)\cosh(2)+i\sin(2)\sinh(2)]\\
&= \frac{1}{2}[1-\cos(2)\cosh(2)]+\frac{i}{2}[\sin(2)\sinh(2)].
\end{aligned}$$

8.

$$\begin{aligned}
&\cos(2-i)-\sin(2-i)\\
&= \cos(2)\cosh(1)+i\sin(2)\sinh(1)-\sin(2)\cosh(1)-i\cos(2)\sinh(1)\\
&= \cosh(1)[\cos(2)-\sin(2)]-i\sinh(1)[\cos(2)-\sin(2)]
\end{aligned}$$

9.

$$e^{\pi i/2} = \cos(\pi/2)+i\sin(\pi/2) = i$$

10.

$$\begin{aligned}
\sin(e^i) &= \sin(\cos(1)+i\sin(1))\\
&= \sin(\cos(1))\cosh(\sin(1))+i\cos(\cos(1))\sinh(\sin(1))
\end{aligned}$$

11. First,

$$\begin{aligned}
e^{z^2} &= e^{(x+iy)^2} = e^{x^2-y^2+2ixy}\\
&= e^{x^2-y^2}[\cos(2xy)+i\sin(2xy)].
\end{aligned}$$

Then

$$u(x,y) = e^{x^2-y^2}\cos(2xy) \text{ and } v(x,y) = e^{x^2-y^2}\sin(2xy).$$

Compute

$$\begin{aligned}
\frac{\partial u}{\partial x} &= e^{x^2-y^2}[2x\cos(2xy) - 2y\sin(2xy)],\\
\frac{\partial u}{\partial y} &= e^{x^2-y^2}[-2y\cos(2xy) - 2x\sin(2xy)],\\
\frac{\partial v}{\partial x} &= e^{x^2-y^2}[2x\sin(2xy) + 2y\cos(2xy)],\\
\frac{\partial v}{\partial y} &= e^{x^2-y^2}[-2y\sin(2xy) + 2x\cos(2xy)].
\end{aligned}$$

The Cauchy-Riemann equations are satisfied for all $x + iy$.

12. Write

$$\frac{1}{z} = \frac{1}{x+iy} = \frac{x-iy}{x^2+y^2},$$

so

$$e^{1/z} = e^{x/(x^2+y^2)}\left[\cos\left(\frac{y}{x^2+y^2}\right) - i\sin\left(\frac{y}{x^2+y^2}\right)\right].$$

Then

$$u(x,y) = e^{x/(x^2+y^2)}\cos\left(\frac{y}{x^2+y^2}\right), v(x,y) = -e^{x/(x^2+y^2)}\sin\left(\frac{y}{x^2+y^2}\right).$$

Then

$$\begin{aligned}
\frac{\partial u}{\partial x} &= \frac{e^{x/(x^2+y^2)}}{(x^2+y^2)^2}\left[(y^2-x^2)\cos\left(\frac{y}{x^2+y^2}\right) + 2xy\sin\left(\frac{y}{x^2+y^2}\right)\right],\\
\frac{\partial u}{\partial y} &= \frac{e^{x/(x^2+y^2)}}{(x^2+y^2)^2}\left[-2xy\cos\left(\frac{y}{x^2+y^2}\right) - (x^2-y^2)\sin\left(\frac{y}{x^2+y^2}\right)\right],\\
\frac{\partial v}{\partial x} &= \frac{e^{x/(x^2+y^2)}}{(x^2+y^2)^2}\left[(x^2-y^2)\sin\left(\frac{y}{x^2+y^2}\right) + 2xy\cos\left(\frac{y}{x^2+y^2}\right)\right],\\
\frac{\partial v}{\partial y} &= \frac{e^{x/(x^2+y^2)}}{(x^2+y^2)^2}\left[2xy\sin\left(\frac{y}{x^2+y^2}\right) - (x^2-y^2)\cos\left(\frac{y}{x^2+y^2}\right)\right].
\end{aligned}$$

The Cauchy-Riemann equations hold for all $z \neq 0$.

13.

$$\begin{aligned}
f(z) = ze^z &= (x+iy)e^x(\cos(y) + i\sin(y))\\
&= xe^x\cos(y) - ye^x\sin(y) + i(ye^x\cos(y) + xe^x\sin(y)),
\end{aligned}$$

so

$$u(x,y) = xe^x\cos(y) - ye^x\sin(y), v(x,y) = ye^x\cos(y) + xe^x\sin(y).$$

Then

$$\frac{\partial u}{\partial x} = e^x(\cos(y) + x\cos(y) - y\sin(y)) = \frac{\partial v}{\partial y}$$

and
$$\frac{\partial u}{\partial y} = e^x(-x\sin(y) - \sin(y) - y\cos(y)) = -\frac{\partial v}{\partial x}.$$
The Cauchy-Riemann equations hold for all z.

14. Write
$$\begin{aligned} f(z) = \cos^2(z) &= \frac{1}{2}(1-\cos(2z)) \\ &= \frac{1}{2} - \frac{1}{2}(\cos(2x)\cosh(2y) - i\sin(2x)\sinh(2y)). \end{aligned}$$
Then
$$u(x,y) = \frac{1}{2} - \frac{1}{2}\cos(2x)\cosh(2y), v(x,y) = \frac{1}{2}\sin(2x)\sinh(2y).$$
Then
$$\begin{aligned} \frac{\partial u}{\partial x} &= \sin(2x)\cosh(2y), \\ \frac{\partial u}{\partial y} &= -\cos(2x)\sinh(2y), \\ \frac{\partial v}{\partial x} &= \cos(2x)\sinh(2y), \\ \frac{\partial v}{\partial y} &= \sin(2x)\cosh(2y). \end{aligned}$$
The Cauchy-Riemann equations are satisfied at every z.

15. If $e^z = e^{x+iy} = 2i$, then
$$e^x[\cos(y) + i\sin(y)] = 2i.$$
Equating real and imaginary parts, we obtain
$$e^x\cos(y) = 0, e^x\sin(y) = 2.$$
Since $e^x \neq 0$ for all real x, then $\cos(y) = 0$, so
$$y = \frac{2n+1}{2}\pi$$
in which n can be any integer. This means that we need
$$e^x\sin\left(\frac{2n+1}{2}\pi\right) = 2.$$
Now $e^x > 0$ for all real x, so we must have
$$\sin\left(\frac{2n+1}{2}\pi\right) > 0.$$

But the quantity on the left is equal to 1 if n is even, and to -1 if n is odd. This means that n must be an even integer, say $n = 2m$, with m any integer. Therefore

$$y = \frac{4m+1}{2}\pi.$$

Then $\sin(y) = 1$ and we are left with $e^x = 2$, so $x = \ln(2)$. All the solutions of $e^z = 2i$ are

$$\ln(2) + \frac{4m+1}{2}\pi i,$$

with m any integer.

16. To prove the first identity, write

$$\begin{aligned}
&\sin(z)\cos(w) + \cos(z)\sin(w)\\
&= \frac{1}{4i}\left[(e^{iz} - e^{-iz})(e^{iw} + e^{-iw}) + (e^{iz} + e^{-iz})(e^{iw} - e^{-iw})\right]\\
&= \frac{1}{4i}\left[e^{i(z+w)} - e^{-i(z+w)} - e^{i(w-z)} + e^{i(z-w)} + e^{i(z+w)} - e^{-i(z+w)} + e^{i(w-z)} - e^{i(z-w)}\right]\\
&= \frac{1}{2i}\left(e^{i(z+w)} - e^{-i(z+w)}\right)\\
&= \sin(z+w).
\end{aligned}$$

For the second identity, argue similarly:

$$\begin{aligned}
&\cos(z)\cos(w) - \sin(z)\sin(w)\\
&= \frac{1}{4}\left[(e^{iz} + e^{-iz})(e^{iw} + e^{-iw}) + (e^{iz} - e^{-iz})(e^{iw} - w^{-iw})\right]\\
&= \frac{1}{4}\left[e^{i(z+w)} + e^{-i(z+w)} + e^{i(w-z)} + e^{-i(w-z)} + e^{i(z+w)} + e^{-i(z+w)} - e^{i(w-z)} - e^{i(z-w)}\right]\\
&= \frac{1}{2}\left(e^{i(z+w)} + e^{-i(z+w)}\right)\\
&= \cos(z+w).
\end{aligned}$$

17. It is convenient to use the polar form of the given equation:

$$e^z = e^r e^{i\theta} = -2.$$

Since $|e^{i\theta)}| = 1$ if θ is real, then

$$|e^z| = e^r = |-2| = 2,$$

so $r = \ln(2)$. Further, we must have

$$e^{i\theta} = -1 = \cos(\theta) + i\sin(\theta),$$

so $\sin(\theta) = 0$ and $\cos(\theta) = -1$. Then $\theta = (2n+1)\pi$, for any integer n. The solutions are therefore

$$z = \ln(2) + (2n+1)\pi, \text{ with } n \text{ any integer}.$$

18. We want all solutions of $\sin(z) = i$. Write this equation as

$$\sin(x + iy) = \sin(x)\cosh(y) + i\cos(x)\sinh(y) = i.$$

Then

$$\sin(x)\cosh(y) = 0, \cos(x)\sinh(y) = 1.$$

Since $\cosh(y) \neq 0$ for real y, the first equation requires that $\sin(x) = 0$, so $x = n\pi$ for any integer n. The second equation becomes

$$\cos(x)\sinh(y) = \cos(n\pi)\sinh(y) = (-1)^n \sinh(y) = 1.$$

Then

$$\sinh(y) = (-1)^n = \frac{1}{2}\left(e^y - e^{-y}\right).$$

Then

$$e^y - e^{-y} = 2(-1)^n,$$

so

$$e^{2y} - 2(-1)^n - 1 = 0.$$

This is a quadratic equation for e^y, which we will solve. Consider cases on n.

If n is even, then the quadratic equation is

$$e^{2y} - 2e^y - 1 = 0$$

with roots

$$e^y = 1 \pm \sqrt{2}.$$

Since $1 - \sqrt{2} < 0$ and $e^y > 0$, discard one root and write

$$e^y = 1 + \sqrt{2}.$$

Then $y = \ln(1 + \sqrt{2})$. With $n = 2m$ in this case, we have obtained the solutions

$$z = 2m + i\ln(1 + \sqrt{2}),$$

with m any integer.

The second case is that n is odd. Now

$$e^{2y} + 2e^y - 1 = 0,$$

with roots

$$e^y = -1 \pm \sqrt{2}.$$

Again, $-1 - \sqrt{2} < 0$, so discard this root and write

$$e^y = -1 + \sqrt{2}.$$

In this case that n is odd, write $n = 2m + 1$ to obtain the solutions

$$z = (2m + 1) + i\ln(-1 + \sqrt{2}),$$

in which m can be any integer.

19.4 The Complex Logarithm

In these problems, $\ln(x)$ denotes the real natural logarithm of x, if x is a positive number. The complex logarithm of z is denoted $\log(z)$.

1. In polar form,
$$z = -4i = 4e^{3n\pi i/2}.$$
Then
$$\log(-4i) = \ln(4) + \left(\frac{3\pi}{2} + 2n\pi\right)i.$$

2. Write
$$2 - 2i = 2\sqrt{2}e^{7\pi i/4}$$
so
$$\log(2-2i) = \ln(2\sqrt{2}) + \left(\frac{7\pi}{4} + 2n\pi\right)i.$$

3. Since $-5 = 5e^{\pi i}$, then
$$\log(-5) = \ln(5) + (2n+1)\pi i.$$

4. Write
$$1 + 5i = \sqrt{26}e^{i\arctan(5)}$$
to obtain
$$\log(1+5i) = \frac{1}{2}\ln(26) + (\arctan(5) + 2n\pi)i.$$

5. Write
$$-9 + 2i = \sqrt{85}e^{(\arctan(-2/9)+\pi)i}$$
to obtain
$$\log(-9+2i) = \frac{1}{2}\ln(85) + (-\arctan(2/9) + (2n+1)\pi)i.$$

6. Since $5 = 5e^{i\theta}$ with $\theta = 0$, then
$$\log(5) = \ln(5) + 2n\pi i.$$
Notice that there are infinitely many complex logarithms of 5, even though 5 is a positive real number with a natural logarithm.

7. Because the complex logarithm of a nonzero number has infinitely many values, we cannot expect to have $\log(zw)$ equal to $\log(z) + \log(w)$. What we claim is that each value of $\log(zw)$ is equal to some value of $\log(z)$ added to some value of $\log(w)$.

To verify this, let z and w be nonzero numbers. Let θ_z be any argument of z and θ_w any argument of w. Then

$$z = |z|e^{(\theta_1+2n\pi)i}, w = |w|e^{(\theta_2+2m\pi)i}$$

and

$$zw = |zw|e^{(\theta_1+\theta_2+2k\pi)i}.$$

Thus

$$\log(zw) = \ln(|zw|) + (\theta_1 + \theta_2 + 2k\pi)i,$$

while

$$\log(z) + \log(w) = ln(|z|) + \ln(|w|) + i(\theta_1 + \theta_2 + 2(n+m)\pi)i.$$

This means that, for any choice of n and m, we can choose $k = n + m$ to obtain a value of $\log(zw)$ that is equal to $\log(z) + \log(w)$.

8. The argument is almost identical to that of the solution to Problem 7, except now we have

$$\frac{z}{w} = \frac{|z|}{|w|}e^{(\theta_1-\theta_2+2(n-m)\pi)i}.$$

19.5 Powers

In these problems, n always denotes an arbitrary integer.

1.

$$\begin{aligned} i^{1+i} &= e^{(1+i)\log(i)} = e^{(1+i)((\pi/2+2n\pi)i)} \\ &= e^{-(\pi/2+2n\pi)}\left[\cos\left(\frac{\pi}{2}+2n\pi\right) + i\sin\left(\frac{\pi}{2}+2n\pi\right)\right] \\ &= ie^{-(\pi/2+2n\pi)}, \end{aligned}$$

since $\sin(2n\pi + \pi/2) = 1$ for each integer n

2.

$$\begin{aligned} (1+i)^{2i} &= e^{2i\log(1+i)} = e^{2i[\ln(\sqrt{2})+i(\pi/4+2n\pi)]} \\ &= e^{-(\pi/2+4n\pi)}[\cos(\ln(2)) + i\sin(\ln(2))] \end{aligned}$$

3.

$$i^i = e^{i\log(i)} = e^{i(i(\pi/2+2n\pi))} = e^{-(\pi/2+2n\pi)}$$

This is consistent with Problem 1, since $i^{1+i} = i(i^i)$.

4.

$$\begin{aligned}(1+i)^{2-i} &= e^{(2-i)\log(1+i)}\\ &= e^{(2-i)(\ln(\sqrt{2})+i(\pi/4+2n\pi))}\\ &= e^{\ln(2)+\pi/4+2n\pi}\left[\cos\left(\frac{\pi}{2}+4n\pi-\ln(\sqrt{2})\right)+i\sin\left(\frac{\pi}{2}+4n\pi-\ln(\sqrt{2})\right)\right]\\ &= 2e^{\pi/4+2n\pi}[\sin(\ln(\sqrt{2}))+i\cos(\ln(\sqrt{2}))]\end{aligned}$$

5.

$$\begin{aligned}(-1+i)^{-3i} &= e^{-3i\log(1+i)}\\ &= e^{-3i(\ln(\sqrt{2})+i(3\pi/4+2n\pi))}\\ &= e^{(9\pi/4+6n\pi)}[\cos(3\ln(\sqrt{2}))-i\sin(3\ln(\sqrt{2}))]\end{aligned}$$

6.

$$\begin{aligned}(1-i)^{1/3} &= \left(\sqrt{2}e^{-i(\pi/4+2n\pi)}\right)^{1/3}\\ &= 2^{1/6}e^{-i(\pi/12+2n\pi/3)}\end{aligned}$$

We obtain distinct powers only for $n = 0, 1, 2$. Other choices of n repeat the powers obtained for $n = 0, 1, 2$.

7.

$$\begin{aligned}i^{1/4} &= \left(e^{i(\pi/2+2n\pi)}\right)^{1/4}\\ &= e^{i(\pi/8+n\pi/2)}\end{aligned}$$

We obtain distinct values only for $n = 0, 1, 2, 3$. Other choices of n repeat these values.

8.

$$16^{1/4} = \left(16e^{2n\pi i}\right)^{1/4} = 2e^{n\pi/2},$$

with all distinct values obtained with $n = 0, 1, 2, 3$. These values are ± 2 and $\pm 2i$.

9.

$$\begin{aligned}(-4)^{2-i} &= e^{(2-i)\log(-4)} = e^{(2-i)(\ln(4)+i(\pi+2n\pi))}\\ &= e^{2\ln(4)+\pi+2n\pi}e^{i(2\pi+4n\pi-\ln(4))}\\ &= 16e^{(2n+1)\pi}[\cos(\ln(4))-i\sin(\ln(4))]\end{aligned}$$

10.

$$\begin{aligned}6^{-2-3i} &= e^{(-2-3i)\log(6)} \\ &= e^{(-2-3i)(\ln(6)+2n\pi i)} \\ &= e^{-2\ln(6)+6n\pi}e^{-(3\ln(6)+4n\pi)i} \\ &= \frac{1}{36}e^{6n\pi}[\cos(3\ln(6)) - i\sin(3\ln(6))]\end{aligned}$$

11.

$$(-16)^{1/4} = \left(16e^{i(\pi+2n\pi)}\right)^{1/4} = 2e^{i(\pi/4+n\pi/2)},$$

for $n = 0, 1, 2, 3$. These values are

$$\sqrt{2}(1+i), \sqrt{2}(-1+i), \sqrt{2}(-1-i), \sqrt{2}(1-i).$$

12. First compute

$$\frac{1+i}{1-i} = \frac{(1+i)(1+i)}{(1-i)(1+i)} = i.$$

Therefore we want

$$i^{1/3} = \left(e^{i(\pi/2+2n\pi)}\right)^{1/3} = e^{i(\pi/6+2n\pi/3)},$$

which has the values (for $n = 0, 1, 2$)

$$\frac{1}{2}(\sqrt{3}+i), \frac{1}{2}(-\sqrt{3}+i), -i.$$

13. These are the sixth roots of unity:

$$1^{1/6} = \left(e^{2n\pi i}\right)^{1/6} = e^{n\pi i/3}$$

for $n = 0, 1, 2, 3, 4, 5$. These values are

$$1, \frac{1}{2}(1+\sqrt{3}i), \frac{1}{2}(-1+\sqrt{3}i), -1, \frac{1}{2}(-1-\sqrt{3}i), \frac{1}{2}(1-\sqrt{3}i).$$

14.

$$\begin{aligned}(7i)^{3i} &= e^{3i\log(7i)} = e^{3i(\ln(7)+i(\pi/2+2n\pi))} \\ &= e^{-3(\pi/2+2n\pi)}[\cos(3\ln(7)) + i\sin(3\ln(7))]\end{aligned}$$

15. Let ω be any nth root of unity different from 1. The numbers ω^j, for $j = 0, 1, \cdots, n-1$, are distinct, hence are all the nth roots of unity. Thus it is enough to show that

$$\sum_{j=0}^{n-1} \omega^j = 0.$$

But,

$$\sum_{j=0}^{n-1} \omega^j = \frac{1-\omega^n}{1-\omega} = 0.$$

We could also reason as follows. Let $\omega_1, \cdots, \omega_n$ be the nth roots of unity. Suppose $\omega_1 \neq 1$ and let $S = \sum_{j=1}^{n} \omega_j$. Then

$$\omega_1 S = \sum_{j=1}^{n} \omega_1 \omega_j = S$$

because the n numbers $\omega_1\omega_1, \cdots, \omega_1\omega_n$ are also the nth roots of unity. But then

$$S(1-\omega_1) = 0.$$

Since $\omega_1 \neq 1$, then $S = 0$.

16. Since, for $a \neq 1$,

$$\sum_{j=0}^{n-1} a^j = \frac{1-a^n}{1-a},$$

we can replace a with $-a$ to obtain

$$\sum_{j=0}^{n-1} (-a)^j = \sum_{j=0}^{n-1} (-1)^j a^j = \frac{1-(-1)^n a^n}{1+a}.$$

Apply this with $a = e^{2\pi i/n}$ and use the fact that $a^n = e^{2\pi i} = 1$ to write

$$\sum_{j=0}^{n-1} (-1)^j e^{2\pi ij/n} = \frac{1-(-1)^n}{1+e^{2\pi i/n}} = \begin{cases} 0 & \text{if } n \text{ is even,} \\ 2/(1+e^{2\pi i/n}) & \text{if } n \text{ is odd.} \end{cases}$$

Chapter 20

Complex Integration

20.1 The Integral of a Complex Function

1. Since $f(z) = 1$ is differentiable for all z, we can also write the antiderivative $F(z) = z$. Since $\gamma(1) = 1 - i$ and $\gamma(3) = 9 - 3i$,

$$\int_\gamma f(z)dz = F(9-3i) - F(1-i) = (9-3i) - (1-i) = 8 - 2i.$$

Alternatively, we can use the parametric equations of γ to obtain

$$\begin{aligned}\int_\gamma dz &= \int_1^3 \gamma'(t)\,dt \\ &= \int_1^3 (2t - i)\,dt = \left[t^2 - ti\right]_1^3 \\ &= (9-3i) - (1-i) = 8 - 2i\end{aligned}$$

2. $f(z) = z^2 - iz$ has the antiderivative $F(z) = \frac{1}{3}z^3 - i\frac{z^2}{2}$ for all z, so

$$\int_\gamma f(z)dz = F(2i) - F(2) = -\frac{8}{3} + \frac{4}{3}i.$$

If we want to do this by parametrizing the curve, one way is to write $\gamma(t) = 2e^{it}$, for $0 \le t \le \pi/2$. Then

$$\begin{aligned}\int_\gamma (z^2 - iz)\,dz &= \int_0^{\pi/2} (4e^{2i\theta} - 2ie^{i\theta})(2ie^{i\theta})\,d\theta \\ &= \int_0^{\pi/2} (8ie^{3i\theta} + 4e^{2i\theta})\,d\theta \qquad = \left[\frac{8}{3}e^{3i\theta} - 2ie^{2i\theta}\right]_0^{\pi/2} \\ &= \frac{8}{3}(-i) + 2i - \left(\frac{8}{3} - 2i\right) \\ &= -\frac{8}{3} + \frac{4}{3}i.\end{aligned}$$

3. $f(z) = \mathrm{Re}(z)$ does not have an antiderivative because f is not differentiable, so we must proceed by parametrizing C. This can be done in many ways, but one is to write $\gamma(t) = 1 + (1+i)t$ for $0 \le t \le 1$. Then, on the curve, $\mathrm{Re}(z) = 1 + t$ and

$$\int_\gamma f(z)\,dz = \int_0^1 (1+t)(1+i)\,dt = \frac{3}{2}(1+i).$$

4. f does not have an antiderivative on an open set containing γ, so parametrize C by $z = 4e^{it}$ for $\pi/2 \le t \le 3\pi/2$. These are polar coordinates in complex notation. Then

$$\int_\gamma \frac{1}{z}\,dz = \int_{\pi/2}^{3\pi/2} \frac{1}{4e^{it}}(4ie^{it})\,dt = \pi i.$$

5. An antiderivative is $F(z) = (z-1)^2/2$, so

$$\int_\gamma f(z)\,dz = F(1-4i) - F(2i) = -\frac{13}{2} + 2i.$$

6. An antiderivative of f is $F(z) = iz^3/3$, so

$$\int_\gamma f(z)\,dz = \left.\frac{i}{3}z^3\right]_{1+2i}^{3+i} = \frac{1}{3}(-28+29i).$$

7. An antiderivative is $F(z) = -\frac{1}{2}\cos(2z)$, so

$$\int_\gamma \sin(2z)\,dz = \left.-\frac{1}{2}\cos(2z)\right]_{-i}^{-4i}$$
$$= -\frac{1}{2}(\cos(-4i) - \cos(-i)) = -\frac{1}{2}[\cosh(8) - \cosh(2)].$$

8. An antiderivative is $F(z) = z + z^3/3$, so

$$\int_\gamma (1+z^2)\,dz = F(3i) - F(-3i) = -12i.$$

9. An antiderivative is $F(z) = -i\sin(z)$, so

$$\int_\gamma -i\cos(z)\,dz = \left.-\sin(z)\right]_0^{2+i}$$
$$= -i\sin(2+i) = -i[\sin(2)\cosh(1) + i\cos(2)\sinh(1)]$$
$$= -\cos(2)\sinh(1) - i\sin(2)\cosh(1).$$

10. f has no antiderivative, so proceed by parametrizing the curve. Describe γ by $\gamma(t) = -4 + t(4+i)$ for $0 \le t \le 1$. On this curve,

$$|z|^2 = 16(t-1)^2 + t^2.$$

Therefore

$$\begin{aligned}\int_\gamma |z|^2\,dz &= \int_0^1 [16(t-1)^2 + t^2](4+i)\,dt \\ &= \frac{4+i}{3}\left[16(t-1)^3 + t^3\right]_0^1 \\ &= \frac{17}{3}(4+i).\end{aligned}$$

11. An antiderivative is $F(z) = (z-i)^4/4$, so

$$\int_\gamma (z-i)^3\,dz = F(2-4i) - F(0) = 10 + 210i$$

12.

$$\begin{aligned}\int_\gamma e^{iz}\,dz &= \left[-ie^{iz}\right]_{-2}^{-4-i} \\ &= -i(e^{1-4i} - e^{-2i}) \\ &= -e\sin(4) + \sin(2) + [\cos(2) - e\cos(4)]i\end{aligned}$$

13. $i\overline{z}$ has no antiderivative, so proceed by parametrizing γ. One way is to write $\gamma(t) = (-4+3i)t$, $0 \le t \le 1$. Then

$$\begin{aligned}\int_\gamma i\overline{z}\,dz &= \int_0^1 -i(4t - 3ti)(-4+3i)\,dt \\ &= (-4+3i)\left(\frac{3}{2} - 2i\right) = \frac{25}{2}i\end{aligned}$$

14. Since $\text{Im}(z)$ has no antiderivative, parametrize the curve by $\gamma(t) = 4e^{it}$, $0 \le t \le 2\pi$. Then

$$\begin{aligned}\int_\gamma \text{Im}(z)\,dz &= \int_0^{2\pi} 4\sin(t)4ie^{it}\,dt \\ &= \int_0^{2\pi} 16(-\sin^2(t) + i\cos(t)\sin(t))\,dt = -16\pi\end{aligned}$$

15. Since f has no antiderivative, write the curve as $\gamma(t) = (1+i)t - i$ for $0 \le t \le 1$. Then

$$\int_\gamma |z|^2\,dz = \int_0^1 [t^2 + (t-1)^2](1+i)\,dt = \frac{2}{3}(1+i)$$

16. Use the fact that

$$\left|\int_\gamma \cos(z^2)\,dz\right| \le 8\pi M,$$

where 8π is the length of γ and M is a positive number such that

$$|\cos(z^2)| \le M \text{ for } z \text{ on } \gamma.$$

With $z = x + iy$, $z^2 = x^2 - y^2 + 2ixy$, so

$$\begin{aligned}|\cos(z^2)| &= |\cos(x^2 - y^2 + 2ixy)| \\ &= |\cos(x^2-y^2)\cosh(2xy) - i\sin(x^2-y^2)\sinh(2xy)| \\ &\le \cosh(2xy) + |\sinh(2xy)| = e^{2xy}.\end{aligned}$$

For points on γ, $x = 4\cos(t)$ and $y = 4\sin(t)$ for $0 \le t \le 2\pi$, so

$$\begin{aligned}e^{2xy} &= e^{2(4\cos(t))(4\sin(t))} = e^{32\sin(t)\cos(t)} \\ &= e^{16\sin(2t)} \le e^8.\end{aligned}$$

We can choose $M = e^{16}$ to obtain

$$\left|\int_\gamma \cos(z^2)\,dz\right| \le 8\pi e^{16}.$$

There is no claim that this huge upper bound is close to the actual value of $|\int_\gamma \cos(z^2)\,dz|$. We made very crude but quick estimates to derive a number M, but much smaller numbers might also work. Often the point to obtaining a bound is to take a limit in which the integral appears, and then it is often enough to know that the integral is bounded.

17. The length of γ is $\sqrt{5}$. Now we need a number M so that

$$\left|\frac{1}{1+z}\right| \le M \text{ for } z \text{ on } \gamma.$$

Now, the point on γ closest to $z = -1$ is $2 + i$, so for z on γ,

$$|z+1| = |z-(-1)| \ge |2+i+1| = \sqrt{10}.$$

Then

$$\left|\frac{1}{z+1}\right| = \frac{1}{|z+1|} \le \frac{1}{\sqrt{10}}$$

and we can choose $M = 1/\sqrt{10}$. Then

$$\left|\int_\gamma \frac{1}{1+z}\,,dz\right| \le \frac{\sqrt{5}}{\sqrt{10}} = \frac{1}{\sqrt{2}}.$$

20.2 Cauchy's Theorem

1. Since $\sin(3z)$ is differentiable everywhere, hence on the curve and at all points enclosed by γ, then

$$\oint_\gamma \sin(3z)\,dz = 0$$

by Cauchy's theorem.

2. The circle encloses i, at which $f(z)$ is not defined. Thus Cauchy's theorem does not apply. Evaluate the integral by parametrizing the curve $z = i + 3e^{it}$ for $0 \le t \le 2\pi$. Then

$$\begin{aligned}\oint_\gamma \frac{2z}{z-i}\,dz &= \int_0^{2\pi} \left(\frac{2i+6e^{it}}{3e^{it}}\right) 3ie^{it}\,dt \\ &= \int_0^{2\pi} (-2+6ie^{it})\,dt = -4\pi.\end{aligned}$$

3. γ encloses $2i$, at which the function is not defined, hence not differentiable. Parametrize γ by

$$\gamma(t) = 2i + 2e^{it} \text{ for } 0 \le t \le 2\pi.$$

Then

$$\begin{aligned}\oint_\gamma \frac{1}{(z-2i)^3}\,dt &= \int_0^{2\pi} \frac{1}{(2e^{it})^2} 2ie^{it}\,dt \\ &= \frac{i}{4}\int_0^{2\pi} e^{-2it}\,dt = 0.\end{aligned}$$

This integral turns out to be 0, but we could not have concluded this from Cauchy's theorem, which does not apply to this integral.

4. Since $z^2\sin(z)$ is differentiable everywhere, this function is differentiable on the curve and at all points enclosed by the curve, so

$$\oint_\gamma z^2\sin(z)\,dz = 0.$$

5. $f(z) = \overline{z}$ is not differentiable, so Cauchy's theorem does not apply. Write $\gamma(t) = e^{it}$ for $0 \le t \le 2\pi$. Then

$$\oint_\gamma \overline{z}\,dz = \int_0^{2\pi} e^{-it} ie^{it}\,dt = 2\pi i.$$

6. $f(z) = 1/\overline{z}$ is not differentiable, so Cauchy's theorem does not apply. Parametrize $\gamma(t) = 5e^{it}$ for $0 \leq t \leq 2\pi$ to obtain

$$\begin{aligned}\oint_\gamma \frac{1}{\overline{z}}\,dz &= \int_0^{2\pi} \frac{1}{5e^{-it}} 5ie^{it}\,dt \\ &= \int_0^{2\pi} ie^{2it}\,dt = 0.\end{aligned}$$

7. Since f is differentiable on the circle and at all of the points it encloses, then

$$\oint_\gamma ze^z\,dz = 0$$

by Cauchy's theorem.

8. A polynomial is differentiable everywhere, so by Cauchy's theorem,

$$\oint_\gamma (z^2 - 4z + i)\,dz = 0.$$

9. $f(z) = |z|^2$ is not differentiable, so Cauchy's theorem does not apply. Parametrize $\gamma(t) = 7e^{it}$ for $0 \leq t \leq 2\pi$. Since $|z| = 7$ on the curve, then

$$\oint_\gamma |z|^2\,dz = \int_0^{2\pi} 49(7)ie^{it}\,dt = 0.$$

10. $f(z) = \sin(1/z)$ is differentiable at all z except 0, which is not on or enclosed by γ. Therefore Cauchy's theorem applies and

$$\oint_\gamma \sin\left(\frac{1}{z}\right)\,dz = 0.$$

11. $f(z) = \mathrm{Re}(z)$ is not differentiable, so parametrize $\gamma(t) = 2e^{it}$ for $0 \leq t \leq 2\pi$. Then

$$\begin{aligned}\oint_\gamma \mathrm{Re}(z)\,dz &= \int_0^{2\pi} 2\cos(t)(2ie^{it})\,dt \\ &= \int_0^{2\pi} [4i\cos^2(t) - 4\cos(t)\sin(t)]\,dt \\ &= 4\pi i.\end{aligned}$$

12. z^2 is differentiable for all z, so by Cauchy's theorem,

$$\oint_C z^2\,dz = 0.$$

We need only evaluate $\oint_C \text{Im}(z)\,dz$. Cauchy's theorem does not apply to this integral because $\text{Im}(z)$ is not a differentiable function. Parametrize each side of the square. Let S_1 bes the left side (0 to $-2i$), S_2 the lower side ($-2i$ to $2-2i$), S_3 the right side ($2-2i$ to 2), and S_4 the top side (2 to 0), oriented counterclockwise. We can parametrize

$$\begin{aligned}
&S_1 : z = -2it,\\
&S_2 : z = 2t - 2i,\\
&S_3 : z = 2 - 2i(1-t),\\
&S_4 : z = 2(1-t),
\end{aligned}$$

all preserving counterclockwise orientation as t increases from 0 to 1. Now

$$\begin{aligned}
\oint_\gamma \text{Im}(z))\,dz = &\int_0^1 (-2t)(-2i)\,dt\\
&+ \int_0^1 (-2)(2)\,dt + \int_0^2 -2(1-t)(2i)\,dt + \int_0^1 0\,dt\\
&= 2i - 4 - 2i = -4.
\end{aligned}$$

Therefore

$$\oint_C f(z)\,dz = -4.$$

20.3 Consequences of Cauchy's Theorem

For some of these problems, be on the alert to the possibility of using Cauchy's integral formula, or Cauchy's integral formula for derivatives.

1. Since $2i$ is the center of the circle γ, we can apply the Cauchy integral formula with $f(z) = z^4$ to write

$$\oint_\gamma \frac{z^4}{z-2i}\,dz = 2\pi i f(2i) = 2\pi i(2i)^4 = 32\pi i.$$

2. By Cauchy's integral formula, with $f(z) = \sin(z^2)$,

$$\oint_\gamma \frac{\sin(z^2)}{z-5}\,dz = 2\pi i f(5) = 2\pi i \sin(25).$$

3. By Cauchy's integral formula, with $f(z) = z^2 - 5z + i$,

$$\begin{aligned}
\oint_\gamma \frac{z^2-5z+i}{z-1+2i}\,dz &= 2\pi i f(1-2i)\\
= 2\pi i[(1-2i)^2 - 5(1-2i) + i] &= 2\pi i(-8+7i)\\
&= -14\pi - 16\pi i.
\end{aligned}$$

4. Apply the Cauchy integral formula for derivatives, with $n = 1$ and $f(z) = 2z^3$, to write
$$\oint_\gamma \frac{2z^3}{(z-2)^2}\,dz = 2\pi i f'(2) = 48\pi i.$$

5. We can use the Cauchy integral formula for the derivative of a function ($n = 1$). With $f(z) = ie^z$, we have
$$\oint_\gamma \frac{ie^z}{(z-2+i)^2}\,dz = 2\pi i f'(2-i)$$
$$= 2\pi i(ie^{2-i}) = -2\pi e^2[\cos(1) - \sin(1)i].$$

6. Apply Cauchy's integral formula for derivatives with $n = 2$ and $f(z) = \cos(z-i)$ to write
$$\oint_\gamma \frac{\cos(z-i)}{(z+2i)^3}\,dz = \frac{2\pi i}{2} f''(-2i)$$
$$= -\pi i\cos(-3i) = -\pi i\cosh(3).$$

7. With $f(z) = z\sin(3z)$ and $n = 2$ in Cauchy's formula for derivatives,
$$\oint_\gamma \frac{z\sin(3z)}{(z+4)^3} = \frac{2\pi i}{2} f''(-4)$$
$$= \pi i[6\cos(12) - 36\sin(12)].$$

8. γ is not a closed curve and the Cauchy integral formulas do not apply. Parametrize γ by $\gamma(t) = 1 - t - it$ for $0 \le t \le 1$. On the curve,
$$f(z) = 2i\overline{z}|z| = 2i[(1-t)+it]\sqrt{1-2t+2t^2}$$
so
$$\int_\gamma 2i\overline{z}|z|\,dz = \int_0^1 2i[1-t+it]\sqrt{1-2t+2t^2}(-1-i)\,dt$$
$$= 2\int_0^1 \sqrt{1-2t+2t^2}\,dt + 2i\int_0^1 (2t-1)\sqrt{1-2t+2t^2}\,dt$$
$$= 1 + \sqrt{2}4\ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right).$$

These integrations can be done using MAPLE, or the indefinite integrals
$$\int \sqrt{1-2t+2t^2}\,dt = \frac{-1+4t}{8}\sqrt{1-2t+2t^2}$$
$$+\frac{7\sqrt{2}}{32}\operatorname{arcsinh}\left(\frac{4}{\sqrt{7}}\left(t-\frac{1}{4}\right)\right)$$
and
$$\int (2t-1)\sqrt{1-2t+2t^2}\,dt = \frac{1}{3}(1-2t+2t^2)^{3/2}.$$

9.

$$\begin{aligned}\oint_\gamma \frac{-(2+i)\sin(z^4)}{(z+4)^2}\,dz &= -2\pi i(2+i)\frac{d}{dz}(\sin(z^4)\Big|_{z=-4}\\ &= 2\pi i(1-2i)\left[4z^3\cos(z^4)\right]_{z=-4}\\ &= -512\pi(1-2i)\cos(256)\end{aligned}$$

10. γ is not a closed curve. An antiderivative of $(z-i)^2$ is $(z-i)^3/3$, so

$$\begin{aligned}\int_\gamma (z-i)^2\,dz &= \frac{1}{3}(z-i)^3\Big]_i^{-i}\\ &= \frac{1}{3}(-2i)^3 = \frac{8}{3}i\end{aligned}$$

11. Parametrize the curve by $\gamma(t) = 3 - t + (1-6t)i$. Then

$$\begin{aligned}\int_\gamma \mathrm{Re}(z+4)\,dz &= \int_0^1 (7-t)(-1-6i)\,dt\\ &= (-1-6i)\frac{13}{2} = -\frac{13}{2} - 39i.\end{aligned}$$

12.

$$\begin{aligned}\oint_\gamma \frac{3z^2\cosh(z)}{(z+2i)^2} &= 2\pi i\frac{d}{dz}(3z^2\cosh(z))\Big|_{z=-2i}\\ &= 2\pi i\left[6\cosh(z) - 3z^2\sinh(z)\right]_{z=-2i}\\ &= 2\pi i[-12i\cosh(2i) + 12\sinh(2i)]\\ &= 24\pi[\cos(2) - \sin(2)]\end{aligned}$$

13. First evaluate

$$\oint_\gamma \frac{e^z}{z}\,dz$$

by the Cauchy integral formula to obtain

$$\oint_\gamma \frac{e^z}{z}\,dz = 2\pi i\,[e^z]_{z=0} = 2\pi i.$$

Now evaluate this integral by parametrizing $\gamma(t) = e^{it}$ for $0 \le t \le 2\pi$. We obtain

$$\begin{aligned}\oint_\gamma \frac{e^z}{z}\,dz &= \int_0^{2\pi} \frac{e^{(\cos(t)+i\sin(t))}}{e^{it}} ie^{it}\,dt\\ &= i\int_0^{2\pi} e^{\cos(t)}\cos(\sin(t))\,dt - \int_0^{2\pi} e^{\cos(t)}\sin(\sin(t))\,dt\\ &= 2\pi i.\end{aligned}$$

Equate the real part of the left side of this equation to the real part of the right side to conclude that

$$\int_0^{2\pi} e^{\cos(t)}\cos(\sin(t))\,dt = 2\pi.$$

If we equate imaginary parts, we also obtain

$$\int_0^{2\pi} e^{\cos(t)}\sin(\sin(t))\,dt = 0.$$

However, we did not need this calculation to evaluate this integral, because this integral, from 0 to π, is the negative of the integral from π to 2π, hence the integral is zero.

14. First observe that

$$f(z) = \frac{z-4i}{z^3+4z} = \frac{z-4i}{z(z-2i)(z+2i)}.$$

Now let γ_1, γ_2 and γ_3 be nonintersecting circles enclosed by γ, which also do not intersect γ, and having centers, respectively, 0, $2i$ and $-2i$. By the extended deformation theorem,

$$\oint_\gamma f(z)\,dz = \sum_{j=1}^{3}\int_{\gamma_j} f(z)\,dz.$$

Consider each term in the sum on the right. On and in the interior of γ_1, write

$$f(z) = \frac{(z-4i)/(z-2i)(z+2i)}{z}$$

is differentiable, and we can apply the Cauchy integral formula to write

$$\oint_{\gamma_1} f(z)\,dz = 2\pi i\left[\frac{z-4i}{(z-2i)(z+2i)}\right]_{z=0}$$

$$= 2\pi i\left[\frac{-4i}{(-2i)(2i)}\right] = 2\pi i(-i) = 2\pi.$$

Similarly, for the integral over γ_2, write

$$f(z) = \frac{(z-4i)/z(z+2i)}{z-2i}$$

and apply the Cauchy integral formula to write

$$\oint_{\gamma_2} f(z)\,dz = 2\pi i\left[\frac{-2i}{(2i)(4i)}\right] = -\frac{\pi}{2}.$$

And, for the integral over γ_3, write

$$f(z) = \frac{(z-4i)/z(z-2i)}{z+2i}$$

to write

$$\oint_{\gamma_3} f(z)\,dz = 2\pi i\left[\frac{-6i}{(-2i)(-4i)}\right] = \frac{3\pi}{2}.$$

Then

$$\oint_{\gamma} f(z)\,dz = 2\pi - \frac{\pi}{2} - \frac{3\pi}{2} = 0.$$

15. We will use the notation of the theorem, Figure 20.14 of the text, and the hint outlined in the problem. In the text it was shown that it is sufficient to show that

$$\oint_{C^*} \frac{f(z)}{z-z_0}\,dz$$

can be made arbitrarily small by choosing b larger. Recall that, for some positive integer n and some positive number M $|z^n f(z)| \leq M$ for z sufficiently large. By choosing z far enough away from z_0, we can make

$$\left|\frac{z^n f(z)}{z-z_0}\right| \leq |z^n f(z)| \leq M.$$

Then, for z on C^*,

$$\left|\frac{f(z)}{z-z_0}\right| \leq \frac{M}{|z^n(z-z_0)}| = \frac{M}{|z^{n+1}||1-z_0/z|}.$$

But

$$\left|1-\frac{z_0}{z}\right| \geq 1-\left|\frac{z_0}{z}\right| > \frac{1}{2}$$

if $|z_0/z| < 1/2$, that is, if $|z_0| < 2|z|$, so

$$\frac{1}{|1-z_0/z|} < 2$$

and then

$$\left|\frac{f(z)}{z-z_0}\right| \leq \frac{2M}{z^{n+1}} < \frac{2M}{b^{n+1}}.$$

Now, the length of C^* is

$$b+ib-(\sigma+ib)+2b+(b-ib)-(\sigma-ib) = 4b-2\sigma.$$

Then,

$$\left|\int_{C^*} \frac{f(z)}{z-z_0}\,dz\right| \leq \frac{2M}{b^{n+1}}\,(4b-2\sigma)\,,$$

or, equivalently,

$$\left|\int_{C^*} \frac{f(z)}{z-z_0}\,dz\right| \leq \frac{2M}{b^n}\left(4-\frac{2\sigma}{b}\right),$$

and this approaches 0 as $b \to \infty$.

Chapter 21

Series Representations of Functions

21.1 Power Series

In Problems 1 - 6, we use the ratio test. Take

$$\lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}(z-\zeta)\right|$$

and determine those values of z for which this limit is less than 1. The technique fails if infinitely many $c_n\,'s$ are zero, or if $|c_{n+1}/c_n|$ has no limit.

1.

$$\left|\frac{(n+2)/2^{n+1}}{(n+1)/2^n}(z+3i)\right| = \frac{1}{2}\frac{n+2}{n+1}|z+3i|$$
$$\to \frac{1}{2}|z+3i| < 1$$

if

$$|z+3i| < 2.$$

The radius of convergence of this series is 2 and the open disk of convergence is $|z+3i| < 2$, the open disk of radius 2 about $-3i$.

2.

$$\left|\frac{1/(2n+3)^2}{1/(2n+1)^2}\frac{(z-i)^{2n+2}}{(z-i)^{2n}}\right| = \left(\frac{2n+1}{2n+3}\right)^2 |(z-i)^2|$$
$$\to |(z-i)^2| < 1$$

if

$$|z-i| < 1.$$

The radius of convergence is 1 and the open disk of convergence is $|z-i| < 1$.

3.

$$\begin{aligned}
&\left|\frac{(n+1)^{n+1}/(n+2)^{n+1}}{n^n/(n+1)^n}(z-1+3i)\right| \\
&\frac{(n+1)^{2n+1}}{n^n(n+2)^{n+1}}|z-1+3i| \\
&=\left(\frac{n+1}{n}\right)^n\left(\frac{n+1}{n+2}\right)^{n+1}|z-1+2i| \\
&=\left(1+\frac{1}{n}\right)^n\left(\frac{1+1/n}{1+2/n}\right)^{n+1}|z-1+3i| \\
&\to e|z-1+3i|<1
\end{aligned}$$

if

$$|z-1+3i|<\frac{1}{e}.$$

The radius of convergence is $1/e$ and the open disk of convergence is $|z-1+3i|<1/e$.

4.

$$\begin{aligned}
&\left|\frac{(2i/(5+i))^{n+1}}{(2i/(5+i))n}(z+3-4i)\right| \\
&=\left|\frac{2i}{5+i}(z+3-4i)\right| \\
&=\frac{2}{\sqrt{26}}|z+3-4i|<1
\end{aligned}$$

if

$$|z+3-4i|<\frac{\sqrt{26}}{2}.$$

The radius of convergence is $\sqrt{26}/2$ and the open disk of convergence is $|z+3-4i|<\sqrt{26}/2$, the open disk of radius $\sqrt{26}/2$ about $-3+4i$.

5. Form

$$\left|\frac{i^{n+1}/2^{n+2}}{i^n/2^{n+1}}(z+8i)\right|\to\frac{1}{2}|z+8i|<1$$

if $|z+8i|<2$. This series has radius of convergence 2 and the open disk of convergence is $|z+8i|<2$, the open disk of radius 2 about the center $-8i$.

6.

$$\begin{aligned}
\left|\frac{(1-i)^{n+1}/(n+3)}{(1-i)^n/(n+2)}(z-3)\right| &=\left(\frac{n+2}{n+3}\right)\sqrt{2}|z-3| \\
&\to\sqrt{2}|z-3|<1
\end{aligned}$$

if

$$|z - 3| < \frac{1}{\sqrt{2}}.$$

The radius of convergence is $1/\sqrt{2}$ and the open disk of convergence is $|z - 3| < 1/\sqrt{2}$.

7. No. The the power series has center $2i$. If the series converged at 0, it would converge also at the point i that is closer to $2i$ than 0 is.

8. The center of this power series is $4 - 2i$. Now $1 + i$ is closer to $4 - 2i$ than i is, so if the series converged at i, it would also have to converge at $1 + i$.

In Problems 9 through 14, we attempt to use known series to derive the requested series.

9. Since we know the series for $\cos(z)$ about 0, replace z with $2z$ to obtain

$$\cos(2z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}(2z)^{2n},$$

or

$$\cos(2z) = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n}}{(2n)!}(z)^{2n}.$$

This series has infinite radius of convergence, since the series for $\cos(z)$ as infinite radius of convergence. This means that both series are valid for all complex z.

10. Using the series for e^z, we obtain the series for e^{-z}, and then

$$\begin{aligned} e^{-z} &= e^{(-z+3-3i)} = e^{3i}e^{-(z+3i)} \\ &= e^{3i}\sum_{n=0}^{\infty} \frac{1}{n!}(-(z+3i))^n \\ &= e^{3i}\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}(z+3i)^n. \end{aligned}$$

This series converges for all z.

11. This is just the rearrangement of a polynomial into powers of $z^2 - 3z + i$. This can be done algebraically, but it is also easy in this case to write the Taylor coefficients:

$$c_0 = f(2-i), c_1 = f'(2-i), c_2 = \frac{1}{2}f''(2-i),$$

in which $f(z) = z^2 - 3z + i$. We obtain

$$z^2 - 3z + i = -3 + (1-2i)(z-2+i) + (z-2+i)^2.$$

12. Using the Maclaurin series for e^z and $\sin(z)$, write

$$e^z - i\sin(z) = \sum_{n=0}^{\infty} \frac{1}{n!} z^n + \sum_{n=0}^{\infty} \frac{(-1)^{n+1} i}{(2n+1)!} z^{2n+1}.$$

Since the sine series contains only odd powers of z while the exponential series contains all powers, it is awkward to attempt to combine these two series under one summation. Both series converge for all z.

13. Like Problem 11, this is an algebraic rearrangement of a second degree polynomial in powers of $z-1-i$. If we use the Taylor coefficients, with $f(z) = (z-9)^2$, then

$$c_0 = f(1+i), c_1 = f'(1+i) \text{ and } c_2 = \frac{1}{2} f''(1+i).$$

We get

$$(z-9)^2 = (63-16i) + (-16+2i)(z-1-i) + (z-1-i)^2.$$

14. Replace z with $z+i$ in the Maclaurin series for $\sin(z)$ to obtain

$$\sin(z+i) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} (z+i)^{2n+1}.$$

This series converges for all z.

15. We know that

$$f(0) = 1, f'(0) = i.$$

Further,

$$f''(z) = 2f(z) + 1,$$

so

$$\begin{aligned} f''(0) &= 2f(0) + 1 = 3, \\ f^{(3)}(0) &= 2f'(0) = 2i, \\ f^{(4)}(0) &= 2f''(0) = 6, \\ f^{(5)}(0) &= 2f^{(3)}(0) = 4i. \end{aligned}$$

For the first six terms of the Maclaurin expansion, these numbers enable us to compute the Taylor coefficients $f^k(0)/k!$. We obtain

$$f(z) = 1 + iz + \frac{3}{2} z^2 + \frac{2i}{3!} z^3 + \frac{6}{4!} z^4 + \frac{4i}{5!} z^5 + \cdots.$$

In this problem we can write the entire Maclaurin expansion, since it is not difficult to show by induction that

$$f^{(2n)}(0) = 2^n + 2^{n-1} \text{ and } f^{(2n+1)}()) = 2^n i.$$

16. With $f(z) = \sin^2(z)$, compute

$$f'(z) = 2\sin(z)\cos(z) = \sin(2z), f''(z) = 2\cos(2z)$$
$$f^{(3)}(z) = -4\sin(2z), f^{(4)}(z) = -8\cos(2z), f^{(5)}(z) = 16\sin(2z), f^{(6)}(z) = 32\cos(2z).$$

(a) From these derivatives, compute the first seven terms of the Maclaurin expansion, obtaining

$$\sin^2(z) = z^2 - \frac{1}{3}z^4 + \frac{2}{45}z^6 + \cdots .$$

(b) Multiply

$$\begin{aligned}\sin^2(z) &= \left(z - \frac{1}{6}z^3 + \frac{1}{120}z^5 + \cdots\right)\left(z - \frac{1}{6}z^3 + \frac{1}{120}z^5 + \cdots\right)\\ &= z^2 - \left(\frac{1}{6} + \frac{1}{6}\right)z^4 + \left(\frac{1}{120} + \frac{1}{36} + \frac{1}{120}\right)z^6 + \cdots\\ &= z^2 - \frac{1}{3}z^4 + \frac{2}{45}z^6 + \cdots .\end{aligned}$$

(c) Use the definition of $\sin(z)$ to write

$$\begin{aligned}\sin^2(z) &= -\frac{1}{4}\left(e^{iz} - e^{-iz}\right)^2\\ &= -\frac{1}{4}\left(e^{2iz} + e^{-2iz} - 2\right)\\ &= \frac{1}{2} - \frac{1}{4}\left(1 + 2iz + \frac{(2iz)^2}{2!} + \cdots + \frac{(2iz)^7}{7!} + \cdots\right)\\ &- \frac{1}{4}\left(1 - 2iz + \frac{(2iz)^2}{2!} + \cdots - \frac{(2iz)^7}{7!} + \cdots\right)\\ &= \frac{1}{2} - \frac{1}{4}\left(1 - 4z^2 + \frac{4}{3}z^4 - \frac{8}{45}z^6 + \cdots\right)\\ &= z^2 - \frac{1}{3}z^4 + \frac{2}{45}z^6 + \cdots .\end{aligned}$$

(d) We can also write

$$\begin{aligned}\sin^2(z) &= \frac{1}{2} - \frac{1}{2}\cos(2z)\\ &= \frac{1}{2} - \frac{1}{2}\left(1 - \frac{(2z)^2}{2!} + \frac{(2z)^4}{4!} - \frac{(2z)^6}{6!} + \cdots\right)\\ &= \frac{1}{2} - \frac{1}{2}\left(1 - z^2 + \frac{2}{3}z^4 - \frac{4}{45}z^6 + \cdots\right)\\ &= z^2 - \frac{1}{3}z^4 + \frac{2}{45}z^6 - \cdots .\end{aligned}$$

17. Begin with z a given complex number, and consider the integral

$$\frac{1}{2\pi i}\oint_\gamma \frac{z^n}{n!w^{n+1}}e^{zw}\,dw$$

with γ the unit circle about the origin (oriented counterclockwise). First expand e^{zw} in its Maclaurin series and then parametrize γ by $\gamma(t) = e^{i\theta}$ to write

$$\begin{aligned}
\oint_\gamma \frac{z^n}{n!w^{n+1}}e^{zw}\,dw &= \frac{1}{2\pi i}\oint_\gamma \frac{z^n}{n!w^{n+1}}\sum_{k=0}^{\infty}\frac{(zw)^k}{k!}\,dw \\
&= \frac{1}{2\pi i}\oint_\gamma \sum_{k=0}^{\infty}\frac{z^{n+k}w^{k-n-1}}{n!k!}\,dw \\
&= \frac{1}{2\pi i}\int_0^{2\pi}\sum_{k=0}^{\infty}\frac{z^{n+k}e^{i(k-n-1)\theta}}{n!k!}ie^{i\theta}\,d\theta \\
&= \sum_{k=0}^{\infty}\frac{1}{2\pi}\int_0^{2\pi}\frac{z^{n+k}}{n!k!}e^{i(k-n)\theta}\,d\theta.
\end{aligned}$$

Now,

$$\int_0^{2\pi} e^{i(k-n)\theta}\,d\theta = \begin{cases} 0 & \text{if } k \neq n, \\ 2\pi & \text{if } k - n. \end{cases}$$

Therefore we have

$$\frac{1}{2\pi i}\oint_\gamma \frac{z^n}{n!w^{n+1}}e^{zw}\,dw = \frac{(z^n)^2}{(n!)^2}.$$

Finally, we can write

$$\begin{aligned}
\sum_{n=0}^{\infty}\frac{1}{(n!)^2}z^{2n} &= \sum_{n=0}^{\infty}\frac{1}{2\pi i}\frac{z^n}{n!w^{n+1}}e^{zw}\,dw \\
&= \frac{1}{2\pi i}\int_0^{2\pi}\sum_{n=0}^{\infty}\frac{z^n}{n!e^{i(n+1)\theta}}e^{ze^{i\theta}}e^{i\theta}\,d\theta \\
&= \frac{1}{2\pi}\left[\sum_{n=0}^{\infty}\frac{(ze^{-i\theta})^n}{n!}\right]e^{ze^{i\theta}}\,d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} e^{ze^{-i\theta}}e^{ze^{i\theta}}\,d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} e^{z(e^{-i\theta}+e^{i\theta})}\,d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} e^{2\pi\cos(\theta)}\,d\theta.
\end{aligned}$$

18. $f(z)$ has a zero of order 3 at 0, because z^3 has a zero of order 3 at 0, and $\cos(0) \neq 0$.

19. $f(z)$ has a zero of order 4 at 0, because z^2 has a zero of order 2, and $\sin(z)$ has a zero of order 1 at 0, so $\sin^2(z)$ has a zero of order 2 there.

20. $f(z) = (z - \pi/2)^2 \cos(z)$ has a zero of order 3 at $\pi/2$ because $\cos(z)$ has a simple zero there, and $(z - \pi/2)^2$ has a zero of order 2 at $\pi/2$.

21. $f(z)$ has a zero of order 3 at $3\pi/2$ because $\cos(z)$ has a simple zero there.

22. $f(z)$ has a zero of order 4 at 0 because $\cos(z)$ has a simple zero at $-\pi/2$, so $\cos^4(z)$ has a fourth order zero there.

23. $f(z)$ is not defined at zero. However, as we will see in the next section, we can write, for $z \neq 0$,

$$\begin{aligned} f(z) = \sin(z^4)/z^2 &= \frac{1}{z^2}\left(z^4 - \frac{1}{6}z^{12} + \cdots\right) \\ &= z^2 - \frac{1}{6}z^{10} + \cdots . \end{aligned}$$

Since the right side of this equation is a power series about 0, it defines a differentiable function $g(z)$ which is equal to $f(z)$ if $z \neq 0$. We can therefore extend $f(z)$ to a differentiable function by setting $f(z) = g(z)$ if $z \neq 0$, and $f(0) = g(0) = 0$. If we do this, then the extended function $g(z)$ has a zero of order 2 at 0. This is the idea behind a removable singularity, which is discussed in the next chapter.

24. We can treat this function like that of Problem 23, since

$$\sin(z - \pi) = (z - \pi) - \frac{1}{3!}(z - \pi)^3 + \frac{1}{5!}(z - \pi)^5 + \cdots ,$$

so if $(z - \pi)^5$ is divided by $\sin^2(z - \pi)$ we obtain a Maclaurin expansion whose first term is $(z - \pi)^3$. We can extend $f(z)$ to this function, defining $f(0) = 0$, and this extended function has a zero of order 3 at π.

25. If we compute the kth derivative of $f(z)$ at z_0 by differentiating each series term by term, we obtain

$$\begin{aligned} f^{(k)}(z_0) &= \left[\sum_{n=0}^{\infty} a_n(n)(n-1)\cdots(n-k+1)(z - z_0)^{n-k}\right]_{z=z_0} \\ &= a_k k! \\ &= \left[\sum_{n=0}^{\infty} b_n(n)(n-1)\cdots(n-k+1)(z - z_0)^{n-k}\right]_{z=z_0} \\ &= b_k k!. \end{aligned}$$

Then

$$a_k = \frac{f^{(k)}(z_0)}{k!} = b_k.$$

The coefficients of the power series expansion of $f(z)$ about z_0 must be the Taylor coefficients at z_0.

21.2 The Laurent Expansion

For these problems, use known expansions, such as power series of exponential and trigonometric functions, and geometric series. This sometimes requires some ingenuity in rewriting functions in ways suited to the task at hand.

1. We want an expansion in powers of $z - i$. To this end, first write

$$\frac{2z}{1+z^2} = \frac{1}{z-i} + \frac{1}{z+i}.$$

The first term is a series (with just one term) in powers of $z - i$. For the second term, rearrange the denominator and use a geometric series:

$$\begin{aligned}\frac{1}{z+i} &= \frac{1}{2i + (z-i)} = \frac{1}{2i\left(1 + \frac{z-i}{2i}\right)} \\ &= \frac{1}{2i}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z-i}{2i}\right)^n \\ &= \sum_{n=0}^{\infty}\frac{(-1)^n}{(2i)^{n+1}}(z-i)^n.\end{aligned}$$

This expansion is valid for

$$\left|\frac{z-i}{2i}\right| < 1$$

or

$$|z - 2i| < 2.$$

The Laurent expansion of $2z/(1+z^2)$ is therefore

$$\frac{1}{z-i} + \sum_{n=0}^{\infty}\frac{(-1)^n}{(2i)^{n+1}}(z-i)^n$$

and this is a valid representation of $f(z)$ in the annulus (punctured disk) $0 < |z-i| < 2$.

2. For $z \neq 0$, write

$$\begin{aligned}\frac{\sin(z)}{z^2} &= \frac{1}{z^2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}z^{2n+1} \\ &= \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}z^{2n-1}.\end{aligned}$$

This expansion represents of $f(z)$ in the annulus $0 < |z| < \infty$, which is the complex plane with the origin removed.

3. If $z \neq 0$, then

$$\frac{1-\cos(2z)}{z^2} = \frac{1}{z^2}\left[1 - \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}(2z)^{2n}\right]$$
$$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}4^n}{(2n)!} z^{2n-2}.$$

This holds for $0 < |z| < \infty$.

4. Write

$$z^2 \cos\left(\frac{1}{z}\right) = z^2 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\left(\frac{i}{z}\right)^{2n} = \sum_{n=0}^{\infty} \frac{1}{(2n)!} z^{2-2n}$$

for all $z \neq 0$ (that is, for $0 < |z| < \infty$).

5. We want a series in powers of $z-1$. The denominator is already a power of $-(z-1)$, so all we need to do is fix the numerator:

$$\frac{z^2}{1-z} = \frac{((z-1)+1)^2}{1-z} = -\frac{1+2(z-1)+(z-1)^2}{z-1}$$
$$= -\frac{1}{z-1} - 2 - (z-1),$$

for $0 < |z-1| < \infty$ (the complex plane with 1 removed).

6. Write

$$\frac{z^2-1}{2z-1} = \frac{1}{2}\left[\frac{z^2+1}{z-1/2}\right] = \frac{1}{2}\left[\frac{1+[(z-1/2)+1/2]^2}{z-1/2}\right]$$
$$= \frac{5}{8}\frac{1}{z-1/2} + \frac{1}{2} + \frac{1}{2}\left(z - \frac{1}{2}\right)$$

for $0 < |z-1/2| < \infty$.

7. Using the exponential series, we have

$$\frac{e^{z^2}}{z^2} = \frac{1}{z^2}\sum_{n=0}^{\infty} \frac{1}{n!} z^{2n} = \sum_{n=-}^{\infty} \frac{1}{n!} z^{2n-2}$$

for $0 < |z| < \infty$.

8. Using the Maclaurin expansion of the sine function,

$$\frac{\sin(4z)}{z} = \frac{1}{z}\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} 4^{2n+1} z^{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n r^{2n+1}}{(2n+1)!} z^{2n},$$

for $0 < |z| < \infty$. Notice that in this case the Laurent expansion about 0 is actually a Taylor series about 0 and converges at 0.

9. The denominator is already a power of $z - i$, so

$$\frac{z+i}{z-i} = \frac{2i + (z-i)}{z-i} = 1 + \frac{2i}{z-i}$$

for $0 < |z - i| < \infty$.

10. Use the Maclaurin expansion of $\sinh(z)$, or, if this is not familiar, write $\sinh(z) = (1/2)(e^z - e^{-z})$ and use the exponential series. Either way,

$$\sinh\left(\frac{1}{z^2}\right) = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!}\left(\frac{1}{z^3}\right)^{2n+1} = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} z^{-6n-3}$$

for $0 < |z| < \infty$.

11. In the discussion, we will refer to Figures 21.2 and 21.3.

f is differentiable on and in the interior of Γ_1, so by Cauchy's integral formula, for any z enclosed by Γ_1.

$$f(z) = \frac{1}{2\pi i}\oint_{\Gamma_1} \frac{f(w)}{w-z}\,dw.$$

Since Γ_2 does not enclose z, then by Cauchy's theorem,

$$\frac{1}{2\pi i}\oint_{\Gamma_2} \frac{f(w)}{w-z}\,dw = 0,$$

in which the factor $1/2\pi i$ was introduced for the next part of the argument. Add these two equations to obtain

$$f(z) = \frac{1}{2\pi i}\left[\oint_{\Gamma_1} \frac{f(w)}{w-z}\,dz + \oint_{\Gamma_2} \frac{f(w)}{w-z}\,dw\right].$$

Orientation on both curves is counterclockwise. In this sum of integrals, each of L_1 and L_2 is traversed in both directions, so integrals over these segments cancel. This sum therefore gives integrals over γ_1 and γ_2, counterclockwise on γ_2 but clockwise on γ_1. Reversing this orientation on γ_1 so that all orientations are counterclockwise, we have

$$f(z) = \frac{1}{2\pi i}\left[\oint_{\gamma_2} \frac{f(w)}{w-z}\,dz - \oint_{\gamma_1} \frac{f(w)}{w-z}\,dw\right].$$

We will manipulate $1/(w-z)$ differently in each of these integrals. For the integral over γ_2, write

$$\begin{aligned}\frac{1}{w-z} &= \frac{1}{w-z_0-(z-z_0)} = \frac{1}{w-z_0}\frac{1}{1-(z-z_0)/(w-z_0)}\\ &= \frac{1}{w-z_0}\sum_{n=0}^{\infty}\left(\frac{z-z_0}{w-z_0}\right)^n\\ &= \sum_{n=0}^{\infty}\frac{1}{(w-z_0)^{n+1}}(z-z_0)^n.\end{aligned}$$

This geometric series is valid because, for w on γ_2,

$$\left|\frac{z-z_0}{w-z_0}\right| < 1.$$

For the integral over γ_1, we have, for w on γ_1,

$$\left|\frac{w-z_0}{z-z_0}\right| < 1,$$

so now write

$$\frac{1}{w-z} = \frac{1}{w-z_0-(z-z_0)} = \frac{-1}{z-z_0}\frac{1}{1-(w-z_0)/(z-z_0)}$$
$$= -\frac{1}{z-z_0}\sum_{n=0}^{\infty}\left(\frac{w-z_0}{z-z_0}\right)^n$$
$$= -\sum_{n=0}^{\infty}(w-z_0)^n\frac{1}{(z-z_0)^{n+1}}.$$

Substitute these into the integrals in the sum representing $f(z)$ and interchange the integrals and the summations to obtain

$$f(z) = \frac{1}{2\pi i}\oint_{\gamma_2}\left(\sum_{n=0}^{\infty}\frac{f(w)}{(w-z_0)^{n+1}}\,dw\right)(z-z_0)^n$$
$$+\frac{1}{2\pi i}\oint_{\gamma_1}\left(\sum_{n=0}^{\infty}f(w)(w-z_0)^n\,dw\right)\frac{1}{(z-z_0)}^{n+1}$$
$$= \sum_{n=0}^{\infty}\left(\frac{1}{2\pi i}\oint_{\gamma_2}\frac{f(w)}{(w-z_0)^{n+1}}\,dw\right)(z-z_0)^n$$
$$+\sum_{n=0}^{\infty}\left(\frac{1}{2\pi i}\oint_{\gamma_1}f(w)(w-z_0)^n\,dw\right)\frac{1}{(z-z_0)^{n+1}}.$$

Put $n=-m-1$ in the last summation to obtain

$$f(z) = \sum_{n=0}^{\infty}\left(\frac{1}{2\pi i}\oint_{\gamma_2}\frac{f(w)}{(w-z_0)^{n+1}}\,dw\right)(z-z_0(^n$$
$$+\sum_{m=-1}^{-\infty}\left(\frac{1}{2\pi i}\oint_{\gamma_1}\frac{f(w)}{(w-z_0)^{m+1}}\right)(z-z_0)^m.$$

Finally, use the deformation theorem to replace these integrals over γ_1 and γ_2 with integrals over Γ, which is any path in the annulus and enclosing z_0. This gives us

$$f(z) = \sum_{n=-\infty}^{\infty}c_n(z-z_0)^n,$$

where

$$c_n = \frac{1}{2\pi i} \oint_\Gamma \frac{f(w)}{(w - z_0)^{n+1}}\, dw,$$

completing the proof.

Chapter 22

Singularities and the Residue Theorem

22.1 Singularities

1. $\cos(z)/z^2$ has just one singularity, $z = 0$, and this is a pole of order 2 because $\cos(0) \neq 0$.

2. $f(z)$ has a double pole at $-i$ and a simple pole at i, since the numerator is differentiable and nonzero at these points.

3. $e^{1/z}(z + 2i)$ has a essential singularity at $z = 0$. The factor $z + 2i$ makes no contribution to singularities of this function.

4. The function is differentiable at all $z \neq \pi$, so we need only concern ourselves with what is happening at π. Now

$$\sin(z - \pi) = \sin(z)\cos(\pi) - \cos(z)\sin(\pi) = -\sin(z)$$

so

$$\frac{\sin(z)}{z - \pi} = -\frac{\sin(z - \pi)}{z - \pi}.$$

Then

$$\begin{aligned}\lim_{z \to \pi} \frac{\sin(z)}{z - \pi} &= -\lim_{z \to \pi} \frac{\sin(z - \pi)}{z - \pi} \\ &= -\lim_{z \to 0} \frac{\sin(z)}{z} = -1.\end{aligned}$$

The fact that this limit is nonzero means that $\sin(z)/(z - \pi)$ has a removable singularity at π.

5. The only singularities are $1, i, -i$, and 1 is a double pole, while i and $-i$ are simple poles.

6. $z/(z+1)^2$ has a double pole at -1.

7. Write
$$\frac{z-i}{z^2+1} = \frac{z-i}{(z+i)(z-i)} = \frac{1}{z+i}.$$
Then $f(z)$ has a removable singularity at i and a simple pole at $-i$.

8. To analyze singularities of $\sin(z)/\sinh(z)$, we must know the zeros of $\sinh(z)$. Of course, the real hyperbolic sine function $\sinh(x)$ is zero only for $x = 0$, but the complex hyperbolic sine function might have zeros in the complex plane. To answer this question, set
$$\sinh(z) = \frac{1}{2}\left(e^z - e^{-z}\right) = 0.$$
Then $e^z - e^{-z} = 0$, so
$$e^{2z} = 1.$$
We know where the complex exponential function equals 1. $e^{2z} = 1$ exactly when $2z = 2n\pi i$, or $z = n\pi i$, with n any integer. Since $\sin(n\pi i) \neq 0$ unless $n = 0$, then the numbers $n\pi i$ $n \neq 0$ are singularities of $\sin(z)/\sinh(z)$. Further, $\cosh(n\pi i) \neq 0$, so these are simple poles of $\sin(z)/\sinh(z)$. If $n = 0$, then we have the origin 0. But $\sin(z)$ and $\sinh(z)$ have simple zeros at 0, so 0 is a removable singularity of this function.

9. Write
$$\frac{z}{z^4-1} = \frac{z}{(z-1)(z+1)(z-i)(z+i)}.$$
This function has simple poles at $\pm 1, \pm i$.

10. $\tan(z) = \sin(z)/\cos(z)$ has simple poles at the simple zeros $(2n+1)\pi/2$ of $\cos(z)$, in which n is any integer.

11. $\sec(z) = 1/\cos(z)$ has simple poles at the zeros of $\cos(z)$. These are $(2n+1)\pi/2$, with n any integer.

12. $e^{1/z(z+1)}$ has essential singularities at 0 and -1. One way to see this is to write
$$\frac{1}{z(z+1)} = \frac{1}{z} - \frac{1}{z+1},$$
so
$$e^{1/z(z+1)} = e^{1/z}e^{-1/(z+1)}.$$
Now look at $z = 0$, to be specific. We know that $e^{1/z}$ has an essential singularity at 0, and $e^{-1/(z+1)}$ is differentiable at 0, so the product will have an essential singularity there. Similar reasoning applies at -1, since $e^{1/z}$ is differentiable at -1.

13. Suppose f is differentiable at z_0 and $f(z_0) \neq 0$, while g has a pole of order m at z_0. We want to show that fg has a pole of order m at z_0.

    Since g has a pole of order m at z_0, then $g(z)$ has a Laurent expansion about z_0 of the form

$$g(z) = \frac{k}{(z-z_0)^m} + \sum_{n=-m+1}^{\infty} c_n(z-z_0)^n,$$

    with m the highest power of $1/(z-z_0)$ appearing in the series. Then $(z-z_0)^m g(z)$ is a series containing only nonnegative powers of $z-z_0$, hence is a power series expansion about z_0. If we denote this series as $h(z)$, then

$$g(z) = \frac{h(z)}{(z-z_0)^m},$$

    with h differentiable at z_0 and $h(z_0) \neq 0$. We now have, in some annulus about z_0,

$$f(z)g(z) = \frac{f(z)h(z)}{(z-z_0)^m},$$

    with $f(z_0)h(z_0) \neq 0$. Therefore fg has a pole of order m at z_0.

22.2 The Residue Theorem

1. $f(z)$ has a pole of order 2 at $z = 1$ and a simple pole at $z = -2i$. Both are enclosed by γ (recall that singularities not enclosed by γ are irrelevant for evaluating an integra of the function about γ). Then

$$\oint_\gamma \frac{1+z^2}{(z-1)^2(z+2i)}\,dz = 2\pi i \operatorname{Res}(f,1) + 2\pi i \operatorname{Res}(f,2i).$$

    Compute

$$\begin{aligned}\operatorname{Res}(f,1) &= \lim_{z\to 1} \frac{d}{dz}\left(\frac{1+z^2}{z+2i}\right) \\ &= \lim_{z\to 1} \frac{(z+2i)(2z)-(1+z^2)}{(z+2i)^2} \\ &= \frac{4i}{-3+4i}\end{aligned}$$

    and

$$\operatorname{Res}(f,-2i) = \lim_{z\to -2i} \frac{1+z^2}{(z-1)^2} = \frac{-3}{-3+4i}.$$

    Then

$$\oint_\gamma \frac{1+z^2}{(z-1)^2(z+2i)}\,dz = 2\pi i\left[\frac{4i}{-3+4i} - \frac{3}{-3+4i}\right] = 2\pi i.$$

2. γ encloses i, which is a double pole and the only singularity of $f(z)$. Then

$$\oint_\gamma \frac{2z}{(z-i)^2}\,dz = 2\pi i \operatorname{Res}(f,i) = \lim_{z\to i}\frac{d}{dz}(2z) = 4\pi i.$$

3. γ does not enclose 0, the only singularity of $f(z)$, so by Cauchy's theorem,

$$\oint_\gamma \frac{e^z}{z}\,dz = 0.$$

4. The only singularities of $f(z)$ are $\pm 2i$, which are simple poles enclosed by γ. Then

$$\begin{aligned}\oint_\gamma \frac{\cos(z)}{4+z^2}\,dz &= 2\pi i \operatorname{Res}(f,2i) + 2\pi i \operatorname{Res}(f,-2i)\\ &= 2\pi i\frac{\cos(2i)}{4i} + 2\pi i\frac{\cos(-2i)}{-4i}\\ &= 0.\end{aligned}$$

5. The function being integrated has simple poles at $\sqrt{6}i$ and $-\sqrt{6}i$, both enclosed by γ. Then

$$\begin{aligned}\oint_\gamma \frac{z+i}{z^2+6}\,dz &= 2\pi i \operatorname{Res}(f,\sqrt{6}i) + 2\pi i \operatorname{Res}(f,-\sqrt{6}i)\\ &= 2\pi i\frac{\sqrt{6}+1}{2\sqrt{6}} + 2\pi i\frac{\sqrt{6}-1}{2\sqrt{6}} = 2\pi i.\end{aligned}$$

6. $f(z) = (z-i)/(2z+1)$ has a simple pole at $-1/2$, which is enclosed by γ. Then

$$\oint_\gamma \frac{z-i}{2z+1}\,dz = 2\pi i \operatorname{Res}(f,-1/2) = \frac{1}{2}\left(-\frac{1}{2}-i\right).$$

7. $z/\sinh^2(z)$ has a simple pole at $z=0$, and double poles at the nonzero zeros of $\sinh(z)$, which occur at $z = n\pi i$ for integer values of n. The only pole of $z/\sinh^2(z)$ enclosed by γ is $z=0$. Therefore

$$\oint_\gamma \frac{z}{\sinh^2(z)}\,dz = 2\pi i \operatorname{Res}(f,0)$$

Compute this residue as

$$\begin{aligned}\operatorname{Res}(f,0) = \lim_{z\to 0}(zf(z)) &= \lim_{z\to 0}\left(\frac{z^2}{\sinh^2(z)}\right)\\ &= \lim_{z\to 0}\frac{z^2}{z^2+\frac{1}{6}z^4+\cdots}\\ &= \lim_{z\to 0}\frac{1}{1+\frac{1}{6}z^2+\cdots} = 1.\end{aligned}$$

Therefore
$$\oint_\gamma \frac{z}{\sinh^2(z)}\,dz = 2\pi i.$$

8. The only singularity of $\cos(z)/ze^z$ enclosed by γ is a simple pole at $z = 0$. Compute
$$\text{Res}(f, 0) = \lim_{z\to 0} \frac{\cos(z)}{e^z} = 1,$$
so
$$\oint_\gamma \frac{\cos(z)}{ze^z}\,dz = 2\pi i\text{Res}(f, 0) = 2\pi i.$$

9. The integrand has simple poles at $i, 3i$ and $-3i$. Only the pole at $-3i$ is enclosed by γ, so
$$\oint_\gamma \frac{iz}{(z^2+9)(z-i)}\,dz = 2\pi i\text{Res}(f, -3i)$$
$$= 2\pi i \lim_{z\to -3i} \frac{iz}{(z-3i)(z-i)} = 2\pi i\left(-\frac{1}{8}\right) = -\frac{\pi i}{4}.$$

10. e^{2/z^2} has an essential singularity at $z = 0$. Using the exponential series,
$$e^{2/z^2} = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{2}{z^2}\right)^n.$$
The coefficient of $1/z$ in this Laurent expansion about 0 is 0, so
$$\oint_\gamma e^{2/z^2}\,dz = 2\pi i\text{Res}(f, 0) = 2\pi i(0) = 0.$$

11. The integrand has a simple pole at $z = -4i$, which is outside γ. Since f is differentiable on and in the region bounded by γ, then by Cauchy's theorem,
$$\oint_\gamma \frac{8z - 4i + 1}{z + 4i}\,dz = 0.$$

12. $f(z)$ has a simple pole at $z = 1 - 2i$, which is enclosed by γ, so
$$\oint_\gamma \frac{z^2}{z - 1 + 2i}\,dz = 2\pi i\text{Res}(f, 1-2i) = 2\pi i(1-2i)^2 = 2\pi(4-3i).$$

13. $\coth(z) = \cosh(z)/\sinh(z)$, so singularities of $\coth(z)$ are zeros of $\sinh(z)$, which are $n\pi i$, with n any integer. Only the simple pole at $z = 0$ is enclosed by γ, so
$$\oint_\gamma \coth(z)\,dz = 2\pi i\text{Res}(f, 0) = 2\pi i\frac{\cosh(0)}{\cosh(0)} = 2\pi i.$$

14. The integrand has simple poles at the cube roots of 8, which are 2, $2e^{2\pi i/3}$, $2e^{4\pi i/3}$. Only 2 is enclosed by γ, so

$$\begin{aligned}\oint_\gamma \frac{(1-z)^2}{z^3-8}\,dz &= 2\pi i \operatorname{Res}(f,2)\\ &= 2\pi i \lim_{z\to 2}\frac{(1-z)^2}{3z^2}\\ &= 2\pi i\left(\frac{1}{12}\right) = \frac{\pi i}{6}.\end{aligned}$$

15. 0 and $4i$ are both simple poles of $f(z)$, so

$$\begin{aligned}\oint_\gamma \frac{e^{2z}}{z(z-4i)}\,dz &= 2\pi i\left[\operatorname{Res}(f,0)+\operatorname{Res}(f,4i)\right]\\ &= 2\pi i\left[-\frac{1}{4i}+\frac{e^{8i}}{4i}\right]\\ &= \frac{\pi}{2}[\cos(8)-1+i\sin(8)].\end{aligned}$$

16. $f(z) = z^2/(z-1)^2$ has a double pole at $z=1$, and

$$\operatorname{Res}(f,1) = \lim_{z\to 1}\frac{d}{dz}(z^2) = 2,$$

so

$$\oint_\gamma \frac{z^2}{(z-1)^2}\,dz = 4\pi i.$$

17. We are supposing that z_0 is a zero of h of order 2, but is not a zero of g. We want to show that

$$\operatorname{Res}(g/h, z_0) = \frac{2g'(z_0)}{h''(z_0)} - \frac{2}{3}\frac{g(z_0)h^{(3)}(z_0)}{(h''(z_0))^2}.$$

To do this, write

$$h(z) = (z-z_0)^2\varphi(z)$$

where $\varphi(z_0)\neq 0$. Then

$$\begin{aligned}\operatorname{Res}(g/h,z_0) &= \lim_{z\to z_0}\frac{d}{dz}\left((z-z_0)^2\frac{g(z)}{h(z)}\right)\\ &= \lim_{z\to z_0}\frac{d}{dz}\left((z-z_0)^2\frac{g(z)}{(z-z_0)^2\varphi(z)}\right)\\ &= \lim_{z\to z_0}\frac{d}{dz}\left(\frac{g(z)}{\varphi(z)}\right)\\ &= \frac{\varphi(z_0)g'(z_0)-\varphi'(z_0)g(z_0)}{(\varphi(z_0))^2}.\end{aligned}$$

Now,

$$h'(z) = 2(z - z_0)\varphi(z) + (z - z_0)^2\varphi'(z),$$

$$h''(z) = 2\varphi(z) + 4(z - z_0)\varphi'(z) + (z - z_0)^2\varphi''(z).$$

and

$$h^{(3)}(z) = 6\varphi'(z) + 6(z - z_0)\varphi''(z) + (z - z_0)^2\varphi^{(3)}(z).$$

Therefore

$$\varphi(z_0) = \frac{1}{2}h''(z_0) \text{ and } \varphi'(z_0) = \frac{1}{6}h^{(3)}(z_0).$$

Substituting these into the above expression for the residue, we obtain

$$\operatorname{Res}(g/h, z_0) = \frac{2g'(z_0)}{h''(z_0)} - \frac{2}{3}\frac{g(z_0)h^{(3)}(z_0)}{(h''(z_0))^2}.$$

18. Suppose first that f has a zero of order k at z_0 in G. We will investigate the residue of f'/f at z_0. Because f has a zero of order k at z_0, then there is a differentiable function g such that $g(z_0) \neq 0$ and, in some open disk about z_0,

$$f(z) = (z - z_0)^k g(z).$$

Then

$$\begin{aligned}\frac{f'(z)}{f(z)} &= \frac{k(z - z_0)^{k-1}g(z) + (z - z_0)^k g'(z)}{(z - z_0)^k g(z)} \\ &= \frac{k}{z - z_0} + \frac{g'(z)}{g(z)}.\end{aligned}$$

Since g'/g is differentiable at z_0, so the last equation implies that f'/f has a simple pole at z_0, and

$$\operatorname{Res}(f'/f, z_0) = k.$$

Next, suppose f has a pole of order m at z_1. In some annulus about z_1, $f(z)$ has Laurent expansion

$$f(z) = \sum_{n=-m}^{\infty} d_n(z - z_1)^n$$

with $d_{-m} \neq 0$. Then

$$(z - z_1)^m f(z) = \sum_{n=-m}^{\infty} d_n(z - z_1)^{n+m} = \sum_{n=0}^{\infty} d_{n-m}(z - z_1)^n = h(z)$$

with h differentiable at z_1 and $h(z_1) = d_{-m} \neq 0$. Then

$$f(z) = (z - z_1)^{-m}h(z),$$

so on some annulus about z_1,

$$\begin{aligned}\frac{f'(z)}{f(z)} &= \frac{-m(z-z_1)^{-m-1}h(z)+(z-z_1)^{-m}h'(z)}{(z-z_1)^{-m}h(z)}\\ &= \frac{-m}{z-z_1}+\frac{h'(z)}{h(z)}.\end{aligned}$$

Since h'/h is differentiable at z_1, then f'/f has a simple pole at z_1, and

$$\operatorname{Res}(f'/f, z_1) = -m.$$

Therefore, the sum of the residues of f'/f at poles of this function enclosed by γ in G counts each zero of f enclosed by γ, according to its multiplicity, and each pole of f enclosed by γ, according to the negative of its multiplicity. If Z is the total number of zeros of f enclosed by γ, including multiplicities, and P the total number of poles of f enclosed by γ, counting multiplicities, then

$$\oint_\gamma \frac{f'(z)}{f(z)}\,dz = 2\pi i(Z-P),$$

and this is equivalent to the argument principle.

19. By the residue theorem, with $g(z) = z/(2+z^2)$

$$\begin{aligned}\oint_\gamma \frac{z}{2+z^2}\,dz &= 2\pi i\left[\operatorname{Res}(g,\sqrt{2}i)+\operatorname{Res}(g,-\sqrt{2}i)\right]\\ &= 2\pi i\left[\frac{\sqrt{2}i}{2\sqrt{2}i}+\frac{-\sqrt{2}i}{-2\sqrt{2}i}\right]\\ &= 2\pi i\left(\frac{1}{2}+\frac{1}{2}\right) = 2\pi i.\end{aligned}$$

For the argument principle, we need to write

$$g(z) = \frac{f'(z)}{f(z)} = \frac{1}{2}\frac{2z}{2+z^2}$$

with $f(z) = 2+z^2$. Then $f'/f = 2g$.

Now $f(z)$ has two simple zeros enclosed by γ, and no poles, so $Z = 2$ and $P = 0$. By the argument principle,

$$\begin{aligned}\oint_\gamma \frac{z}{2+z^2}\,dz &= \frac{1}{2}\oint_\gamma \frac{2z}{2+z^2}\,dz\\ &= \pi i(Z-P) = 2\pi i.\end{aligned}$$

It is important in this calculation of an integral to be clear on the difference between $\oint_\gamma g(z)\,dz$, and $\oint_\gamma (f'(z)/f(z))\,dz$. In this example $f'(z)/f(z) = 2g(z)$.

20. Let $g(z) = \tan(z)$. g has simple poles at $\pm\pi/2$, enclosed by γ. By the residue theorem,

$$\begin{aligned}\oint_\gamma \tan(z)\,dz &= 2\pi i\left[\mathrm{Res}(g,\pi/2) + \mathrm{Res}(g,-\pi/2)\right]\\ &= 2\pi i\left[\frac{\sin(\pi/2)}{-\sin(\pi/2)} + \frac{\sin(-\pi/2)}{-\sin(-\pi/2)}\right]\\ &= 2\pi i\left[-1-1\right] = -4\pi i.\end{aligned}$$

To use the argument principle, notice that $g(z) = -f'(z)/f(z)$, where $f(z) = \cos(z)$. Now f has no poles, and simple zeros at $\pm\pi/2$ enclosed by γ. Then

$$\begin{aligned}\oint_\gamma \tan(z)\,dz &= \oint_\gamma \frac{\sin(z)}{\cos(z)}\,dz\\ &= -\oint_\gamma \frac{f'(z)}{f(z)}\,dz\\ &= -2\pi i(Z-P) = -4\pi i.\end{aligned}$$

21. First, $g(z) = (z+1)/(z^2+2z+4)$ has simple poles at $-1\pm\sqrt{3}i$, enclosed by γ. Then

$$\begin{aligned}\oint_\gamma \frac{z+1}{z^2+2z+4}\,dz &= 2\pi i\left[\mathrm{Res}(g,-1-\sqrt{3}i) + \mathrm{Res}(g,-1+\sqrt{3}i)\right]\\ &= 2\pi i\left[\frac{1-1-\sqrt{3}i}{2(-1-\sqrt{3}i)+2} + \frac{-1+\sqrt{3}i+1}{2(-1+\sqrt{3}i)+2}\right]\\ &= 2\pi i\left(\frac{1}{2}+\frac{1}{2}\right) = 2\pi i.\end{aligned}$$

To use the argument principle, note that

$$\frac{z+1}{z^2+2z+4} = \frac{1}{2}\frac{f'(z)}{f(z)},$$

where $f(z) = z^2+2z+4$. f has $Z = 2$ simple zeros enclosed by γ and no poles ($P = 0$), so

$$\frac{1}{2}\oint_\gamma \frac{2z+2}{z^2+2z+4}\,dz = \pi i(Z-P) = 2\pi i.$$

22. Because p has exactly n simple zeros enclosed by γ, then $p'(z)/p(z)$ has simple poles at $z_1, \cdots, c_n$, and

$$\mathrm{Res}(p'/p, z_j) = \frac{p'(z_j)}{p'(z_j)} = 1.$$

By the residue theorem

$$\oint_\gamma \frac{p'(z)}{p(z)}\,dz = 2\pi i \sum_{j=1}^{n} \operatorname{Res}(p'/p, z_j) = 2n\pi i.$$

If we use the argument principle, then $p(z)$ has exactly n simple zeros enclosed by γ, so $Z = n$, and a polynomial has no poles, so $P = 0$, hence

$$\oint_\gamma \frac{p'(z)}{p(z)}\,dz = 2\pi i(Z - P) = 2n\pi i.$$

22.3 Evaluation of Real Integrals

Most of these problems are done using one of equations (22.3), (22.4) or (22.6). In problems involving rational functions of sine and cosine, γ always denotes the unit circle about the origin.

1. With $z = e^{i\theta}$, we have

$$\cos(\theta) = \frac{1}{2}\left(z + \frac{1}{z}\right) \text{ and } d\theta = \frac{1}{iz}\,dz,$$

so

$$\begin{aligned}\int_0^{2\pi} \frac{1}{2-\cos(\theta)}\,d\theta &= \oint_\gamma \frac{1}{2 - \frac{1}{2}(z+1/z)}\frac{1}{iz}\,dz \\ &= 2i\oint_\gamma \frac{1}{z^2-4z+1}\,dz.\end{aligned}$$

Now $f(z) = 1/(z^2-4z+1)$ has simple poles at $z_1 = 2-\sqrt{3}$ and $z_2 = 2+\sqrt{3}$. Only z_1 is enclosed by γ, and

$$\operatorname{Res}(f, 2-\sqrt{3}) = \frac{1}{2(2-\sqrt{3})-4} = -\frac{1}{2\sqrt{3}}.$$

Then

$$\int_0^{2\pi} \frac{1}{2-\cos(\theta)}\,d\theta = 2i(2\pi i)\left(\frac{-1}{2\sqrt{3}}\right) = \frac{2\pi}{\sqrt{3}}.$$

Note that the integral must be real and positive, since the integrand is positive, and this checks out.

2. $f(z) = 1/(z^4+1)$ has two simple poles in the upper half-plane, at fourth roots of -1 having positive imaginary parts. these are

$$z_1 = \frac{1}{\sqrt{2}}(1+i) \text{ and } z_2 = \frac{1}{\sqrt{2}}(-1+i).$$

Compute the residues at these points. If $z^4 + 1 = 0$, then

$$\frac{1}{z^3} = -z.$$

Then

$$\mathrm{Res}(f, z_1) = \frac{1}{4z_1^3} = -\frac{1}{4}z_1 \text{ and } \mathrm{Res}(f, z_2) = -\frac{1}{4}z_2.$$

so

$$\int_{-\infty}^{\infty} \frac{1}{x^4+1}\,dx = -\frac{2\pi i}{4}(z_1 + z_2) = \frac{\pi}{\sqrt{2}}.$$

3. $f(z) = 1/(1 + z^6)$ has simple poles in the upper half-plane at $z_1 = i$, $z_2 = (\sqrt{3} + i)/2$ and $z_3 = (-\sqrt{3} + i)/2$. At each pole, compute

$$\mathrm{Res}(f, z_j) = \frac{1}{6z_j^5} = -\frac{1}{6}z_j,$$

so

$$\int_{-\infty}^{\infty} \frac{1}{1+x^6}\,dx = 2\pi i\left[\frac{1}{6}(z_1 + z_2 + z_3)\right] = \frac{2\pi}{3}.$$

Then

$$\int_0^{\infty} \frac{1}{1+x^6}\,dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{1}{1+x^6}\,dx = \frac{\pi}{3}.$$

4.

$$\begin{aligned}\int_0^{2\pi} \frac{1}{6+\sin(\theta)}\,d\theta &= \oint_\gamma \frac{1}{6 + \frac{i}{2}(z - 1/z)}\frac{1}{iz}\,dz\\ &= 2\oint_\gamma \frac{1}{z^2 + 12iz + 1}\,dz.\end{aligned}$$

The integrand has simple poles at $z_1 = (-6+\sqrt{37})i$ and $z_2 = (-6-\sqrt{37})i$. Of these, only z_1 is enclosed by γ, and

$$\mathrm{Res}(f, z_1) = \frac{1}{2z_1 + 12i} = \frac{1}{2\sqrt{37}i}.$$

Then, recalling the factor of 2 in front of the integral as written above, we have

$$\int_0^{2\pi} \frac{1}{6+\sin(\theta)}\,d\theta = 2\pi i\frac{1}{\sqrt{37}i} = \frac{2\pi}{\sqrt{37}}.$$

5. Let

$$f(z) = \frac{ze^{2iz}}{z^4 + 16}.$$

Then f has simple poles in the upper half-plane at $z_1 = (1 + i)\sqrt{2}$ and $z_2 = (-1 + i)\sqrt{2}$. Compute

$$\mathrm{Res}(f, z_1) = \frac{e^{2\sqrt{2}(-1+i)}}{16i} \text{ and } \mathrm{Res}(f, z_2) = \frac{2^{2\sqrt{2}(-1-i)}}{-16i}$$

to obtain

$$\int_{-\infty}^{\infty} \frac{x\sin(2x)}{x^4+16}\,dx = \operatorname{Im}\left[2\pi i\left(\frac{e^{-2\sqrt{2}}}{8}\right)\left(\frac{e^{2\sqrt{2}i}-e^{-2\sqrt{2}i}}{2i}\right)\right]$$
$$= \frac{\pi e^{-2\sqrt{2}}}{4}\sin(2\sqrt{2}).$$

6. $f(z) = 1/(z^2-2z+6)$ has one simple pole in the upper half-plane and it is $z_1 = 1+\sqrt{5}i$. Compute

$$\operatorname{Res}(f, z_1) = \frac{1}{2z_1-1} = \frac{1}{2\sqrt{5}i}.$$

Then

$$\int_{-\infty}^{\infty} \frac{1}{x^2-2x+6}\,dx = \frac{\pi}{\sqrt{5}}.$$

7. First use the identity

$$\cos^2(x) = \frac{1}{2}(1+\cos(2x))$$

to write the integral as

$$\int_{-\infty}^{\infty} \frac{\cos^2(x)}{(x^2+4)^2}\,dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{1+\cos(2x)}{(x^2+4)^2}\,dx.$$

Let

$$f(z) = \frac{1+e^{2iz}}{(z^2+4)^2}.$$

Then f has a pole of order 2 in the upper half-plane at $2i$, and

$$\operatorname{Re}(f, 2i) = \lim_{z\to 2i}\frac{d}{dz}\left[\frac{1+e^{2iz}}{(z+2i)^2}\right] = \frac{1+5e^{-4}}{32i}.$$

Then

$$\int_{-\infty}^{\infty} \frac{\cos^2(x)}{(x^2+4)^2}\,dx = \frac{1}{2}\operatorname{Re}\left[2\pi i\left(\frac{1+5e^{-4}}{32i}\right)\right]$$
$$= \frac{\pi}{3}(1+5e^{-4}).$$

8. Complex methods will work for this integral, but it is easier to observe that, with the change of variables $\theta = 2\pi - \varphi$,

$$\int_{\pi}^{2\pi} \frac{\sin(\theta)}{2+\sin(\theta)}\,d\theta = -\int_{0}^{\pi} \frac{\sin(\varphi)}{2+\sin(\varphi)}\,d\varphi.$$

Then

$$\int_{0}^{2\pi} \frac{\sin(\theta)}{2+\sin(\theta)}\,d\theta = 0.$$

9. Let $f(z) = z^2/(z^2+4)^2$. The only singularity of f in the upper half-plane is $2i$, which is a double pole. Compute

$$\text{Res}(f, 2i) = \lim_{z\to 2i} \frac{d}{dz}\left[\frac{z^2}{(z+2i)^2}\right] = -\frac{i}{8}.$$

Then

$$\int_{-\infty}^{\infty} \frac{x^2}{(x^2+4)^2}\,dx = 2\pi i\left(-\frac{i}{8}\right) = \frac{\pi}{4}.$$

10. Let

$$f(z) = \frac{e^{i\beta x}}{(z^2+\alpha^2)^2}.$$

Then f has only one singularity in the upper half-plane, a double pole at αi. Compute

$$\begin{aligned}\text{Res}(f, \alpha i) &= \lim_{z\to \alpha i} \frac{d}{dz}\left(\frac{e^{i\beta z}}{(z+\alpha i)^2}\right)\\ &= -\frac{(\alpha\beta+1)e^{-\alpha\beta}}{4\alpha^3} i.\end{aligned}$$

Then

$$\begin{aligned}\int_{\infty}^{\infty} \frac{\cos(\beta x)}{(x^2+\alpha^2)^2}\,dx &= 2\pi i\left[-\frac{(\alpha\beta+1)e^{-\alpha\beta}}{4\alpha^3} i\right]\\ &= \frac{(\alpha\beta+1)e^{-\alpha\beta}\pi}{2\alpha^3}.\end{aligned}$$

11. Let

$$f(z) = \frac{e^{i\alpha z}}{z^2+1}.$$

The only singularity f has in the upper half-plane is a simple pole at i. Compute

$$\text{Res}(f, i) = \frac{e^{-\alpha}}{2i}.$$

Then

$$\int_{-\infty}^{\infty} \frac{\cos(\alpha x)}{x^2+1}\,dx = 2\pi i\left(\frac{e^{-\alpha}}{2i}\right) = \pi e^{-\alpha}.$$

12. Let

$$f(z) = \frac{z^2 e^{i\alpha z}}{(z^2+\beta^2)^2}.$$

Then $f(z)$ has a double pole in the upper half-plane at βi. Compute

$$\begin{aligned}\operatorname{Res}(f,\beta i) &= \lim_{z\to\beta i}\frac{d}{dz}\frac{z^2e^{i\alpha z}}{(z+\beta i)^2}\\ &= \lim_{z\to\beta i}\left(2ze^{i\alpha z}(z+\beta i)^{-2}+iz^2\alpha z(z+\beta i)^{-2}-2z^2e^{i\alpha z}(z+\beta i)^{-3}\right)\\ &= 2\beta ie^{-\alpha\beta}(2\beta i)^{-2}+i\alpha(-\beta^2)e^{-\alpha\beta}(2\beta i)^{-2}-2(-\beta^2)e^{-\alpha\beta}(2\beta i)^{-3}\\ &= \frac{e^{-\alpha\beta}}{4\beta}i(\alpha\beta-1).\end{aligned}$$

Then

$$\begin{aligned}\int_{-\infty}^{\infty}\frac{x^2\cos(\alpha x)}{(x^2+\beta^2)^2}\,dx &= 2\pi i\left(\frac{e^{-\alpha\beta}}{4\beta}i(\alpha\beta-1)\right)\\ &= \frac{\pi}{2\beta}e^{-\alpha\beta}(1-\alpha\beta).\end{aligned}$$

13. Begin with

$$\begin{aligned}\int_0^{2\pi}\frac{1}{\alpha^2\cos^2(\theta)+\beta^2\sin^2(\theta)}\,d\theta &= \oint_\gamma\frac{1}{\alpha^2(z+1/z)^2/4-\beta^2(z-1/z)^2/4}\frac{1}{iz}\,dz\\ &= \frac{4}{i}\oint_\gamma\frac{z}{(\alpha^2-\beta^2)z^4+2(\alpha^2+\beta^2)z^2+(\alpha^2-\beta^2)}\,dz.\end{aligned}$$

Solving for the zeros of the denominator of the integrand, we find that the singularities satisfy

$$z^2=\frac{\beta-\alpha}{\beta+\alpha}\text{ or }z^2=\frac{\beta+\alpha}{\beta-\alpha}.$$

Since $\alpha>0$ and $\beta>0$,

$$\left|\frac{\beta-\alpha}{\beta+\alpha}\right|<1\text{ and }\left|\frac{\beta+\alpha}{\beta-\alpha}\right|>1.$$

The simple poles enclosed by the unit circle are the square roots z_1 and z_2 of $(\beta-\alpha)/(\beta+\alpha)$. The residue of the integrand at each of these poles are obtained by a straightforward computation using Corollary 22.1. After some computation, we obtain

$$\operatorname{Res}(f,z_j)=\frac{1}{8\alpha\beta}$$

for $j=1,2$. Then

$$\int_0^{2\pi}\frac{1}{\alpha^2\cos^2(\theta)+\beta^2\sin^2(\theta)}\,d\theta=\frac{4}{i}(2\pi i)\frac{2}{8\alpha\beta}=\frac{2\pi}{\alpha\beta}.$$

14. Let

$$g(\theta) = \frac{1}{\alpha + \sin^2(\theta)}.$$

Write

$$\int_0^{2\pi} g(\theta)\,d\theta = \int_0^{\pi/2} g(\theta)\,d\theta + \int_{\pi/2}^{\pi} g(\theta)\,d\theta$$
$$+ \int_{\pi}^{3\pi/2} g(\theta)\,d\theta + \int_{3\pi/2}^{2\pi} g(\theta)\,d\theta.$$

In the second, third and fourth integrals on the right, put $\theta = \pi - u$, $\theta = \pi + u$ and $\theta = 2\pi - u$, respectively, to obtain

$$\int_0^{\pi/2} \frac{1}{\alpha + \sin^2(\theta)}\,d\theta = \frac{1}{4}\int_0^{2\pi} \frac{1}{\alpha + \sin^2(\theta)}\,d\theta$$
$$= \frac{1}{4}\oint_\gamma \frac{1}{\alpha - (z - 1/z)^2/4}\frac{1}{iz}\,dz$$
$$= i\oint_\gamma \frac{z}{z^4 - (2+4\alpha)z^2 + 1}\,dz.$$

The integrand of the last integral has simple poles at z_1 and z_2, where

$$z_j = (1 + 2\alpha) - 2\sqrt{\alpha^2 + \alpha}.$$

Compute the residues:

$$\mathrm{Res}(f, z_k) = \left[\frac{z}{4z^3 - (4 + 8\alpha)z}\right]_{z_k} = \frac{-1}{8\sqrt{\alpha^2 + \alpha}}.$$

Then

$$\int_0^{\pi/2} \frac{1}{\alpha + \sin^2(\theta)}\,d\theta = i(2\pi i)\left[\frac{-2}{8\sqrt{\alpha^2 + \alpha}}\right]$$
$$= \frac{\pi}{2\sqrt{\alpha^2 + \alpha}}.$$

15. Let Γ denote the suggested rectangular path. The four sides are

$\Gamma_1 : z = x, -R \le x \le R$ (lower side of the rectangle),
$\Gamma_2 : z = R + it, 0 \le \beta \le R$ (right side),
$\Gamma_3 : z = x + i\beta, x : R \to -R$ (top),
$\Gamma_4 : z = -R + it, t : \beta \to 0$ (right side).

The intervals for the parameters on the sides are chosen to maintain counterclockwise orientation around Γ. Now observe that e^{-z^2} is differentiable on and in the region bounded by Γ, and use Cauchy's theorem and the parametrization on each side of the rectangle to write

$$\oint_\Gamma e^{-z^2}\,dz = 0 = \sum_{j=1}^{4}\int_{\Gamma_j} e^{-z^2}\,dz.$$

Look at each of the integrals on the right. First,

$$\int_{\Gamma_1} e^{-z^2}\,dz = \int_{-R}^{R} e^{-x^2}\,dx.$$

Next,

$$\int_{\Gamma_2} e^{-z^2}\,dz = \int_0^\beta e^{-(R^2+2Rti-t^2)}i\,dt = ie^{-R^2}\int_0^\beta e^{t^2}[\cos(2Rt) - i\sin(2Rt)]\,dt.$$

For the third side,

$$\int_{\Gamma_3} e^{-z^2}\,dz = \int_R^{-R} e^{-(x^2+2x\beta i-\beta^2)}\,dx = e^{-\beta^2}\int_{-R}^{R} e^{-x^2}[\cos(2\beta x) - i\sin(2\beta x)]\,dx.$$

Finally, on the fourth side,

$$\int_{\Gamma_4} e^{-z^2}\,dz = \int_\beta^0 e^{-(R^2-2Rti-t^2)}i\,dt = ie^{-R^2}\int_0^\beta e^{t^2}[-\cos(2Rt) - i\sin(2Rt)]\,dt.$$

The integrals having factors of e^{-R^2} tend to zero as $R \to \infty$. Thus, upon adding these four integrals and letting $R \to \infty$, we obtain

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx - e^{\beta^2}\int_{-\infty}^{\infty}[\cos(2\beta x) - i\sin(2\beta x)]\,dx = 0.$$

Now $e^{-x^2}\sin(2\beta x)$ is an odd function on the real line, so

$$\int_{-\infty}^{\infty} e^{-x^2}\sin(2\beta x)\,dx = 0.$$

Therefore,

$$e^{\beta^2}\int_{-\infty}^{\infty} e^{-x^2}\cos(2\beta x)\,dx = \int_0^\infty e^{-x^2}\,dx.$$

Finally, use the known result that

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$$

to obtain

$$\int_{-\infty}^{\infty} e^{-x^2}\cos(2\beta x)\,dx = \sqrt{\pi}e^{-\beta^2}.$$

Finally, because the integrand is an even function, then

$$\int_0^\infty e^{-x^2}\cos(2\beta x)\,dx = \frac{\sqrt{\pi}}{2}e^{-\beta^2}.$$

16. Let Γ be the path indicated in Figure 22.3 of the text. By Cauchy's theorem,
$$\oint_\Gamma e^{iz^2}\,dz = 0.$$
Now examine the integral on the left over the three pieces of Γ consisting of the segment on the $x-$ axis (Γ_1), the circular arc (Γ_2), then the segment from the end of this arc back to the origin (Γ_3).

On Γ_1, $z = x$ and
$$\int_{\Gamma_1} e^{iz^2}\,dz = \int_0^R e^{ix^2}\,dx = \int_0^R [\cos(x^2) + i\sin(x^2)]\,dx.$$
On Γ_2, $z = Re^{i\theta}$ and
$$\int_{\Gamma_2} e^{iz^2}\,dz = \int_0^{\pi/4} e^{iR^2} e^{2i\theta}\,d\theta.$$
On Γ_3, $z = re^{i\pi/4}$, so
$$\int_{\Gamma_3} e^{iz^2}\,dz = \int_R^0 e^{-r^2} e^{i\pi/4}\,dr.$$
Notice the integration from R to 0 here to maintain counterclockwise orientation on Γ.

We want to take the limit on these integrals as $R \to \infty$. The integral over Γ_3 clearly has limit zero, because of the factor e^{-r^2} in the integral. The integral over Γ only has R in the upper limit of integration. The integral over Γ_2 is less obvious. In this integral, first make the change of variable $u = 2\theta$ to obtain
$$\int_{\Gamma_2} e^{iz^2}\,dz = \frac{1}{2}\int_0^{\pi/2} e^{iR^2\cos(u) - R^2\sin(u)} iRe^{iu/2}\,du.$$
Then
$$\begin{aligned}\Big|\int_{\Gamma_2} e^{iz^2}\,dz\Big| &\le \frac{R}{2}\int_0^{\pi/2} |e^{iR^2\cos(u)}||e^{iu/2}|e^{-R^2\sin(u)}\,du \\ &= \frac{R}{2}\int_0^{\pi/2} e^{-R^2\sin(u)}\,du \\ &\le \frac{R}{2}\left(\frac{\pi}{2\pi R^2}\right) \\ &= \frac{\pi}{4R} \to 0\end{aligned}$$
as $R \to \infty$. Thus, when we form the sum of the integrals over Γ_1, Γ_2 and Γ_3, and take the limit as $R \to \infty$, we obtain
$$\int_0^\infty [\cos(x^2) + i\sin(x^2)]\,dx - e^{i\pi/4}\int_0^\infty e^{-r^2}\,dr = 0.$$

Since we know that
$$\int_0^\infty e^{-r^2}\,dr = \frac{\sqrt{\pi}}{2},$$
and
$$e^{i\pi/4} = \frac{1}{\sqrt{2}}(1+i),$$
we obtain
$$\int_0^\infty \cos(x^2)\,dx + i\int_0^\infty \sin(x^2)\,dx = \frac{\sqrt{\pi}}{2\sqrt{2}}(1+i).$$
Equating real parts and then imaginary parts on both sides of this equation, we have Fresnel's integrals,
$$\int_0^\infty \cos(x^2)\,dx = \int_0^\infty \sin(x^2)\,dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}.$$

17. First observe that, because the integrand is an even function,
$$\int_0^\infty \frac{x\sin(\alpha x)}{x^4+\beta^4}\,dx = \frac{1}{2}\int_{-\infty}^\infty \frac{x\sin(\alpha x)}{x^4+\beta^4}\,dx.$$
Now
$$f(z) = \frac{ze^{i\alpha z}}{z^4+\beta^4}$$
has simple poles in the upper half-plane at $z_1 = \beta e^{i\pi/4}$ and $z_2 = \beta e^{3\pi i/4}$. Compute the residues of f at these poles. In general,
$$\operatorname{Res}(f, z_k) = \left[\frac{ze^{i\alpha z}}{4z^3}\right]_{z=z_k} = \frac{e^{i\alpha z_k}}{4z_k^2}.$$
Then,
$$\operatorname{Res}(f, z_1) = \frac{e^{i\alpha\beta e^{i\pi/4}}}{4\beta^2 i} \text{ and } \operatorname{Res}(f, z_2) = \frac{e^{i\alpha\beta e^{3\pi i/4}}}{-4\beta^2 i}.$$
Then
$$\begin{aligned}\int_0^\infty \frac{x\sin(\alpha x)}{x^4+\beta^4}\,dx &= \frac{1}{2}\mathrm{Im}\left[\frac{2\pi i}{4\beta^2}\left(e^{i\alpha\beta(1+i)/\sqrt{2}} - e^{i\alpha\beta(-1+i)/\sqrt{2}}\right)\frac{1}{i}\right]\\ &= \frac{\pi e^{-\alpha\beta/\sqrt{2}}}{2\beta^2}\sin\left(\frac{\alpha\beta}{\sqrt{2}}\right).\end{aligned}$$

18. First write
$$\int_0^{2\pi} \frac{1}{(\alpha+\beta\cos(\theta))^2}\,d\theta = \int_0^{\pi} \frac{1}{(\alpha+\beta\cos(\theta))^2}\,d\theta + \int_\pi^{2\pi} \frac{1}{(\alpha+\beta\cos(\theta))^2}\,d\theta.$$

In the last integral on the right, put $\theta = 2\pi - u$ to show that the two integrals on the right are equal, hence

$$\begin{aligned}\int_0^{\pi} \frac{1}{(\alpha+\beta\cos(\theta))^2}\,d\theta &= \frac{1}{2}\int_0^{2\pi} \frac{1}{(\alpha+\beta\cos(\theta))^2}\,d\theta \\ &= \frac{1}{2}\oint_{\gamma} \frac{1}{(\alpha+\beta(z+1/z)/2)^2}\frac{1}{iz}\,dz \\ &= \frac{2}{i}\oint_{\gamma} \frac{z}{(\beta z^2+2\alpha z+\beta)^2}\,dz.\end{aligned}$$

Now

$$f(z) = \frac{z}{(\beta z^2+2\alpha z+\beta)^2}$$

has double poles at the zeros of $\beta z^2 + 2\alpha z + \beta$, which are

$$z_1 = \frac{-\alpha+\sqrt{\alpha^2-\beta^2}}{\beta} \text{ and } z_2 = \frac{-\alpha-\sqrt{\alpha^2-\beta^2}}{\beta}.$$

Since z_2 is outside the unit disk, we need only the residue at z_1:

$$\begin{aligned}\operatorname{Res}(f, z_1) &= \lim_{z\to z_1}\frac{d}{dz}\left(\frac{z}{\beta^2(z-z_2)^2}\right) \\ &= \frac{1}{\beta^2}\lim_{z\to z_1}\left[\frac{(z-z_2)^2-2z(z-z_2)}{(z-z_2)^4}\right] \\ &= \frac{1}{\beta^2}\left(\frac{\alpha\beta^2}{4(\alpha^2-\beta^2)^{3/2}}\right) \\ &= \frac{\alpha}{4(\alpha^2-\beta^2)^{3/2}}.\end{aligned}$$

Then

$$\int_0^{\pi}\frac{1}{(\alpha+\beta\cos(\theta))^2}\,d\theta = \frac{2}{i}(2\pi i)\frac{\alpha}{4(\alpha^2-\beta^2)^{3/2}} = \frac{\pi\alpha}{(\alpha^2-\beta^2)^{3/2}}.$$

22.4 Residues and the Inverse Laplace Transform

1. $F(z) = z/(z^2+9)$ has simple poles at $\pm 3i$, so compute

$$\operatorname{Res}(e^{tz}F(z), 3i) = \frac{1}{2}e^{3i} \text{ and } \operatorname{Res}(e^{tz}F(z), -3i) = \frac{1}{2}e^{-3i}.$$

Then

$$\mathcal{L}^{-1}[F(s)](t) = \frac{1}{2}(e^{3i}+e^{-3i}) = \cos(3t).$$

2. $F(z) = 1/(z+3)^2$ has a double pole at -3, and

$$\operatorname{Res}(e^{tz}F(z), -3) = \lim_{z \to -3} \frac{d}{dz}(e^{tz}) = te^{-3t}.$$

Then

$$\mathcal{L}^{-1}[F(s)](t) = te^{-3t}.$$

3. Let

$$F(z) = \frac{1}{(z-2)^2(z+4)}.$$

F has a double pole at 2 and simple pole at -4. Compute

$$\begin{aligned}\operatorname{Res}(e^{tz}F(z), 2) &= \lim_{z \to 2} \frac{d}{dz}\left(\frac{e^{tz}}{z+4}\right) \\ &= \lim_{z \to 2}\left[\frac{te^{tz}}{z+4} - \frac{e^{tz}}{(z+4)^2}\right] \\ &= \frac{1}{6}te^{2t} - \frac{1}{36}e^{2t}.\end{aligned}$$

Next,

$$\operatorname{Res}(e^{tz}F(z), -4) = \frac{e^{-4t}}{36}.$$

Then

$$\mathcal{L}^{-1}[F(s)](t) = \left(\frac{1}{6}t - \frac{1}{36}\right)e^{2t} + \frac{1}{36}e^{-4t}.$$

4. Let

$$F(z) = \frac{1}{(z^2+9)(z-2)^2}.$$

F has simple poles at $\pm 3i$ and a double pole at 2. Compute

$$\begin{aligned}\operatorname{Res}(e^{tz}F(z), 3i) &= \left(\frac{2}{169} - \frac{5}{1014}i\right)e^{3it}, \\ \operatorname{Res}(e^{tz}F(z), -3i) &= \frac{1}{72+30i}e^{-3it}, \\ \operatorname{Res}(e^{tz}F(z), 2) &= \frac{1}{13}te^{2t} - \frac{4}{169}e^{2t}.\end{aligned}$$

A routine but lengthy (by hand) calculation of the sum of these residues yields

$$\begin{aligned}\mathcal{L}^{-1}[F(s)](t) = {} & \left(\frac{-4}{169} + \frac{1}{13}t\right)e^{2t} \\ & + \frac{4}{169}\cos(3t) - \frac{5}{507}\sin(3t).\end{aligned}$$

5. Let $F(z) = 1/(z+5)^3)$. Then F has a pole of order 3 at -5 and we compute
$$\mathrm{Res}(e^{tz}F(z), -5) = \frac{1}{2}t^2 e^{-5t} = \mathcal{L}^{-1}[F(s)](t).$$

6. Let $F(z) = 1/(z^3+8)$. Then F has simple poles at the cube roots of -8, which are
$$z_1 = 1+\sqrt{3}i, z_2 = -2, z_3 = 1-\sqrt{3}i.$$
Compute the residues:
$$\begin{aligned}
\mathrm{Res}(e^{tz}F(z), z_1) &= \frac{1}{-6+6i\sqrt{3}}e^{(1+\sqrt{3}i)t},\\
\mathrm{Res}(e^{tz}F(z), z_2) &= \frac{1}{12}e^{-2t},\\
\mathrm{Res}(e^{tz}F(z), z_3) &= \frac{-6-6i\sqrt{3}}{e}^{(1-\sqrt{3}i)t}.
\end{aligned}$$
Upon adding these residues and rearranging terms, we obtain
$$\mathcal{L}^{-1}[F(s)](t) = \frac{1}{12}e^{-t} + \frac{1}{12}e^{t}\left(-\cos(\sqrt{3}t) + \sqrt{3}\sin(\sqrt{3}t)\right).$$

7. Let $F(z) = 1/(1+z^4$. Then F has simple poles at the fourth roots of -1, which are
$$\begin{aligned}
z_1 &= \frac{1}{\sqrt{2}}(1+i), z_2 = \frac{1}{\sqrt{2}}(-1+i),\\
z_3 &= \frac{1}{\sqrt{2}}(1-i), z_4 = \frac{1}{\sqrt{2}}(-1-i).
\end{aligned}$$
The residues are
$$\begin{aligned}
\mathrm{Res}(e^{tz}F(z), z_1) &= \frac{1}{2\sqrt{2}(-1+i)}e^{(1+i)t/\sqrt{2}},\\
\mathrm{Res}(e^{tz}F(z), z_2) &= \frac{1}{2\sqrt{2}(1+i)}e^{(-1+i)t/\sqrt{2}},\\
\mathrm{Res}(e^{tz}F(z), z_3) &= \frac{1}{2\sqrt{2}(-1-i)}e^{(1-i)t/\sqrt{2}},\\
\mathrm{Res}(e^{tz}F(z), z_4) &= \frac{1}{2\sqrt{2}(1-i)}e^{(-1-i)t/\sqrt{2}}.
\end{aligned}$$
Upon rearranging the sum of these residues, we obtain
$$\mathcal{L}^{-1}[F(s)](t) = -\frac{1}{\sqrt{2}}\sinh\left(\frac{t}{\sqrt{2}}\right)\cos\left(\frac{t}{\sqrt{2}}\right) + \frac{1}{\sqrt{2}}\cosh\left(\frac{t}{\sqrt{2}}\right)\sin\left(\frac{t}{\sqrt{2}}\right).$$

8. By equation (3.7), we immediately have
$$\mathcal{L}^{-1}[F(s)](t) = H(t-1)e^{t-1}.$$

9. Let $F(z) = z^2/(z-2)^3$. Then F has a pole of order 3 at 2. The residue is

$$\begin{aligned}\operatorname{Res}(e^{tz}F(z), 2) &= \lim_{z\to 2} \frac{1}{2}\frac{d^2}{dz^2}(z^2e^{tz}) \\ &= \lim_{z\to 2}(2e^{2t} + 4tze^{2t} + t^2z^2e^{2t}) \\ &= (1 + 4t + 2t^2)e^{2t}.\end{aligned}$$

This is $\mathcal{L}^{-1}[F(s)](t)$.

10. Let
$$F(z) = \frac{z+3}{(z^3-1)(z+2)}.$$

F has simple poles at the cube roots of 1 and a simple pole at -2. The cube roots of -1 are

$$z_1 = 1, z_2 = \frac{1}{2}(-1+\sqrt{3}i), \frac{1}{2}(-1-\sqrt{3}i).$$

The cube roots of 1 are

$$z_1 = 1, z_2 = \frac{1}{2}(-1+\sqrt{3}i), z_3 = \frac{1}{2}(-1-\sqrt{3}i).$$

Compute the residues

$$\begin{aligned}\operatorname{Res}(e^{tz}F(z), z_1) &= \frac{4}{9}e^t, \\ \operatorname{Res}(e^{tz}F(z), z_2 &= \frac{\sqrt{3}}{18}e^{(-1+\sqrt{3}i)t/2}(-\sqrt{3}+5i), \\ \operatorname{Res}(e^{tz}F(z), z_3 &= \frac{\sqrt{3}}{18}e^{(-1-\sqrt{3}i)t/2}(\sqrt{3}+5i), \\ \operatorname{Res}(e^{tz}F(z), z=-2) &= -\frac{1}{9}e^{-2t}.\end{aligned}$$

A rearrangement of the sum of these residues yields

$$\begin{aligned}\mathcal{L}^{-1}[F(s)](t) &= -\frac{1}{9}e^{-2t} + \frac{4}{9}e^t \\ &\quad - \frac{1}{3}e^{-t/2}\cos\left(\frac{\sqrt{3}}{2}t\right) - \frac{5\sqrt{3}}{9}e^{-t/2}\sin\left(\frac{\sqrt{3}}{2}t\right).\end{aligned}$$

Chapter 23

Conformal Mappings and Applications

23.1 Conformal Mappings

In each part of Problems 1 - 3, we use the MAPLE plotting routine *conformal* to generate the image of the given rectangle under the mapping. The rectangles themselves are not shown in this plot but are easily sketched separately.

1. The mapping is $w = e^z$, which was discussed in Example 23.2. The images of the rectangles of Parts (a) through (e) are shown in Figures 23.1 - 23.5, respectively.

2. The mapping is $w\cos(z)$. Write

$$w = u + iv = \cos(x + iy) = \cos(x)\cosh(y) - i\sin(x)\sinh(y)$$

so

$$u = \cos(x)\cosh(y), v = -\sin(x)\sinh(y).$$

We will examine the image of a vertical or horizontal line under this mapping. First consider the vertical line $x = a$. An image point has the form $(\cos(a)\cosh(y) - \sin(a)\sinh(y))$. If a is not a zero of $\cos(x)$ or $\sin(x)$, then

$$\frac{u^2}{\cos^2(a)} - \frac{v^2}{\sin^2(a)} = 1.$$

This is the equation of a hyperbola in the $w-$ plane, but the image is only one branch of this hyperbola, because $\cosh(y) > 0$ for all real y.

If $a = n\pi$ for an integer n, then $\sin(a) = 0$ and the image point of a point on the line is $(\cos(n\pi)\cosh(y), 0)$, or $((-1)^n\cosh(y), 0)$. Now, $\cosh(y) \geq 1$ for all real y. Therefore, depending on whether n is even or odd, the image

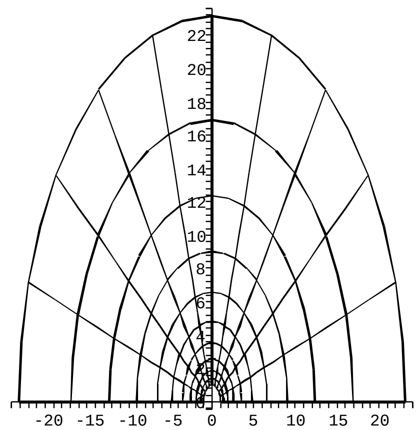

Figure 23.1: Problem 1(a).

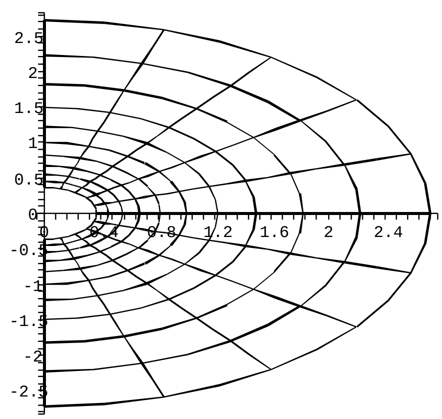

Figure 23.2: Problem 1(b).

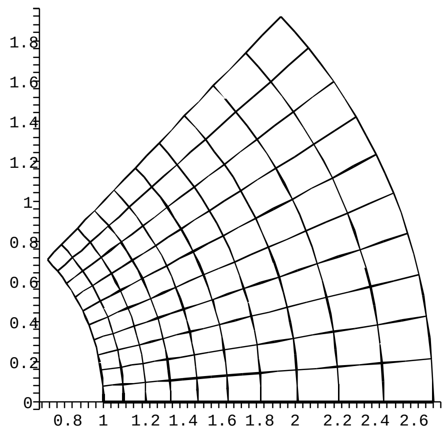

Figure 23.3: Problem 1(c).

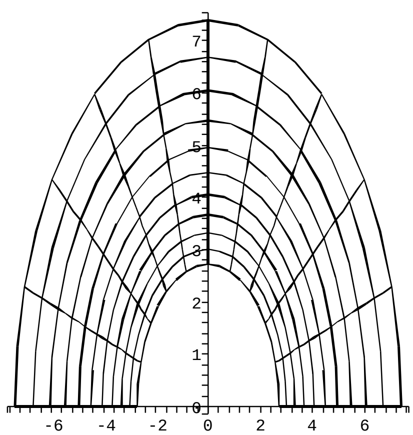

Figure 23.4: Problem 1(d).

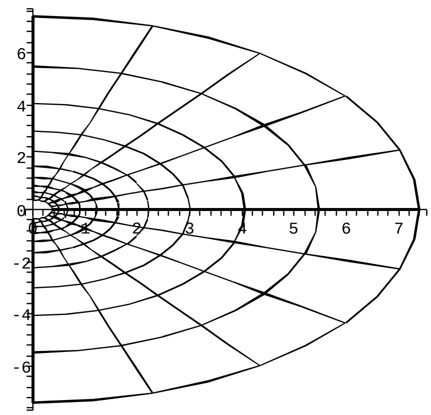

Figure 23.5: Problem 1(e).

of the line $x = n\pi$) is either the interval $[1, \infty)$ or the interval $(-\infty, -1)$ on the real axis in the $w-$ plane.

If $a = (2n + 1)\pi/2$, for n an integer, then $\cos(a) = 0$, so the image of a point on the line is $(0, -\sin((2n+1)\pi/2)\sinh(y))$. The image of the line is the imaginary axis in the $w0$ plane, since $\sinh(y)$ varies over all real values as y varies from $-\infty$ to ∞.

For a horizontal line $y = b$, if $b \neq 0$, the image of the line $y = b$ is given by points

$$w = \cos(x)\cosh(b) - i\sin(x)\sinh(b).$$

If $b \neq 0$, this is the ellipse

$$\frac{u^2}{\cosh^2(b)} + \frac{v^2}{\sinh^2(b)} = 1.$$

If $b = 0$, then $w = \cos(x)$, so w maps the line $y = b$ to the interval $[-1, 1]$ on the real axis in the $w-$ plane.

The images of the rectangles of Parts (a) through (e) are shown in Figures 23.6 - 23.10.

3. The mapping is $w = 4\sin(z)$, which was discussed in Example 23.2. The images of the specified rectangles are shown in Figures 23.11 through 23.15.

4. Write $z = re^{i\theta}$ in polar form. Then $w = z^2 = r^2e^{2i\theta}$. If r varies from 0 to ∞, so does r^2. And as θ varies from $\pi/4$ to $5\pi/4$, 2θ varies over

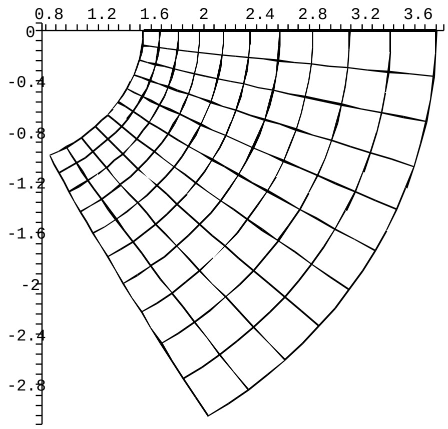

Figure 23.6: Problem 2(a).

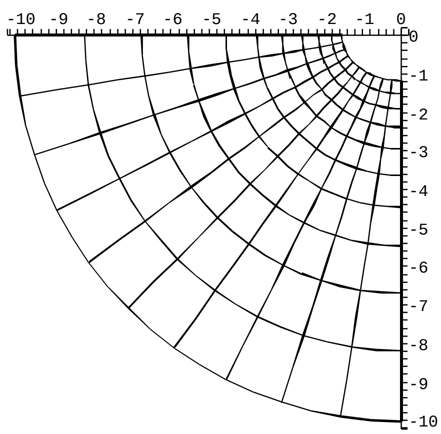

Figure 23.7: Problem 2(b).

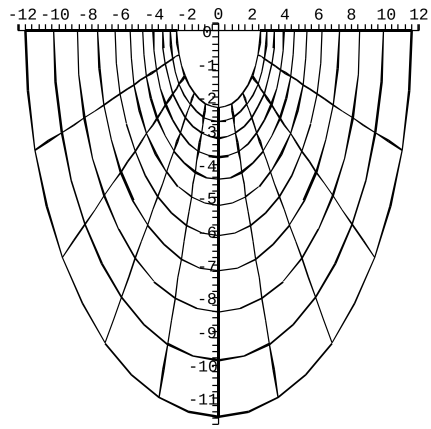

Figure 23.8: Problem 2(c).

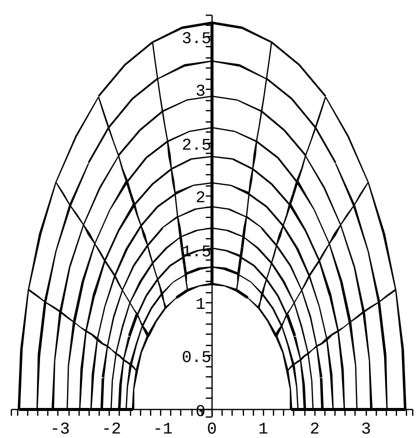

Figure 23.9: Problem 2(d).

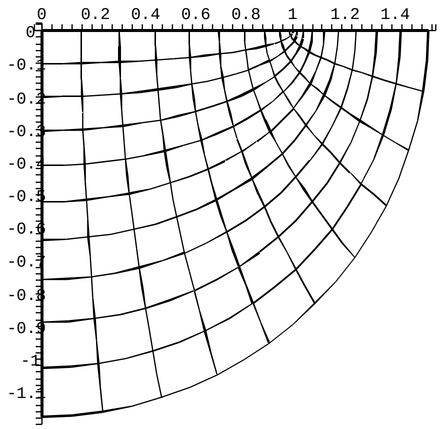

Figure 23.10: Problem 2(e).

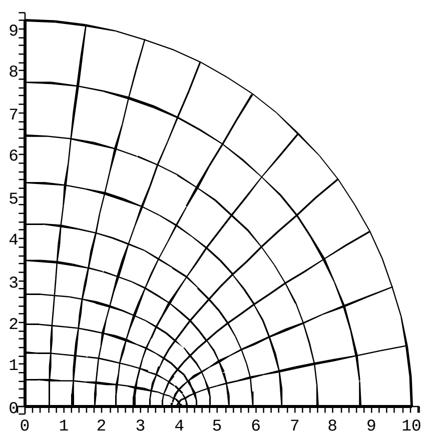

Figure 23.11: Problem 3(a).

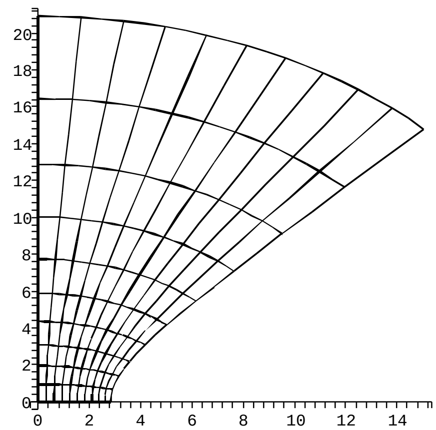

Figure 23.12: Problem 3(b).

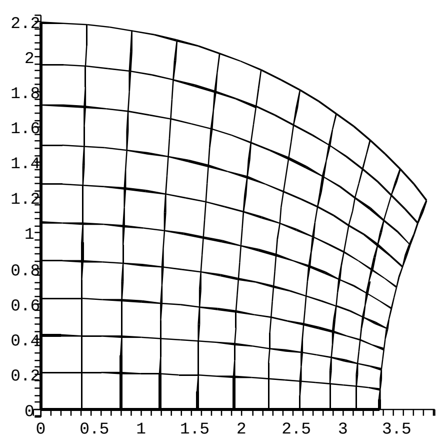

Figure 23.13: Problem 3(c).

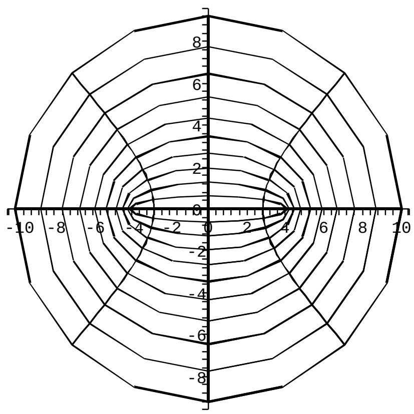

Figure 23.14: Problem 3(d).

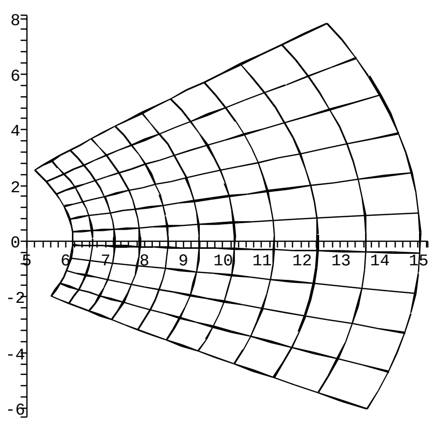

Figure 23.15: Problem 3(e).

$\pi/2$ to $5\pi/2$, an interval of length 2π. Therefore the image of the sector $\pi/4 \le \theta \le 5\pi/4$ is the entire $w-$ plane.

5. The analysis is like that of Problem 4. If $z = re^{i\theta}$, then $w = z^3 = r^3 e^{3i\theta}$. If $\pi/6 \le \theta \le \pi/3$, then $\pi/2 \le 3\theta \le \pi$. The given sector is mapped to the second quadrant in the $w-$ plane.

6. Let $z = re^{i\theta}$. Then

$$w = u + iv = \frac{1}{2}\left(re^{i\theta} + \frac{1}{r}e^{-i\theta}\right).$$

Using Euler's formula for $e^{i\theta}$ and $e^{-i\theta}$, we obtain

$$u = \frac{1}{2}\left(r + \frac{1}{r}\right)\cos(\theta), v = \frac{1}{2}\left(r - \frac{1}{r}\right)\sin(\theta).$$

Since $\sin^2(\theta) + \cos^2(\theta) = 2$, then

$$\left(\frac{u}{\frac{1}{2}(r+1/r)}\right)^2 + \left(\frac{v}{\frac{1}{2}(r-1/r)}\right)^2 = 1,$$

assuming that $r \ne 1$. This is an ellipse in the $w-$ plane. Because $r+1/r > r-1/r$, the foci are $(\pm c, 0)$, where

$$c^2 = \frac{1}{4}\left[\left(r + \frac{1}{r}\right)^2 - \left(r - \frac{1}{r}\right)^2\right] = 1.$$

This means that a circle $z = r \ne 1$ maps to an ellipse with foci $(\pm 1, 0)$ in the $w-$ plane.

If $r = 1$, so we have the unit circle about the origin in the $z-$ plane, then $v = 0$ and $u = 2\cos(\theta)$, so the image of this circle is the interval $[-2, 2]$ in the $w-$ plane.

7. Using some of the analysis done for Problem 6, a half-line $\theta = k$ maps to points $u + iv$ with

$$u = \frac{1}{2}\left(r + \frac{1}{r}\right)\cos(k), v = \frac{1}{2}\left(r - \frac{1}{r}\right)\sin(k).$$

Assuming that $\cos(k)$ and $\sin(k)$ are not zero, then a little algebraic manipulation gives us

$$\frac{u^2}{\cos^2(k)} - \frac{v^2}{\sin^2(k)} = 1$$

which is the equation of a hyperbola. The foci are $(\pm c, 0)$, where

$$c^2 = \cos^2(k) + \sin^2(k) = 1.$$

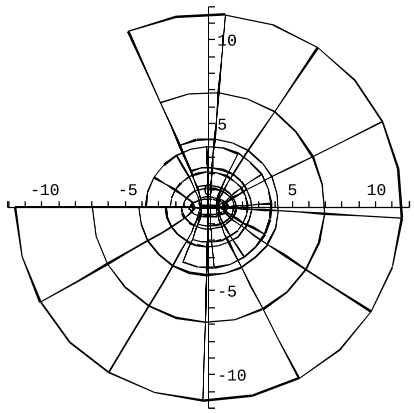

Figure 23.16: Problem 8(a).

We must separately consider the cases the $\sin(k) = 0$, so $k = n\pi$, or $\cos(k) = 0$, so $k = (2n+1)\pi/2$, for n any integer.

The case $k = n\pi$ gives us

$$u = \frac{1}{2}\left(r + \frac{1}{r}\right)(-1)^n, v = 0,$$

which is the half-interval $u \geq 1, v = 0$ if n is even and the half-interval $u \leq -1, v = 0$ if n is odd.

The case $k = (2n+1)\pi/2$ gives us $u = 0, -\infty < v < \infty$, which is the imaginary axis in the $w-$ plane.

8. (a) First let $w = \cos(z)$. We can use the analysis of the solution to Problem 2 for the images of vertical and horizontal lines. Figure 23.16 shows the image for $\alpha = 2$. Different choices of α will of course change the image.

   (b) For $w = \sin(z)$, we can use some of the analysis done in the solution to Problem 3. Figure 23.17 shows the image for $\alpha = 2$.

9. Write
$$w = 2z^2 = 2(x+iy)^2 = 2(x^2 - y^2) + 4ixy.$$

   The vertical line $x = 0$ maps to $u = -2y^2, v = 0$, which is the negative $u-$ axis. Other vertical lines $x = a$ map to parabolas

$$u = 2a^2 - \frac{v^2}{8a^2}$$

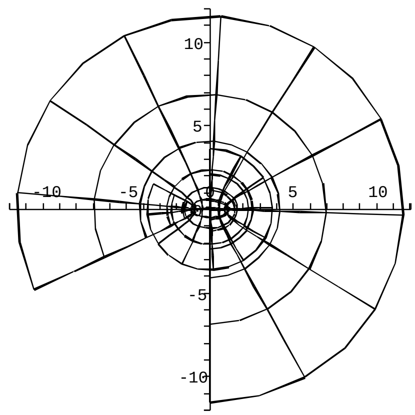

Figure 23.17: Problem 8(b).

having intercepts at $(2a^2, 0)$ and opening to the left. The horizontal line $y = 0$ maps to $u = 2y^2 \geq 0, v = 0$, the positive $u-$ axis. Other horizontal lines $y = b$ map onto the parabolas

$$u = \frac{v^2}{8b^2} - 2b^2$$

having intercepts $(-2b^2, 0)$ and opening right.

Figure 23.18 shows the image of the rectangle defined by $0 \leq x \leq 3/2, -3/2 \leq y \leq 3/2$.

10. Let $w = e^z = e^{x+iy}$ for all real x and for $0 \leq y \leq 2\pi$. If we write

$$w = e^x \cos(y) + ie^x \sin(y)$$

then the fact that y varies over an entire period of the sine and cosine functions means that every point of the $w-$ plane, except 0, is the image of a point in the $z-$ plane (let $z = \log(w)$. Thus e^z maps the $z-$ plane to the entire $w-$ plane with the origin removed.

11. If $\text{Re}(z) = -4$, then $(z + \overline{z})2 = -4$, so $z + \overline{z} = -8$. Now, if $w = 2i/z$, then $z = 2i/w$, so

$$z + \overline{z} = \frac{2i}{w} - \frac{2i}{\overline{w}} = -8.$$

Multiply this by $w\overline{w}$ and rearrange terms to obtain

$$8w\overline{w} - 2i(w - \overline{w}) = 0.$$

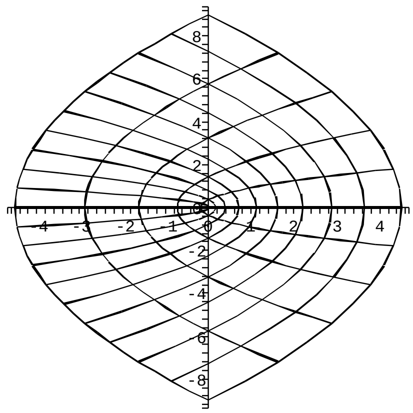

Figure 23.18: Problem 9.

Now put $w = u + iv$ to obtain

$$2(u^2 + v^2) + v = 0.$$

Complete the square to write

$$u^2 + \left(v + \frac{1}{4}\right)^2 = \frac{1}{4}.$$

This is the equation of a circle of radius $1/2$ centered at $(0, -1/4)$ in the $w-$ plane, and is the image of the vertical line $x = -4$ under the given mapping.

12. Solve $w = 2iz - 4$ to write

$$w = \frac{w+4}{2i}.$$

Now $\text{Re}(z) = (z + \overline{z})/2 = 5$ becomes

$$\frac{w+4}{2i} - \frac{\overline{w}+4}{2i} = 10.$$

Set $w = u + iv$ to obtain

$$\frac{w - \overline{w}}{2i} = \text{Im}(w) = v = 10,$$

a horizontal line in the $w-$ plane.

13. Solve the mapping for z in terms of w:

$$z = \frac{-1}{w+i}.$$

Substitute this into the given line to obtain

$$\frac{1}{2}\left(\frac{-1}{w+i} - \frac{1}{\overline{w}-i}\right) + \frac{1}{2i}\left(\frac{-1}{w+i} + \frac{1}{\overline{w}-i}\right) = 4.$$

After multiplying this by $2i(w+i)(\overline{w}-i)$ and rearranging terms, put $w = u+iv$ to obtain

$$4(u^2+v^2) + 7v + u = 3.$$

Complete the square in this equation to obtain

$$\left(u+\frac{1}{8}\right)^2 + \left(v+\frac{7}{8}\right)^2 = \frac{1}{32},$$

the equation of a circle with radius $\sqrt{2}/8$ and center $(-1/8, -7/8)$.

14. Solve the mapping for z in terms of w and set

$$|z| = 4 = \left|\frac{-w-1+i}{2w-1}\right|.$$

Then

$$|w+1-i| = 4|2w-1|.$$

Put $w = u+iv$ to

$$(u+1)^2 + (v-1)^2 = 16(2u-1)^2 + 64v^2.$$

Rearrange terms in this equation to obtain

$$\left(u-\frac{11}{21}\right)^2 + \left(v+\frac{1}{63}\right)^2 = \frac{208}{3969}.$$

This is the equation of a circle of radius $\sqrt{208/3969}$ and center $(11/21, -1/63)$.

15. Invert the mapping to obtain

$$z = \frac{5+iw}{2-w}.$$

Then

$$\begin{aligned} z - \overline{z} = 2\text{Re}(z) &= 2\text{Re}\left(\frac{5-v+iu}{2-u-iv}\right) \\ &= \frac{2((5-v)(2-u)-uv)}{(u-2)^2+v^2} = \frac{20-4v-10u}{(u-2)^2+v^2}. \end{aligned}$$

Next,

$$\frac{1}{2i}(z-\overline{z}) = \text{Im}(z) = \frac{(2-u)u+(5-v)v}{(u-2)^2+v^2}.$$

Substitute these into the equation of the given line and clear fractions to obtain

$$20-4v-10u-3(2u-u^2+5v-v^2)-5(u^2-4u+4+v^2)=0.$$

Simplify this expression and complete the square to write

$$(u-1)^2+\left(v+\frac{19}{4}\right)^2=\frac{377}{16}.$$

This is the equation of a circle of radius $\sqrt{377}/4$ and having center $(1,-19/4)$.

16. From the mapping, obtain

$$z=\frac{-2}{w-3i-1}.$$

Substitute this into $|z-i|=1$ to get

$$\left|\frac{-2}{w-3i-1}-i\right|=1,$$

or

$$|w-3i-1|=|-2-iw-3+i|.$$

Put $w=u+iv$ and simplify this expression to obtain

$$(u-1)^2+(v-3)^2=(u-1)^2+(v-5)^2,$$

from which we obtain $v=4$. The mapping is a translation followed by an inversion, and maps the given circle to a horizontal line.

17. Substitute the given values into equation (3.1) to obtain

$$(1-w)(1+2i)(-1)(3-z)=(1-z)(1)(1+i)(1+i-w).$$

Solve for w:

$$w=\frac{(1+4i)z-(3+8i)}{(2+3i)z-(4+7i)}.$$

18. Substitute the given values in equation (23.1) and solve for w to obtain

$$w=\frac{(1+i)z-(2+2i)}{(3-i)z-2}.$$

19. Since $w_3 = \infty$, substitute the given values into equation (23.1), but leave out the terms involving w_3 to obtain

$$(1 + i - w)(1 - 2i)(4 - z) = (1 - z)(-2 + 2i)(4 - 2i).$$

Solve for w:

$$w = \frac{(33 + i)z - (48 + 16i)}{5(z - 4)}.$$

20. Omitting terms in equation (23.1) that involve w_3, we obtain

$$w = \frac{(9 - 7i)z - (21 + 27i)}{13(z + 1)}.$$

21. Substitute these values into equation (23.1) and solve for w to obtain

$$w = \frac{(3 + 22i)z + 4 - 75i}{(2 + 3i)z - (21 - 4i)}.$$

22. Let f be a conformal mapping from the $z-$ to the $w-$ plane, and g a conformal mapping from the $w-$ plane to the $Z-$ plane. The $g \circ f$ is a differentiable mapping from the $z-$ plane to the $Z-$ plane.

Let C_1 and C_2 be paths in the $z-$ plane intersection at P at an angle of θ (angle between their tangents at P). Because f is conformal, $f(C_1)$ and $f(C_2)$ are paths in the $w-$ plane intersecting at $f(P)$ at an angle of θ. Because g is conformal, $g(f(C_1))$ and $g(f(C_2))$ are paths in the $Z-$ plane intersecting at the same angle θ. Therefore $g \circ f$ preserves angles.

Further, $g \circ f$ preserves orientation. If θ is measured as the angle between C_1 and C_2 going counterclockwise sense, then this sense of orientation is preserved by f, and then by g.

Therefore $g \circ f$ is conformal.

23. If we require that a conformal mapping be differentiable, then immediately $T(z) = \overline{z}$ is disqualified, because we have seen that this function is not differentiable. It is also easy to show that the conjugation mapping reverses sense of orientation. For example, let C_1 be the nonnegative real axis and C_2 the nonnegative imaginary axis in the $z-$ plane. The sense of rotation from C_1 to C_2 is counterclockwise, and the angle between these curves is $\pi/2$. However, T maps C_1 to C_1, and C_2 to the negative imaginary axis, a clockwise rotation. Therefore again T is not conformal.

24. Suppose T is a bilinear transformation that is not a translation and is not the identity mapping $T(z) = z$ that leaves every point unmoved. Write

$$T(z) = \frac{az + b}{cz + d}.$$

Then z is a fixed point of T if and only if

$$T(z) = z = \frac{az+b}{cz+d}.$$

But then

$$cz^2 + (d-a)z - b = 0.$$

This is a quadratic equation if $c \neq 0$. T has two fixed points (if this quadratic equation has distinct roots), or one fixed point (if the quadratic equation has repeated roots).

This leaves the case that $c = 0$. In this case

$$T(z) = \frac{a}{d}z + \frac{b}{d}.$$

If $b \neq 0$, then this is a translation, contrary to our assumption. Therefore in this case $b = 0$ and $T(z) = kz$, where $k = a/d$. If $a = d$, this is the identity mapping, and we assumed that it is not. Therefore $a \neq d$ and T is a magnification/rotation, which has exactly one fixed point, $z = 0$.

Therefore every bilinear transformation that is neither a translation nor the identity mapping has one or two fixed points. A translation has the form $T(z) = az + b$ with $b \neq 0$, and leaves no point unmoved. Thus a translation has no fixed point. The identity mapping $T(z) = z$ leaves every point unmoved, so every point is a fixed point.

25. Let

$$T(z) = \frac{az+b}{cz+d}.$$

If T is not a translation or the identity mapping, then by the argument used for Problem 24, T can have at most two fixed points. Therefore, if T has three fixed points, then either T is a translation or the identity mapping. But a translation has no fixed point, hence T is the identity mapping.

26. First make a preliminary observation. If

$$T(z) = \frac{az+b}{cz+d}$$

is a bilinear mapping, then $ad - bc \neq 0$, which guarantees that T has an inverse mapping. It is routine to solve $T(z) = w$ for z in terms of w to obtain the inverse transformation

$$T^{-1}(w) = \frac{-wd+b}{wc-a},$$

which is again bilinear. The composition $T^{-1} \circ T$ is the identity mapping I in the $z-$ plane, where $I(z) = z$ for all z.

Now suppose $T(z_j) = S(z_j)$ for $j = 1, 2, 3$. Then

$$\begin{aligned}(S^{-1} \circ T)(z_j) &= S^{-1}(T(z_j)) = S^{-1}(S(z_j)) \\ &= (S^{-1} \circ S)(z_j) = I(z_j = z_j\end{aligned}$$

for $j = 1, 2, 3$. Theen $S^{-1} \circ T$ has three fixed points, and is therefore the identity mapping

$$S^{-1} \circ T = I.$$

Then

$$\begin{aligned}S &= S \circ I = S \circ (S^{-1} \circ T) \\ &= (S \circ S^{-1}) \circ T = I \circ T = T.\end{aligned}$$

27. Given z_2, z_3, z_4, let P be the unique bilinear transformation that maps

$$z_2 \to 1, z_3 \to 0, z_4 \to \infty.$$

Then

$$[z_1, z_2, z_3, z_4] = P(z_1).$$

Now let T be any bilinear transformation. Then

$$[T(z_1), T(z_2), T(z_3), T(z_4)] = R(T(z_1)),$$

where R is the unique bilinear mapping that sends

$$T(z_2) \to 1, T(z_3) \to 0, T(z_4) \to \infty.$$

Then $R \circ T = P$. Then

$$\begin{aligned}&[T(z_1), T(z_2), T(z_3), T(z_4)] = R(T(z_1)) = R(T(z_1)) \\ &= P(z_1) = [z_1, z_2, z_3, z_4].\end{aligned}$$

28. Let

$$w = T(z) = 1 - \frac{z_3 - z_4}{z_3 - z_2} \frac{z - z_2}{z - z_4}.$$

A routine calculation yields

$$w_2 = T(z_2) = 1, w_3 = T(z_3) = 0, w_4 = T(z_4) = \infty.$$

Since three points and their images uniquely determine a bilinear transformation, as noted in the solution to Problem 26. T is this unique bilinear transformation, so

$$[z_1, z_2, z_3, z_4] = T(z_1).$$

29. In the definition of cross ratio, w_2, w_3, w_4 all lie on an (extended) line, the real axis. Since circles/lines map to circles/lines under bilinear transformations, then $[z_1, z_2, z_3, z_4]$ is real if and only if z_1, z_2, z_3, z_4 all lie on the same line or circle.

23.2 Construction of Conformal Mappings

There may be many conformal mappings between two given domains. In these solutions presented below, a conformal mapping is produced, together with some of the thought that went into its construction, but many other solutions are possible.

In particular, note that when circles and/or lines are involved as boundaries of domains, a bilinear transformation may serve in producing a conformal mapping. In other circumstances, we may have to construct a conformal mapping using other differentiable functions.

1. Both domains are circles, having different radii (3 and 6) and centers. Thus map $|z| < 3$ onto $|w - 1 + i| < 6$ by using a scaling factor of 2 and then a translation to superimpose the center of the initial domain onto the center of the target domain. These two effects are achieved by the bilinear transformation
$$w = 2z + 1 - i.$$

2. We can construct this mapping in three stages. First invert $|z| = 3$ by $w_1 = 1/z$. Now expand by a factor of 18 so the radii match, $w_2 = 18w_1 = 18/z$. Finally translate centers to match by $w_3 = w_2 + 1 - i$. Putting these together, we have
$$w = \frac{18}{z} + 1 - i = \frac{(1-i)z + 18}{z}.$$

3. We will need an inversion (at some stage) because we are mapping the interior of a disk to the exterior of a disk. First translate by using $w_1 = z + 2i$, so the image disk in the w_1- plane has center $(0, 0)$. Next invert by
$$w_2 = \frac{1}{z + 2i}.$$
Next scale by a factor of 2 to match radii of boundaries,
$$w_3 = 2w_2 = \frac{2}{z + 2i}.$$
Finally, translate the center by 3 to form
$$w = w_3 + 3 = \frac{2}{z + i} + 3 = \frac{3z + 2 + 6i}{z + 2i}.$$

4. A mapping of the half-plane $\mathrm{Re}(z) > 1$ onto the half-plane $\mathrm{Im}(w) > -1$ can be achieved by first rotating counterclockwise by $\pi/2$ by $w_1 = iz$, then shifting down 2 units by $w_2 = w_1 - 2i$. Thus form
$$w = iz - 2i = i(z - 2).$$
Notice that the form of this mapping suggests another mapping that will also work, namely, shift to the left by two units, then rotate by $\pi/2$ radians counterclockwise.

5. We can map the line $\mathrm{Re}(z) = 0$ onto the circle $|w| = 4$ by a bilinear transformation. The domain $\mathrm{Re}(z) < 0$ consists of all numbers to the left of the imaginary axis, which is the boundary. Choose three points on this axis, ordered upward so the region $\mathrm{Re}(z) < 0$ is on the left as we walk up the line. Choose three points on the image circle $|w| = 4$, counterclockwise so the interior of this circle is on our left as we walk around it in this order. Convenient choices are

$$z_1 = -i, z_2 = 0, z_3 = i, w_1 = -4i, w_2 = 4, w_3 = 4i.$$

The bilinear transformation mapping $z_j \to w_j$

$$w = T(z) = 4\left(\frac{1+z}{1-z}\right).$$

As a check, $z = -1$, which has negative real part, maps to 0, interior to the circle $|w| < 4$. Thus w maps $\mathrm{Re}(z) < 0$ to $|w| < 4$.

Of course, other choices for the $z_j{'s}$ and $w_j{'s}$ may result in different mappings between the two given domains.

6. The domain $\mathrm{Im}(z) > -4$ consists of all $x + iy$ lying above the horizontal line $y = -4$. This has as boundary the line $y = -4$. We want to make this domain to $|w - i| > 2$, the exterior of the circle of radius 2 centered at i. This domain has boundary $|w - i| = 2$. Choose three points on the line $y = -4$ in the $z-$ plane, ordered from left to right so that the domain is to the left. Choose three points on the circle in the $w-$ plane, clockwise so as we walk around the boundary the region (exterior to the circle) is on the left. Convenient choices are

$$z_1 = -1 - 4i, z_2 = -4i, z_3 = 1 - 4i, w_1 = 3i, w_2 = 2 + i, w_3 = -i.$$

Find the bilinear transformation mapping $z_j \to w_j$ by solving for w in the equation

$$(3i - w)(-2 - 2i)(-1)(1 - 4i - z) = (-1 - 4i - z)(-2 + 2i)(-i - w)$$

to obtain

$$w = \frac{(-2 + i)z - (3 + 10i)}{z + 3i}.$$

7. Because the boundary of the wedge in the $w-$ plane is not a line or circle, we cannot construct a bilinear mapping to solve this problem. However, wedges suggest using polar representations. Let $z = re^{i\theta}$ for $0 < \theta < \pi$. These are points in the upper half-plane. Let

$$w = z^{1/3} = r^{1/3}e^{i\theta/3} = \rho e^{i\varphi},$$

where $\rho > 0$ and $0 < \varphi < \pi/3$. This mapping is conformal because

$$\frac{dw}{dz} = \frac{1}{3}z^{-2/3} \neq 0$$

for z in the upper half-plane, and the mapping takes the open upper half-plane onto the open wedge $0 < \theta < \pi/2$.

8. Let $z = x + iy = re^{i\theta}$, with $y > 0$. Then $\arg(z) = \theta$ is unique (restricted to $0 \leq \theta < 2\pi$), and
$$w = \ln(r) + i\theta.$$
Since r can be any positive number, $\ln(r)$ varies over all real numbers. Further,
$$\mathrm{Im}(w) = \theta \text{ in } (0, \pi).$$
Thus $\log(z)$ is in the strip $0 < \mathrm{Im}(w) < \pi$.

To show that the mapping is onto, choose any $w = u + iv$ in this strip. Let $z = e^w$. Then
$$\mathrm{Im}(z) = e^u > 0$$
and
$$\log(z) = u + iv = w.$$
Thus the mapping is onto. Finally, the mapping is conformal because
$$\frac{d}{dz}(\log(z)) = \frac{1}{z} \neq 0.$$

9. The solution to this problem requires some familiarity with the gamma and beta functions, which are discussed in Section 15.3.

To show that f maps the upper half-plane onto the given rectangle, we will evaluate the function at -1, 0, 1 and ∞ and then show that these are the vertices of that rectangle.

First, it is obvious that $f(0) = 0$. Next,
$$f(1) = 2i\int_0^1 (\xi^2 - 1)^{-1/2}\xi^{-1/2}\,d\xi$$
$$2i\int_0^1 \frac{(1-\xi^2)^{-1/2}}{i}\xi^{-1/2}\,d\xi = 2\int_0^1 (1-\xi^2)^{-1/2}\xi^{-1/2}\,d\xi.$$
Let $\xi = u^{1/2}$ to obtain (in terms of the beta and gamma functions)
$$\begin{aligned} f(1) &= \int_0^1 (1-u)^{-1/2}u^{-3/4}\,du \\ &= B(1/4, 1/2) = \frac{\Gamma(1/4)\Gamma(1/2)}{\Gamma(3/4)} = c. \end{aligned}$$
Next calculate
$$f(-1) = 2i\int_0^{-1} (\xi^2 - 1)^{-1/2}\xi^{-1/2}\,d\xi.$$

Let $\xi = -u$ to obtain

$$f(-1) = 2i \int_0^1 (1-u^2)^{-1/2} u^{-1/2}\, du$$
$$= iB(1/4, 1/2) = \frac{i\Gamma(1/4)\Gamma(1/2)}{\Gamma(3/4)} = ic.$$

Finally, calculate

$$f(\infty) = 2i \int_0^\infty (\xi+1)^{-1/2}(\xi-1)^{-1/2}\xi^{-1/2}\, d\xi$$
$$= 2i \int_0^1 (\xi+1)^{-1/2}(\xi-1)^{-1/2}\xi^{-1/2}\, d\xi$$
$$+ 2i \int_1^\infty (\xi+1)^{-1/2}(\xi-1)^{-1/2}\xi^{-1/2}\, d\xi.$$

The first integral on the last line is $B(1/4, 1/2)$. In the second integral, put $\xi = 1/u$ to obtain

$$f(\infty) = c + 2i \int_1^0 \left(\frac{1+u}{u}\right)^{-1/2} \left(\frac{1-u}{u}\right)^{-1/2} u^{1/2} \left(\frac{1}{u^2}\right) du$$
$$= c + 2i \int_0^1 (1-u^2)^{-1/2} u^{-1/2}\, du = (1+i)c.$$

23.3 Conformal Mapping Solutions of Dirichlet Problems

In this section we use conformal mappings between domains to solve certain Dirichlet problems. In each case, one could use other conformal mappings than those used in these solutions.

1. Begin by mapping the upper half-plane $\mathrm{Im}(z) > 0$ to the unit disk $|w| < 1$. One such mapping is
$$w = T(z) = \frac{i-z}{i+z}.$$
The solution of this dirichlet problem for the upper half-plane is
$$u(x, y) = \mathrm{Re}(f(z)),$$
where C is the boundary of the upper half-plane (the real line) and
$$f(z) = \frac{1}{2\pi i} \int_C g(\xi) \left(\frac{T(\xi) + T(z)}{T(\xi) - T(z)}\right) \frac{T'(\xi)}{T(\xi)}\, d\xi.$$

On C, parametrize $\xi = t$ for $-\infty < t < \infty$, going from left to right to preserve positive orientation. Now all we must do is compute the quantity to be integrated. First,

$$T'(z) = \frac{-2i}{(i+z)^2}.$$

Next,

$$\begin{aligned}&\frac{T(\xi)+T(z)}{T(\xi)-T(z)}\frac{T'(\xi)}{T(\xi)}\\&= \frac{(i-t)/(i+t)+(i-z)/(i+z)}{(i-t)(i+t)-(i-t)(i+z)}\left(\frac{i+t}{i-t}\right)\left(\frac{-2i}{(i+t)^2}\right).\end{aligned}$$

After some algebra this simplifies to

$$\frac{-2(1+tz)}{(z-t)(1+t^2)}.$$

Put $z = x+iy$ to simplify this expression further to write it as

$$\frac{(1+tx)(x-t)-ty^2-iy(1+t^2)}{(x-t)^2+y^2}\frac{-2}{1+t^2}.$$

Substitute this into the integral and extract the real part, recalling that $g(t)$ is real-valued, to obtain

$$u(x,y) = \frac{y}{\pi}\int_{-\infty}^{\infty}\frac{g(t)}{(x-t)^2+y^2}\,dt.$$

This agrees with the solution of the Dirichlet problem for the upper half-plane obtained in Chapter 18.

2. The mapping

$$w = T(z) = \frac{i-z^2}{i+z^2}$$

takes the first quadrant onto the unit disk. (Note that this is not a bilinear mapping). Compute

$$\frac{T'(z)}{T(z)} = \frac{2iz}{1+z^4}.$$

The boundary of the right quarter-plane (first quadrant) consists of L_1, the nonnegative real axis, and L_2, the nonnegative imaginary axis. On L_2, $\xi = it$ for t varying from ∞ to 0 (down this axis to maintain positive orientation on the boundary of the first quadrant). Put $\xi = it$ on L_2 to compute

$$\frac{T(\xi)+T(z)}{T(\xi)-T(z)} = \frac{(i+t^2)/(i-t^2)+(i-z^2)/(i+z^2)}{(i+t^2)/(i-t^2)-(i-z^2)/(i+z^2)} = \frac{t^2z^2-1}{i(t^2+z^2)}.$$

On L_1, $\xi = t$ as t varies from 0 to ∞. Putting $xi = t$, compute

$$\frac{T(\xi)+T(z)}{T(\xi)-T(z)} = \frac{t^2z^2+1}{i(t^2-z^2)}.$$

Put these into the integral formula for the solution of the Dirichlet problem to obtain

$$u(x,y) = \mathrm{Re}\left[\frac{1}{2\pi i}\int_\infty^0 g(it)\frac{t^2z^2-1}{i(t^2+z^2)}\left(\frac{-2t}{1+t^4}\right) i\,dt + \frac{1}{2\pi i}\int_0^\infty g(t)\frac{t^2z^2+1}{i(t^2-z^2)}\left(\frac{2it}{1+t^4}\right) dt\right].$$

To determine this real part of these integrals, recall that $g(it) = g(0,t)$ and $g(t) = g(t,0)$ are both real-valued. Further, both integrals have a factor of i^2 in the denominator (including the $2\pi i$ factor). Thus each integral is left with a factor i, and we must find:

$$\mathrm{Im}\left(\frac{t^2(x+iy)^2-1}{t^2+(x+iy)^2}\right) = \mathrm{Im}\left(\frac{t^2(x^2-y^2)-1+2xyt^2i}{t^2+x^2-y^2+2xyi}\right) = \frac{2xy(1+t^4)}{(t^2+x^2+y^2)^2+4x^2y^2}$$

and, omitting some computational details,

$$\mathrm{Im}\left(\frac{t^2z^2+1}{t^2-z^2}\right) = \frac{2xy(1+t^4)}{t^2-x^2+y^2+4xy}.$$

We therefore obtain the solution

$$u(x,y) = \frac{2xy}{\pi}\int_0^\infty \frac{tg(0,t)}{(t^2+x^2-y^2)^2+4x^2y^2}\,dt + \frac{2xy}{\pi}\int_0^\infty \frac{tg(t,0)}{t^2-x^2+y^2+4x^2y^2}\,dt.$$

3. The bilinear mapping

$$w = T(z) = \frac{1}{R}(z-z_0)$$

takes the disk $|z-z_0| < R$ to the unit disk $|w| < 1$. On C, the boundary of $|z-z_0| < R$, we can write $\xi = z_0 + Re^{it}$ as t varies from 0 to 2π. Compute

$$\frac{T(\xi)+T(z)}{T(\xi)-T(z)}\,d\xi = \frac{Re^{it}+(z-z_0)}{Re^{it}-(z-z_0)}\,\frac{ie^{it}}{e^{it}}\,dt.$$

Since $g(\xi) = g(z_0 + Re^{it})$ is real-valued, we can write the solution

$$u(x, y) = \frac{1}{2\pi} \int_0^{2\pi} g(x_0 + R\cos(t), y_0 + R\sin(t))K(x, y, t)\, dt,$$

where

$$K(x, y, t) = \operatorname{Re}\left[\frac{R\cos(t) + x - x_0 + i(R\sin(t) + y - y_0)}{R\cos(t) - x + x_0 + i(R\sin(t) - y + y_0)}\right]$$
$$= \frac{R^2 - (x - x_0)^2 - (y - y_0)^2}{R^2 + (x - x_0)^2 + (y - y_0)^2 - 2R(x - x_0)\cos(t) - 2R(y - y_0)\sin(t)}.$$

4. From Example 23.19, the integral solution for the right half-plane is

$$u(x, y) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{xg(it)}{x^2 + (t - y)^2}\, dt.$$

Substituting in the given boundary function, we obtain

$$u(x, y) = \frac{x}{\pi} \int_{-1}^{1} \frac{1}{x^2 + (t - y)^2}\, dt.$$

5. Use Poisson's integral formula to obtain

$$u(r\cos(\theta), r\sin(\theta)) = \frac{1}{2\pi} \int_0^{2\pi} \frac{r(\cos(\varphi) - \sin(\varphi))(1 - r^2)}{1 + r^2 - 2r\cos(\varphi - \theta)}\, d\varphi.$$

6. By Poisson's formula,

$$u(r\cos(\theta), r\sin(\theta)) = \frac{1}{2\pi} \int_0^{\pi/4} \frac{1 - r^2}{1 + r^2 - 2r\cos(\varphi - \theta)}\, d\varphi.$$

7. First construct a conformal mapping of the strip S onto the unit circle. Begin with $w_1 = \pi iz/2$, which rotates the strip $\pi/2$ radians counterclockwise and expands it to the strip $-\pi/2 \leq \operatorname{Re}(w_1) \leq \pi/2$. The reason for doing this is to exploit the mapping of Example 23.3. From this, put $w_2 = \sin(w_1)$. This maps the w_1- strip onto the upper half-plane. Finally, find the bilinear mapping that maps

$$-1 \to -i, 0 \to 1, 1 \to i$$

to obtain

$$w = \frac{i - w_2}{i + w_2},$$

mapping the upper half-plane of the w_2- plane to the unit disk in the $w-$ plane. The end result of this sequence of mappings is

$$w = T(z) = \frac{i - \sin(\pi iz/2)}{i + \sin(\pi iz/2)}.$$

We can write this as

$$w = \frac{1 - \sinh(\pi z/2)}{1 + \sinh(\pi z/2)}.$$

The solution of the Dirichlet problem is

$$u(x, y) = \mathrm{Re}\left[\frac{1}{2\pi i}\int_C g(\xi)\left(\frac{T(\xi) + T(z)}{T(\xi) - T(z)}\right)\frac{T'(\xi)}{T(\xi)}\,d\xi\right].$$

Since $g(\xi) = 0$ along the upper and lower edges of S, the solution simplifies to

$$u(x, y) = \mathrm{Re}\left[\frac{1}{2\pi i}\int_K g(\xi)\left(\frac{T(\xi) + T(z)}{T(\xi) - T(z)}\right)\frac{T'(\xi)}{T(\xi)}\,d\xi\right],$$

where K is the segment of the imaginary axis from i to $-i$. On K,

$$g(\xi) = g(it) = g(0, t) = 1 - |t|$$

and

$$T(\xi) = T(it) = \frac{i + \sin(\pi t/2)}{i - \sin(\pi t/2)}.$$

We need to compute

$$\frac{T'(it)}{T(it)}\,d(it) = \frac{\pi\cos(\pi t/2)}{1 + \sin^2(\pi t/2)}\,dt,$$

and

$$\frac{T(\xi) + T(z)}{T(\xi) - T(z)} = \frac{|T(\xi)|^2 + 2i\mathrm{Im}(T(z)\overline{T(\xi)}) + |T(z)|^2}{|T(\xi)|^2 - 2\mathrm{Re}(T(z)\overline{T(\xi)}) + |T(z)|^2}.$$

Now $|T(\xi)|^2 = 1$, since T maps the boundary of S onto the unit circle $|w| = 1$. Finally, we can write the solution

$$u(x, y) = \int_1^{-1}\frac{(1 - |t|)\cos(\pi t/2)}{1 + \sin^2(\pi t/2)}\,\frac{\mathrm{Im}(T(z)\overline{T(it)})}{1 - 2\mathrm{Re}(T(z)\overline{T(it)}) + |T(z)|^2}\,dt.$$

23.4 Models of Plane Fluid Flow

1. Write $a = Ke^{i\theta}$ and $z = x + iy$ to compute

$$\begin{aligned} f(z) = az &= Ke^{i\theta}(x + iy) \\ &= K[x\cos(\theta) - y\sin(\theta)] + iK[x\sin(\theta) + y\cos(\theta)]. \end{aligned}$$

With $f(z) = \varphi(x, y) + i\psi(x, y)$, we can identify equipotential curves as graphs of

$$\varphi(x, y) = K[x\cos(\theta) - y\sin(\theta)] = \text{ constant}$$

and streamlines as graphs of

$$\psi(x, y) = K[x\sin(\theta) + y\cos(\theta)] = \text{ cosntant}.$$

Since θ is a given constant, the equipotential lines are straight lines

$$y = \cot(\theta)x + b$$

having slope $\cot(\theta)$, while the streamlines are straight ines

$$y = -\tan(\theta)x + c,$$

of slope $-\tan(\theta)$. The equipotential lines and streamlines form orthogonal families, since $-\cot(\theta)\tan(\theta) = -1$, so the slopes of equipotential lines and streamlines are negative reciprocals of each other.

The velocity is

$$V(z) = V(x, y) = \overline{f'(z)} = \overline{a} = Ke^{-i\theta},$$

a constant velocity. Since $f'(z) \neq 0$, there are no stagnation points, hence no source or sink.

2. Write

$$f(z) = z^3 = (x + iy)^3 = (x^3 - 3xy^2) + i(3x^2y - y^3).$$

Then

$$\varphi(x, y) = x^3 - 3xy^2 \text{ and } \psi(x, y) = 3x^2y - y^3.$$

Equipotential curves are graphs of curves

$$x^3 - 3xy^2 = c$$

and streamlines are graphs of curves

$$3x^2y - y^3 = k.$$

If $c = 0$, then $x = 0$ (the $y-$ axis) or $y = \pm(1/\sqrt{3})x$. These lines divide the plane into six wedge-shaped regions meeting at the origin. Equipotential curves occur in these regions and are asymptotic to its boundary lines. Figure 23.19 shows some of these equipotential curves, and Figure 23.20 some streamlines.

Note that the streamlines can be obtained by rotating equipotential curves $\pi/2$ radians clockwise.

The velocity of this flow is

$$\begin{aligned} V(x, y) &= \overline{f'(z)} \\ &= \overline{3z^2} = 3(x^2 - y^2) - 6xyi = u(x, y) + iv(x, y) \end{aligned}$$

with

$$u(x, y) = 3(x^2 - y^2) \text{ and } v(x, y) = -6xy.$$

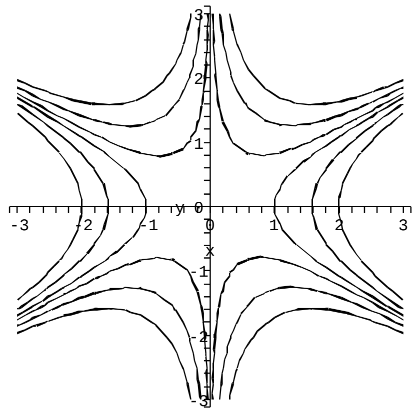

Figure 23.19: Equipotential curves in Problem 2.

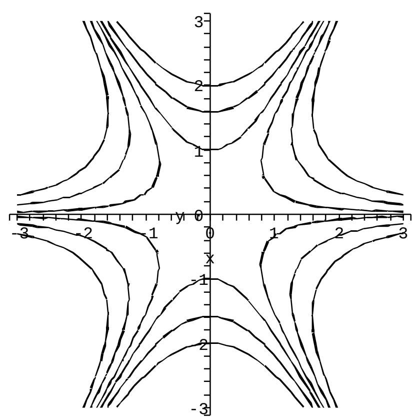

Figure 23.20: Streamlines in Problem 2.

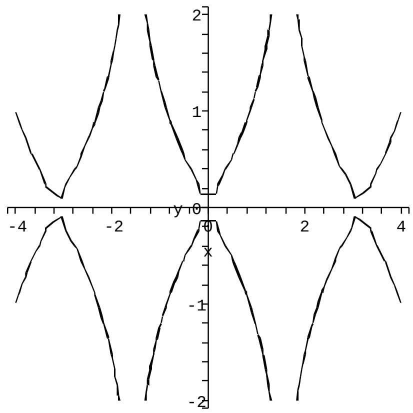

Figure 23.21: Equipotential curves in Problem 3.

Since $f'(0) = 0$, the origin is a stagnation point. The flow is irrotational because the divergence of the velocity is zero. The flow is also solenoidal. To see this, use Green's theorem to calculate

$$\oint_{|z|=r} -v(x,y)\,dx + u(x,y)\,dy = \iint_{|z|\leq r} (-6y + 6y)\,dA = 0.$$

The origin is neither a source nor a sink. Finally, on $|z| = r$,

$$|\mathbf{V}(x,y)| = 3r^2$$

so the velocity is increasing with distance from the origin. We can envision the potential $f(z) = z^3$ as describing fluid motion along the streamlines, with the straight lines $y = \pm(1/\sqrt{3})x$ and $y = 0$ acting as barriers of the flow (such as sides of a container). As fluid particles near the origin they slow down, and speed up again as they move away from the origin.

3. Begin with

$$f(z) = \cos(z) = \cos(x)\cosh(y) - i\sin(x)\sinh(y) = \varphi(x,y) + i\psi(x,y).$$

Equipotential curves are graphs of $\cos(x)\cosh(y) = c$ (Figure 23.21) and streamlines (Figure 23.22) are graphs of $\sin(x)\sinh(y) = k$.

Since $f'(z) = -\sin(z) = 0$ if $z = n\pi$, with n any integer, this flow has infinitely many stagnation points.

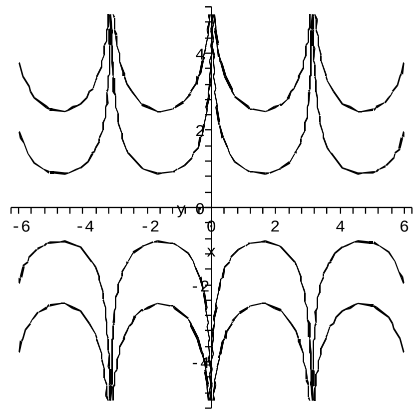

Figure 23.22: Streamlines in Problem 3.

The velocity is

$$V(x,y) = \overline{f'(z)} = \overline{-\sin(z)}$$
$$= -\sin(x)\cosh(y) + i\cos(x)\sinh(y) = u(x,y) + iv(x,y).$$

Then

$$u(x,y) = -\sin(x)\cosh(y), v(x,y) = \cos(x)\sinh(y).$$

This has divergence zero. Further, using Green's theorem, it is routine to check that the flux of the flow across any closed path is zero, so the flow is solenoidal.

The circulation is also zero about any closed path, so there is no source or sink for this flow.

4. First write

$$f(z) = z + iz^2 = (x - 2xy) + i(y + x^2 - y^2),$$

so

$$\varphi(x,y) = x - 2xy \text{ and } \psi(x,y) = y + x^2 - y^2.$$

Equipotential lines (Figure 23.23) are graphs of curves

$$x - 2xy = c$$

and streamlines (Figure 23.24) are graphs of curves

$$y + x^2 - y^2 = k.$$

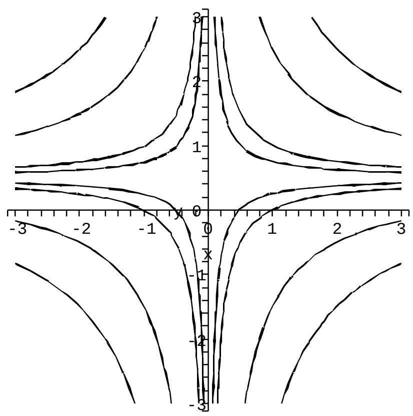

Figure 23.23: Equipotential curves in Problem 4.

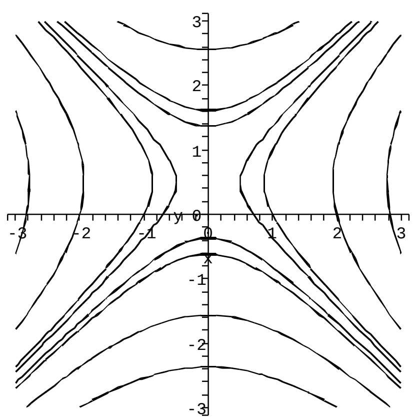

Figure 23.24: Streamlines in Problem 4.

For the velocity, compute

$$\begin{aligned} V(x,y) &= \overline{f'(z)} = \overline{1+2iz} \\ &= (1-2y) - 2xi = u(x,y) + iv(x,y). \end{aligned}$$

Since u and v satisfy the Cauchy-Riemann equations, the flow is both irrotational and solenoidal. Further, $f'(z) = 0$ if $z = i/2$, so $(0, 1/2)$ is a stagnation point.

On circles $|z - 1/2| = r$,

$$|f'(z)| = 2\sqrt{x^2 + (y-1/2)^2} = 2r,$$

so the velocity decreases near the stagnation point. We can envision the flow as fluid confined to one of the regions between lines $y = 1/2 \pm x$, with fluid motion along hyperbolic streamlines.

5. Write

$$f(z) = K\log(z - z_0) = K\ln|z - z_0| + iK\arg(z - z_0),$$

so equipotential curves are graphs of

$$\varphi(x,y) = K\ln|z - z_0| = c,$$

which are concentric circles about z_0, and streamlines are graphs of

$$\psi(x,y) = iK\arg(z - z_0) = k.$$

These are half-lines emanating from z_0.

This flow has no stagnation points. The velocity is

$$\overline{f'(z)} = \frac{\overline{K}}{|z - z_0|^2}\,(x - x_0 + i(y - y_0)) = u(x,y) + iv(x,y).$$

For any circle $\gamma : |z - z_0| = r$, compute

$$\begin{aligned} &\oint_\gamma -v\,dx + u\,dy \\ &= \int_0^{2\pi} \left[-\frac{K}{r^2}(r\sin(t))(-r\sin(t)) + \frac{K}{r^2}(r\cos(t))(r\cos(t)) \right] dt \\ &= 2\pi K. \end{aligned}$$

Therefore z_0 is a source if $K > 0$ and a sink if $K < 0$.

6. Write

$$\begin{aligned} f(z) &= K\mathrm{Log}\left(\frac{z-a}{z-b}\right) \\ &= \frac{K}{2}\ln\left(\frac{|z|^2 + |a|^2 - 2\mathrm{Re}(a\overline{z})}{|z|^2 + |b|^2 - 2\mathrm{Re}(b\overline{z})}\right) + iK\arg\left(\frac{z-a}{z-b}\right) \\ &= \varphi(x,y) + i\psi(x,y). \end{aligned}$$

To analyze the equipotential curves, let $a = a_1 + ia_2$, $b = b_1 + ib_2$ and $z = x + iy$. An equipotential curve $\varphi(x, y) = c$ is the graph of

$$(x^2 + y^2)(1 - c) - 2(a_1 x + a_2 y - c(b_1 x + b_2 y)) + a_1^2 + a_2^2 - c(b_1^2 + b_2^2) = 0.$$

If $c = 1$, this is the line

$$(a_1 - b_1)x + (a_2 - b_2)y = \frac{1}{2}[(a_1^2 + a_2^2) - (b_1^2 + b_2^2)].$$

If $c \neq 1$, we get an equation

$$\left(x - \frac{a_1 - cb_1}{1 - c}\right)^2 + \left(y - \frac{a_2 - cb_2}{1 - c}\right)^2 = r^2,$$

where

$$r^2 = \frac{c}{(1 - c)^2}[(a_1 - b_1)^2 + (a_2 - b_2)^2].$$

These are circles if $c > 0$. Notice that the centers of these circles all lie on the line

$$(a_2 - b_2)x - (a_1 - b_1)y + a_1 b_2 - a_2 b_1 = 0.$$

This line connects $a = a_1 + a_2 i$ and $b = b_1 + b_2 i$ in the complex plane. This line containing the centers of the equipotential curves (for $c \neq 1$) is orthogonal to the equipotential curve obtained when $c = 1$, and these two lines intersect at $((a_1 + b_1)/2, (a_2 + b_2)/2)$, which is the midpoint of the segment connecting a and b in the complex plane.

For the streamlines, write

$$\begin{aligned}
\frac{z - a}{z - b} &= \frac{(z - a)(\overline{z} - \overline{b})}{|z - b|^2} \\
&= \frac{|z|^2 - (a\overline{z} + \overline{b}z) + a\overline{b}}{|z - b|^2} \\
&= \frac{x^2 + y^2 - [(a_1 + b_1)x + (a_2 + b_2)y] + a_1 b_1 + a_2 b_2}{(x - b_1)^2 + (y - b_2)^2} \\
&\quad + i\frac{a_2 b_1 - a_1 b_2 - x(a_2 - b_2) + y(a_1 - b_1)}{(x - b_1)^2 + (y - b_2)^2}.
\end{aligned}$$

A streamline $\arg((z - a)/(z - b)) = k$ has the form

$$\frac{a_2 b_1 - a_1 b_2 - (a_2 - b_2)x + y(a_1 - b_1)}{x^2 + y^2 - [(a_1 + b_1)x + (a_2 + b_2)y] + a_1 b_1 + a_2 b_2} = k.$$

If $k = 0$ we obtain the line

$$(a_2 - b_2)x - (a_1 - b_1)y + a_1 b_2 - a_2 b_1 = 0,$$

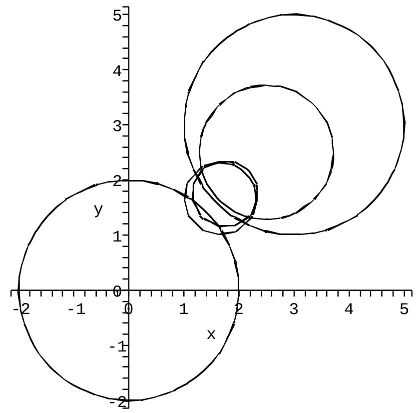

Figure 23.25: Equipotential curves in Problem 6.

which connects a to b in the complex plane. If $k \neq 0$, we obtain

$$\left(x + \frac{a_2 - b_2 - k(a_1 + b_1)}{2k}\right)^2 + \left(y - \frac{a_1 - b_1 + k(a_2 + b_2)}{2k}\right)^2 = r^2,$$

where

$$r^2 = \frac{1 + k^2}{4k^2}\left[(a_1 - b_1)^2 + (a_2 - b_2)^2\right].$$

This is the equation of a circle of radius r. The centers of these circles lie on the line

$$(a_1 - b_1)x + (a_2 - b_2)y - \frac{1}{2}[(a_1 - b_1)^2 + (a_2 - b_2)^2] = 0.$$

This is the perpendicular bisector of the segment between a and b, and each circle passes through both a and b.

Figure 23.25 shows some equipotential curves for the case $a = 1 + i$ and $b = 2 + 2i$. Now these curves are are circles with centers on the line $y = x$. Figure 23.26 shows some streamlines for this case. Centers of the streamlines lie on the perpendicular bisector of the line $y = x$ between $1 + i$ and $2 + 2i$.

7. Write

$$\begin{aligned} f(z) &= K\left(x + iy + \frac{1}{x + iy}\right) \\ &= \frac{Kx(x^2 + y^2 + 1)}{x^2 + y^2} + i\frac{Ky(x^2 + y^2 - 1)}{x^2 + y^2}. \end{aligned}$$

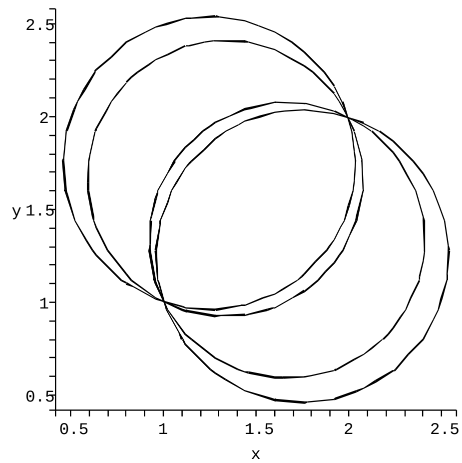

Figure 23.26: Streamlines in Problem 6.

Equipotential curves are graphs of

$$\varphi(x, y) = \frac{Kx(x^2 + y^2 + 1)}{x^2 + y^2} = c_1$$

Some equipotential curves are shown in Figure 23.27 for $K = 1$.

Streamlines are graphs of

$$\psi(x, y) = \frac{Ky(x^2 + y^2 - 1)}{x^2 + y^2} = c_2.$$

For $c_1 = 0$, we get the equipotential curve $x = 0$, the imaginary axis. For $c_1 \neq 0$, set $c_1 = kb$ we can write

$$y^2 = -\frac{x(x^2 - bx + 1)}{x - b}.$$

Figure 23.28 shows some streamlines for $K = 1$.

The velocity of the flow is

$$\overline{f'(z)} = K\left(1 - \frac{1}{\overline{z}^2}\right).$$

There is a stagnation point at $z = 1$ and at $z = -1$.

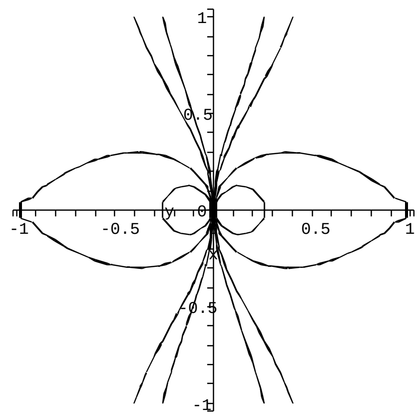

Figure 23.27: Equipotential curves in Problem 7.

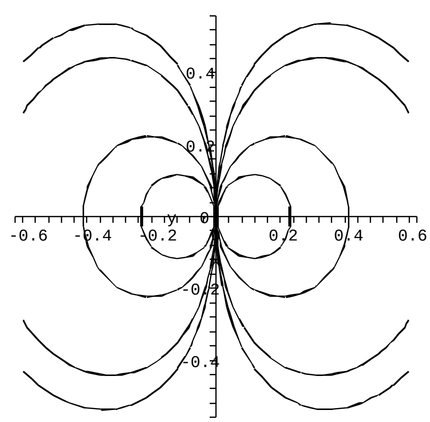

Figure 23.28: Streamlines in Problem 7.

8. Using part of the solution of Problem 6, we can write

$$\begin{aligned} f(z) &= \frac{m-ik}{2\pi}\left[\frac{|z|^2+|a|^2-2\mathrm{Re}(a\overline{z})}{|z|^2+|b|^2-2\mathrm{Re}(b\overline{z})}+i\arg\left(\frac{z-a}{z-b}\right)\right] \\ &= \varphi(x,y)+i\psi(x,y). \end{aligned}$$

Equipotential curves are graphs of

$$\varphi(x,y)=\frac{m}{2\pi}\left[\frac{|z|^2+|a|^2-2\mathrm{Re}(a\overline{z})}{|z|^2+|b|^2-2\mathrm{Re}(b\overline{z})}\right]+\frac{k}{2\pi}\arg\left(\frac{z-a}{z-b}\right)=c_1$$

and streamlines are graphs of

$$\psi(x,y)=\frac{m}{2\pi}\arg\left(\frac{z-a}{z-b}\right)-\frac{k}{2\pi}\left[\frac{|z|^2+|a|^2-2\mathrm{Re}(a\overline{z})}{|z|^2+|b|^2-2\mathrm{Re}(b\overline{z})}\right]=c_2.$$

Compute

$$f'(z)=\frac{m-ik}{2\pi}\left(\frac{a-b}{(z-a)(z-b)}\right).$$

Since the velocity is

$$V(x,y)=\overline{f'(z)}=u(x,y)+iv(x,y),$$

then

$$f'(z)=u(x,y)-iv(x,y)$$

so

$$f'(z)\,dz=(u-iv)(dx+i\,dy)=(u\,dx+v\,dy)+i(-v\,dx+u\,dy).$$

Therefore, for any closed path γ,

$$\oint_\gamma f'(z)\,dz=\oint_\gamma(u\,dx+v\,dy)+i\oint_\gamma(-v\,dx+u\,dy).$$

The integral on the left is easily evaluated using the residue theorem. First write

$$\oint_\gamma f'(z)\,dz=2\pi i\left(\frac{m-ik}{2\pi}\right)\sum\mathrm{Res}\left(\frac{a-b}{(z-a)(z-b)}\right),$$

where the sum is over residues at the poles enclosed by γ. The integrand has simple poles at a and b, and

$$\mathrm{Res}\left(\frac{a-b}{(z-a)(z-b)},a\right)=1,\mathrm{Res}\left(\frac{a-b}{(z-a)(z-b)},b\right)=-1.$$

Now consider cases.

If γ encloses only a, then

$$\oint_\gamma (u\,dx + v\,dy) + i\oint_\gamma (-v\,dx + u\,dy) = k + im,$$

so $z = a$ is a source of strength m and a vortex of strength k.

If γ encloses only b, then

$$\oint_\gamma (u\,dx + v\,dy) + i\oint_\gamma (-v\,dx + u\,dy) = -k - im,$$

so b is a sink of strength m and a vortex of strength $-k$.

9. Let $z = x + iy$ to obtain

$$\begin{aligned} f(z) &= K\left(x + iy + \frac{1}{x+iy}\right) + \frac{ib}{2\pi}\mathrm{Log}(x+iy) \\ &= \frac{Kx(x^2+y^2+1)}{x^2+y^2} - \frac{b}{2\pi}\arg(x+iy) \\ &\quad + i\left[\frac{Ky(x^2+y^2-1)}{x^2+y^2} + \frac{b}{4\pi}\ln(x^2+y^2)\right]. \end{aligned}$$

Equipotential curves are graphs of

$$\varphi(x,y) = \frac{Kx(x^2+y^2+1)}{x^2+y^2} - \frac{b}{2\pi}\arg(x+iy) = c_1.$$

Streamlines are graphs of

$$\psi(x,y) = \frac{Ky(x^2+y^2-1)}{x^2+y^2} + \frac{b}{4\pi}\ln(x^2+y^2) = c_2.$$

Some equipotential lines are shown in Figure 23.29 for $K = 1$ and $b = 2\pi$. Streamlines are shown in Figure 23.30 for $K = 1$ and $b = 4\pi$.

Compute

$$f'(z) = K\left(1 - \frac{1}{z^2}\right) + \frac{ib}{2\pi z} = \frac{1}{z^2}\left[kz^2 + \frac{ib}{2\pi}z - k\right].$$

Stagnation points occur where $f'(z) = 0$. These points are

$$z = -\frac{ib}{4\pi k} \pm \sqrt{1 - \frac{b^2}{16\pi^2K^2}}.$$

These points lie on the unit circle symmetrically across the imaginary axis.

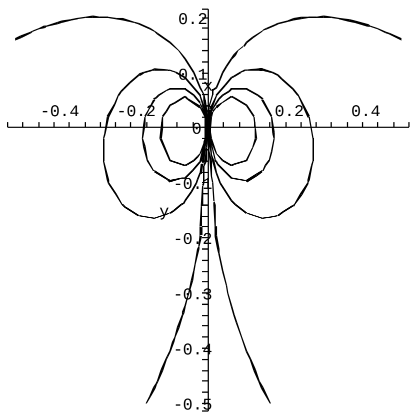

Figure 23.29: Equipotential curves in Problem 9.

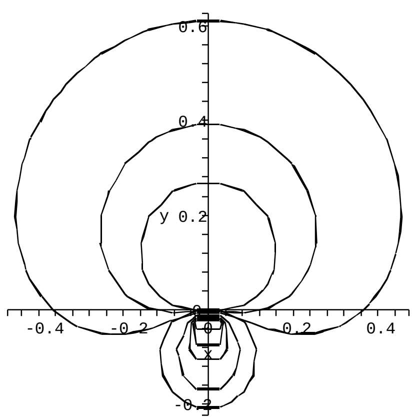

Figure 23.30: Streamlines in Problem 9.

10. Because for this Log function, the argument is restricted to $-\pi \leq \arg(z) < \pi$, we can write

$$f(z) = iKa\sqrt{3}\left[\mathrm{Log}\left(z - \frac{ia\sqrt{3}}{2}\right) - \mathrm{Log}\left(z + \frac{ia\sqrt{3}}{2}\right)\right].$$

Compute

$$f'(z) = iKa\sqrt{3}\frac{ia\sqrt{3}}{(z - ia\sqrt{3}/2)(z + ia\sqrt{3}/2)} = \frac{-3Ka^2((\overline{z})^2 + 3a^2/4)}{(z^2 + 3a^2/4)((\overline{z})^2 + 3a^2/4)}.$$

Parametrize the boundary of $4x^2 + 4(y-a)^2 = a^2$ by setting

$$x = \frac{a}{2}\cos(\theta), y = a + \frac{a}{2}\sin(\theta) \text{ for } 0 \leq \theta \leq 2\pi.$$

Then

$$f'(z(\theta)) = \frac{6K[\sin(\theta) + i\cos(\theta)]}{2 + \sin(\theta)}.$$

Then

$$\overline{f'(z(\theta))} = u(x(\theta), y(\theta)) + iv(x(\theta), y(\theta)),$$

where

$$u(x(\theta), y(\theta)) = \frac{6K\sin(\theta)}{2 + \sin(\theta)}, v(x(\theta), y(\theta)) = -\frac{6K\cos(\theta)}{3 + \sin(\theta)}.$$

Now compute the circulation of the flow about a closed curve γ:

$$\begin{aligned}
&\oint_\gamma u\,dx + v\,dy \\
&= \int_0^{2\pi}\left(\frac{6K\sin(\theta)}{2 + \sin(\theta)}\left[-\frac{a}{2}\sin(\theta)\right] - \frac{6K\cos(\theta)}{2 + \sin(\theta)}\left[\frac{a}{2}\cos(\theta)\right]\right)d\theta \\
&= -3Ka\int_0^{2\pi}\frac{1}{2 + \sin(\theta)}\,d\theta < 0,
\end{aligned}$$

because the integral is positive. Since the circulation of the flow about $(0, a)$ is not zero, the flow is not irrotational.

Next compute the flux

$$\begin{aligned}
&\oint_\gamma -v\,dx + u\,dy \\
&= \int_0^{2\pi}\left\{\frac{6K\cos(\theta)}{2 + \sin(\theta)}\left[-\frac{a}{2}\sin(\theta)\right] + \frac{6K\sin(\theta)}{2 + \sin(\theta)}\left[\frac{a}{2}\cos(\theta)\right]\right\}d\theta \\
&= 0.
\end{aligned}$$

Therefore the flow is solenoidal.

11. From the solution to Problem 10, we have

$$(f'(z))^2 = \frac{9a^4K^2}{(z - ia\sqrt{3}/2)^2(z + ia\sqrt{3}/2)^2}.$$

By Blasius's theorem, the thrust of the fluid outside the barrier $4x^2 + 4(y-a)^2 = a^2$ is the vector $A\mathbf{i} + B\mathbf{j}$, where

$$\begin{aligned}
A - Bi &= \frac{1}{2} i\rho \oint_\gamma (f'(z))^2 \, dz \\
&= \frac{i\rho}{2} \oint_\gamma \frac{9a^4K^2}{(z - ia\sqrt{3})/2^2(z + ia\sqrt{3}/2)^2} \, dz \\
&= \pi\rho \operatorname{Res}\left((f'(z))^2, ia\sqrt{3}/2\right) \\
&= -\pi\rho(9a^4K^2)\frac{d}{dz}\left[\left(z + \frac{ia\sqrt{3}}{2}\right)^{-2}\right]_{z=ia\sqrt{3}/2} \\
&= -9\pi a^4K^2\rho(-2(ia\sqrt{3})^{-3}) \\
&= -\frac{18\pi a^4K^2\rho}{3\sqrt{3}a^3} i.
\end{aligned}$$

The vertical component of the thrust is

$$B = 2\sqrt{3}\pi a\rho K^2.$$